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Elementary Abelian p-groups of rank greater than or equal to $4p-2$ are not CI-groups

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Abstract In this paper we prove that an elementary Abelian p-group of rank $4p-2$ is not a $\mathrm{CI}^{(2)}$-group, i.e. there exists a 2-closed transitive permutation group containing two non-conjugate regular elementary Abelian p-subgroups of rank $4p-2$, see Hirasaka and Muzychuk (J. Comb. Theory Ser. A **94**(2), 339–362, 2001). It was shown in Hirasaka and Muzychuk (loc cit) and Muzychuk (Discrete Math. **264**(1–3), 167–185, 2003) that this is related to the problem of determining whether an elementary Abelian p-group of rank n is a CI-group.

As a strengthening of this result we prove that an elementary Abelian p-group E of rank greater or equal to $4p-2$ is not a CI-group, i.e. there exist two isomorphic Cayley digraphs over E whose corresponding connection sets are not conjugate in $\mathrm{Aut}\, E$.

Keywords Cayley graph · CI-group · Schur ring · 2-closure

1 Introduction

Let H be a group and S a subset of H. The Cayley digraph of H with connection set S, denoted $\mathrm{Cay}(H,S)$, is the graph with vertex set H and edge set $\{(h_1,h_2)\mid h_1h_2^{-1}\in S\}$. Two Cayley digraphs $\mathrm{Cay}(H,S)$ and $\mathrm{Cay}(H,T)$ are said to be *Cayley isomorphic* if there exists an element $g\in \mathrm{Aut}\, H$ such that $S^g=T$. A subset S of a group H is said to be a CI-subset (or Cayley isomorphic subset) if for each $T\subseteq H$ the digraphs $\mathrm{Cay}(H,S)$ and $\mathrm{Cay}(H,T)$ are isomorphic if and only if they are Cayley isomorphic. Finally, a group H is said to be a CI-group if each subset of

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H is a CI-subset. We refer the interested reader to the survey article [5] for details, examples and the main results on CI-groups. We remark that the classification of CI-groups is not known and at the time of this writing it seems that the classification problem strongly depends on whether an elementary Abelian p-group of rank n is a CI-group, see [5].

It was proved in [1] that $S \subseteq H$ is a CI-subset if and only if any regular subgroup of $\mathrm{Aut}(\mathrm{Cay}(H, S))$ isomorphic to H is conjugate to H in $\mathrm{Aut}(\mathrm{Cay}(H, S))$. This result is the starting point of most of the results on CI-groups.

In [3] the authors proved that if V is a regular elementary Abelian p-subgroup of rank n ($n \leq 4$, $p > 2$) of a 2-closed permutation group G, then any regular subgroup W of G isomorphic to V is conjugate to V in G. In particular, by the former paragraph, as the automorphism group of a digraph is a 2-closed group we have that an elementary Abelian p-group of rank less than or equal to 4 is a CI-group. Motivated by this result the authors of [3] gave the following definition, see [3] page 341.

Definition 1 The group H is said to be a $\mathrm{CI}^{(2)}$-group if for any 2-closed permutation group G containing the right regular permutation representation of H, we have that any two regular subgroups of G isomorphic to H are conjugate in G.

Clearly if H is a $\mathrm{CI}^{(2)}$-group, then H is a CI-group. The authors of [3] asked for a complete classification of the $\mathrm{CI}^{(2)}$-elementary Abelian p-groups. At the time of this writing it is known that if an elementary Abelian p-group of rank n is a $\mathrm{CI}^{(2)}$-group, then $n < 2p - 1 + \binom{2p-1}{p}$, see [6], and, if $n \leq 4$, then an elementary Abelian p-group of rank n is a $\mathrm{CI}^{(2)}$-group, see [3].

In Sect. 3 we prove the following theorem.

Theorem 1 *An elementary Abelian p-group of rank $n \geq 4p - 2$ is not a* $\mathrm{CI}^{(2)}$*-group.*

We remark that a proof of Theorem 1 could be given using cohomological arguments. As a matter of consistency we present a proof closely related to the arguments in [6].

Finally, in Sect. 4, we prove Theorem 2, which is the main result of this paper.

Theorem 2 *An elementary Abelian p-group of rank n is not a* CI*-group if $n \geq 4p-2$.*

We remark that although Theorem 1 is a corollary of Theorem 2, the proof of the former is needed in order to prove the latter.

We refer to the precious book [9] for the main results on Schur rings and to [3] and [6] for their connections to $\mathrm{CI}^{(2)}$-groups and for the notation. We strongly advise the reader to use [6] as a crib.

2 Preliminaries

Let p be a prime number, let V, W be non-trivial elementary Abelian p-groups and let $U = V \mathrm{wr}_W W$ be the wreath product of V with W acting on $\Omega = W \times V$ in its natural imprimitive action. We recall that $(x, y)^{wf} = (x + w, y + f(x + w))$ for any

$(x, y) \in \Omega$ and $wf \in U$. We denote by $B = V^W$ the base group of U. We denote by $\xi_m(U)$ then mth term of the upper central series of U. In particular, the centre $\xi_1(U)$ of U consists of the constant functions of B. We shall often identify a constant function of B, i.e. an element of $\xi_1(U)$, with its image. Let G be a subgroup of U containing $W\xi_1(U)$. Note that $W\xi_1(U)$ is an elementary Abelian regular subgroup of U. Furthermore $G = WL$ where $L = G \cap B$. We denote by L_0 the stabilizer in G of the element $(0, 0)$ of Ω. The group L_0 defines a map $H : W \to 2^V = \{S \mid S \subseteq V\}$ by $H(w) = \{f(w) \mid f \in L_0\}$.

Lemma 1 *H satisfies the following properties*:

(i) *$H(w)$ is a subspace of V*;
(ii) $H(0) = \{0\}$;
(iii) *$H(\beta w) = H(w)$ for any $\beta \neq 0$ in $\mathbb{F}_p$*;
(iv) *$H(w_1 + w_2) \subseteq H(w_1) + H(w_2)$ for any $w_1, w_2 \in W$.*

Proof (i) L_0 is an $\mathbb{F}_p$-vector subspace of B so $H(w)$ is an $\mathbb{F}_p$-vector subspace of V.

