J Algebr Comb (2007) 26: 383–414
DOI 10.1007/s10801-007-0062-7

Canonical bases of higher-level q-deformed Fock spaces

Xavier Yvonne

Received: 12 June 2006 / Accepted: 29 January 2007 /
Published online: 17 April 2007
© Springer Science+Business Media, LLC 2007

Abstract We show that the transition matrices between the standard and the canonical bases of infinitely many weight subspaces of the higher-level q-deformed Fock spaces are equal.

1 Introduction

The q-deformed higher-level Fock spaces were introduced in [6] in order to compute the crystal graph of any irreducible integrable representation of level $l \geq 1$ of $U_q(\widehat{\mathfrak{sl}}_n)$. More precisely, the Fock representation $\mathbf{F}_q[\boldsymbol{s}_l]$ depends on a parameter $\boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$ called multi-charge. It contains as a submodule the irreducible integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module with highest weight $\Lambda_{s_1} + \cdots + \Lambda_{s_l}$. The representation $\mathbf{F}_q[\boldsymbol{s}_l]$ is a generalization of the level-one Fock representation of $U_q(\widehat{\mathfrak{sl}}_n)$ ([4, 17], see also [14, 15]).

The canonical bases are bases of the Fock representations that are invariant under a certain involution $^-$ of $U_q(\widehat{\mathfrak{sl}}_n)$ and that give at $q = 0$ and $q = \infty$ the crystal bases. They were constructed for $l = 1$ in [14, 15] and for $l \geq 1$ by Uglov [19]. In [19], Uglov provides an algorithm for computing these canonical bases. He also gives an expression of the transition matrices between the standard and the canonical bases in terms of Kazhdan-Lusztig polynomials for affine Hecke algebras of type A.

In this article we prove three theorems.

1. The first one (Theorem 3.9) is a generalization to $l \geq 1$ of a result of [13]. It compares the transition matrices of the canonical bases of some weight subspaces inside a given Fock space $\mathbf{F}_q[\boldsymbol{s}_l]$. The weights involved are conjugated under the action of the Weyl group of $U_q(\widehat{\mathfrak{sl}}_n)$. This action leads to bijections σ_i that can

X. Yvonne (✉)
Institut Camille Jordan (Mathématiques), Université Lyon I, 43 Bd du 11 Novembre 1918,
69622 Villeurbanne Cedex, France
e-mail: yvonne@math.univ-lyon1.fr

be described in a combinatorial way by adding/removing as many i-nodes as possible to the l-multi-partitions indexing the canonical bases. These bijections are generalizations of the Scopes bijections introduced in [18] in order to study, when $n = p$ is a prime number, the p-blocks of symmetric groups of a given defect.
2. In a dual manner, our second result (Theorem 4.4) gives some sufficient conditions on multi-charges $\boldsymbol{s}_l$ and $\boldsymbol{t}_l$ with given residues modulo n that ensure that the transition matrices of the canonical bases of some weight subspaces of $\mathbf{F}_q[\boldsymbol{s}_l]$ and $\mathbf{F}_q[\boldsymbol{t}_l]$ coincide.
3. Our third result (Theorem 5.2) is an application of Theorem 4.4 to the case when the multi-charges $\boldsymbol{s}_l = (s_1, \dots, s_l)$ and $\boldsymbol{t}_l = (t_1, \dots, t_l)$ are dominant, that is $s_1 \gg \cdots \gg s_l$ and $t_1 \gg \cdots \gg t_l$. It shows that the transition matrices of the canonical bases of the Fock spaces $\mathbf{F}_q[\boldsymbol{s}_l]$ stabilize when $\boldsymbol{s}_l$ becomes dominant (with a given sequence of residues modulo n). This supports the following conjecture (see [22]). We conjecture that if $\boldsymbol{s}_l = (s_1, \dots, s_l)$ is dominant, then the transition matrix of the homogeneous component of degree m of the canonical basis of the Fock space $\mathbf{F}_q[\boldsymbol{s}_l]$ is equal to the decomposition matrix of the cyclotomic v-Schur algebra $\mathcal{S}_{\mathbb{C},m}(\zeta; \zeta^{s_1}, \dots, \zeta^{s_l})$ of [2], where ζ is a complex primitive n-th root of unity. This conjecture generalizes both Ariki's theorem for Ariki-Koike algebras (see [1]) and a result of Varagnolo and Vasserot (see [20]) which relates the canonical basis of the level-one Fock space and the decomposition matrix of v-Schur algebras with parameter a complex n-th root of unity.

Notation Let $\mathbb{N}$ (respectively $\mathbb{N}^*$) denote the set of nonnegative (respectively positive) integers, and for $a, b \in \mathbb{R}$ denote by $[\![a; b]\!]$ the discrete interval $[a, b] \cap \mathbb{Z}$. For $X \subset \mathbb{R}$, $t \in \mathbb{R}$, $N \in \mathbb{N}^*$, put

$$X^N(t) := \{(s_1, \dots, s_N) \in X^N \mid s_1 + \cdots + s_N = t\}. \tag{1}$$

Throughout this article, we fix 3 integers $n, l \geq 1$ and $s \in \mathbb{Z}$. Let Π denote the set of all integer partitions, and for $N \in \mathbb{N}^*$, let Π^N denote the set of all N-multi-partitions. The empty partition (respectively empty N-multi-partition) will be denoted by $\emptyset$ (respectively $\boldsymbol{\emptyset}_N$).

2 Higher-level q-deformed Fock spaces

In this section, we introduce the higher-level Fock spaces and their canonical bases. We follow here [19], to which we refer the reader for more details. All definitions and results given here are due to Uglov.

2.1 The quantum algebras $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$

In this section, we assume that $n \geq 2$ and $l \geq 2$. Let $\widehat{\mathfrak{sl}}_n$ be the Kac-Moody algebra of type $A^{(1)}_{n-1}$ defined over the field $\mathbb{Q}$ [7]. Let $\mathfrak{h}^*$ be the dual of the Cartan subalgebra of $\widehat{\mathfrak{sl}}_n$. Let $\Lambda_0, \dots, \Lambda_{n-1} \in \mathfrak{h}^*$ be the fundamental weights, $\alpha_0, \dots, \alpha_{n-1} \in \mathfrak{h}^*$ be the simple roots and $\delta := \alpha_0 + \cdots + \alpha_{n-1}$ be the null root. It will be convenient to extend the index set of the fundamental weights by setting $\Lambda_i := \Lambda_{i \bmod n}$ for all $i \in \mathbb{Z}$. The simple roots are related to the fundamental weights by

$$\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} + \delta_{i,0}\delta \qquad (0 \le i \le n-1). \tag{2}$$

For $0 \le i, j \le n-1$, let $a_{i,j}$ be the coefficient of Λ_j in α_i. The space

$$\mathfrak{h}^* = \bigoplus_{i=0}^{n-1} \mathbb{Q}\Lambda_i \oplus \mathbb{Q}\delta = \bigoplus_{i=0}^{n-1} \mathbb{Q}\alpha_i \oplus \mathbb{Q}\Lambda_0$$

is equipped with a non-degenerate bilinear symmetric form $(.\,,\,.)$ defined by

$$(\alpha_i, \alpha_j) = a_{i,j}, \quad (\Lambda_0, \alpha_i) = \delta_{i,0}, \quad (\Lambda_0, \Lambda_0) = 0 \qquad (0 \le i, j \le n-1). \tag{3}$$

Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the q-deformed universal enveloping algebra of $\widehat{\mathfrak{sl}}_n$. This is an algebra over $\mathbb{Q}(q)$ with generators $e_i, f_i, t_i^{\pm 1}$ $(0 \le i \le n-1)$ and ∂. Let $U_q'(\widehat{\mathfrak{sl}}_n)$ be the subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by $e_i, f_i, t_i^{\pm 1}$ $(0 \le i \le n-1)$. The relations in $U_q'(\widehat{\mathfrak{sl}}_n)$ are standard and will be omitted (see *e.g.* [10]). The relations among the degree generator ∂ and the generators of $U_q'(\widehat{\mathfrak{sl}}_n)$ can be found in [19, §2.1]. If M is a $U_q(\widehat{\mathfrak{sl}}_n)$-module, denote by $\mathcal{P}(M)$ the set of weights of M and let $M\langle w\rangle$ denote the subspace of M of weight w. If $x \in M\langle w\rangle \setminus \{0\}$ is a weight vector, denote by

$$\mathrm{wt}(x) := w \tag{4}$$

the weight of x. The *Weyl group* of $\widehat{\mathfrak{sl}}_n$ (or $U_q(\widehat{\mathfrak{sl}}_n)$), denoted by W_n, is the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the simple reflections σ_i defined by

$$\sigma_i(\Lambda) = \Lambda - (\Lambda, \alpha_i)\alpha_i \qquad (\Lambda \in \mathfrak{h}^*, 0 \le i \le n-1). \tag{5}$$

Note that W_n is isomorphic to $\widetilde{\mathfrak{S}}_n$, the affine symmetric group which is a Coxeter group of type $A_{n-1}^{(1)}$.

We also introduce the algebra $U_p(\widehat{\mathfrak{sl}}_l)$ with

$$p := -q^{-1}. \tag{6}$$

In order to distinguish the elements related to $U_q(\widehat{\mathfrak{sl}}_n)$ from those related to $U_p(\widehat{\mathfrak{sl}}_l)$, we put dots over the latter. For example, $\dot{e}_i, \dot{f}_i, \dot{t}_i^{\pm 1}$ $(0 \le i \le l-1)$ and $\dot{\partial}$ are the generators of $U_p(\widehat{\mathfrak{sl}}_l)$, $\dot{\alpha}_i$ $(0 \le i \le l-1)$ are the simple roots for $U_p(\widehat{\mathfrak{sl}}_l)$, $\dot{W}_l = \langle \dot{\sigma}_0, \dots, \dot{\sigma}_{l-1}\rangle$ is the Weyl group of $U_p(\widehat{\mathfrak{sl}}_l)$ and so on. Similarly, if M is a $U_p(\widehat{\mathfrak{sl}}_l)$-module, denote by $\dot{\mathcal{P}}(M)$ the set of weights of M.

2.2 The space Λ^s

2.2.1 The vector space Λ^s and its standard basis

Following [19], we now recall the definition of Λ^s, the space of semi-infinite q-wedge products of charge s (this space is denoted by $\Lambda^{s+\frac{\infty}{2}}$ in [19]). First, let $r \ge 2$ be an integer, and $\Lambda_q^r V$ be the space of q-wedge products of finite length r (this space is denoted by Λ^r in [19]; we hope that this does not make any confusion with our Λ^s). As a vector space over $\mathbb{Q}(q)$, $\Lambda_q^r V$ is spanned by the *q-wedge products*

$$u_{\boldsymbol{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r}, \qquad \boldsymbol{k} = (k_1, k_2, \ldots, k_r) \in \mathbb{Z}^r, \tag{7}$$

with relations given in [19, Prop. 3.16]. These relations are called *straightening rules* (we will not need them in this article). Now, define Λ^s as the inductive limit

$$\Lambda^s = \varinjlim \Lambda_q^r V, \tag{8}$$

where maps $\Lambda_q^r V \to \Lambda_q^{r'} V$ $(r' > r)$ are given by $v \mapsto v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-r'+1}$. Less formally, Λ^s is spanned by q-wedge products of infinite length

$$u_{\boldsymbol{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots, \qquad \boldsymbol{k} = (k_1, k_2, \ldots) \in P(s), \tag{9}$$

where $P(s)$ is the set of all sequences of integers $(k_1, k_2, \ldots)$ such that $k_i = s - i + 1$ for i large enough. The straightening rules given in [19, Prop. 3.16 (i)] still hold for any pair of adjacent factors of a q-wedge product $u_{\boldsymbol{k}} \in \Lambda^s$. From now on, we shall assume without further comment that all q-wedge products lie in Λ^s (in particular, they have infinitely many factors). Using the straightening rules, one can express a q-wedge product as a linear combination of so-called *ordered q-wedge products*, namely q-wedge products $u_{\boldsymbol{k}}$ with $\boldsymbol{k} \in P^{++}(s)$, where

$$P^{++}(s) := \{(k_1, k_2, \ldots) \in P(s) \mid k_1 > k_2 > \cdots\}. \tag{10}$$

In fact, the ordered q-wedge products $\{u_{\boldsymbol{k}} \mid \boldsymbol{k} \in P^{++}(s)\}$ form a basis of Λ^s, called the *standard basis*. In this article, it will be convenient to use different indexations of this basis which we give now.

* *Indexation $u_{\boldsymbol{k}}$*. This is the indexation we have just described.

* *Indexation λ*. To the ordered q-wedge product $u_{\boldsymbol{k}}$ corresponds a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ defined by

$$\lambda_i := k_i - (s + 1 - i) \qquad (i \geq 1). \tag{11}$$

If $u_{\boldsymbol{k}}$ and λ are related this way, write

$$|\lambda, s\rangle := u_{\boldsymbol{k}}. \tag{12}$$

* *Indexation $\boldsymbol{\lambda}_n$*. Recall the definition of $\mathbb{Z}^n(s)$ from (1). Uglov constructed a bijection

$$\tau_n' : \Pi \to \Pi^n \times \mathbb{Z}^n(s), \quad \lambda \mapsto (\boldsymbol{\lambda}_n, \boldsymbol{s}_n) \tag{13}$$

(see [19, §4.1], where this map is denoted by τ_n^s). With the notation above, $\boldsymbol{\lambda}_n$ is the n-quotient of λ and $\boldsymbol{s}_n$ is a variation of the n-core of λ (see *e.g.* [16, Ex.8, p.12]). Write

$$|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^\bullet := |\lambda, s\rangle \tag{14}$$

if $(\boldsymbol{\lambda}_n, \boldsymbol{s}_n) = \tau'_n(\lambda)$. Note that this indexation coincides with the indexation λ if $n = 1$.

* Indexation $\boldsymbol{\lambda}_l$. Uglov constructed a bijection

$$\tau_l : \Pi \to \Pi^l \times \mathbb{Z}^l(s), \quad \lambda \mapsto (\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \tag{15}$$

(see again [19, §4.1], where this map is denoted by τ_l^s). The map τ_l is a variation of the map τ'_n defined above. Write

$$|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle := |\lambda, s\rangle \tag{16}$$

if $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) = \tau_l(\lambda)$. Note that this indexation coincides with the indexation λ if $l = 1$.

