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A census of highly symmetric combinatorial designs

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Abstract As a consequence of the classification of the finite simple groups, it has been possible in recent years to characterize Steiner t-designs, that is t-$(v,k,1)$ designs, mainly for $t=2$, admitting groups of automorphisms with sufficiently strong symmetry properties. However, despite the finite simple group classification, for Steiner t-designs with $t>2$ most of these characterizations have remained long-standing challenging problems. Especially, the determination of all flag-transitive Steiner t-designs with $3 \leq t \leq 6$ is of particular interest and has been open for about 40 years (cf. Delandtsheer (Geom. Dedicata **41**, p. 147, 1992 and Handbook of Incidence Geometry, Elsevier Science, Amsterdam, 1995, p. 273), but presumably dating back to 1965).

The present paper continues the author's work (see Huber (J. Comb. Theory Ser. A **94**, 180–190, 2001; Adv. Geom. **5**, 195–221, 2005; J. Algebr. Comb., 2007, to appear)) of classifying all flag-transitive Steiner 3-designs and 4-designs. We give a complete classification of all flag-transitive Steiner 5-designs and prove furthermore that there are no non-trivial flag-transitive Steiner 6-designs. Both results rely on the classification of the finite 3-homogeneous permutation groups. Moreover, we survey some of the most general results on highly symmetric Steiner t-designs.

Keywords Steiner designs · Flag-transitive group of automorphisms · 3-homogeneous permutation groups

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1 Introduction

For positive integers $t \leq k \leq v$ and λ, we define a t-(v, k, λ) *design* to be a finite incidence structure $\mathcal{D} = (X, \mathcal{B}, I)$, where X denotes a set of *points*, $|X| = v$, and $\mathcal{B}$ a set of *blocks*, $|\mathcal{B}| = b$, with the properties that each block $B \in \mathcal{B}$ is incident with k points, and each t-subset of X is incident with λ blocks. A *flag* of $\mathcal{D}$ is an incident point-block pair $(x, B) \in I$ with $x \in X$ and $B \in \mathcal{B}$. We consider automorphisms of $\mathcal{D}$ as pairs of permutations on X and $\mathcal{B}$ which preserve incidence, and call a group $G \leq \mathrm{Aut}(\mathcal{D})$ of automorphisms of $\mathcal{D}$ *flag-transitive* (respectively *block-transitive*, *point t-transitive*, *point t-homogeneous*) if G acts transitively on the flags (respectively transitively on the blocks, t-transitively on the points, t-homogeneously on the points) of $\mathcal{D}$. For short, $\mathcal{D}$ is said to be, e.g., flag-transitive if $\mathcal{D}$ admits a flag-transitive group of automorphisms. For historical reasons, a t-(v, k, λ) design with $\lambda = 1$ is called a *Steiner t-design* (sometimes also known as a *Steiner system*). We note that in this case each block is determined by the set of points which are incident with it, and thus can be identified with a k-subset of X in a unique way. If $t < k < v$ holds, then we speak of a *non-trivial* Steiner t-design.

As a consequence of the classification of the finite simple groups, it has been possible in recent years to characterize Steiner t-designs, mainly for $t = 2$, admitting groups of automorphisms with sufficiently strong symmetry properties. However, despite the classification of the finite simple groups, for Steiner t-designs with $t > 2$ most of these characterizations have remained long-standing challenging problems. Especially, the determination of all flag-transitive Steiner t-designs with $3 \leq t \leq 6$ is of particular interest and has been open for about 40 years (cf. [12, p. 147] and [13, p. 273], but presumably dating back to 1965).

The present paper continues the author's work [21, 22, 24] of classifying all flag-transitive Steiner 3-designs and 4-designs. We give a complete classification of all flag-transitive Steiner 5-designs in Section 4 and prove furthermore in Section 5 that there are no non-trivial flag-transitive Steiner 6-designs. Both results rely on the classification of the finite 3-homogeneous permutation groups, which itself depends on the finite simple group classification. Summarizing our results in this paper, we state:

The classification of all non-trivial Steiner t-designs with $t = 5$ or 6 admitting a flag-transitive group of automorphisms is as follows.

Main Theorem *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial Steiner t-design with $t = 5$ or 6. Then $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ if and only if one of the following occurs*:

(1) *$\mathcal{D}$ is isomorphic to the Witt* 5-(12, 6, 1) *design, and $G \cong M_{12}$*,
(2) *$\mathcal{D}$ is isomorphic to the Witt* 5-(24, 8, 1) *design, and $G \cong PSL(2, 23)$ or $G \cong M_{24}$.*

Referring to the author's work mentioned above, we present the complete determination of all flag-transitive Steiner t-designs with $t \geq 3$ in Section 2. Moreover, we give in this context a survey on some of the most general results on highly symmetric Steiner t-designs.

2 Classifications of highly symmetric combinatorial designs

In the sequel, we survey classification results of highly symmetric Steiner t-designs. For detailed descriptions of the respective designs and their groups of automorphisms as well as for further surveys concerning in particular highly symmetric Steiner 2-designs, we refer to [5, Sect. 1, 2], [15, Ch. 2.3, 2.4, 4.4], [28] and [30].

As presumably one of the first most general results, all point 2-transitive Steiner 2-designs were characterized by W. M. Kantor [27, Thm. 1], using the classification of the finite 2-transitive permutation groups.

Theorem 1 (Kantor 1985). *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial Steiner 2-design, and let $G \leq \mathrm{Aut}(\mathcal{D})$ act point 2-transitively on $\mathcal{D}$. Then one of the following holds*:

(1) *$\mathcal{D}$ is isomorphic to the 2-$(\frac{q^d-1}{q-1}, q+1, 1)$ design whose points and blocks are the points and lines of the projective space $PG(d-1, q)$, and $PSL(d, q) \leq G \leq P\Gamma L(d, q)$, or $(d-1, q) = (3, 2)$ and $G \cong A_7$,*
(2) *$\mathcal{D}$ is isomorphic to a Hermitian unital $U_H(q)$ of order q, and $PSU(3, q^2) \leq G \leq P\Gamma U(3, q^2)$,*
(3) *$\mathcal{D}$ is isomorphic to a Ree unital $U_R(q)$ of order q with $q = 3^{2e+1} > 3$, and $Re(q) \leq G \leq \mathrm{Aut}(Re(q))$,*
(4) *$\mathcal{D}$ is isomorphic to the 2-$(q^d, q, 1)$ design whose points and blocks are the points and lines of the affine space $AG(d, q)$, and one of the following holds (where G_0 denotes the stabilizer of $0 \in X$)*:
  (i) $G \leq A\Gamma L(1, q^d)$,
  (ii) $G_0 \trianglerighteq SL(\frac{d}{a}, q^a)$, $d \geq 2a$,
  (iii) $G_0 \trianglerighteq Sp(\frac{2d}{a}, q^a)$, $d \geq 2a$,
  (iv) $G_0 \trianglerighteq G_2(q^a)'$, *$q$ even*, $d = 6a$,
  (v) $G_0 \trianglerighteq SL(2, 3)$ *or* $SL(2, 5)$, $v = q^2$, $q = 5, 7, 9, 11, 19, 23, 29$ *or* 59,
  (vi) $G_0 \trianglerighteq SL(2, 5)$, *or G_0 contains a normal extraspecial subgroup E of order 2^5 and G_0/E is isomorphic to a subgroup of S_5*, $v = 3^4$,
  (vii) $G_0 \cong SL(2, 13)$, $v = 3^6$,
(5) *$\mathcal{D}$ is isomorphic to the affine nearfield plane A_9 of order 9, and G_0 as in* (4)(vi),
(6) *$\mathcal{D}$ is isomorphic to the affine Hering plane A_{27} of order 27, and G_0 as in* (4)(vii),
(7) *$\mathcal{D}$ is isomorphic to one of the two Hering spaces 2-$(9^3, 9, 1)$, and G_0 as in* (4)(vii).

As an easy implication, W. M. Kantor [27, Thm. 3] obtained moreover the classification of all point t-transitive Steiner t-designs with $t > 2$.

Certainly, among the highly symmetric properties of incidence structures, flag-transitivity is a particularly important and natural one. Even long before the aforementioned classification of the finite simple groups, a general study of flag-transitive Steiner 2-designs was introduced by D. G. Higman and J. E. McLaughlin [20] proving that a flag-transitive group $G \leq \mathrm{Aut}(\mathcal{D})$ of automorphisms of a Steiner 2-design $\mathcal{D}$ is necessarily primitive on the points of $\mathcal{D}$. They posed the problem of classifying all finite flag-transitive projective planes, and showed that such planes are Desarguesian if its orders are suitably restricted. Much later W. M. Kantor [29] determined all such planes apart from the still open case when the group of automorphisms is a Frobenius group of prime degree. His proof involves detailed knowledge of primitive permutation groups of odd degree based on the classification of the finite simple groups. In a big common effort, F. Buekenhout, A. Delandtsheer, J. Doyen, P. B. Kleidman, M. W. Liebeck, and J. Saxl [6, 14, 31, 34, 38] essentially characterized all finite flag-transitive linear spaces, that is flag-transitive Steiner 2-designs. Their result, which also relies on the finite simple group classification, starts with the result of Higman and McLaughlin and uses the O'Nan-Scott Theorem for finite primitive permutation groups. For the incomplete case with a 1-dimensional affine group of automorphisms, we refer to [6, Sect. 4] and [30, Sect. 3].