(ii) By definition of L_0 we get that $H(0) = \{0\}$.

(iii) Let f be in L_0 and w in W. If $\beta \in \mathbb{F}_p$, then $g_\beta = f^{-\beta w} - f(\beta w)$ (here we are identifying $f(\beta w) \in V$ with the constant function $x \mapsto f(\beta w)$) is an element in L_0 (in fact $g_\beta(0) = 0$) and $g_\beta(w) = f((\beta + 1)w) - f(\beta w)$. Thus $f(w)$, $f(2w) - f(w), \dots, f((p-1)w) - f((p-2)w)$ and $-f((p-1)w)$ are elements of $H(w)$. So $f(\beta w) \in H(w)$ for any $\beta \in \mathbb{F}_p$. In particular $H(\beta w) \subseteq H(w)$ for any $\beta \in \mathbb{F}_p$. If $\beta \neq 0$, we have $H(w) = H(\beta w)$.

(iv) Now let f be in L_0 and $w_1, w_2 \in W$. Consider $g = f^{-w_2} - f(w_2)$ (as usual $f(w_2)$ denotes the element of $\xi_1(U)$ with image $f(w_2)$). Now $g \in L$ and $g(0) = (f^{-w_2} - f(w_2))(0) = f(w_2) - f(w_2) = 0$, so $g \in L_0$. Moreover $g(w_1) = f(w_1 + w_2) - f(w_2)$, therefore $f(w_1 + w_2) = g(w_1) + f(w_2) \in H(w_1) + H(w_2)$. □

We denote by $\mathrm{Der}(W, L)$ the group of derivations from W to L and by $\mathrm{Inn}(W, L)$ the group of inner derivations, see [8].

Lemma 2 *Any regular elementary Abelian subgroup of G is conjugate to $W\xi_1(U)$ if and only if* $\mathrm{Der}(W, L) = \mathrm{Hom}(W, \xi_1(U)) + \mathrm{Inn}(W, L)$.

Proof Let δ be an element in $\mathrm{Der}(W, L)$. Then $H = X_\delta \xi_1(U)$ is a regular elementary Abelian subgroup of G, where $X_\delta = \{ww^\delta \mid w \in W\}$. In particular if H is conjugate to $W\xi_1(U)$, then there exists $l \in L$ such that $X_\delta \subseteq W^l \xi_1(U)$. In other words, for any $w \in W$ there exists a unique $c_w \in \xi_1(U)$ such that $w^\delta = [w, l] + c_w$. The map $c_- : w \mapsto c_w$ is an homomorphism of W into $\xi_1(U)$. In fact

$$
\begin{aligned}
[w_1, l]^{w_2} + [w_2, l] + c_{w_1} + c_{w_2} &= ([w_1, l] + c_{w_1})^{w_2} + [w_2, l] + c_{w_2} \\
&= (w_1^\delta)^{w_2} + w_2^\delta \\
&= (w_1 + w_2)^\delta = [w_1 + w_2, l] + c_{w_1 + w_2}
\end{aligned}
$$

and so $c_{w_1+w_2} = c_{w_1} + c_{w_2}$. Thus $\delta = c_- + [-, l] \in \mathrm{Hom}(W, \xi_1(U)) + \mathrm{Inn}(W, L)$.

As any regular elementary Abelian subgroup H of G is of the form $X_\delta\xi_1(U)$, for some $\delta\in \mathrm{Der}(W,L)$, the other side of the implication is totally trivial. □

Assume that the group G is 2-closed. Every orbital of G corresponds to an orbit of the point stabilizer L_0, i.e. every orbital of G corresponds to a suborbit of the form $(w,v)^{L_0}=\{(w,f(w)+v)\mid f\in L_0\}=(w,v+H(w))$. The group G contains the regular subgroup $W\xi_1(U)$ and so every orbital is a Cayley digraph over $W\times V$. In particular the orbital corresponding to the suborbit $(w,v)^{L_0}$ is $\{((w_1,v_1),(w_2,v_2))\mid (w_1-w_2,v_1-v_2)\in(w,v+H(w))\}=\mathrm{Cay}(W\times V,(w,v+H(w)))$. Furthermore $G=\bigcap_{(w,v)}\mathrm{Aut}(\Gamma_{(w,v)})$, where $\Gamma_{(w,v)}=\mathrm{Cay}(W\times V,(w,v+H(w)))$.

Let us define the $\mathbb{F}_p$-vector space

$$\begin{aligned}\mathrm{Hom}_H(W,V)=\{\psi\in B\mid \psi(w_1+w_2)-\psi(w_1)-\psi(w_2)\in H(w_1)\cap H(w_2)\\ \text{for any } w_1,w_2\in W,\ \psi(0)=0\},\end{aligned}$$

see [6].

Next if $\delta\in\mathrm{Der}(W,L)$, we define $\theta(\delta):W\to V$ by $\theta(\delta)(w)=(w^\delta)(0)$.

Lemma 3 *θ is an injective homomorphism of* $\mathrm{Der}(W,L)$ *into* $\mathrm{Hom}_H(W,V)$. *If G is 2-closed, then θ is an isomorphism.*

Proof Clearly $\theta(\delta)(0)=(0^\delta)(0)=0$. Let w_1,w_2 be elements of W. Set $w_1^\delta=f+g$ where $f\in L_0$ and $g\in\xi_1(U)$. We have

$$\begin{aligned}\theta(\delta)(w_1+w_2)-\theta(\delta)(w_1)-\theta(\delta)(w_2)&=(w_1+w_2)^\delta(0)-w_1^\delta(0)-w_2^\delta(0)\\&=((w_1)^\delta)^{w_2}(0)+w_2^\delta(0)-w_1^\delta(0)-w_2^\delta(0)\\&=w_1^\delta(-w_2)-w_1^\delta(0)=f(-w_2)\in H(w_2).\end{aligned}$$

By symmetry (the group W is commutative) we get that

$$\theta(\delta)(w_1+w_2)-\theta(\delta)(w_1)-\theta(\delta)(w_2)\in H(w_1).$$

Therefore $\theta(\delta)\in\mathrm{Hom}_H(W,V)$.