Example 2.1 Take $n = 2$, $l = 3$ and $s = -1$. Then we have

$$\begin{aligned} u_3 \wedge u_1 \wedge u_0 \wedge u_{-2} \wedge u_{-4} \wedge u_{-6} \wedge u_{-7} \wedge \cdots &= \big|(4, 3, 3, 2, 1), -1\big\rangle \\ &= \big|\big((3, 3), \emptyset\big), (-1, 0)\big\rangle^\bullet \\ &= \big|\big((1, 1), (1, 1), (1)\big), (0, 0, -1)\big\rangle. \end{aligned}$$

◇

2.2.2 Three actions on Λ^s

Following [3, 6, 19], the vector space Λ^s can be made into an integrable representation of level l of the quantum algebra $U_q(\widehat{\mathfrak{sl}}_n)$. This representation can be described in a nice way if we use the indexation $\boldsymbol{\lambda}_l$. In order to recall the explicit formulas, let us first introduce some notation. Fix $\boldsymbol{\lambda}_l = (\lambda^{(1)}, \ldots, \lambda^{(l)}) \in \Pi^l$ and $\boldsymbol{s}_l = (s_1, \ldots, s_l) \in \mathbb{Z}^l$. Identify the multi-partition $\boldsymbol{\lambda}_l$ with its Young diagram $\{(i, j, b) \in \mathbb{N}^* \times \mathbb{N}^* \times [\![1; l]\!] \mid 1 \le j \le \lambda_i^{(b)}\}$, whose elements are called *nodes* of $\boldsymbol{\lambda}_l$. For each node $\gamma = (i, j, b)$ of $\boldsymbol{\lambda}_l$, define its *residue modulo n* by

$$\operatorname{res}_n(\gamma) = \operatorname{res}_n(\gamma, \boldsymbol{s}_l) := (s_b + j - i) \bmod n \in \mathbb{Z}/n\mathbb{Z} \cong [\![0; n-1]\!]. \tag{17}$$

If $\operatorname{res}_n(\gamma) = c$, we say that γ is a *c-node*. If $\boldsymbol{\mu}_l \in \Pi^l$ is such that $\boldsymbol{\mu}_l \supset \boldsymbol{\lambda}_l$ and $\gamma := \boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l$ is a c-node of $\boldsymbol{\mu}_l$, we say that γ is a *removable c-node of* $\boldsymbol{\mu}_l$ or that γ is an *addable c-node of* $\boldsymbol{\lambda}_l$. For $0 \le c \le n-1$, denote by

$$M_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n) \quad (\text{respectively } A_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n), \quad \text{respectively } R_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n)) \tag{18}$$

the number of c-nodes (respectively of addable c-nodes, respectively of removable c-nodes) of $\boldsymbol{\lambda}_l$. Put

$$N_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n) := A_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n) - R_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n). \tag{19}$$

For $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l$, $\boldsymbol{s}_l \in \mathbb{Z}^l$, $c \in [\![0; n-1]\!]$ and $k \in \mathbb{N}^*$, write

$$\boldsymbol{\lambda}_l \xrightarrow{c:k} \boldsymbol{\mu}_l \tag{20}$$

if there exists a sequence of l-multi-partitions $\boldsymbol{\nu}_l^{(0)} \subset \boldsymbol{\nu}_l^{(1)} \subset \cdots \subset \boldsymbol{\nu}_l^{(k)}$ such that $\boldsymbol{\lambda}_l = \boldsymbol{\nu}_l^{(0)}$, $\boldsymbol{\mu}_l = \boldsymbol{\nu}_l^{(k)}$ and for all $1 \le j \le k$, $\boldsymbol{\nu}_l^{(j)} \setminus \boldsymbol{\nu}_l^{(j-1)}$ is an addable c-node of $\boldsymbol{\nu}_l^{(j-1)}$.

Given a multi-charge $(s_1, \dots, s_l)$ and two nodes $\gamma = (i, j, b)$ and $\gamma' = (i', j', b')$, write

$$\gamma < \gamma' \tag{21}$$

if either $s_b + j - i < s_{b'} + j' - i'$ or $s_b + j - i = s_{b'} + j' - i'$ and $b < b'$. This defines a total ordering on the set of the addable and removable c-nodes of a given multi-partition. If $\boldsymbol{\lambda}_l \xrightarrow{c:k} \boldsymbol{\mu}_l$, put

$$N_c^{>}(\boldsymbol{\lambda}_l; \boldsymbol{\mu}_l; \boldsymbol{s}_l; n) = \sum_{\gamma \in \boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l} \Big(\sharp\{\beta \in \mathbb{N}^3 \mid \beta \text{ is an addable } c\text{-node of } \boldsymbol{\mu}_l \text{ and } \beta > \gamma\} - \sharp\{\beta \in \mathbb{N}^3 \mid \beta \text{ is a removable } c\text{-node of } \boldsymbol{\lambda}_l \text{ and } \beta > \gamma\}\Big), \tag{22}$$

and define similarly $N_c^{<}(\boldsymbol{\lambda}_l; \boldsymbol{\mu}_l; \boldsymbol{s}_l; n)$.

Example 2.2 Take $\boldsymbol{s}_l = (5, 0, 2, 1)$, $\boldsymbol{\lambda}_l = \big((5, 3, 3, 1), (3, 2), (4, 3, 1), (2, 2, 2, 1)\big)$, $n = 3$ and $c = 0$. Then we have

$$M_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n) = 11, \quad A_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n) = R_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n) = 5 \quad \text{and} \quad N_c(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n) = 0.$$

The addable c-nodes of $\boldsymbol{\lambda}_l$ are $(5, 1, 4)$, $(4, 2, 1)$, $(1, 4, 2)$, $(1, 3, 4)$ and $(1, 5, 3)$. The removable c-nodes of $\boldsymbol{\lambda}_l$ are $(2, 2, 2)$, $(3, 1, 3)$, $(3, 2, 4)$, $(2, 3, 3)$ and $(1, 5, 1)$. The list of all these nodes arranged with respect to the ordering described above is

$$(5,1,4) < (2,2,2) < (3,1,3) < (3,2,4) < (4,2,1) < (1,4,2) < (2,3,3) < (1,3,4) < (1,5,3) < (1,5,1).$$

Take also $\boldsymbol{\mu}_l = \big((5, 3, 3, 1), (3, 2), (5, 3, 1), (2, 2, 2, 1)\big)$, so that $\boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l = \{(1, 5, 3)\}$ is a single c-node. Then $N_c^{>}(\boldsymbol{\lambda}_l; \boldsymbol{\mu}_l; \boldsymbol{s}_l; n) = 0 - 1 = -1$ and $N_c^{<}(\boldsymbol{\lambda}_l; \boldsymbol{\mu}_l; \boldsymbol{s}_l; n) = 4 - 4 = 0$. ◇

For $\boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$, define

$$\Delta(\boldsymbol{s}_l, n) := \frac{1}{2} \sum_{b=1}^{l} \Big(\frac{s_b^2}{n} - s_b\Big) - \Big(\frac{(s_b \bmod n)^2}{n} - (s_b \bmod n)\Big). \tag{23}$$

Now we can state the following result.