Theorem 2 (Buekenhout et al. 1990). *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a Steiner 2-design, and let $G \leq \mathrm{Aut}(\mathcal{D})$ act flag-transitively on $\mathcal{D}$. Then one of the following occurs*:

(1) *$\mathcal{D}$ is isomorphic to the 2-$(q^d, q, 1)$ design whose points and blocks are the points and lines of the affine space $AG(d, q)$, and one of the following holds*:
  (i) *G is 2-transitive (hence as in* Theorem 1 (4)),
  (ii) *$d = 2$, $q = 11$ or 23, and G is one of the three solvable flag-transitive groups given in* [17, *Table II*],
  (iii) *$d = 2$, $q = 9, 11, 19, 29$ or 59, $G_0^{(\infty)} \cong SL(2, 5)$ (where $G_0^{(\infty)}$ denotes the last term in the derived series of G_0), and G is given in* [17, *Table II*],
  (iv) *$d = 4$, $q = 3$, and $G_0 \cong SL(2, 5)$*,

(2) *$\mathcal{D}$ is isomorphic to a non-Desarguesian affine translation plane. More precisely, one of the following holds*:
  (i) *$\mathcal{D}$ is isomorphic to a Lüneburg-Tits plane $\mathrm{Lue}(q^2)$ of order q^2 with $q = 2^{2e+1} > 2$, and $Sz(q) \leq G_0 \leq \mathrm{Aut}(Sz(q))$*,
  (ii) *$\mathcal{D}$ is isomorphic to the affine Hering plane A_{27} of order 27, and $G_0 \cong SL(2, 13)$*,
  (iii) *$\mathcal{D}$ is isomorphic to the affine nearfield plane A_9 of order 9, and G is one of the seven flag-transitive subgroups of $\mathrm{Aut}(A_9)$, described in* [18, §5],

(3) *$\mathcal{D}$ is isomorphic to one of the two Hering spaces 2-$(9^3, 9, 1)$, and $G_0 \cong SL(2, 13)$*,

(4) *$\mathcal{D}$ is isomorphic to the 2-$(\frac{q^d-1}{q-1}, q+1, 1)$ design whose points and blocks are the points and lines of the projective space $PG(d-1, q)$, and $PSL(d, q) \leq G \leq P\Gamma L(d, q)$, or $(d-1, q) = (3, 2)$ and $G \cong A_7$*,

(5) *$\mathcal{D}$ is isomorphic to a Hermitian unital $U_H(q)$ of order q, and $PSU(3, q^2) \leq G \leq P\Gamma U(3, q^2)$*,

(6) *$\mathcal{D}$ is isomorphic to a Ree unital $U_R(q)$ of order q with $q = 3^{2e+1} > 3$, and $Re(q) \leq G \leq \mathrm{Aut}(Re(q))$*,

(7) *$\mathcal{D}$ is isomorphic to a Witt-Bose-Shrikhande space $W(q)$ with $q = 2^d \geq 8$, and $PSL(2, q) \leq G \leq P\Gamma L(2, q)$*,

(8) *$G \leq A\Gamma L(1, q)$.*

Investigating t-designs $\mathcal{D}$ for arbitrary λ, but large t, P. J. Cameron and C. E. Praeger [9, Thm. 1.1 and 2.1] showed that for $t \geq 7$ the flag-transitivity, respectively for $t \geq 8$ the block-transitivity of $G \leq \mathrm{Aut}(\mathcal{D})$ implies at least its point 4-homogeneity and proved the following result:

Theorem 3 (Cameron and Praeger 1993). *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a t-(v, k, λ) design. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively on $\mathcal{D}$, then $t \leq 7$, while if $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$, then $t \leq 6$.*

However, especially the determination of all flag-transitive Steiner t-designs with $3 \leq t \leq 6$ has remained of particular interest, and even the classification of all flag-transitive Steiner 3-designs has been known as “a long-standing and still open problem” (cf. [12, p. 147] and [13, p. 273]). Presumably, H. Lüneburg [36] in 1965 has been the first dealing with part of this problem characterizing flag-transitive Steiner quadruple systems (i.e., Steiner 3-designs with block size $k = 4$) under the additional strong assumption that every non-identity element of the group of automorphisms fixes at most two distinct points. This result has been generalized in 2001 by the author [21], omitting the additional assumption on the number of fixed points. Recently, the author [22, 24] completely determined all flag-transitive Steiner 3-designs and 4-designs using the classification of the finite 2-transitive permutation groups. In the present paper, the remaining investigations of all flag-transitive Steiner 5-designs and 6-designs are given, utilizing the classification of the finite 3-homogeneous permutation groups. Summarizing the author’s results, the complete determination of all non-trivial Steiner t-designs with $t \geq 3$ admitting a flag-transitive group of automorphisms can now be stated as follows.

Theorem 4 (Huber 2005/06). *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial Steiner t-design with $t \geq 3$. Then $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ if and only if one of the following occurs*:

(1) *$\mathcal{D}$ is isomorphic to the 3-$(2^d, 4, 1)$ design whose points and blocks are the points and planes of the affine space $AG(d, 2)$, and one of the following holds*:
   (i) *$d \geq 3$, and $G \cong AGL(d, 2)$,*
   (ii) *$d = 3$, and $G \cong AGL(1, 8)$ or $A\Gamma L(1, 8)$,*
   (iii) *$d = 4$, and $G_0 \cong A_7$,*
   (iv) *$d = 5$, and $G \cong A\Gamma L(1, 32)$,*

(2) *$\mathcal{D}$ is isomorphic to a 3-$(q^e + 1, q + 1, 1)$ design whose points are the elements of the projective line $GF(q^e) \cup \{\infty\}$ and whose blocks are the images of $GF(q) \cup \{\infty\}$ under $PGL(2, q^e)$ (respectively $PSL(2, q^e)$, e odd) with a prime power $q \geq 3$, $e \geq 2$, and the derived design at any given point is isomorphic to the 2-$(q^e, q, 1)$ design whose points and blocks are the points and lines of $AG(e, q)$, and $PSL(2, q^e) \leq G \leq P\Gamma L(2, q^e)$,*

(3) *$\mathcal{D}$ is isomorphic to a 3-$(q + 1, 4, 1)$ design whose points are the elements of $GF(q) \cup \{\infty\}$ with a prime power $q \equiv 7 \pmod{12}$ and whose blocks are the images of $\{0, 1, \varepsilon, \infty\}$ under $PSL(2, q)$, where ε is a primitive sixth root of unity in $GF(q)$, and the derived design at any given point is isomorphic to the Netto triple system $N(q)$, and $PSL(2, q) \leq G \leq P\Sigma L(2, q)$,*

(4) *$\mathcal{D}$ is isomorphic to one of the following Witt designs*:
   (i) *the 3-$(22, 6, 1)$ design, and $G \trianglerighteq M_{22}$,*
   (ii) *the 4-$(11, 5, 1)$ design, and $G \cong M_{11}$,*
   (iii) *the 4-$(23, 7, 1)$ design, and $G \cong M_{23}$,*
   (iv) *the 5-$(12, 6, 1)$ design, and $G \cong M_{12}$,*
   (v) *the 5-$(24, 8, 1)$ design, and $G \cong PSL(2, 23)$ or $G \cong M_{24}$.*

We remark that the Steiner 3-designs in Part (1) (ii) with $G \cong AGL(1,8)$ and (iv) with $G \cong A\Gamma L(1,32)$ as well as the Steiner 5-design in Part (4) with $G \cong PSL(2,23)$ are sharply flag-transitive, and furthermore, concerning Part (4) (v), that M_{24} as the full group of automorphisms of $\mathcal{D}$ contains only one conjugacy class of subgroups isomorphic to $PSL(2,23)$.

3 Definitions and preliminary results

If $\mathcal{D} = (X, \mathcal{B}, I)$ is a t-(v,k,λ) design with $t \geq 2$, and $x \in X$ arbitrary, then the *derived* design with respect to x is $\mathcal{D}_x = (X_x, \mathcal{B}_x, I_x)$, where $X_x = X \backslash \{x\}$, $\mathcal{B}_x = \{B \in \mathcal{B} : (x, B) \in I\}$ and $I_x = I|_{X_x \times \mathcal{B}_x}$. In this case, $\mathcal{D}$ is also called an *extension* of $\mathcal{D}_x$. Obviously, $\mathcal{D}_x$ is a $(t-1)$-$(v-1, k-1, \lambda)$ design.

Let G be a permutation group on a non-empty set X. We call G *semi-regular* if the identity is the only element that fixes any point of X. If additionally G is transitive, then it is said to be *regular*. Furthermore, for $x \in X$, the orbit x^G containing x is called *regular* if it has length $|G|$. If $\{x_1, \ldots, x_m\} \subseteq X$, let $G_{\{x_1,\ldots,x_m\}}$ be its setwise stabilizer and $G_{x_1,\ldots,x_m}$ its pointwise stabilizer (for short, we often write $G_{x_1\ldots x_m}$ in the latter case).

For $\mathcal{D} = (X, \mathcal{B}, I)$ a Steiner t-design with $G \leq \mathrm{Aut}(\mathcal{D})$, let G_B denote the setwise stabilizer of a block $B \in \mathcal{B}$, and for $x \in X$, we define $G_{xB} = G_x \cap G_B$.

Let $\mathbb{N}$ be the set of positive integers (in this article, $0 \notin \mathbb{N}$). For integers m and n, let (m,n) denote the greatest common divisor of m and n, and we write $m \mid n$ if m divides n.

For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest positive integer which is at most x.

All other notation is standard.

When considering a Steiner t-design $\mathcal{D}$ with $t = 2$, it is elementary that the point 2-transitivity of $G \leq \mathrm{Aut}(\mathcal{D})$ implies its flag-transitivity. However, for $t \geq 3$, it can be deduced from a result of R. E. Block [3, Thm. 2] that the converse holds:

Proposition 5 (cf. [4, 22]). *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a Steiner t-design with $t \geq 3$. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$, then G also acts point 2-transitively on $\mathcal{D}$.*

For $t \geq 5$, the flag-transitivity of $G \leq \mathrm{Aut}(\mathcal{D})$ has an even stronger implication due to the following assertion, which follows from Block's theorem and a combinatorial result of D. K. Ray-Chaudhuri and R. M. Wilson [37, Thm. 1].