The map θ is an homomorphism. If $\delta_1,\delta_2\in\mathrm{Der}(W,L)$ and $w\in W$, then $\theta(\delta_1+\delta_2)(w)=w^{\delta_1+\delta_2}(0)=w^{\delta_1}(0)+w^{\delta_2}(0)=(\theta(\delta_1)+\theta(\delta_2))(w)$. Moreover $\theta(0)=0$. Finally $w^{\alpha\delta}=\alpha w^\delta$ for any $\alpha\in\mathbb{F}_p$ and so $\theta(\alpha\delta)=\alpha\theta(\delta)$.

The map θ is injective. $\theta(\delta)=0$ if and only if $w^\delta(0)=0$ for any $w\in W$. So $w^\delta\in L_0$ for any $w\in W$. In particular $0=(w+w_1)^\delta(0)=((w^\delta)^{w_1}+w_1^\delta)(0)=w^\delta(-w_1)$ for any w_1,w. Thus $w^\delta=0$ for any w. Therefore $\delta=0$.

Assume that G is 2-closed. Let ψ be an element of $\mathrm{Hom}_H(W,V)$. Define $\eta(w)=\psi(-w)$ for any $w\in W$. The function η lies in the base group B. Furthermore η lies in $\mathrm{Hom}_H(W,V)$, in fact $\eta(0)=\psi(0)=0$ and $\eta(w_1+w_2)-\eta(w_1)-\eta(w_2)=\psi(-w_1-w_2)-\psi(-w_1)-\psi(-w_2)\in H(-w_1)\cap H(-w_2)=H(w_1)\cap H(w_2)$. We claim that η normalizes G, i.e. $[W,\eta]\subseteq L$. Let w' be in W. We have to prove that $[w',\eta]\in L$, i.e. $g=\eta-\eta^{w'}+\eta(-w')\in L_0$. We claim that $\Gamma_{(w,v)}^g=\Gamma_{(w,v)}$ for every

$(w, v) \in \Omega$. Let $((w_1, v_1), (w_2, v_2))$ be in $\Gamma_{(w,v)}$. In particular $w_1 - w_2 = w$ and $v_1 - v_2 = v + x$ for some x in $H(w)$. We have

$$((w_1, v_1), (w_2, v_2))^g = ((w_1, v_1 + g(w_1)), (w_2, v_2 + g(w_2)))$$

and

$$v_1 + g(w_1) - (v_2 + g(w_2)) = v + x + \eta(w_1) - \eta(w_1 - w') - \eta(w_2) + \eta(w_2 - w').$$

Now, the function η lies in $\mathrm{Hom}_H(W, V)$, therefore

$$\eta(w_2 + w) - \eta(w_2) - \eta(w) \in H(w)$$
$$-\eta(w_2 + w - w') + \eta(w_2 - w') + \eta(w) \in H(w)$$

so $\eta(w_1) - \eta(w_1 - w') - \eta(w_2) + \eta(w_2 - w') \in H(w)$. Hence $v_1 + g(w_1) - (v_2 + g(w_2)) \in v + H(w)$. This yields $\Gamma^g_{(w,v)} = \Gamma_{(w,v)}$ for every $(w, v) \in \Omega$. So, $g \in G^{(2)} = G$. Therefore $[W, \eta] \subseteq L$ and $[-, (-\eta)] \in \mathrm{Der}(W, L)$. Finally $\theta([-, (-\eta)])(w) = [w, -\eta](0) = (\eta^w - \eta)(0) = \eta(-w) - \eta(0) = \psi(w)$, so $\theta([-, (-\eta)]) = \psi$. This proves θ is surjective and $\mathrm{Der}(W, L) \cong \mathrm{Hom}_H(W, V)$. □

Lemma 4 *Assume that G is a 2-closed group. Any two regular elementary Abelian subgroups of G are conjugate if and only if for any $\psi \in \mathrm{Hom}_H(W, V)$ there exists $\Lambda \in \mathrm{Hom}(W, V)$ such that $(\psi - \Lambda)(w) \in H(w)$ for every $w \in W$.*

Proof Assume that any two regular elementary Abelian subgroups of G are conjugate and let $\psi \in \mathrm{Hom}_H(W, V)$. So, by Lemmas 2 and 3 we have $\psi = \theta([-, l] + \Lambda)$ for some $l \in L_0$ and $\Lambda \in \mathrm{Hom}(W, \xi_1(U))$. Clearly

$$\begin{aligned}(\theta([-, l] + \Lambda) - \Lambda)(w) &= w^{[-,l]+\Lambda}(0) - \Lambda(w) \\ &= [w, l](0) = (-l)^w(0) = (-l)(-w) \in H(w)\end{aligned}$$

for any $w \in W$. Conversely, by Lemma 3 it is enough to prove that $\mathrm{Der}(W, L) = \mathrm{Hom}(W, \xi_1(U)) + \mathrm{Inn}(W, L)$. Let δ be an element of $\mathrm{Der}(W, L)$. By hypothesis there exists $\Lambda \in \mathrm{Hom}(W, V)$ such that $(\theta(\delta) - \Lambda)(w) \in H(w)$ for any $w \in W$. Set $g(w) = (\theta(\delta) - \Lambda)(-w)$ for any $w \in W$. Clearly $\Gamma^g_{(w,v)} = \Gamma_{(w,v+g(w))} = \Gamma_{(w,v)}$ for any $(w, v) \in \Omega$. This yields $g \in L$. We leave the reader to check that $\delta = [-, (-g)] + \Lambda$. Thus the proof is complete. □

Summing up so far, a 2-closed subgroup G of U contains two non-conjugate elementary Abelian regular subgroups if and only if there exists $\psi \in \mathrm{Hom}_H(W, V)$ such that there does not exist $\Lambda \in \mathrm{Hom}(W, V)$ such that $(\psi - \Lambda)(w) \in H(w)$ for every $w \in W$, see Proposition 7 in [6]. Note that this description is based on the knowledge of H and in particular the group G does not appear.