Theorem 2.3 [3, 6, 19] *The following formulas define on Λ^s a structure of an integrable representation of level l of the quantum algebra $U_q(\widehat{\mathfrak{sl}}_n)$.*

$$e_i.|\boldsymbol{\nu}_l, \boldsymbol{s}_l\rangle = \sum_{\boldsymbol{\lambda}_l \xrightarrow{i:1} \boldsymbol{\nu}_l} q^{-N_i^{<}(\boldsymbol{\lambda}_l; \boldsymbol{\nu}_l; \boldsymbol{s}_l; n)} |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle, \tag{24}$$

$$f_i.|\boldsymbol{\nu}_l, \boldsymbol{s}_l\rangle = \sum_{\boldsymbol{\nu}_l \xrightarrow{i:1} \boldsymbol{\mu}_l} q^{N_i^{>}(\boldsymbol{\nu}_l; \boldsymbol{\mu}_l; \boldsymbol{s}_l; n)} |\boldsymbol{\mu}_l, \boldsymbol{s}_l\rangle, \tag{25}$$

$$t_i.|\boldsymbol{\nu}_l, \boldsymbol{s}_l\rangle = q^{N_i(\boldsymbol{\nu}_l;\boldsymbol{s}_l;n)}\,|\boldsymbol{\nu}_l, \boldsymbol{s}_l\rangle, \tag{26}$$

$$\partial.|\boldsymbol{\nu}_l, \boldsymbol{s}_l\rangle = -\big(\Delta(\boldsymbol{s}_l, n) + M_0(\boldsymbol{\nu}_l; \boldsymbol{s}_l; n)\big)\,|\boldsymbol{\nu}_l, \boldsymbol{s}_l\rangle. \tag{27}$$

□

Note that these formulas involve no straightening of q-wedge products. They are therefore handy to use for computations.

In a completely similar way, Λ^s can be made into an integrable representation of level n of the quantum algebra $U_p(\widehat{\mathfrak{sl}}_l)$. This action can be described using the indexation $\boldsymbol{\lambda}_n$. Namely, we have (with obvious notation) the following result.

Theorem 2.4 [3, 6, 19] *The following formulas define on Λ^s a structure of an integrable representation of level n of the quantum algebra $U_p(\widehat{\mathfrak{sl}}_l)$.*

$$\dot{e}_i.|\boldsymbol{\nu}_n, \boldsymbol{s}_n\rangle^\bullet = \sum_{\boldsymbol{\lambda}_n \xrightarrow{i:1} \boldsymbol{\nu}_n} p^{-N_i^<(\boldsymbol{\lambda}_n;\boldsymbol{\nu}_n;\boldsymbol{s}_n;l)}\,|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^\bullet, \tag{28}$$

$$\dot{f}_i.|\boldsymbol{\nu}_n, \boldsymbol{s}_n\rangle^\bullet = \sum_{\boldsymbol{\nu}_n \xrightarrow{i:1} \boldsymbol{\mu}_n} p^{N_i^>(\boldsymbol{\nu}_n;\boldsymbol{\mu}_n;\boldsymbol{s}_n;l)}\,|\boldsymbol{\mu}_n, \boldsymbol{s}_n\rangle^\bullet, \tag{29}$$

$$\dot{t}_i.|\boldsymbol{\nu}_n, \boldsymbol{s}_n\rangle^\bullet = p^{N_i(\boldsymbol{\nu}_n;\boldsymbol{s}_n;l)}\,|\boldsymbol{\nu}_n, \boldsymbol{s}_n\rangle^\bullet, \tag{30}$$

$$\dot{\partial}.|\boldsymbol{\nu}_n, \boldsymbol{s}_n\rangle^\bullet = -\big(\Delta(\boldsymbol{s}_n, l) + M_0(\boldsymbol{\nu}_n; \boldsymbol{s}_n; l)\big)\,|\boldsymbol{\nu}_n, \boldsymbol{s}_n\rangle^\bullet. \tag{31}$$

□

Theorems 2.3 and 2.4 show in particular that the vectors of the standard basis of Λ^s are weight vectors for the actions of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$, and the weights are given by:

Corollary 2.5 [19], (27–30) *With obvious notation, we have*

$$\mathrm{wt}(|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle) = -\Delta(\boldsymbol{s}_l, n)\delta + \Lambda_{s_1} + \cdots + \Lambda_{s_l} - \sum_{i=0}^{n-1} M_i(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n)\,\alpha_i, \tag{32}$$

$$\begin{aligned}\dot{\mathrm{wt}}(|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle) &= -\big(\Delta(\boldsymbol{s}_l, n) + M_0(\boldsymbol{\lambda}_l; \boldsymbol{s}_l; n)\big)\dot{\delta} + (n - s_1 + s_l)\dot{\Lambda}_0 \\ &\quad + \sum_{i=1}^{l-1} (s_i - s_{i+1})\,\dot{\Lambda}_i,\end{aligned} \tag{33}$$

$$\dot{\mathrm{wt}}(|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^\bullet) = -\Delta(\boldsymbol{s}_n, l)\dot{\delta} + \dot{\Lambda}_{s_1} + \cdots + \dot{\Lambda}_{s_n} - \sum_{i=0}^{l-1} M_i(\boldsymbol{\lambda}_n; \boldsymbol{s}_n; l)\,\dot{\alpha}_i, \tag{34}$$

$$\mathrm{wt}(|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^\bullet) = -\big(\Delta(\boldsymbol{s}_n, l) + M_0(\boldsymbol{\lambda}_n; \boldsymbol{s}_n; l)\big)\delta + (l - s_1 + s_n)\Lambda_0 + \sum_{i=1}^{n-1} (s_i - s_{i+1})\,\Lambda_i. \tag{35}$$

□

Definition 2.6 For $m \in \mathbb{Z}^*$, define an endomorphism B_m of Λ^s by

$$B_m(u_{k_1} \wedge u_{k_2} \wedge \cdots) := \sum_{j=1}^{+\infty} u_{k_1} \wedge \cdots \wedge u_{k_{j-1}} \wedge u_{k_j - nlm} \wedge u_{k_{j+1}} \wedge \cdots \quad \big((k_1, k_2, \ldots) \in P^{++}(s)\big). \tag{36}$$

◇

Using a variation of [19, Lemma 3.18] for q-wedge products with infinitely many factors, one sees that the sum above involves only finitely many nonzero terms, hence B_m is well-defined. This definition comes from a passage to the limit $r \to \infty$ in the action of the center of the Hecke algebra of $\widehat{\mathfrak{S}}_r$ on q-wedge products of r factors. However, the operators B_m do not commute, but by [19, Prop. 4.4], they span a Heisenberg algebra

$$\mathcal{H} := \langle B_m \mid m \in \mathbb{Z}^* \rangle. \tag{37}$$

We now recall some results concerning the actions of $U_q(\widehat{\mathfrak{sl}}_n)$, $U_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$ on Λ^s.

Proposition 2.7 [19], Prop. 4.6 *Recall that $p = -q^{-1}$. Then the actions of $U'_q(\widehat{\mathfrak{sl}}_n)$, $U'_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$ on Λ^s pairwise commute.* □

For L, $N \in \mathbb{N}^*$, introduce the finite set

$$A_{L,N}(s) := \{(r_1, \ldots, r_L) \in \mathbb{Z}^L(s) \mid r_1 \geq \cdots \geq r_L,\ r_1 - r_L \leq N\}. \tag{38}$$

Using [19, §4.1], it is not hard to see that if $\boldsymbol{r}_l \in \mathbb{Z}^l(s)$ and $\boldsymbol{r}_n \in \mathbb{Z}^n(s)$ are such that $|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle = |\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet$, then $\boldsymbol{r}_l \in A_{l,n}(s)$ and $\boldsymbol{r}_n \in A_{n,l}(s)$. Conversely, if $\boldsymbol{r}_l \in A_{l,n}(s)$, then there exists a unique $\boldsymbol{r}_n \in A_{n,l}(s)$ such that $|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet = |\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle$, and if $\boldsymbol{r}_n \in A_{n,l}(s)$, then there exists a unique $\boldsymbol{r}_l \in A_{l,n}(s)$ such that $|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle = |\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet$. Therefore,

$$\{|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle \mid \boldsymbol{r}_l \in A_{l,n}(s)\} = \{|\boldsymbol{\emptyset}_n, \boldsymbol{r}_n\rangle^\bullet \mid \boldsymbol{r}_n \in A_{n,l}(s)\}$$

is a set of highest weight vectors simultaneously for the actions of $U'_q(\widehat{\mathfrak{sl}}_n)$ and $U'_p(\widehat{\mathfrak{sl}}_l)$. It is easy to see that these vectors are also singular for the action of $\mathcal{H}$, that is, they are annihilated by the B_m, $m > 0$. It turns out that these vectors are the only singular vectors simultaneously for the actions of $U'_q(\widehat{\mathfrak{sl}}_n)$, $U'_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$, and we have the following theorem.