Proposition 6 (cf. [9]). *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a Steiner t-design with $t \geq 2$. Then, the following holds*:

(a) *If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively on $\mathcal{D}$, then G also acts point $\lfloor t/2 \rfloor$-homogeneously on $\mathcal{D}$.*
(b) *If $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$, then G also acts point $\lfloor (t+1)/2 \rfloor$-homogeneously on $\mathcal{D}$.*

We note that Propositions 5 and 6 hold also for arbitrary λ, whereas for a 2-(v,k,λ) design the implication that the point 2-transitivity yields its flag-transitivity is only true if $(r,\lambda)=1$ (see, e.g., [15, Ch. 2.3, Lemma 8]).

In order to investigate in the following all flag-transitive Steiner 5-designs and 6-designs, we can as a consequence of Proposition 6 (b) make use of the classification of all finite 3-homogeneous permutation groups, which itself relies on the classification of all finite simple groups (cf. [8, 19, 26, 33, 35]).

The list of groups is as follows.

Let G be a finite 3-homogeneous permutation group on a set X with $|X|\geq 4$. Then G is either of

(A) Affine Type: G contains a regular normal subgroup T which is elementary Abelian of order $v=2^d$. If we identify G with a group of affine transformations

$$x \mapsto x^g + u$$

of $V=V(d,2)$, where $g\in G_0$ and $u\in V$, then particularly one of the following occurs:

(1) $G\cong AGL(1,8)$, $A\Gamma L(1,8)$, or $A\Gamma L(1,32)$
(2) $G_0\cong SL(d,2)$, $d\geq 2$
(3) $G_0\cong A_7$, $v=2^4$

or

(B) Almost Simple Type: G contains a simple normal subgroup N, and $N\leq G\leq \mathrm{Aut}(N)$. In particular, one of the following holds, where N and $v=|X|$ are given as follows:

(1) A_v, $v\geq 5$
(2) $PSL(2,q)$, $q>3$, $v=q+1$
(3) M_v, $v=11,12,22,23,24$ (Mathieu groups)
(4) M_{11}, $v=12$

We note that if q is odd, then $PSL(2,q)$ is 3-homogeneous for $q\equiv 3 \pmod 4$, but not for $q\equiv 1 \pmod 4$, and hence not every group G of almost simple type satisfying (2) is 3-homogeneous on X. For required basic properties of the listed groups, we refer, e.g., to [11], [25], [32, Ch. 2, 5].

We will now recall some standard combinatorial results on which we rely in the sequel. Let r (respectively λ_2) denote the total number of blocks incident with a given point (respectively pair of distinct points), and let all further parameters be as defined at the beginning of Section 1.

Lemma 7 *Let $\mathcal{D}=(X,\mathcal{B},I)$ be a Steiner t-design. If $G\leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$, then*

$$r \,\Big|\, |G_x|$$

for any $x\in X$.

Lemma 8 *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a t-(v, k, λ) design. Then the following holds*:

(a) $bk = vr$.
(b) $\binom{v}{t}\lambda = b\binom{k}{t}$.
(c) $r(k-1) = \lambda_2(v-1)$ *for* $t \geq 2$, *where* $\lambda_2 = \lambda\frac{\binom{v-2}{t-2}}{\binom{k-2}{t-2}}$.

Proposition 9 (cf. [7, 39]). *If $\mathcal{D} = (X, \mathcal{B}, I)$ is a non-trivial Steiner t-design, then the following holds*:

(a) $v \geq (t+1)(k-t+1)$.
(b) $v - t + 1 \geq (k-t+2)(k-t+1)$ *for* $t > 2$. *If equality holds, then* $(t, k, v) = (3, 4, 8), (3, 6, 22), (3, 12, 112), (4, 7, 23)$, *or* $(5, 8, 24)$.

We note that (a) is stronger for $k < 2(t-1)$, while (b) is stronger for $k > 2(t-1)$. For $k = 2(t-1)$ both assert that $v \geq t^2 - 1$.

As we are in particular interested in the case when $3 \leq t \leq 6$, we deduce from (b) the following upper bound for the positive integer k.

Corollary 10 *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial Steiner t-design with $t = 3 + i$, where $i = 0, 1, 2, 3$. Then*

$$k \leq \left\lfloor \sqrt{v} + \tfrac{3}{2} + i \right\rfloor.$$

Remark 11 If $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on any Steiner t-design $\mathcal{D}$ with $t \geq 3$, then applying Proposition 5 and Lemma 8 (b) yields the equation

$$b = \frac{\binom{v}{t}}{\binom{k}{t}} = \frac{v(v-1)\left|G_{xy}\right|}{|G_B|},$$

where x and y are two distinct points in X and B is a block in $\mathcal{B}$, and thus

$$\binom{v-2}{t-2} = (k-1)\binom{k-2}{t-2}\frac{\left|G_{xy}\right|}{|G_{xB}|} \text{ if } x \in B.$$

4 The classification of flag-transitive Steiner 5-designs

The classification of all non-trivial Steiner 5-designs admitting a flag-transitive group of automorphisms is as follows.

Main Theorem 1 *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial Steiner 5-design. Then $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ if and only if one of the following occurs*:

(1) *$\mathcal{D}$ is isomorphic to the Witt* 5-(12, 6, 1) *design, and* $G \cong M_{12}$,
(2) *$\mathcal{D}$ is isomorphic to the Witt* 5-(24, 8, 1) *design, and* $G \cong PSL(2, 23)$ *or* $G \cong M_{24}$.

We remark that in Part (2), $G \cong PSL(2, 23)$ acts sharply flag-transitively on $\mathcal{D}$, and furthermore that M_{24} as the full group of automorphisms of $\mathcal{D}$ contains only one conjugacy class of subgroups isomorphic to $PSL(2, 23)$ (cf. [11]).

4.1 Groups of automorphisms of affine type

In this subsection, we start with the proof of Main Theorem 1. Using the notation as before, *let* $\mathcal{D} = (X, \mathcal{B}, I)$ *be a non-trivial Steiner* 5*-design with* $G \leq \mathrm{Aut}(\mathcal{D})$ *acting flag-transitively on* $\mathcal{D}$ *throughout the proof*. We recall that due to Proposition 6, we may restrict ourselves to the consideration of the finite 3-homogeneous permutation groups listed in Section 3. Clearly, in the following we may assume that $k > 5$ as trivial Steiner 5-designs are excluded. Let us first assume that G is of affine type.

Case (1): $G \cong AGL(1,8)$, $A\Gamma L(1,8)$, or $A\Gamma L(1,32)$.

We may assume that $k > 5$. For $v = 8$, we obtain $k = 6$ by Corollary 10, which is not possible in view of Lemma 8 (b). If $v = 32$, then $|G| = 5v(v-1)$, and Lemma 7 immediately yields that $G \leq \mathrm{Aut}(\mathcal{D})$ cannot act flag-transitively on any non-trivial Steiner 5-design $\mathcal{D}$.

Case (2): $G_0 \cong SL(d,2)$, $d \geq 2$.

Let e_i denote the i-th standard basis vector of the vector space $V = V(d,2)$, and $\langle e_i \rangle$ the 1-dimensional vector subspace spanned by e_i. We will prove by contradiction that $G \leq \mathrm{Aut}(\mathcal{D})$ cannot act flag-transitively on any non-trivial Steiner 5-design $\mathcal{D}$.

We may assume that $v = 2^d > k > 5$. For $d = 3$, we have $v = 8$ and $k = 6$ by Corollary 10, which is not possible in view of Lemma 8 (b) again. So, we may assume that $d > 3$. We remark that clearly any five distinct points are non-coplanar in $AG(d,2)$ and hence generate an affine subspace of dimension at least 3. Let $\mathcal{E} = \langle e_1, e_2, e_3 \rangle$ denote the 3-dimensional vector subspace spanned by e_1, e_2, e_3. Then by linear algebra $SL(d,2)_{\mathcal{E}}$, and therefore also $G_{0,\mathcal{E}}$, acts point-transitively on $V \setminus \mathcal{E}$. If the unique block $B \in \mathcal{B}$ which is incident with the 5-subset $\{0, e_1, e_2, e_3, e_1 + e_2\}$ contains some point outside $\mathcal{E}$, then it would already contain all points of $V \setminus \mathcal{E}$. But then, we would have $k \geq 2^d - 8 + 5 = 2^d - 3$, a contradiction to Corollary 10. Hence, B lies completely in $\mathcal{E}$, and by the flag-transitivity of G, it follows that each block must be contained in a 3-dimensional affine subspace. Thus, clearly $k \leq 8$. But, on the other hand, for $\mathcal{D}$ to be a block-transitive 5-design admitting $G \leq \mathrm{Aut}(\mathcal{D})$, we obtain from [1] the necessary (and sufficient) condition that $2^d - 3$ must divide $\binom{k}{4}$, and hence it follows for each respective value of k that $d = 3$, contradicting our assumption.

Case (3): $G_0 \cong A_7$, $v = 2^4$.

Since $v = 2^4$, we obtain from Corollary 10 that $k \leq 7$. But, Lemma 7 easily rules out the cases when $k = 6$ or 7.

4.2 Groups of automorphisms of almost simple type

Before we consider in this subsection successively those cases where G is of almost simple type, we indicate some lemmas which will be required for Case (2).