Conversely, we show that a map H satisfying $(i), (ii), (iii), (iv)$ is intimately related to a 2-closed subgroup of U. For $(w, v) \in W \times V$ define $\Gamma_{(w,v)} = \{((w_1, v_1), (w_2, v_2)) \mid w_1 - w_2 = w, v_1 - v_2 \in H(w) + v\}$. Finally define, only for the remainder of this section, $A = \bigcap_{(w,v)} \mathrm{Aut}(\Gamma_{(w,v)})$. Note that $W \times V$ acting via right multiplication is a regular subgroup of $\mathrm{Aut}(\Gamma_{(w,v)})$. In particular $\Gamma_{(w,v)} = \mathrm{Cay}(W \times V, (w, v + H(w)))$.

We claim that A is a subgroup of U. Firstly we prove that $\Sigma = \{\{w\} \times V\}_{w\in W}$ is a block system for A. Let g be an element of A and Δ be $\{w\} \times V$. Assume that $\Delta^g \cap \Delta \neq \emptyset$. So, $(w, v_1)^g = (w, v_2)$ for some $v_1, v_2 \in V$. Let $(w, v) \in \Delta$. Then $((w, v), (w, v_1)) \in \Gamma_{(0,v-v_1)}$ so $((w, v), (w, v_1))^g = ((w, v)^g, (w, v_2)) \in \Gamma_{(0,v-v_1)}$. This implies that $(w, v)^g \in \Delta$, so $\Delta^g = \Delta$. Next, we prove that A acts regularly on Σ. The group A is transitive on Ω and so A is transitive on Σ. Assume that g fixes $\{w\} \times V$ and let $w_1 \in W$. Now, $(w, 0)^g = (w, v)$ for some $v \in V$ and $(w_1, 0)^g = (w_2, v_2)$ for some $w_2 \in W$ and $v_2 \in V$. Now $g \in \mathrm{Aut}(\Gamma_{(w_1-w,0)})$ and $((w_1, 0), (w, 0)) \in \Gamma_{(w_1-w,0)}$ so $((w_1, 0)^g, (w, 0)^g) = ((w_2, v_2), (w, v)) \in \Gamma_{(w_1-w,0)}$, so, $w_2 - w = w_1 - w$. Thus $w_1 = w_2$. This says that A acts regularly on Σ. Let us denote by L the kernel of the permutation representation of A on Σ (i.e. the stabilizer of the set $\{0_W\} \times V$). It remains to prove that L acts regularly on $\{0_W\} \times V$. The group L contains $\{0_W\} \times V$ and so L acts transitively on each set of the form $\{w\} \times V$. Let g be in L. Assume that $(0, 0)^g = (0, 0)$. Let v be in V and let us prove that $(0, v)^g = (0, v)$. Note that $((0, v), (0, 0)) \in \Gamma_{(0,v)}$ and so $((0, v), (0, 0))^g = ((0, v_1), (0, 0))$ is an element of $\Gamma_{(0,v)} = \mathrm{Cay}(W \times V, (0, v))$, so, $v_1 = v$. This proves that A is a subgroup of U.

3 The construction

Since the elementary Abelian 2-group of rank 6 is known not to be a CI-group, see [7], we assume that $p \geq 3$.

We strongly use the well-known structure of the upper central series of the group U, see [4].

Let V be the Galois field $\mathbb{F}_{p^n}$, where $n = 2p - 1$, and let $\varepsilon_1, \dots, \varepsilon_n$ be an $\mathbb{F}_p$-basis of V. Also, let W be an $\mathbb{F}_p$-vector space of dimension n with basis $e_1, \dots, e_n$ and dual basis $e_1^*, \dots, e_n^*$. Let $\mathcal{S}$ be the symmetric V-algebra on $e_1^*, \dots, e_n^*$ and natural filtration $\mathcal{S} = \oplus_{m\in\mathbb{N}}\mathcal{S}_m$ (recall that $\mathcal{S}_m$ is spanned by the monomials $e_{i_1}^{*j_1} \otimes \cdots \otimes e_{i_k}^{*j_k}$ of degree m). Let $\mathcal{M}_m$ be the V-subspace of $\mathcal{S}_m$ spanned by the monomials $e_{i_1}^{*j_1} \otimes \cdots \otimes e_{i_k}^{*j_k}$ of degree m such that $j_1, \dots, j_k \leq p - 1$. Let $\pi : \mathcal{S} \to B = V^W$ be the valuation map and $Z_i = \pi(\mathcal{M}_i)$. For instance $\pi(e_{i_1}^{*j_1} \otimes \cdots \otimes e_{i_k}^{*j_k})(w)$ is the element of V defined by $(e_{i_1}^*(w))^{j_1} \cdots (e_{i_k}^*(w))^{j_k}$ for any $w \in W$. It is known that $B \cap \xi_m(U) = \oplus_{i\leq m-1} Z_i$ and that $\pi_{|\mathcal{M}_i}$ is injective. Without loss of generality we identify $e_{i_1}^{*j_1} \otimes \cdots \otimes e_{i_k}^{*j_k}$ in $\mathcal{M}_m$ with its image under π.

Let X be the set $\{1, \dots, n\}$ and $X_i = \{A \subseteq X \mid |A| = i\}$ for $i = 0, \dots, p$. If A is a subset of X, then we denote by e_A^* the element of B defined by $\otimes_{i\in A} e_i^*$. Similarly ε_A denotes $\sum_{i\in A} \varepsilon_i \in V$. We let ε and e denote ε_X and $\sum_{i\in X} e_i$, respectively.

Consider

$$f = \sum_{A\in X_p} \varepsilon_{X\setminus A} e_A^* \in B.$$

Define $L = [W, f]$. If $w_1, w_2 \in W$, then $[w_1 + w_2, f] = [w_1, f]^{w_2} + [w_2, f]$. This shows that $G = \langle W, L\rangle = WL$. Since $[G, f] = [W, f] \subseteq G$, we have that f normalizes G.