Theorem 2.8 [19], Thm. 4.8 *We have*

$$\begin{aligned}\Lambda^s &= \bigoplus_{\boldsymbol{r}_l \in A_{l,n}(s)} U'_q(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{H} \otimes U'_p(\widehat{\mathfrak{sl}}_l).|\emptyset_l, \boldsymbol{r}_l\rangle \\ &= \bigoplus_{\boldsymbol{r}_n \in A_{n,l}(s)} U'_q(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{H} \otimes U'_p(\widehat{\mathfrak{sl}}_l).|\emptyset_n, \boldsymbol{r}_n\rangle^\bullet.\end{aligned}$$

□

2.2.3 The involution $^-$ *of* Λ^s

Following [19], the space Λ^s can be endowed with an involution $^-$. Instead of recalling the definition of this involution, we give its main properties (by [23, Thm. 3.11], they turn out to characterize it completely).

Proposition 2.9 [19] *There exists an involution* $^-$ *of* Λ^s *such that*:

(i) $^-$ *is a* $\mathbb{Q}$*-linear map of* Λ^s *such that for all* $u \in \Lambda^s$, $k \in \mathbb{Z}$, *we have* $\overline{q^k u} = q^{-k}\,\overline{u}$.
(ii) (Unitriangularity property). *For all* $\lambda \in \Pi$, *we have*

$$\overline{|\lambda, s\rangle} \in |\lambda, s\rangle + \bigoplus_{\mu \lhd \lambda} \mathbb{Z}[q, q^{-1}]\,|\mu, s\rangle,$$

where $\lhd$ *stands for the dominance ordering on partitions.*
(iii) *For all* $\lambda \in \Pi$, *we have* $\mathrm{wt}(\overline{|\lambda, s\rangle}) = \mathrm{wt}(|\lambda, s\rangle)$ *and* $\dot{\mathrm{w}}\mathrm{t}(\overline{|\lambda, s\rangle}) = \dot{\mathrm{w}}\mathrm{t}(|\lambda, s\rangle)$.
(iv) *For all* $0 \le i \le n-1$, $0 \le j \le l-1$, $m < 0$, $v \in \Lambda^s$, *we have*

$$\overline{f_i.v} = f_i.\overline{v}, \quad \overline{\dot{f}_j.v} = \dot{f}_j.\overline{v} \quad \text{and} \quad \overline{B_m.v} = B_m.\overline{v}.$$

Proof Let $^-$ be the involution of Λ^s defined in [19, Prop. 3.23 & Eqn. (39)]. By construction, (i) holds. The other statements come from [19, Prop. 4.11 & 4.12] and Corollary 2.5. □

2.3 q-deformed higher-level Fock spaces

2.3.1 Definition

By Theorem 2.3, the space

$$\mathbf{F}_q[\boldsymbol{s}_l] := \bigoplus_{\boldsymbol{\lambda}_l \in \Pi^l} \mathbb{Q}(q)|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \subset \Lambda^s \qquad (\boldsymbol{s}_l \in \mathbb{Z}^l(s)) \tag{39}$$

is a $U_q(\widehat{\mathfrak{sl}}_n)$-submodule of Λ^s. The reader should be aware that $\mathbf{F}_q[\boldsymbol{s}_l]$ is *not* a $U_p(\widehat{\mathfrak{sl}}_l)$-submodule of Λ^s. In a similar way, by Theorem 2.4, the space

$$\mathbf{F}_p[\boldsymbol{s}_n]^\bullet := \bigoplus_{\boldsymbol{\lambda}_n \in \Pi^n} \mathbb{Q}(q)|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^\bullet \subset \Lambda^s \qquad (\boldsymbol{s}_n \in \mathbb{Z}^n(s)) \tag{40}$$

is a $U_p(\widehat{\mathfrak{sl}}_l)$-submodule of Λ^s.

Definition 2.10 [19] The representations $\mathbf{F}_q[\boldsymbol{s}_l]$ and $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$) are called *(q-deformed) Fock spaces*. When $l > 1$ and $n > 1$, we speak of higher-level Fock spaces. ◇

Since the maps τ_l and τ'_n are bijections, we have the following decompositions:

$$\Lambda^s = \bigoplus_{\boldsymbol{s}_l \in \mathbb{Z}^l(s)} \mathbf{F}_q[\boldsymbol{s}_l] = \bigoplus_{\boldsymbol{s}_n \in \mathbb{Z}^n(s)} \mathbf{F}_p[\boldsymbol{s}_n]^\bullet. \tag{41}$$

Neither of these decompositions is compatible with the decompositions of Λ^s given in Theorem 2.8.

2.3.2 Fock spaces as weight subspaces of Λ^s. Actions of the Weyl groups.

Let $N, L \in \mathbb{N}^*$. Recall the definition of $\mathbb{Q}^L(s)$ and $\mathbb{Q}^L(N)$ from (1) and define a map

$$\theta_{L,N} : \mathbb{Q}^L(s) \to \mathbb{Q}^L(N), \quad (s_1, \dots, s_L) \mapsto (N - s_1 + s_L, s_1 - s_2, \dots, s_{L-1} - s_L). \tag{42}$$

(Note that this map also depends on the charge $s \in \mathbb{Z}$ that we have fixed. However, since s will not vary in this paper, we do not keep it in our notation.) It is easy to see that $\theta_{L,N}$ is bijective. Moreover, for $(a_1, \dots, a_L) \in \mathbb{Q}^L(N)$, the l-tuple $(s_1, \dots, s_L) := \theta_{L,N}^{-1}(a_1, \dots, a_L)$ is given by

$$s_i = \frac{1}{L}\left(s - \sum_{j=1}^{L-1} j a_{j+1}\right) + \sum_{j=i+1}^{L} a_j \qquad (1 \le i \le L). \tag{43}$$

The next result shows that the Fock spaces are sums of certain weight subspaces of Λ^s. The proof follows easily from Corollary 2.5.