Let q be a prime power p^e, and U a subgroup of $PSL(2,q)$. Furthermore, let N_l denote the number of orbits of length l and let $n = (2, q-1)$. In [23, Ch. 5], we have in particular determined the orbit-lengths from the action of subgroups of $PSL(2,q)$

on the points of the projective line. For the list of subgroups of $PSL(2,q)$, we thereby refer to [16, Ch. 12, p. 285f.] or [25, Ch. II, Thm. 8.27].

Lemma 12 *Let U be the cyclic group of order c with $c \mid \frac{q\pm1}{n}$. Then*

(a) *if $c \mid \frac{q+1}{n}$, then $N_c = (q+1)/c$,*
(b) *if $c \mid \frac{q-1}{n}$, then $N_1 = 2$ and $N_c = (q-1)/c$.*

Lemma 13 *Let U be the dihedral group of order $2c$ with $c \mid \frac{q\pm1}{n}$. Then*

(i) *for $q \equiv 1 \pmod 4$:*
 (a) *if $c \mid \frac{q+1}{2}$, then $N_c = 2$ and $N_{2c} = (q+1-2c)/(2c)$,*
 (b) *if $c \mid \frac{q-1}{2}$, then $N_2 = 1$, $N_c = 2$, and $N_{2c} = (q-1-2c)/(2c)$, unless $c = 2$, in which case $N_2 = 3$ and $N_4 = (q-5)/4$,*
(ii) *for $q \equiv 3 \pmod 4$:*
 (a) *if $c \mid \frac{q+1}{2}$, then $N_{2c} = (q+1)/(2c)$,*
 (b) *if $c \mid \frac{q-1}{2}$, then $N_2 = 1$ and $N_{2c} = (q-1)/(2c)$,*
(iii) *for $q \equiv 0 \pmod 2$:*
 (a) *if $c \mid q+1$, then $N_c = 1$ and $N_{2c} = (q+1-c)/(2c)$,*
 (b) *if $c \mid q-1$, then $N_2 = 1$, $N_c = 1$, and $N_{2c} = (q-1-c)/(2c)$.*

Lemma 14 *Let U be the elementary Abelian group of order $\overline{q} \mid q$. Then $N_1 = 1$ and $N_{\overline{q}} = q/\overline{q}$.*

Lemma 15 *Let U be a semi-direct product of the elementary Abelian group of order $\overline{q} \mid q$ and the cyclic group of order c with $c \mid \overline{q}-1$ and $c \mid q-1$. Then $N_1 = 1$, $N_{\overline{q}} = 1$, and $N_{c\overline{q}} = (q-\overline{q})/(c\overline{q})$.*

Lemma 16 *Let U be $PSL(2,\overline{q})$ with $\overline{q}^m = q$, $m \geq 1$. Then $N_{\overline{q}+1} = 1$, $N_{\overline{q}(\overline{q}-1)} = 1$ if m is even, and all other orbits are regular.*

Lemma 17 *Let U be $PGL(2,\overline{q})$ with $\overline{q}^m = q$, $m > 1$ even. Then $N_{\overline{q}+1} = 1$, $N_{\overline{q}(\overline{q}-1)} = 1$, and all other orbits are regular.*

Lemma 18 *Let U be isomorphic to A_4. Then*

(i) *for $q \equiv 1 \pmod 4$:*
 (a) *if $3 \mid \frac{q+1}{2}$, then $N_6 = 1$ and $N_{12} = (q-5)/12$,*
 (b) *if $3 \mid \frac{q-1}{2}$, then $N_4 = 2$, $N_6 = 1$, and $N_{12} = (q-13)/12$,*
 (c) *if $3 \mid q$, then $N_4 = 1$, $N_6 = 1$, and $N_{12} = (q-9)/12$,*
(ii) *for $q \equiv 3 \pmod 4$:*
 (a) *if $3 \mid \frac{q+1}{2}$, then $N_{12} = (q+1)/12$,*
 (b) *if $3 \mid \frac{q-1}{2}$, then $N_4 = 2$ and $N_{12} = (q-7)/12$,*
 (c) *if $3 \mid q$, then $N_4 = 1$ and $N_{12} = (q-3)/12$,*
(iii) *for $q = 2^e$, $e \equiv 0 \pmod 2$: $N_1 = 1$, $N_4 = 1$, and $N_{12} = (q-4)/12$.*

Lemma 19 *Let U be isomorphic to S_4. Then*

(i) *for* $q \equiv 1 \pmod 8$:
  (a) *if* $3 \mid \frac{q+1}{2}$, *then* $N_6 = 1$, $N_{12} = 1$, *and* $N_{24} = (q-17)/24$,
  (b) *if* $3 \mid \frac{q-1}{2}$, *then* $N_6 = 1$, $N_8 = 1$, $N_{12} = 1$, *and* $N_{24} = (q-25)/24$,
  (c) *if* $3 \mid q$, *then* $N_4 = 1$, $N_6 = 1$, *and* $N_{24} = (q-9)/24$,
(ii) *for* $q \equiv -1 \pmod 8$:
  (a) *if* $3 \mid \frac{q+1}{2}$, *then* $N_{24} = (q+1)/24$,
  (b) *if* $3 \mid \frac{q-1}{2}$, *then* $N_8 = 1$ *and* $N_{24} = (q-7)/24$.

Lemma 20 *Let U be isomorphic to A_5. Then*

(i) *for* $q \equiv 1 \pmod 4$:
  (a) *if* $q = 5^e$, $e \equiv 1 \pmod 2$, *then* $N_6 = 1$ *and* $N_{60} = (q-5)/60$,
  (b) *if* $q = 5^e$, $e \equiv 0 \pmod 2$, *then* $N_6 = 1$, $N_{20} = 1$, *and* $N_{60} = (q-25)/60$,
  (c) *if* $15 \mid \frac{q+1}{2}$, *then* $N_{30} = 1$ *and* $N_{60} = (q-29)/60$,
  (d) *if* $3 \mid \frac{q+1}{2}$ *and* $5 \mid \frac{q-1}{2}$, *then* $N_{12} = 1$, $N_{30} = 1$, *and* $N_{60} = (q-41)/60$,
  (e) *if* $3 \mid \frac{q-1}{2}$ *and* $5 \mid \frac{q+1}{2}$, *then* $N_{20} = 1$, $N_{30} = 1$, *and* $N_{60} = (q-49)/60$,
  (f) *if* $15 \mid \frac{q-1}{2}$, *then* $N_{12} = 1$, $N_{20} = 1$, $N_{30} = 1$, *and* $N_{60} = (q-61)/60$,
  (g) *if* $3 \mid q$ *and* $5 \mid \frac{q+1}{2}$, *then* $N_{10} = 1$ *and* $N_{60} = (q-9)/60$,
  (h) *if* $3 \mid q$ *and* $5 \mid \frac{q-1}{2}$, *then* $N_{10} = 1$, $N_{12} = 1$, *and* $N_{60} = (q-21)/60$,
(ii) *for* $q \equiv 3 \pmod 4$:
  (a) *if* $15 \mid \frac{q+1}{2}$, *then* $N_{60} = (q+1)/60$,
  (b) *if* $3 \mid \frac{q+1}{2}$ *and* $5 \mid \frac{q-1}{2}$, *then* $N_{12} = 1$ *and* $N_{60} = (q-11)/60$,
  (c) *if* $3 \mid \frac{q-1}{2}$ *and* $5 \mid \frac{q+1}{2}$, *then* $N_{20} = 1$ *and* $N_{60} = (q-19)/60$,
  (d) *if* $15 \mid \frac{q-1}{2}$, *then* $N_{12} = 1$, $N_{20} = 1$, *and* $N_{60} = (q-31)/60$.

We shall now turn to the examination of those cases where $G \leq \mathrm{Aut}(\mathcal{D})$ is of almost simple type.

Case (1): $N = A_v$, $v \geq 5$.

We may assume that $v \geq 7$. But then A_v, and hence also G, is 5-transitive and does not act on any non-trivial Steiner 5-design $\mathcal{D}$ in view of [27, Thm. 3].

Case (2): $N = PSL(2,q)$, $v = q+1$, $q = p^e > 3$.

Here $\mathrm{Aut}(N) = P\Gamma L(2,q)$, and $|G| = (q+1)q\frac{(q-1)}{n}a$ with $n = (2, q-1)$ and $a \mid ne$. We may assume that $q \geq 5$. We will show that only the flag-transitive design given in Part (2) of Main Theorem 1 with $G \cong PSL(2,23)$ can occur.

We will first assume that $N = G$. Then, by Remark 11, we obtain

$$(q-2)(q-3)\,|PSL(2,q)_{0B}| \cdot n = (k-1)(k-2)(k-3)(k-4). \qquad (1)$$

In view of Proposition 9 (b), we have

$$q-3 \geq (k-3)(k-4), \tag{2}$$

and thus it follows from equation (1) that

$$(q-2)\,|PSL(2,q)_{0B}| \cdot n \leq (k-1)(k-2). \tag{3}$$

If we assume that $k \geq 9$, then clearly

$$(k-1)(k-2) < 2(k-3)(k-4),$$

and hence we obtain

$$(q-2)\,|PSL(2,q)_{0B}| \cdot n < 2(q-3)$$

due to Proposition 9 (b) again, which is obviously only possible when $|PSL(2,q)_{0B}| \cdot n = 1$. Thus, in particular q has to be even. But then, considering equation (1) yields that the left hand side of the equation is not divisible by 4, whereas obviously the right hand side is always divisible by 8, a contradiction. If $k < 9$, then, using equation (1) and inequality (2), the very few remaining possibilities for k can immediately be ruled out by hand, except for the case when $k = 8$, $q = 23$ and $|PSL(2,q)_{0B}| = 1$. It is well-known that for the parameters $t = 5$, $v = 24$ and $k = 8$ there exists (up to isomorphism) only the unique Witt 5-(24, 8, 1) design $\mathcal{D}$, which can be constructed from $PSL(2,23)$ in its natural 3-homogeneous action on the elements of $GF(23) \cup \{\infty\}$. Furthermore, it can be shown that the setwise stabilizer $PSL(2,23)_B$ of an appropriate, unique block $B \in \mathcal{B}$ is a dihedral group of order 8 (see, e.g., [2, Ch. IV, 1.5], [10, Ch. XIV, 115], and [40, Thm. 5] for a uniqueness proof). Thus, using Lemma 8 (b), we obtain $b = 759 = \left[PSL(2,23) : PSL(2,23)_B\right]$, and hence $PSL(2,23)$ acts block-transitively on $\mathcal{D}$. As for $q = 23$, the dihedral group of order 8 has only orbits of length 8 in view of Lemma 13 (ii)(a), clearly $PSL(2,23)_B$ acts transitively on the points of B. Since we have $|PSL(2,23)_{0B}| = 1$, it follows that $PSL(2,23)$ acts even sharply flag-transitively on $\mathcal{D}$.