Let C be a subset of X of size $p - i$. Define

$$g_C = \sum_{B\in X_i, C\cap B=\emptyset} \varepsilon_{X\setminus(B\cup C)} e^*_B.$$

Lemma 5 *The group L is generated by the set $\{g_C \mid C \in X_i, 1 \le i \le p\}$ and L_0 is generated by $\{g_C \mid C \in X_i, 1 \le i \le p - 1\}$.*

Proof If $i \in A$, then $(e^*_A)^{e_i}(w) = \prod_{a\in A} e^*_a(w - e_i) = e^*_A(w) - e^*_{A\setminus\{i\}}(w)$ and so

$$(\dagger) \qquad [e_i, e^*_A] = \begin{cases} 0 & \text{if } i \notin A, \\ e^*_{A\setminus\{i\}} & \text{if } i \in A. \end{cases}$$

Using (†) we get

$$[e_i, f] = \sum_{B\in X_{p-1}, i\notin B} \varepsilon_{X\setminus(B\cup\{i\})} e^*_B = g_{\{i\}} \in L \quad \text{for any } i \in X.$$

More generally we have

$$[e_i, g_C] = \begin{cases} 0 & \text{if } i \in C, \\ g_{C\cup\{i\}} & \text{if } i \notin C. \end{cases}$$

The group L is generated by the left-normed commutators $[f, e_{i_1}, \ldots, e_{i_k}]$ for $k \ge 1$ and therefore L is generated by $\{g_C \mid C \in X_i, 1 \le i \le p\}$. Our claim on L_0 is just an easy remark. □

Corollary 1 *G contains $\xi_1(U)$.*

Proof By Lemma 5 we have that $L \cap \xi_1(U)$ is generated by $\{g_C \mid C \in X_p\} = \{\varepsilon_{X\setminus C} \mid C \in X_p\}$. As usual we are identifying the elements of V with the elements of $\xi_1(U)$. We leave the reader to show that $\{\varepsilon_{X\setminus C} \mid C \in X_p\}$ spans V. □

By Corollary 1 we have that $U \supseteq G \supseteq W\xi_1(U)$ and so we can apply Section 2 to G.

Let φ be the non-degenerate bilinear symmetric form on V defined by $\varphi(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Lemma 6 *$H(e_i) = \varepsilon_i^\perp$ for any $i \in X$.*

Proof We have that $g_C(e_i) = 0$ for any $C \in X_{p-j}$, $j \ge 2$. Therefore, by Lemma 5, $H(e_i) = \langle g_C(e_i) \mid C \in X_{p-1}\rangle = \langle \varepsilon_{X\setminus(C\cup\{i\})} \mid i \notin C, C \in X_{p-1}\rangle$, while it is an easy exercise in linear algebra to prove that this vector space is $\varepsilon_i^\perp$. □

Lemma 7 *If $i \ne j$, then $H(e_i + e_j) = (\varepsilon_i - \varepsilon_j)^\perp$.*

Proof Arguing is the same way as in Lemma 6 we have $H(e_i+e_j)=\langle g_C(e_i+e_j)\mid C\in X_{p-2}\cup X_{p-1}\rangle$. It is routine computation in linear algebra to prove that $\langle g_C(e_i+e_j)\mid C\in X_{p-2}\cup X_{p-1}\rangle=\langle \varepsilon_{X\setminus(C\cup\{i,j\})},\varepsilon_{X\setminus(C'\cup\{i\})}+\varepsilon_{X\setminus(C'\cup\{j\})}\mid C\in X_{p-2}, C'\in X_{p-1}, i,j\notin C\cup C'\rangle=(\varepsilon_i-\varepsilon_j)^{\perp}$. □

Lemma 8 $H(e)=\varepsilon^{\perp}$.

Proof Let C be a set of size $p-i$, where $1\le i\le p-1$. Then

$$\varphi(g_C(e),\varepsilon)=\sum_{B\in X_i,C\cap B=\emptyset}\varphi(\varepsilon_{X\setminus(B\cup C)},\varepsilon)=\sum_{B\in X_i,C\cap B=\emptyset}(p-1)$$
$$=-\binom{p+i-1}{i}=0.$$

If $C\in X_{p-1}$, then we have $g_C(e)=\sum_{i\notin C}\varepsilon_{X\setminus(C\cup\{i\})}=-\varepsilon_{X\setminus C}$. Thus $\varepsilon^{\perp}\supseteq H(e)\supseteq\langle g_C(e)\mid C\in X_{p-1}\rangle=\langle\varepsilon_{X\setminus C}\mid C\in X_{p-1}\rangle=\varepsilon^{\perp}$. □

Lemma 9 $H(e-e_i)=(\varepsilon+\varepsilon_i)^{\perp}$ *for any* $i\in X$.

Proof Let C be a set of size $p-j$, where $1\le j\le p-1$. If $i\notin C, B\in X_j$, $i\notin B$, $B\cap C=\emptyset$, then $\varphi(\varepsilon_{X\setminus(C\cup B)},\varepsilon+\varepsilon_i)=0$. If $i\in C$, $B\in X_j$, $C\cap B=\emptyset$, then $\varphi(\varepsilon_{X\setminus(C\cup B)},\varepsilon+\varepsilon_i)=p-1$. The function e_A^* is not 0 on the element $e-e_i$ if and only if $i\notin A$. Thus

$$(*)\qquad \varphi(g_C(e-e_i),\varepsilon+\varepsilon_i)=\sum_{B\in X_j,i\notin B,C\cap B=\emptyset}\varphi(\varepsilon_{X\setminus(B\cup C)},\varepsilon+\varepsilon_i).$$

So, if $i\notin C$, then $(*)$ is equal to 0. If $i\in C$, then $(*)$ is equal to

$$\sum_{B\in X_j,C\cap B=\emptyset}(p-1)=(p-1)\binom{p+j-1}{j}=0.$$

If $C\in X_{p-1}$ and $i\notin C$, then $g_C(e-e_i)=\sum_{j\notin C,j\neq i}\varepsilon_{X\setminus(C\cup\{j\})}=-\varepsilon_{X\setminus C}-\varepsilon_{X\setminus(C\cup\{i\})}$. If $C\in X_{p-1}$ and $i\in C$, then $g_C(e-e_i)=\sum_{j\notin C}\varepsilon_{X\setminus(C\cup\{j\})}=-\varepsilon_{X\setminus C}$. Now, $(\varepsilon+\varepsilon_i)^{\perp}\supseteq H(e-e_i)\supseteq\langle g_C(e-e_i)\mid C\in X_{p-1}\rangle=\langle\varepsilon_{X\setminus C},\varepsilon_{X\setminus C'}+\varepsilon_{X\setminus(C'\cup\{i\})}\mid C',C\in X_{p-1}, i\in C, i\notin C'\rangle=(\varepsilon+\varepsilon_i)^{\perp}$. □

Lemma 10 $f(e_i)=0$, $f(e_i+e_j)=0$, $f(e)=\varepsilon$, $f(e-e_i)=\varepsilon$, *for any* $i,j\in X$, $i\neq j$.