Proposition 2.11 [19]

(i) *Let $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$. Let $(a_1, \dots, a_n) := \theta_{n,l}(\boldsymbol{s}_n)$ and $w := \sum_{i=1}^n a_i \Lambda_{i-1}$. Then*

$$\mathbf{F}_p[\boldsymbol{s}_n]^\bullet = \bigoplus_{d \in \mathbb{Z}} \Lambda^s\langle w + d\delta\rangle.$$

(ii) *Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. Let $(a_1, \dots, a_l) := \theta_{l,n}(\boldsymbol{s}_l)$ and $\dot{w} := \sum_{i=1}^l a_i \dot{\Lambda}_{i-1}$. Then*

$$\mathbf{F}_q[\boldsymbol{s}_l] = \bigoplus_{d \in \mathbb{Z}} \Lambda^s\langle \dot{w} + d\dot{\delta}\rangle.$$

□

Note that the operator B_m ($m \in \mathbb{Z}^*$) maps the weight subspace $\Lambda^s\langle w\rangle$ (respectively $\Lambda^s\langle \dot{w}\rangle$) into $\Lambda^s\langle w + m\delta\rangle$ (respectively $\Lambda^s\langle \dot{w} + m\dot{\delta}\rangle$). Therefore, by Proposition 2.11, the Fock spaces $\mathbf{F}_q[\boldsymbol{s}_l]$ and $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$) are stable under the action of $\mathcal{H}$.

We now compare some weight subspaces of the Fock spaces. The proof follows again from Corollary 2.5.

Proposition 2.12

(i) *Let* $\boldsymbol{s}_l=(s_1,\dots,s_l)\in\mathbb{Z}^l(s)$ *and* w *be a weight of* $\mathbf{F}_q[\boldsymbol{s}_l]$. *Then there exists a unique pair* $(\boldsymbol{s}_n,\dot{w})$ *such that* $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle=\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{w}\rangle$, *where* $\boldsymbol{s}_n$ *is in* $\mathbb{Z}^n(s)$ *and* $\dot{w}$ *is a weight of* $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$. *More precisely, write* $w=d\delta+\sum_{i=1}^n a_i\Lambda_{i-1}$ *with* $a_1,\dots,a_n,d\in\mathbb{Z}$, *and put* $s_0:=n+s_l$. *Then we have* $\boldsymbol{s}_n=\theta_{n,l}^{-1}(a_1,\dots,a_n)$ *and* $\dot{w}=d\dot{\delta}+\sum_{i=0}^{l-1}(s_i-s_{i+1})\dot{\Lambda}_i$.

(ii) *Let* $\boldsymbol{s}_n=(s_1,\dots,s_n)\in\mathbb{Z}^n(s)$ *and* $\dot{w}$ *be a weight of* $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$. *Then there exists a unique pair* $(\boldsymbol{s}_l,w)$ *such that* $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{w}\rangle=\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$, *where* $\boldsymbol{s}_l$ *is in* $\mathbb{Z}^l(s)$ *and* w *is a weight of* $\mathbf{F}_q[\boldsymbol{s}_l]$. *More precisely, write* $\dot{w}=d\dot{\delta}+\sum_{i=1}^l a_i\dot{\Lambda}_{i-1}$ *with* $a_1,\dots,a_l,d\in\mathbb{Z}$, *and put* $s_0:=l+s_n$. *Then we have* $\boldsymbol{s}_l=\theta_{l,n}^{-1}(a_1,\dots,a_l)$ *and* $w=d\delta+\sum_{i=0}^{n-1}(s_i-s_{i+1})\Lambda_i$. □

Example 2.13 Take $n=3$, $l=2$, $\boldsymbol{s}_l=(1,0)$ and $w=-2\Lambda_0+\Lambda_1+3\Lambda_2-2\delta$. Then by (32), we have $\mathrm{wt}\big(\big|\big((1,1),(1)\big),\boldsymbol{s}_l\big\rangle\big)=w$, so w is a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$. By Proposition 2.12 (i), we have $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle=\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{w}\rangle$ with $\boldsymbol{s}_n=(2,1,-2)$ and $\dot{w}=2\dot{\Lambda}_0+\dot{\Lambda}_1-2\dot{\delta}$. Moreover, using (32) and (34), we see that for all $|\boldsymbol{\lambda}_l,\boldsymbol{s}_l\rangle=|\boldsymbol{\lambda}_n,\boldsymbol{s}_n\rangle^\bullet\in\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle=\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{w}\rangle$, we have $M_0(\boldsymbol{\lambda}_l;\boldsymbol{s}_l;n)=2$, $M_1(\boldsymbol{\lambda}_l;\boldsymbol{s}_l;n)=1$, $M_2(\boldsymbol{\lambda}_l;\boldsymbol{s}_l;n)=0$ and $M_0(\boldsymbol{\lambda}_n;\boldsymbol{s}_n;l)=M_1(\boldsymbol{\lambda}_n;\boldsymbol{s}_n;l)=0$ (this shows *a posteriori* that $\dim(\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle)=1$). ◇

We now deal with the actions of the Weyl groups of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$ on the set of the weight subspaces of Λ^s. Recall that $W_n=\langle\sigma_0,\dots,\sigma_{n-1}\rangle$ is the Weyl group of $U_q(\widehat{\mathfrak{sl}}_n)$. Since the α_i's are the simple roots and the Λ_j's are the fundamental weights for the Kac-Moody algebra $\widehat{\mathfrak{sl}}_n$, we have $(\Lambda_j,\alpha_i)=\delta_{i,j}$ for all $0\le i,\ j\le n-1$. Hence by (5), W_n acts on the weight lattice $\bigoplus_{i=0}^{n-1}\mathbb{Z}\Lambda_i\oplus\mathbb{Z}\delta$ by

$$\sigma_i.\delta=\delta\quad\text{and}$$
$$\sigma_i.\Lambda_j=\begin{cases}\Lambda_j & \text{if } j\neq i,\\ \Lambda_{i-1}+\Lambda_{i+1}-\Lambda_i-\delta_{i,0}\,\delta & \text{if } j=i\end{cases}\qquad(0\le i,j\le n-1).\tag{44}$$

Moreover, it is easy to see that W_n acts faithfully on $\mathbb{Z}^n(s)$ by

$$\begin{cases}\sigma_0.(s_1,\dots,s_n)=(s_n+l,s_2,\dots,s_{n-1},s_1-l),\\ \sigma_i.(s_1,\dots,s_n)=(s_1,\dots,s_{i+1},s_i,\dots,s_n)\qquad(1\le i\le n-1),\end{cases}\tag{45}$$

and the set $A_{n,l}(s)$ defined by (38) is a fundamental domain for this action. In a similar way, one can define two actions of the Weyl group $\dot{W}_l$ of $U_p(\widehat{\mathfrak{sl}}_l)$, one on the weight lattice $\bigoplus_{i=0}^{l-1}\mathbb{Z}\dot{\Lambda}_i\oplus\mathbb{Z}\dot{\delta}$ and one on $\mathbb{Z}^l(s)$. The following lemma will be useful later.