Now, let us assume that $N < G \leq \mathrm{Aut}(N)$. We recall that $q = p^e > 3$, and will distinguish in the following the cases $p > 3$, $p = 2$, and $p = 3$.

First, let $p > 3$. We define $G^* = G \cap (PSL(2,q) \rtimes \langle \tau_\alpha \rangle)$ with $\tau_\alpha \in \mathrm{Sym}(GF(p^e) \cup \{\infty\}) \cong S_v$ of order e induced by the Frobenius automorphism $\alpha : GF(p^e) \longrightarrow GF(p^e)$, $x \mapsto x^p$. Then, by Dedekind's law, we can write

$$G^* = PSL(2,q) \rtimes (G^* \cap \langle \tau_\alpha \rangle). \tag{4}$$

Defining $P\Sigma L(2,q) = PSL(2,q) \rtimes \langle \tau_\alpha \rangle$, it can easily be calculated that $P\Sigma L(2,q)_{0,1,\infty} = \langle \tau_\alpha \rangle$, and $\langle \tau_\alpha \rangle$ has precisely $p+1$ distinct fixed points (cf., e.g., [15, Ch. 6.4, Lemma 2]). As $p > 3$, we conclude therefore that $G^* \cap \langle \tau_\alpha \rangle \leq G^*_{0B}$

for some appropriate, unique block $B \in \mathcal{B}$ by the definition of Steiner 5-designs. Furthermore, clearly $PSL(2,q) \cap (G^* \cap \langle \tau_\alpha \rangle) = 1$. Hence, we have

$$\begin{aligned} \left|(0,B)^{G^*}\right| &= \left[G^* : G^*_{0B}\right] \\ &= \left[PSL(2,q) \rtimes (G^* \cap \langle \tau_\alpha \rangle) : PSL(2,q)_{0B} \rtimes (G^* \cap \langle \tau_\alpha \rangle)\right] \\ &= \left[PSL(2,q) : PSL(2,q)_{0B}\right] \\ &= \left|(0,B)^{PSL(2,q)}\right|. \end{aligned} \tag{5}$$

Thus, if we assume that $G^* \leq \mathrm{Aut}(\mathcal{D})$ acts already flag-transitively on $\mathcal{D}$, then we obtain $\left|(0,B)^{G^*}\right| = \left|(0,B)^{PSL(2,q)}\right| = bk$ in view of Remark 11. Hence, $PSL(2,q)$ must also act flag-transitively on $\mathcal{D}$, and we may proceed as in the case when $N = G$. Therefore, let us assume that $G^* \leq \mathrm{Aut}(\mathcal{D})$ does not act flag-transitively on $\mathcal{D}$. Then, we conclude that $\left[G : G^*\right] = 2$ and G^* has exactly two orbits of equal length on the set of flags. Thus, by equation (5), we obtain for the orbit containing the flag $(0,B)$ that $\left|(0,B)^{G^*}\right| = \left|(0,B)^{PSL(2,q)}\right| = \frac{bk}{2}$. As it is well-known the normalizer of $PSL(2,q)$ in $\mathrm{Sym}(X)$ is $P\Gamma L(2,q)$, and hence in particular $PSL(2,q)$ is normal in G. It follows therefore that we have under $PSL(2,q)$ also precisely one further orbit of equal length on the set of flags. Then, proceeding similarly to the case $N = G$ for each orbit on the set of flags, we obtain (representative for the orbit containing the flag $(0,B)$) that

$$\frac{(q-2)(q-3)\,|PSL(2,q)_{0B}| \cdot n}{2} = (k-1)(k-2)(k-3)(k-4), \tag{6}$$

and as here $n = 2$, this is equivalent to

$$\begin{aligned} (q-2)(q-3)\,|PSL(2,q)_{0B}| &= (k-1)(k-2)(k-3)(k-4) \\ &= k(k^3 - 10k^2 + 35k - 50) + 24. \end{aligned} \tag{7}$$

Hence, we have in particular

$$k \;\Big|\; (q-2)(q-3)\,|PSL(2,q)_{0B}| - 24. \tag{8}$$

Since $PSL(2,q)_B$ can have one or two orbits of equal length on the set of points of B, we have

$$k \text{ or } \frac{k}{2} = \left|0^{PSL(2,q)_B}\right| = \left[PSL(2,q)_B : PSL(2,q)_{0B}\right]. \tag{9}$$

By the same arguments as in case $N = G$, we deduce from equation (7) that

$$(q-2)\,|PSL(2,q)_{0B}| \leq (k-1)(k-2), \tag{10}$$

and assuming that $k \geq 9$, we obtain

$$(q-2)\,|PSL(2,q)_{0B}| < 2(q-3),$$

which is clearly only possible when $|PSL(2,q)_{0B}| = 1$. Hence, it follows that

$$(q-2)(q-3) = (k-1)(k-2)(k-3)(k-4), \tag{11}$$

and $k \mid (q-2)(q-3) - 24$ in view of property (8). On the other hand, for $k \geq 9$, we obtain from equation (9) that k or $\frac{k}{2} = |PSL(2,q)_B| \mid |PSL(2,q)| = \frac{q^3-q}{2}$, and thus in particular $k \mid q^3 - q$. But, it can easily be seen that $(q^3 - q, (q-2)(q-3) - 24) \mid 2^3 \cdot 3 \cdot 11$, and thus we have only a small number of possibilities for k to check, which can easily be eliminated by hand using equation (11). For $k < 9$, the very few remaining possibilities for k can immediately be ruled out by hand using inequality (2) and equation (7), except for the case when $k = 8$, $q = 23$ and $|PSL(2,q)_{0B}| = 2$. But, as involutions are fixed point free on the points of the projective line for $q \equiv 3 \pmod 4$ (cf., e.g., [25, Ch. II, Thm. 8.5]), this is impossible.

Now, let $p = 2$. Then, clearly $N = PSL(2,q) = PGL(2,q)$, and we have $\mathrm{Aut}(N) = P\Sigma L(2,q)$. If we assume that $\langle \tau_\alpha \rangle \leq P\Sigma L(2,q)_{0B}$ for some appropriate, unique block $B \in \mathcal{B}$, then, using the terminology of (4), we have $G^* = G = P\Sigma L(2,q)$ and as clearly $PSL(2,q) \cap \langle \tau_\alpha \rangle = 1$, we can apply equation (5). Thus, $PSL(2,q)$ must also be flag-transitive, which has already been considered. Therefore, we may assume that $\langle \tau_\alpha \rangle \nleq P\Sigma L(2,q)_{0B}$. Let s be a prime divisor of $e = |\langle \tau_\alpha \rangle|$. As the normal subgroup $H := (P\Sigma L(2,q)_{0,1,\infty})^s \leq \langle \tau_\alpha \rangle$ of index s has precisely $p^s + 1$ distinct fixed points (see, e.g., [15, Ch. 6.4, Lemma 2]), we have $G \cap H \leq G_{0B}$ for some appropriate, unique block $B \in \mathcal{B}$ by the definition of Steiner 5-designs. It can then be deduced that $e = s^u$ for some $u \in \mathbb{N}$, since if we assume for $G = P\Sigma L(2,q)$ that there exists a further prime divisor $\overline{s}$ of e with $\overline{s} \neq s$, then $\overline{H} := (P\Sigma L(2,q)_{0,1,\infty})^{\overline{s}} \leq \langle \tau_\alpha \rangle$ and H are both subgroups of $P\Sigma L(2,q)_{0B}$ by the flag-transitivity of $P\Sigma L(2,q)$, and hence $\langle \tau_\alpha \rangle \leq P\Sigma L(2,q)_{0B}$, a contradiction. Furthermore, as $\langle \tau_\alpha \rangle \nleq P\Sigma L(2,q)_{0B}$, we may, by applying Dedekind's law, assume that

$$G_{0B} = PSL(2,q)_{0B} \rtimes (G \cap H).$$

Thus, by Remark 11, we obtain

$$(q-2)(q-3)\,|PSL(2,q)_{0B}|\,|G \cap H| = (k-1)(k-2)(k-3)(k-4)\,|G \cap \langle \tau_\alpha \rangle|.$$

Using that $k = |0^{G_B}| = [G_B : G_{0B}]$, we have more precisely

(A) if $G = PSL(2,q) \rtimes (G \cap H)$:

$$(q-2)(q-3)\,|PSL(2,q)_{0B}| = (k-1)(k-2)(k-3)(k-4)$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \frac{|PSL(2,q)_B|}{k}, \quad \text{or}$$