Proof Clearly the first two identities are trivial. We have $f(e)=\sum_{A\in X_p}\varepsilon_{X\setminus A}=\binom{2p-2}{p}\sum_{i\in X}\varepsilon_i=\varepsilon$. Finally $f(e-e_i)=\sum_{A\in X_p,i\notin A}\varepsilon_{X\setminus A}=\binom{2p-3}{p}\sum_{j\in X,j\neq i}\varepsilon_j+\binom{2p-2}{p}\varepsilon_i=\varepsilon$. □

Let g be the linear map defined by

$$2^{-1}\sum_{i\in X}\varepsilon_{X\setminus\{i\}}e_i^*.$$

We have $g(e_i)=2^{-1}\varepsilon_{X\setminus\{i\}}$, $g(e_i+e_j)=2^{-1}(\varepsilon_{X\setminus\{i\}}+\varepsilon_{X\setminus\{j\}})$, $g(e)=2^{-1}\sum_{j\in X}\varepsilon_{X\setminus\{j\}}=-\varepsilon$ and $g(e-e_i)=-\varepsilon-2^{-1}\varepsilon_{X\setminus\{i\}}$.

Lemma 11 *There exists no linear map $\Lambda: W\to V$ such that $(f-\Lambda)(w)\in H(w)$ for every $w\in W$.*

Proof By the preliminary remark on the map g and by Lemmas 6, 7, 8, 9, 10 we have that $(f+g)(e_i)\in H(e_i)$, $(f+g)(e_i+e_j)\in H(e_i+e_j)$, $(f+g)(e)\in H(e)$. Therefore it suffices to prove that there exists no linear map x such that $x(e_i)\in\varepsilon_i^\perp$, $x(e_i+e_j)\in(\varepsilon_i-\varepsilon_j)^\perp$, $x(e)\in\varepsilon^\perp$ and $\varphi(x(e-e_n)-2^{-1}\varepsilon_{X\setminus\{n\}},\varepsilon+\varepsilon_n)=0$. By way of contradiction let us assume that we have such an $x=\sum_{i\in X}a_ie_i^*$, for $a_i\in V$.

In particular we have $\varphi(a_j,\varepsilon_i)=\varphi(a_i,\varepsilon_j)$ for any $i\neq j$ and $\varphi(a_i,\varepsilon_i)=0$. This yields

$$\varphi(x(e),\varepsilon_i)=\sum_{j\in X}\varphi(a_j,\varepsilon_i)=\sum_{j\in X}\varphi(a_i,\varepsilon_j)=\varphi(a_i,\varepsilon).$$

Furthermore

$$\varphi(x(e-e_n),\varepsilon+\varepsilon_n)=\varphi(x(e),\varepsilon)+\varphi(x(e),\varepsilon_n)-\varphi(a_n,\varepsilon)-\varphi(a_n,\varepsilon_n)=0.$$

So $\varphi(x(e-e_n)-2^{-1}\varepsilon_{X\setminus\{n\}},\varepsilon+\varepsilon_n)=-2^{-1}\varphi(\varepsilon_{X\setminus\{n\}},\varepsilon+\varepsilon_n)=1\neq 0$, a contradiction. Thus the result is proved. □

Theorem 1 *An elementary Abelian p-group of rank $n\geq 4p-2$ is not a* $\mathrm{CI}^{(2)}$*-group.*

Proof By construction the element f normalizes the 2-closure $G^{(2)}$ of G. We have $G^{(2)}=WL^{(2)}$ and clearly G and $G^{(2)}$ induce the same map H. Therefore $[-,f]\in\mathrm{Der}(W,L)$ and $\theta([-,f])\in\mathrm{Hom}_H(W,V)$. We have $\theta([-,f])(w)=[w,f](0)=-f(-w)$. Thus $f\in\mathrm{Hom}_H(W,V)$. The group $W\times V$ has rank $4p-2$. Therefore, by Lemma 4 and Lemma 11, the group $W\times V$ is not a $\mathrm{CI}^{(2)}$-group. It is an easy application of the first paragraph of Sect. 5.1 in [5] to see that any elementary Abelian p-group of rank $n\geq 4p-2$ is not a $\mathrm{CI}^{(2)}$-group. □

Based on some computer evidence we present the following conjecture.

Conjecture The group G is 2-closed.

4 Proof of Theorem 2

As in Section 3, we assume that $p>2$. In this section we shall repetitively use Proposition 22.1 (Schur-Wielandt principle), Propositions 22.4, 23.5 and Theorem 23.9

in [9]. For the computations inside the group algebra $\mathbb{Q}[W \times V]$ we shall stick to the notation and to the terminology of Sect. 2 and 4 in [6].

In the proof of Theorem 1 we have shown that $f \in \mathrm{Hom}_H(W, V)$. Moreover, it was proved in [6] (Proposition 5 page 173) that the linear span, $\mathcal{A}_H$, of the simple quantities $\{(w, H(w) + v)\}_{w\in W, v\in V}$ is a Schur ring.

Let E be $\{e_i\}_{i\in X} \cup \{e_i + e_j\}_{i\neq j} \cup \{e, e - e_n\}$. Note that in the proof of Lemma 11 we proved that there exists no linear function Λ such that $(f - \Lambda)(w) \in H(w)$ for every $w \in E$.