Lemma 2.14 *Let* $\boldsymbol{s}_n\in\mathbb{Z}^n(s)$ *and* $\dot{w}$ *be a weight of* $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$. *Let* $(\boldsymbol{s}_l,w)\in\mathbb{Z}^l(s)\times\mathcal{P}(\Lambda^s)$ *be the unique pair such that* $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{w}\rangle=\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ *(see Proposition 2.12* (i)). *Let* $\dot{\sigma} \in \dot{W}_l$. *In the same way, let* $(\boldsymbol{t}_l, w') \in \mathbb{Z}^l(s) \times \mathcal{P}(\Lambda^s)$ *be the unique pair such that* $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{\sigma}.\dot{w}\rangle = \mathbf{F}_q[\boldsymbol{t}_l]\langle w'\rangle$. *Then we have*

$$\boldsymbol{t}_l = \dot{\sigma}.\boldsymbol{s}_l \qquad \textit{and} \qquad w' = w + \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{t}_l\rangle) - \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle).$$

Proof The proof follows immediately from the formulas given in Proposition 2.12. □

2.3.3 *The lower crystal basis* $(\mathcal{L}[\boldsymbol{s}_l], \mathcal{B}[\boldsymbol{s}_l])$ *of* $\mathbf{F}_q[\boldsymbol{s}_l]$ *at* $q = 0$

Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. Following [8], let $\mathbb{A} \subset \mathbb{Q}(q)$ be the ring of rational functions which are regular at $q = 0$, $\mathcal{L}[\boldsymbol{s}_l] := \bigoplus_{\boldsymbol{\lambda}_l \in \Pi^l} \mathbb{A}\,|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle$ and for $0 \le i \le n-1$, let $\tilde{e}_i^{\,\mathrm{low}}$, $\tilde{f}_i^{\,\mathrm{low}}$, $\tilde{e}_i^{\,\mathrm{up}}$ and $\tilde{f}_i^{\,\mathrm{up}}$ denote Kashiwara's operators acting on $\mathcal{L}[\boldsymbol{s}_l]$. The following lemma shows that sometimes, certain powers of the operators $\tilde{e}_i^{\,\mathrm{low}}$ and $\tilde{e}_i^{\,\mathrm{up}}$ coincide (one has an analogous result for $\tilde{f}_i^{\,\mathrm{low}}$ and $\tilde{f}_i^{\,\mathrm{up}}$).

Lemma 2.15 *Let* $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$, $u \in (\mathrm{Ker}\, e_i) \cap \mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ *and* $k := (w, \alpha_i)$. *Then we have*

$$(\tilde{e}_i^{\,\mathrm{up}})^k.(f_i^{(k)}.u) = (\tilde{e}_i^{\,\mathrm{low}})^k.(f_i^{(k)}.u) = u.$$

Proof The second equality follows easily by induction on k from the definition of $\tilde{e}_i^{\,\mathrm{low}}$. Let us now show that $(\tilde{e}_i^{\,\mathrm{up}})^k.(f_i^{(k)}.u) = u$. Note that for $0 \le j \le k$, we have

$$(\mathrm{wt}(f_i^{(k-j)}.u), \alpha_i) = (\mathrm{wt}(u), \alpha_i) - (k-j)(\alpha_i, \alpha_i) = 2j - k.$$

By induction on $0 \le k' \le k$, we get therefore, by definition of $\tilde{e}_i^{\,\mathrm{up}}$,

$$(\tilde{e}_i^{\,\mathrm{up}})^{k'}.(f_i^{(k)}.u) = \left(\prod_{j=0}^{k'-1} \frac{[(2j-k)+(k-j)+1]}{[k-j]}\right) f_i^{(k-k')}.u \qquad (0 \le k' \le k).$$

As a consequence, we have $(\tilde{e}_i^{\,\mathrm{up}})^k.(f_i^{(k)}.u) = \left(\prod_{j=0}^{k-1} \frac{[j+1]}{[k-j]}\right) u = u.$ □

Put

$$\mathcal{B}[\boldsymbol{s}_l] := \{|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \bmod q\mathcal{L}[\boldsymbol{s}_l] \mid \boldsymbol{\lambda}_l \in \Pi^l\}. \tag{46}$$

From now on, we shall write more briefly $\boldsymbol{\lambda}_l$ for the element in $\mathcal{B}[\boldsymbol{s}_l]$ indexed by the corresponding multi-partition. By [3, 6, 19], the pair $(\mathcal{L}[\boldsymbol{s}_l], \mathcal{B}[\boldsymbol{s}_l])$ is a lower crystal basis of $\mathbf{F}_q[\boldsymbol{s}_l]$ at $q = 0$ in the sense of [8], and the crystal graph $\mathcal{B}[\boldsymbol{s}_l]$ contains the arrow $\boldsymbol{\lambda}_l \xrightarrow{i} \boldsymbol{\mu}_l$ if and only if the multi-partition $\boldsymbol{\mu}_l$ is obtained from $\boldsymbol{\lambda}_l$ by adding a good i-node in the sense of [19, Thm. 2.4]. We shall still denote by $\tilde{e}_i^{\,\mathrm{low}}$ and $\tilde{f}_i^{\,\mathrm{low}}$ Kashiwara's operators acting on $\mathcal{B}[\boldsymbol{s}_l] \cup \{0\}$.

2.3.4 Uglov's canonical bases of the Fock spaces

By Propositions 2.9 (iii) and 2.11, the Fock spaces $\mathbf{F}_q[\boldsymbol{s}_l]$ and $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$) are stable under the involution $^-$. The involution induced on these spaces will still be denoted by $^-$. Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. For $\boldsymbol{\mu}_l \in \Pi^l$, write

$$\overline{|\boldsymbol{\mu}_l, \boldsymbol{s}_l\rangle} = \sum_{\boldsymbol{\lambda}_l \in \Pi^l} a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\, |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \tag{47}$$

with $a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q) \in \mathbb{Z}[q, q^{-1}]$, and let

$$A_{\boldsymbol{s}_l}(q) := \big(a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\big)_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l} \tag{48}$$

denote the matrix of the involution $^-$ of $\mathbf{F}_q[\boldsymbol{s}_l]$. Since the weight subspaces of $\mathbf{F}_q[\boldsymbol{s}_l]$ are stable under the involution $^-$, (32) implies that $a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)$ is zero unless $|\boldsymbol{\lambda}_l| = |\boldsymbol{\mu}_l|$, where $|\boldsymbol{\lambda}_l|$ (respectively $|\boldsymbol{\mu}_l|$) denotes the number of boxes contained in the Young diagram of $|\boldsymbol{\lambda}_l|$ (respectively $|\boldsymbol{\mu}_l|$). By Proposition 2.9 (ii), the matrix $A_{\boldsymbol{s}_l}(q)$ is unitriangular. One can therefore define, by a classical argument, canonical bases of the Fock space $\mathbf{F}_q[\boldsymbol{s}_l]$ as follows.

Theorem 2.16 [19] *Let $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. Then there exists a unique base*

$$\{G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\} \qquad \Big(\textit{respectively } \{G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}\Big)$$

of $\mathbf{F}_q[\boldsymbol{s}_l]$ such that:

(i) $\overline{G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)} = G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ (*respectively* $\overline{G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)} = G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$),
(ii) $G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \equiv |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \bmod q\mathcal{L}^+[\boldsymbol{s}_l]$ (*respectively* $G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \equiv |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \bmod q^{-1}\mathcal{L}^-[\boldsymbol{s}_l]$),

where $\mathcal{L}^\epsilon[\boldsymbol{s}_l] := \bigoplus_{\boldsymbol{\lambda}_l \in \Pi^l} \mathbb{Z}[q^\epsilon]\, |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle$ ($\epsilon = \pm 1$). □

Definition 2.17 The bases $\{G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ and $\{G^-(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ are called *Uglov's canonical bases* of $\mathbf{F}_q[\boldsymbol{s}_l]$. Define entries $\Delta^+_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)$, $\Delta^-_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q) \in \mathbb{Z}[q, q^{-1}]$ ($\boldsymbol{\lambda}_l$, $\boldsymbol{\mu}_l \in \Pi^l$) by

$$G^+(\boldsymbol{\mu}_l, \boldsymbol{s}_l) = \sum_{\boldsymbol{\lambda}_l \in \Pi^l} \Delta^+_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\, |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle, \quad G^-(\boldsymbol{\mu}_l, \boldsymbol{s}_l) = \sum_{\boldsymbol{\lambda}_l \in \Pi^l} \Delta^-_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\, |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle, \tag{49}$$

and denote by

$$\Delta^\epsilon_{\boldsymbol{s}_l}(q) := \big(\Delta^\epsilon_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q)\big)_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l} \qquad (\epsilon = \pm 1) \tag{50}$$

the transition matrices between the standard and the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]$. ◇

By [19], the entries of $\Delta^{+}_{s_l}(q)$ (respectively $\Delta^{-}_{s_l}(q)$) are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type A, so by [11], these polynomials are in $\mathbb{N}[q]$ (respectively $\mathbb{N}[p]$). Moreover, both canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]$ are dual to each other with respect to a certain bilinear form, which gives an inversion formula for Kazhdan-Lusztig polynomials; see [19, Thm. 5.15]. By [19], the basis $\{G^+(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ is a lower global crystal basis (in the sense of [8]) of the integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module $\mathbf{F}_q[\boldsymbol{s}_l]$.