(B) if $G = P\Sigma L(2,q)$:

$$(q-2)(q-3)\,|PSL(2,q)_{0B}| = (k-1)(k-2)(k-3)(k-4)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \frac{|PSL(2,q)_B|}{k} \cdot \begin{cases} s, & \text{if } G_B = PSL(2,q)_B \rtimes \langle \tau_\alpha \rangle \\ 1, & \text{if } G_B = PSL(2,q)_B \rtimes H. \end{cases}$$

As far as condition (A) is concerned, we may argue exactly as in the earlier case $N=G$. Thus, only condition (B) has to be examined, and we will also show that here $G \leq \mathrm{Aut}(\mathcal{D})$ cannot act flag-transitively on any non-trivial Steiner 5-design $\mathcal{D}$. Clearly, there always exists a Klein four-group $V_4 \leq PSL(2,q)$, which fixes some 4-subset S of X and some additional point $x \in X$, and hence must fix the unique block $B \in \mathcal{B}$ which is incident with $S \cup \{x\}$ by the definition of Steiner 5-designs. Examining the list of possible subgroups of $PSL(2,q)$ with their orbits on the projective line (cf. Lemmas 12-20), it follows that we only have to consider the possibility when $PSL(2,q)_B$ is conjugate to $PSL(2,\overline{q})$ with $\overline{q}^m=q$, $m \geq 1$, and by Lemma 16, we conclude that $k=\overline{q}+1$. Applying condition (B) yields then

$$(q-2)(q-3)\,|PSL(2,q)_{0B}| = \overline{q}(\overline{q}-1)(\overline{q}-2)(\overline{q}-3)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \overline{q}(\overline{q}-1)\cdot \begin{cases} s, & \text{or} \\ 1. \end{cases} \tag{12}$$

Since $q=2^{s^u}$, we can write $\overline{q}=2^{s^w}$ for some integer $0 \leq w \leq u$, and $q=\overline{q}^m=\overline{q}^{s^{u-w}}$. As we may assume that $k=\overline{q}+1=2^{s^w}+1>5$, it follows in particular that $w \geq 1$, and hence $s<2^{s^w}=\overline{q}$. Thus, using equation (12), we obtain

$$(\overline{q}^{s^{u-w}}-2)(\overline{q}^{s^{u-w}}-3)=(q-2)(q-3)\leq(\overline{q}-2)(\overline{q}-3)s<(\overline{q}^2-2s)(\overline{q}-3).$$

But, as clearly $u-w \geq 1$ (otherwise, $k=q+1$, a contradiction to Corollary 10), this yields a contradiction for every prime s.

Now, let $p=3$. We have $\mathrm{Aut}(N)=P\Gamma L(2,q)=PGL(2,q)\rtimes\langle\tau_\alpha\rangle$, and as $PGL(2,q)$ is sharply 3-transitive, it follows that $P\Gamma L(2,q)_{0,1,\infty}=\langle\tau_\alpha\rangle$. Again, we define $G^*=G\cap(PSL(2,q)\rtimes\langle\tau_\alpha\rangle)$ and may write $G^*=PSL(2,q)\rtimes(G^*\cap\langle\tau_\alpha\rangle)$ as in equation (4). We distinguish the cases $G=G^*$ and $[G:G^*]=2$. In the following, we will examine the first case in detail, whereas the second may be treated mutatis mutandis. Let $G=G^*$. Then, we have $\mathrm{Aut}(N)=P\Sigma L(2,q)$. If we assume that $\langle\tau_\alpha\rangle \leq P\Sigma L(2,q)_{0B}$ for some appropriate, unique block $B\in\mathcal{B}$, then $G=P\Sigma L(2,q)$, and as clearly $PSL(2,q)\cap\langle\tau_\alpha\rangle=1$, we can apply equation (5). Thus, $PSL(2,q)$ must also be flag-transitive, which has already been considered. Therefore, we may assume that $\langle\tau_\alpha\rangle \nleq P\Sigma L(2,q)_{0B}$. Let s be a prime divisor of $e=|\langle\tau_\alpha\rangle|$. As already mentioned, the normal subgroup $H:=(P\Sigma L(2,q)_{0,1,\infty})^s \leq \langle\tau_\alpha\rangle$ of index s has precisely p^s+1 distinct fixed points, and hence we have $G\cap H \leq G_{0B}$ for some appropriate, unique block $B\in\mathcal{B}$ by the definition of Steiner 5-designs. It can then be deduced exactly as for $p=2$ that $e=s^u$ for some $u\in\mathbb{N}$. As $\langle\tau_\alpha\rangle \nleq P\Sigma L(2,q)_{0B}$, we may, by applying Dedekind's law, assume that

$$G_{0B}=PSL(2,q)_{0B}\rtimes(G\cap H).$$

Thus, by Remark 11, we obtain

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}|\,|G\cap H|=(k-1)(k-2)(k-3)(k-4)\,|G\cap\langle\tau_\alpha\rangle|\,.$$

Using that $k=|0^{G_B}|=\left[G_B:G_{0B}\right]$, we have more precisely

(A*) if $G = PSL(2,q) \rtimes (G \cap H)$:

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (k-1)(k-2)(k-3)(k-4)$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \frac{|PSL(2,q)_B|}{k}, \quad \text{or}$$

(B*) if $G = P\Sigma L(2,q)$:

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (k-1)(k-2)(k-3)(k-4)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \frac{|PSL(2,q)_B|}{k} \cdot \begin{cases} s, & \text{if } G_B = PSL(2,q)_B \rtimes \langle \tau_\alpha \rangle \\ 1, & \text{if } G_B = PSL(2,q)_B \rtimes H. \end{cases}$$

Considering condition (A*), we may argue exactly as in the earlier case $N = G$. Thus, only condition (B*) has to be examined, and we will show in the following that here $G \leq \mathrm{Aut}(\mathcal{D})$ cannot act flag-transitively on any non-trivial Steiner 5-design $\mathcal{D}$. In view of the subgroups of $PSL(2,q)$ with their orbits on the projective line (Lemmas 12-20), we have to examine the following possibilities:

(i) $PSL(2,q)_B$ is conjugate to a cyclic subgroup of order c with $c \mid \frac{q\pm1}{2}$ of $PSL(2,q)$, and $k = c$.
(ii) $PSL(2,q)_B$ is conjugate to a dihedral subgroup of order $2c$ with $c \mid \frac{q\pm1}{2}$ of $PSL(2,q)$, and $k = c$ or $2c$.
(iii) $PSL(2,q)_B$ is conjugate to an elementary Abelian subgroup of order $\overline{q} \mid q$ of $PSL(2,q)$, and $k = \overline{q}$.
(iv) $PSL(2,q)_B$ is conjugate to a semi-direct product of an elementary Abelian subgroup of order $\overline{q} \mid q$ with a cyclic subgroup of order c of $PSL(2,q)$ with $c \mid \overline{q}-1$ and $c \mid q-1$, and $k = \overline{q}$ or $c\overline{q}$.
(v) $PSL(2,q)_B$ is conjugate to $PSL(2,\overline{q})$ with $\overline{q}^m = q$, $m \geq 1$, and $k = \overline{q}+1$, $\overline{q}(\overline{q}-1)$ if m is even, or $k = (\overline{q}+1)\overline{q}(\overline{q}-1)/2$.
(vi) $PSL(2,q)_B$ is conjugate to $PGL(2,\overline{q})$ with $\overline{q}^m = q$, $m > 1$ even, and $k = \overline{q}+1, \overline{q}(\overline{q}-1)$ or $k = (\overline{q}+1)\overline{q}(\overline{q}-1)$.
(vii) $PSL(2,q)_B$ is conjugate to A_4, and $k = 6$ or 12.
(viii) $PSL(2,q)_B$ is conjugate to S_4, and $k = 6$ or 24.
(ix) $PSL(2,q)_B$ is conjugate to A_5, and $k = 10$, 12 or 60.

Since $q = 3^{s^u}$, we can write $\overline{q} = 3^{s^w}$ for some integer $0 \leq w \leq u$, and $q = \overline{q}^m = \overline{q}^{s^{u-w}}$.

ad (i): By condition (B*), we have

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (c-1)(c-2)(c-3)(c-4)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \begin{cases} s, & \text{or} \\ 1. \end{cases}$$

In view of the earlier case $N = G$, it is sufficient to consider the equation

$$(q-2)(q-3) = \frac{(c-1)(c-2)(c-3)(c-4)s}{2}. \tag{13}$$

For $c \mid \frac{q+1}{2}$, equation (13) yields

$$c \mid \frac{(q+1)(q-6)}{2} = \frac{(q-2)(q-3)}{2} - 6 = \frac{(c-1)(c-2)(c-3)(c-4)s}{4} - 6$$
$$= \frac{cs}{4}(c^3 - 10c^2 + 35c - 50) + 6s - 6,$$

and thus $c \mid 6s - 6$ must hold. If $c \mid \frac{q-1}{2}$, then, by equation (13), we have

$$c \mid \frac{(q-1)(q-4)}{2} = \frac{(q-2)(q-3)}{2} - 1 = \frac{(c-1)(c-2)(c-3)(c-4)s}{4} - 1$$
$$= \frac{cs}{4}(c^3 - 10c^2 + 35c - 50) + 6s - 1,$$

and hence $c \mid 6s - 1$ must hold. As clearly $c < 6s$ in both cases, it follows from equation (13) that in particular

$$(3^{s^u} - 2)(3^{s^u - 1} - 1) < \frac{c^4 s}{6} < 6^3 \cdot s^5,$$

which implies that $s^u \leq 7$. As $c \mid 6s - 6$ respectively $c \mid 6s - 1$, this leaves only a very small number of possibilities for k to check, which can easily be ruled out by hand using equation (13).