Proposition 1 *If S is an $\mathcal{A}_H$-subset such that*

$$\underline{(w, H(w))} \in \langle\langle \underline{S} \rangle\rangle \qquad \text{for every } w \in E, \tag{1}$$

then S is not a CI*-subset of $W \times V$. In particular $W \times V$ is not a* CI*-group ($\langle\langle \underline{S} \rangle\rangle$ denotes the Schur ring generated by $\underline{S}$).*

Proof The following argument mimics Proposition 4 in [6]. Assume S is a CI-subset. Since $f \in \mathrm{Hom}_H(W, V)$, it is easy to check that $\mathrm{Cay}(W \times V, T)^f = \mathrm{Cay}(W \times V, T^f)$ for any simple quantity T in $\mathcal{A}_H$ (see Proposition 6 in [6]). So, since S is an $\mathcal{A}_H$-subset, we have $\mathrm{Cay}(W \times V, S^f) = \mathrm{Cay}(W \times V, S)^f \cong \mathrm{Cay}(W \times V, S)$. Therefore, since S is a CI-subset, we get $S^f = S^g$ for some g in $\mathrm{Aut}(W \times V)$. Then, $\mathrm{Cay}(W \times V, S)^f = \mathrm{Cay}(W \times V, S^f) = \mathrm{Cay}(W \times V, S^g) = \mathrm{Cay}(W \times V, S)^g$. Therefore fg^{-1} is an automorphism of $\mathrm{Cay}(W \times V, S)$.

Now, using Equation (1), the reader can verify (see Theorem 2.4 in [6]) that fg^{-1} is an automorphism of $\mathrm{Cay}(W \times V, (w, H(w)))$ for every $w \in E$. Thus $(w, H(w))^f = (w, H(w))^g$ for every $w \in E$. Hence $(w, H(w))$ is a CI-subset for every $w \in E$ and

$$[(w, H(w))]_{w\in E} \cong_{\mathrm{Cay}} [(w, H(w))^f]_{w\in E}.$$

Now, Proposition 7 in [6] yields that there exists $\Lambda \in \mathrm{Hom}(W, V)$ such that $(f - \Lambda)(w) \in H(w)$ for every $w \in E$, a contradiction.

This proves that S is not a CI-subset of $W \times V$. Therefore $W \times V$ is not a CI-group. □

Proof of Theorem 2 Like in Theorem 1, it is enough to prove that $W \times V$ is not a CI-group (see the first paragraph of Sect. 5.1 in [5]). Let S be the $\mathcal{A}_H$-subset

$$\Big(\bigcup_{i\in X}(0, \varepsilon_i)\Big) \bigcup \Big(\bigcup_{i,j\in X, i\neq j} (e_i + e_j, H(e_i + e_j))\Big) \bigcup (e_n, H(e_n))$$

$$\bigcup (e - e_n, H(e - e_n)) \bigcup \Big(\bigcup_{1\le j\le p-1} (je_{2j}, H(e_{2j}))\Big) \bigcup (e, H(e))$$

$$\bigcup \Big(\bigcup_{1\le j\le p-1} (je_{2j-1}, H(e_{2j-1}) + \{\pm\varepsilon\})\Big).$$

By Proposition 1, it remains to prove that Equation (1) holds for the set S. Let us denote by $\mathcal{A}$ the Schur ring $\langle\langle \underline{S} \rangle\rangle$.

Case $\mathbf{p \geq 5}$. The element $(\underline{S}+\underline{S})\circ \underline{S}$ lies in $\mathcal{A}$ and it can be written as

$$\sum_{(a,b)\in S} \lambda(a,b)\underline{(a,b)},$$

where $\lambda(a,b)$ is the number of solutions of the equation

$$(\ddagger) \qquad (u,x)+(v,y)=(a,b)$$

with $(u,x),(v,y)\in S$. It can be easily shown that

$$\begin{aligned}(\underline{S}+\underline{S})\circ \underline{S} = 0T_1 \uplus 2p^{n-2}T_2 \uplus 2(n-2)T_3 \uplus (2(n-2)\uplus 2p^{n-2})T_4 \\ \uplus(2(n-2)\uplus 4p^{n-2})T_5 \uplus 2(n-1)T_6 \uplus (2(n-1)\uplus 6p^{n-2})T_7\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \biguplus_{i\in X} \underline{(0,\varepsilon_i)} \uplus \underline{(e-e_n, H(e-e_n))},\\
T_2 &= \underline{(e,H(e))},\\
T_3 &= \biguplus_{i<j,\{i,j\}\not\subseteq\{1,2,n\}} \underline{(e_i+e_j, H(e_i+e_j))},\\
T_4 &= \underline{(e_2+e_n, H(e_2+e_n))},\\
T_5 &= \underline{(e_1+e_2, H(e_1+e_2))} \uplus \underline{(e_1+e_n, H(e_1+e_n))},\\
T_6 &= \biguplus_{2\leq j\leq p-1} \underline{(je_{2j}, H(e_{2j}))} \uplus \biguplus_{2\leq j\leq p-1} \underline{(je_{2j-1}, H(e_{2j-1})+\{\pm\varepsilon\})},\\
T_7 &= \underline{(e_1, H(e_1)+\{\pm\varepsilon\})} \uplus \underline{(e_2,H(e_2))} \uplus \underline{(e_n, H(e_n))}.
\end{aligned}$$

For instance, if $a=0$, then $(\ddagger)$ has no solution and so $\lambda(0,b)=0$. Also if $a=e_n$, then $(\ddagger)$ has solutions with $u=0, e_n, e_{2p-2}+e_n, -e_{2p-2}, e_{2p-3}+e_n, -e_{2p-3}$. Studying all this possibilities we get $\lambda(a,b)=2(n-1)\uplus 6p^{n-2}$. All the other computations are similar.

Note that $0, 2p^{n-2}, 2(n-2), (2(n-2)+2p^{n-2}), (2(n-2)+4p^{n-2}), 2(n-1), (2(n-1)+6p^{n-2})$ are all distinct. So, by Schur-Wielandt principle, $T_1,\dots,T_7$ lie in $\mathcal{A}$.

If $i, j \neq n$, then $H(e-e_n)+\varepsilon_i = H(e-e_n)+\varepsilon_j$. Thus

$$\begin{aligned}\left(\biguplus_{i\in X}\underline{(0,\varepsilon_i)}\right) + \underline{(e-e_n, H(e-e_n))} &= \biguplus_{i\in X} \underline{(e-e_n, H(e-e_n)+\varepsilon_i)}\\
&= (n-1)\underline{(e-e_n, H(e-e_n)+\varepsilon_1)}\\
&\quad \uplus \underline{(e-e_n, H(e-e_n)+\varepsilon_n)}.\end{aligned}$$

Therefore, we have

$$T_1 + T_1 = \biguplus_{i\in X} \underline{(0, 2\varepsilon_i)} \uplus 2 \biguplus_{i<j} \underline{(0, \varepsilon_i + \varepsilon_j)} \uplus 2(n-1)\underline{(e - e_n, H(e - e_n) + \varepsilon_1)}$$

$$\uplus 2\underline{(e - e_n, H(e - e_n) + \varepsilon_n)} \uplus p^{n-1}\underline{(2(e - e_n), H(e - e_n))}.$$

By Schur-Wielandt principle $t = \underline{(2(e - e_n), H(e - e_n))}$ is an element of $\mathcal{A}$ and so, by Theorem 23.9 in [9], $\underline{(e - e_n, H(e - e_n))} = (2^{-1})t \in \mathcal{A}$.

Note that $(e, H(e)) + (-e + e_n, H(e - e_n)) = (e_n, V)$ is a set of $\mathcal{A}$, so $\underline{(e_n, V)} \circ \underline{S} = \underline{(e_n, H(e_n))} \in \mathcal{A}$.

Similarly, $(e_2 + e_n, H(e_2 + e_n)) + (-e_n, H(e_n)) = (e_2, V)$ is a set of $\mathcal{A}$, so $\underline{(e_2, H(e_2))} = \underline{(e_2, V)} \circ \underline{S} \in \mathcal{A}$.

We have

$$T_6 + T_6 = 2p^{n-2}U_1 \uplus 8p^{n-2}U_2 \uplus 4p^{n-2}U_3 \uplus 2p^{n-1}U_4 \uplus p^{n-1}U_5$$

where

$$U_1 = \biguplus_{2\le j_1<j_2\le p-1} \underline{(j_1e_{2j_1} + j_2e_{2j_2}, V)},$$

$$U_2 = \biguplus_{2\le j_1<j_2\le p-1} \underline{(j_1e_{2j_1-1} + j_2e_{2j_2-1}, V)},$$

$$U_3 = \biguplus_{2\le j_2\le j_1\le p-1} \underline{(j_1e_{2j_1} + j_2e_{2j_2-1}, V)},$$

$$U_4 = \biguplus_{1\le j\le p-1} \underline{(2je_{2j-1}, H(e_{2j-1}))},$$

$$U_5 = \biguplus_{2\le j\le p-1} \underline{(2je_{2j}, H(e_{2j}))} \uplus \biguplus_{2\le j\le p-1} \underline{(2je_{2j-1}, H(e_{2j-1}) + \{\pm 2\varepsilon\})}.$$

The elements $2p^{n-2}, 8p^{n-2}, 4p^{n-2}, 2p^{n-1}, p^{n-1}$ are all distinct and so using Schur-Wielandt principle we have that $U_1, \ldots, U_5$ lie in $\mathcal{A}$.

We have

$$F_l = \underline{(e_{2l-1} + e_{2l}, H(e_{2l-1} + e_{2l}))} = (l^{-1})(U_3 \circ (l)T_3) \in \mathcal{A},$$

for $2 \le l \le p-1$.

We have

$$(U_4 + (-2l)F_l) \circ ((-2)\underline{S}) = p^{n-2}\underline{(-2le_{2l}, H(e_{2l}))} \in \mathcal{A},$$

for every $2 \le l \le p-1$. So, $\underline{(e_{2l}, H(e_{2l}))} \in \mathcal{A}$ for $2 \le l \le p-1$.

Now $(e_{2l-1} + e_{2l}, H(e_{2l} + e_{2l-1})) + (-e_{2l}, H(e_{2l})) = (e_{2l-1}, V)$ is a set of $\mathcal{A}$ for $2 \le l \le p-1$. Therefore $\underline{(e_{2l-1}, H(e_{2l-1}))} = \underline{(e_{2l-1}, V)} \circ (((2l)^{-1})U_4) \in \mathcal{A}$ for $2 \le l \le p-1$.

So far we have proved that $\underline{(e_i, H(e_i))} \in \mathcal{A}$ for $i \geq 2$. In particular, by the definition of T_7, we get $q = \underline{(e_1, H(e_1) + \{\pm\varepsilon\})} \in \mathcal{A}$. We have

$$q + q = 2p^{n-1}\underline{(2e_1, H(e_1))} \uplus p^{n-1}\underline{(2e_1, H(e_1) + \{\pm 2\varepsilon\})}.$$

Therefore, arguing as usual, we get $\underline{(e_1, H(e_1))} \in \mathcal{A}$.

Now $\underline{(e_i, H(e_i))} + \underline{(e_j, H(e_j))} = \underline{(e_i + e_j, V)}$ is a set of $\mathcal{A}$. So $\underline{(e_i + e_j, H(e_i + e_j))} = \underline{(e_i + e_j, V)} \circ \underline{S} \in \mathcal{A}$ for any i, j with $i \neq j$.

This proves that $\mathcal{A}$ is a $\mathbb{Q}$-Schur ring with the required properties and so $W \times V$ is not a CI-group. Once again using the first paragraph of Sect. 5.1 in [5] we have that any elementary Abelian p-group of rank $n \geq 4p - 2$ is not a CI-group.

Case p = 3. We have

$$(\underline{S} + \underline{S}) \circ \underline{S} = 0T' \uplus 2p^{n-1}T^* \uplus 2p^{n-2}T_2 \uplus 2(n-2)T_3 \uplus (2(n-2) \uplus 2p^{n-2})T_4$$
$$\uplus (2(n-2) + 4p^{n-2})T_5 \uplus 2(n-1)T_6 \uplus (2(n-1) + 6p^{n-2})T_7$$

where $T' = \sum_{i \in X} \underline{(0, \varepsilon_i)}$, $T^* = \underline{(e - e_n, H(e - e_n))}$ and $T_2, \dots, T_7$ are defined as in the case $p \geq 5$.

Note that $0, 2p^4, 2p^{n-2}, 2(n-2), (2(n-2) + 2p^{n-2}), (2(n-2) + 4p^{n-2}), 2(n-1), (2(n-1) + 6p^{n-2})$ are all distinct. So, by Schur-Wielandt principle, the elements $T', T^*, T_2, \dots, T_7$ lie in $\mathcal{A}$. Now the computations are exactly the same as in the case $p \geq 5$. □

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