ad (ii): Let $k = c$. Applying condition (B*) yields

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (c-1)(c-2)(c-3)(c-4)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = 2 \cdot \begin{cases} s, & \text{or} \\ 1. \end{cases}$$

First, let $k = c$. Due to the earlier case $N = G$, it is sufficient to consider the equation

$$(q-2)(q-3) = \frac{(c-1)(c-2)(c-3)(c-4)s}{4}. \tag{14}$$

If $c \mid \frac{q+1}{2}$, then, by equation (14), we have

$$c \mid \frac{(q+1)(q-6)}{2} = \frac{(q-2)(q-3)}{2} - 6 = \frac{(c-1)(c-2)(c-3)(c-4)s}{8} - 6$$
$$= \frac{cs}{8}(c^3 - 10c^2 + 35c - 50) + 3s - 6,$$

and hence $c \mid 3s - 6$ must hold. For $c \mid \frac{q-1}{2}$, it follows from equation (14) that

$$c \mid \frac{(q-1)(q-4)}{2} = \frac{(q-2)(q-3)}{2} - 1 = \frac{(c-1)(c-2)(c-3)(c-4)s}{8} - 1$$
$$= \frac{cs}{8}(c^3 - 10c^2 + 35c - 50) + 3s - 1,$$

and thus $c \mid 3s-1$ must hold. Obviously, we have $c < 3s$ in both cases, and therefore equation (14) gives in particular

$$4(3^{s^u}-2)(3^{s^u-1}-1) < \frac{c^4 s}{3} < 3^3 \cdot s^5,$$

which implies that $s^u \leq 5$. Due to the fact that $c \mid 3s-6$ respectively $c \mid 3s-1$, we have only a very small number of possibilities for k to check, which can easily be ruled out by hand using equation (14). Now, let $k = 2c$. Due to condition (B*), we have

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (2c-1)(2c-2)(2c-3)(2c-4)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \begin{cases} s, & \text{or} \\ 1. \end{cases}$$

Again, it suffices to consider the equation

$$\frac{(q-2)(q-3)}{2} = (2c-1)(c-1)(2c-3)(c-2)s. \tag{15}$$

For $c \mid \frac{q+1}{2}$, equation (15) yields

$$c \mid \frac{(q+1)(q-6)}{2} = \frac{(q-2)(q-3)}{2} - 6 = (2c-1)(c-1)(2c-3)(c-2)s - 6$$
$$= cs(4c^3 - 20c^2 + 35c - 25) + 6s - 6,$$

and thus $c \mid 6s-6$ must hold. If $c \mid \frac{q-1}{2}$, then due to equation (15), we have

$$c \mid \frac{(q-1)(q-4)}{2} = \frac{(q-2)(q-3)}{2} - 1 = (2c-1)(c-1)(2c-3)(c-2)s - 1$$
$$= cs(4c^3 - 20c^2 + 35c - 25) + 6s - 1,$$

and hence $c \mid 6s-1$ must hold. As clearly $c < 6s$ in both cases, we deduce from equation (15) that in particular

$$(3^{s^u}-2)(3^{s^u-1}-1) < \frac{(2c)^4 s}{6} < 2^4 \cdot 6^3 \cdot s^5,$$

and hence it follows that $s^u \leq 7$. Since we have $c \mid 6s-6$ respectively $c \mid 6s-1$, this leaves only a very small number of possibilities for k to check, which can easily be ruled out by hand using equation (15).

ad (iii): In view of condition (B*), we have

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (\overline{q}-1)(\overline{q}-2)(\overline{q}-3)(\overline{q}-4)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \begin{cases} s, & \text{or} \\ 1. \end{cases}$$

It suffices to consider the equation

$$2(q-2)(q-3) = (\overline{q}-1)(\overline{q}-2)(\overline{q}-3)(\overline{q}-4)s. \tag{16}$$

As we may assume that $k=\overline{q}=3^{s^w}>5$, we have in particular $w\geq 1$, and hence $s<3^{s^w}=\overline{q}$. Thus, using equation (16), we obtain

$$(\overline{q}^{s^{u-w}}-2)(\overline{q}^{s^{u-w}}-3)=(q-2)(q-3)<\overline{q}^4 s<\overline{q}^5.$$

But, as clearly $u-w\geq 1$ (otherwise, $k=q$, a contradiction to Corollary 10), this yields a contradiction for $s\geq 3$. If $s=2$, then $(\overline{q}^{2^{u-w}}-2)(\overline{q}^{2^{u-w}}-3)<2\overline{q}^4$ must hold, which cannot be true for $u-w>1$. Thus, let $u-w=1$. Hence, it follows from equation (16) that in particular

$$\overline{q}-2 \,\big|\, (q-2)(q-3)=\overline{q}^4-5\overline{q}^2+6.$$

But, it is easily seen that $(\overline{q}^4-5\overline{q}^2+6,\overline{q}-2)=(2,\overline{q}-2)=1$, yielding a contradiction.

ad (iv): Let $k=\overline{q}$. By condition (B*), we have

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}|=(\overline{q}-1)(\overline{q}-2)(\overline{q}-3)(\overline{q}-4)s$$
$$\text{with}\quad |PSL(2,q)_{0B}|=c\cdot\begin{cases} s, & \text{or}\\ 1. \end{cases}$$

As $c\mid\overline{q}-1$, we may argue, mutatis mutandis, as in subcase (iii). For $k=c\overline{q}$, condition (B*) yields

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}|=(c\overline{q}-1)(c\overline{q}-2)(c\overline{q}-3)(c\overline{q}-4)s$$
$$\text{with}\quad |PSL(2,q)_{0B}|=\begin{cases} s, & \text{or}\\ 1. \end{cases}$$

We may consider only the equation

$$2(q-2)(q-3)=(c\overline{q}-1)(c\overline{q}-2)(c\overline{q}-3)(c\overline{q}-4)s. \tag{17}$$

Then, surely $(q-2)(q-3)=q^2-5q+6$ must be divisible by $c\overline{q}-3$. Polynomial division with remainder gives

$$q^2-5q+6=\left(\sum_{i=1}^{m}3^{i-1}\frac{q^2}{(c\overline{q})^i}+\sum_{j=1}^{\overline{m}}3^{j-1}\frac{\left((\frac{3}{c})^m-5\right)q}{(c\overline{q})^j}\right)\left(c\overline{q}-3\right)$$
$$+\left(\frac{3}{c}\right)^{\overline{m}}\frac{\left((\frac{3}{c})^m-5\right)q}{\overline{q}^{\overline{m}}}+6$$

for a suitable $\overline{m}\in\mathbb{N}$ (such that

$$\deg\left(\left(\frac{3}{c}\right)^{\overline{m}}\frac{\left((\frac{3}{c})^m-5\right)q}{\overline{q}^{\overline{m}}}+6\right)<\deg\left(c\overline{q}-3\right)$$

as is well-known). As $c\mid q-1$, clearly c is not divisible by 3. Thus, the remainder can be rewritten as

$$\frac{\left((\frac{3}{c})^m-5\right)}{c^{\overline{m}}}\cdot 3^{s^u-\overline{m}(s^w-1)}+6,$$

and hence in order for the remainder to vanish, necessarily $s^u - \overline{m}(s^w - 1) = 1$ must hold. But then, we obtain $3^m = (-2c^{\overline{m}} + 5)c^m$, a contradiction.

ad (v): Let $k = \overline{q} + 1$. In view of condition (B*), we have

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = \overline{q}(\overline{q}-1)(\overline{q}-2)(\overline{q}-3)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \frac{\overline{q}(\overline{q}-1)}{2} \cdot \begin{cases} s, & \text{or} \\ 1. \end{cases}$$

Again, it suffices to consider the equation

$$(q-2)(q-3) = (\overline{q}-2)(\overline{q}-3)s. \tag{18}$$

As we may assume that $k = \overline{q} + 1 = 3^{s^w} + 1 > 5$, it follows in particular that $w \geq 1$, and hence $s < 3^{s^w} = \overline{q}$. Thus, using equation (18), we obtain

$$(\overline{q}^{s^{u-w}} - 2)(\overline{q}^{s^{u-w}} - 3) = (q-2)(q-3) = (\overline{q}-2)(\overline{q}-3)s < (\overline{q}^2 - 2s)(\overline{q}-3).$$

But, as clearly $u - w \geq 1$ (otherwise, $k = q + 1$, a contradiction to Corollary 10), this yields a contradiction for every prime s. If $m > 1$ even and $k = \overline{q}(\overline{q}-1)$, then, in view of condition (B*), we have

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (\overline{q}^2-\overline{q}-1)(\overline{q}^2-\overline{q}-2)(\overline{q}^2-\overline{q}-3)(\overline{q}^2-\overline{q}-4)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \frac{(\overline{q}+1)}{2} \cdot \begin{cases} s, & \text{or} \\ 1. \end{cases}$$

We may consider only the equation

$$(q-2)(q-3)(\overline{q}+1) = (\overline{q}^2-\overline{q}-1)(\overline{q}^2-\overline{q}-2)(\overline{q}^2-\overline{q}-3)(\overline{q}^2-\overline{q}-4)s.$$

As obviously $(\overline{q}^2 - \overline{q} - 1, \overline{q} + 1) = 1$, it follows that $\overline{q}^2 - \overline{q} - 1 \mid (q-2)(q-3)$ must hold. But, for $m > 1$ even, polynomial division with remainder gives

$$q^2 - 5q + 6 = \left(\sum_{i=1}^{m-1} n_i \frac{q^2}{\overline{q}^{i+1}} + \sum_{j=1}^{m} (n_j \cdot n_m + n_{j-1}(n_{m-1}-5))\frac{q}{\overline{q}^j}\right)\left(\overline{q}^2 - \overline{q} - 1\right)$$
$$+ (n_{m+1} \cdot n_m + n_m(n_{m-1}-5))\overline{q} + n_m^2 + n_{m-1}(n_{m-1}-5) + 6,$$

where n_i denote the i-th Fibonacci number recursively defined via

$$n_0 = 0,\ \ n_1 = n_2 = 1,\ \ n_i = n_{i-1} + n_{i-2}\ \ (i \geq 3).$$

As it can easily be seen the remainder never vanishes, and hence we obtain a contradiction. For $k = (\overline{q}+1)\overline{q}(\overline{q}-1)/2$, condition (B*) yields

$$2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (\frac{\overline{q}^3-\overline{q}}{2}-1)(\frac{\overline{q}^3-\overline{q}}{2}-2)(\frac{\overline{q}^3-\overline{q}}{2}-3)(\frac{\overline{q}^3-\overline{q}}{2}-4)s$$
$$\text{with} \quad |PSL(2,q)_{0B}| = \begin{cases} s, & \text{or} \\ 1. \end{cases}$$

It suffices to consider the equation

$$2(q-2)(q-3) = (\frac{\overline{q}^3-\overline{q}}{2}-1)(\frac{\overline{q}^3-\overline{q}}{2}-2)(\frac{\overline{q}^3-\overline{q}}{2}-3)(\frac{\overline{q}^3-\overline{q}}{2}-4)s. \quad (19)$$

If we assume that $\overline{q}=3$, then $k=12$. Thus, we obtain from equation (19) that $s^u<5$. Hence, there are only a very small number of possibilities to check, which can easily be ruled out by hand. Therefore, let us assume that $\overline{q}>3$. Then, we have in particular $w\geq 1$, and hence $s<3^{s^w}=\overline{q}$. Thus, using equation (19), we obtain

$$2(q-2)(q-3) < (\frac{\overline{q}^3-\overline{q}}{2})^4 s < \frac{1}{16}\overline{q}^{12}s < \frac{1}{16}\overline{q}^{13}.$$

On the other hand, it follows that

$$\begin{aligned} 2(q-2)(q-3) &= (\frac{\overline{q}^3-\overline{q}}{2}-1)(\frac{\overline{q}^3-\overline{q}}{2}-2)(\frac{\overline{q}^3-\overline{q}}{2}-3)(\frac{\overline{q}^3-\overline{q}}{2}-4)s \\ &\geq 2(\frac{\overline{q}^3-\overline{q}}{2}-1)(\frac{\overline{q}^3-\overline{q}}{2}-2)(\frac{\overline{q}^3-\overline{q}}{2}-3)(\frac{\overline{q}^3-\overline{q}}{2}-4) \\ &= \frac{1}{8}\overline{q}^{12}-l \end{aligned}$$

with $l=\frac{1}{2}\overline{q}^{10}+\frac{5}{2}\overline{q}^9-\frac{3}{4}\overline{q}^8-\frac{15}{2}\overline{q}^7-17\overline{q}^6+\frac{15}{2}\overline{q}^5+\frac{279}{8}\overline{q}^4+\frac{95}{2}\overline{q}^3-\frac{35}{2}\overline{q}^2-50\overline{q}-48$. As for $\overline{q}>3$, clearly $l<\frac{1}{16}\overline{q}^{12}$ holds, we obtain

$$2(q-2)(q-3)\geq \frac{1}{16}\overline{q}^{12}.$$

But as $2(q-2)(q-3)=2(\overline{q}^{2m}-5\overline{q}^m+6)$, this leaves at most only $m=6$, which clearly cannot occur since $m=s^{u-w}$.

ad (vi): We may argue, mutatis mutandis, as in subcase (v).

ad (vii): In view of condition (B*), we have

$$\begin{aligned} &2(q-2)(q-3)\,|PSL(2,q)_{0B}| = (k-1)(k-2)(k-3)(k-4)s \\ &\text{with} \quad |PSL(2,q)_{0B}| = \frac{12}{k}\cdot\begin{cases} s, & \text{or} \\ 1. \end{cases} \end{aligned}$$

It is sufficient to consider the equation

$$(3^{s^u}-2)(3^{s^u}-3) = \frac{k(k-1)(k-2)(k-3)(k-4)}{24}\cdot s.$$

Thus, for $k=6$ respectively $k=12$, we obtain $s^u\leq 2$ respectively $s^u<5$, and thus we have only a very small number of possibilities to check, which can easily be ruled out by hand.

ad (viii) and (ix): These subcases can be treated similarly to subcase (vii), completing the examination of condition (B*).

Case (3): $N=M_v$, $v=11,12,22,23,24$.

If $v = 12$ or 24, then $G = M_v$ is always 5-transitive, and thus [27, Thm. 3] yields the designs described in Main Theorem 1. Obviously, flag-transitivity holds as the 5-transitivity of G implies that G_x acts block-transitively on the derived Steiner 4-design $\mathcal{D}_x$ for any $x \in X$. By Corollary 10, we obtain for $v = 11$ that $k \leq 6$, and for $v = 22$ or 23 that $k \leq 8$, and the very small number of cases for k can easily be ruled out by hand using Lemma 7.

Case (4): $N = M_{11}$, $v = 12$.

As it is known, this exceptional permutation action occurs inside the Mathieu group M_{24} in its action on 24 points. This set can be partitioned into two sets X_1 and X_2 of 12 points each such that the setwise stabilizer of X_1 is the Mathieu group M_{12}. The stabilizer in this latter group of a point x in X_1 is isomorphic to M_{11} and operates (apart from its natural 4-transitive action on $X_1 \setminus \{x\}$) 3-transitively on the 12 points of X_2. The geometry preserved by the 3-transitive action of M_{11} is not a Steiner t-design, but a 3-(12, 6, 2) design (e.g. [2, Ch., IV, 5.3]).

This completes the proof of Main Theorem 1.

5 The non-existence of flag-transitive Steiner 6-designs

We prove the following result:

Main Theorem 2 *There are no non-trivial Steiner* 6*-designs* $\mathcal{D}$ *admitting a flag-transitive group* $G \leq \mathrm{Aut}(\mathcal{D})$ *of automorphisms.*

5.1 Groups of automorphisms of affine type

In the following, we begin with the proof of Main Theorem 2. Using the notation as before, *let us assume that* $\mathcal{D} = (X, \mathcal{B}, I)$ *is a non-trivial Steiner* 6*-design with* $G \leq \mathrm{Aut}(\mathcal{D})$ *acting flag-transitively on* $\mathcal{D}$ *throughout the proof*. Clearly, in the sequel we may assume that $k > 6$ as trivial Steiner 6-designs are excluded. We will examine in this subsection successively those cases where G is of affine type.

Case (1): $G \cong AGL(1,8)$, $A\Gamma L(1,8)$, or $A\Gamma L(1,32)$.

We may assume that $k > 6$. If $v = 8$, then Corollary 10 would imply that $k = 6$. For $v = 32$, we have $|G| = 5v(v-1)$ and Lemma 7 immediately yields that $G \leq \mathrm{Aut}(\mathcal{D})$ cannot act flag-transitively on any non-trivial Steiner 6-design $\mathcal{D}$.

Case (2): $G_0 \cong SL(d,2)$, $d \geq 2$.

We may argue, mutatis mutandis, as in the corresponding case in Main Theorem 1.

Case (3): $G_0 \cong A_7$, $v = 2^4$.

As $v = 2^4$, we have $k \leq 8$ by Corollary 10. But, Lemma 8 (c) obviously eliminates the cases when $k = 7$ or 8.

5.2 Groups of automorphisms of almost simple type

We will examine in this subsection successively those cases where G is of almost simple type.

Case (1): $N = A_v$, $v \geq 5$.

We may assume that $v \geq 8$. But then A_v, and hence also G, is 6-transitive and does not act on any non-trivial Steiner 6-design $\mathcal{D}$ due to [27, Thm. 3].

Case (2): $N = PSL(2,q)$, $v = q+1$, $q = p^e > 3$.

For the existence of flag-transitive Steiner 6-designs, necessarily

$$r = \frac{q(q-1)(q-2)(q-3)(q-4)}{(k-1)(k-2)(k-3)(k-4)(k-5)} \,\Big|\, |G_0| \,\Big|\, |P\Gamma L(2,q)_0| = q(q-1)e$$

must hold in view of Lemma 7. Thus, we have in particular

$$(q-2)(q-3)(q-4) \,\big|\, (k-1)(k-2)(k-3)(k-4)(k-5)e, \text{ where } e \leq \log_2 q. \quad (20)$$

But, on the other hand, Corollary 10 yields $k \leq \lfloor \sqrt{q+1} + \frac{9}{2} \rfloor < q^{\frac{1}{2}} + 5$. Hence, in view of property (20), we have only a small number of possibilities to check, which can easily be ruled out by hand using Lemma 8 (c). Therefore, $G \leq \mathrm{Aut}(\mathcal{D})$ cannot act flag-transitively on any non-trivial Steiner 6-design $\mathcal{D}$. This has also been proven in [9, Cor. 4.3], whereas our estimation is slightly better.

Case (3): $N = M_v$, $v = 11, 12, 22, 23, 24$.

Due to Corollary 10, we obtain for $v = 11$ or 12 that $k \leq 7$, and for $v = 22, 23$ or 24 that $k \leq 9$, and the very small number of cases for k can easily be eliminated by hand using Lemma 7.

Case (4): $N = M_{11}$, $v = 12$.

By the same arguments as in the corresponding case in Main Theorem 1, it follows that $G \leq \mathrm{Aut}(\mathcal{D})$ cannot act on any Steiner t-design $\mathcal{D}$.

This completes the proof of Main Theorem 2.

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