Let $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$. In a similar way, one can define canonical bases $\{G^{\epsilon}(\boldsymbol{\lambda}_n, \boldsymbol{s}_n)^{\bullet} \mid \boldsymbol{\lambda}_n \in \Pi^n\}$ ($\epsilon = \pm 1$) of the Fock space $\mathbf{F}_p[\boldsymbol{s}_n]^{\bullet}$. By [19], the basis $\{G^{-}(\boldsymbol{\lambda}_n, \boldsymbol{s}_n)^{\bullet} \mid \boldsymbol{\lambda}_n \in \Pi^n\}$ is a lower global crystal basis of the integrable $U_p(\widehat{\mathfrak{sl}}_l)$-module $\mathbf{F}_p[\boldsymbol{s}_n]^{\bullet}$. For $\boldsymbol{\mu}_n \in \Pi^n$, $\epsilon = \pm 1$, write

$$G^{\epsilon}(\boldsymbol{\mu}_n, \boldsymbol{s}_n)^{\bullet} = \sum_{\boldsymbol{\lambda}_n \in \Pi^n} \dot{\Delta}^{\epsilon}_{\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n; \boldsymbol{s}_n}(q) \, |\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^{\bullet}, \tag{51}$$

where the entries $\dot{\Delta}^{\epsilon}_{\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n; \boldsymbol{s}_n}(q)$ are in $\mathbb{Z}[q, q^{-1}]$. Since the weight subspaces of Λ^s are stable under the involution $^-$, we have $\dot{\Delta}^{\epsilon}_{\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n; \boldsymbol{s}_n}(q) = 0$ unless $\mathrm{wt}(|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^{\bullet}) = \mathrm{wt}(|\boldsymbol{\mu}_n, \boldsymbol{s}_n\rangle^{\bullet})$ and $\dot{\mathrm{wt}}(|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^{\bullet}) = \dot{\mathrm{wt}}(|\boldsymbol{\mu}_n, \boldsymbol{s}_n\rangle^{\bullet})$. In this case, by Proposition 2.12 (ii), there exist $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$ and $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l$ such that $|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle = |\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^{\bullet}$ and $|\boldsymbol{\mu}_l, \boldsymbol{s}_l\rangle = |\boldsymbol{\mu}_n, \boldsymbol{s}_n\rangle^{\bullet}$. It is then not hard to see that

$$\dot{\Delta}^{\epsilon}_{\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n; \boldsymbol{s}_n}(q) = \Delta^{\epsilon}_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q). \tag{52}$$

3 Comparison of canonical bases of weight subspaces of $\mathbf{F}_q[\boldsymbol{s}_l]$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$)

From now on, we shall use the following notation.

Notation 3.1 For $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$ and $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$, put

$$\Pi^l(\boldsymbol{s}_l; w) := \{\boldsymbol{\lambda}_l \in \Pi^l \bigm| |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \in \Lambda^s\langle w\rangle\}, \tag{53}$$

and define similarly $\Pi^n(\boldsymbol{s}_n; \dot{w})$ for $\boldsymbol{s}_n \in \mathbb{Z}^n(s)$ and $\dot{w} \in \dot{\mathcal{P}}(\mathbf{F}_p[\boldsymbol{s}_n]^{\bullet})$. ◇

Definition 3.2 Let $\boldsymbol{s}_l, \boldsymbol{t}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$ and $w' \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{t}_l])$. We say that the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ and $\mathbf{F}_q[\boldsymbol{t}_l]\langle w'\rangle$ are *similar* if there exists a bijection

$$\sigma : \Pi^l(\boldsymbol{s}_l; w) \to \Pi^l(\boldsymbol{t}_l; w')$$

such that for all $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l(\boldsymbol{s}_l; w)$, $\epsilon = \pm 1$, we have

$$\Delta^{\epsilon}_{\sigma(\boldsymbol{\lambda}_l), \sigma(\boldsymbol{\mu}_l); \boldsymbol{t}_l}(q) = \Delta^{\epsilon}_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \boldsymbol{s}_l}(q).$$

In other words, the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ and $\mathbf{F}_q[\boldsymbol{t}_l]\langle w'\rangle$ are similar if the transition matrices between the standard bases and the canonical bases are equal up to a reindexing of rows and columns. ◇

Notation 3.3 Throughout this section we fix a multi-charge $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$. To simplify, we drop the multi-charge $\boldsymbol{s}_l$ in the notation of this section, that is we denote by $\boldsymbol{\lambda}_l$ (respectively $G^{\pm}(\boldsymbol{\lambda}_l)$) the vector of the standard (respectively canonical) basis of $\mathbf{F}_q[\boldsymbol{s}_l]$ indexed by the corresponding multi-partition and so on. In particular, we use the notation $\boldsymbol{\lambda}_l$ either for a vector of the standard basis of $\mathbf{F}_q[\boldsymbol{s}_l]$ or for a vertex in the crystal graph $\mathcal{B} := \mathcal{B}[\boldsymbol{s}_l]$. ◇

3.1 The bijections σ_i

We recall here the definition of the involution σ_i $(0 \le i \le n-1)$ of the crystal graph $\mathcal{B}$. We sometimes view σ_i as a bijection of Π^l.

Definition 3.4 Let $\boldsymbol{\lambda}_l \in \mathcal{B} \cong \Pi^l$ and $i \in [\![0; n-1]\!]$. Let $\mathcal{C}$ be the i-chain in $\mathcal{B}$ containing $\boldsymbol{\lambda}_l$. Let $\sigma_i(\boldsymbol{\lambda}_l) \in \mathcal{B} \cong \Pi^l$ be the unique element in $\mathcal{C}$ such that $\mathrm{wt}(\sigma_i(\boldsymbol{\lambda}_l)) = \sigma_i.(\mathrm{wt}(\boldsymbol{\lambda}_l))$. In other words, $\sigma_i(\boldsymbol{\lambda}_l)$ is obtained from $\boldsymbol{\lambda}_l$ via a central symmetry in the middle of $\mathcal{C}$. This defines an involution σ_i of $\mathcal{B}$. This map induces, for $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$, a bijection

$$\sigma_i : \Pi^l(\boldsymbol{s}_l; w) \xrightarrow{\sim} \Pi^l(\boldsymbol{s}_l; \sigma_i.w). \tag{54}$$

◇

By [9], the definition of $\sigma_0, \dots, \sigma_{n-1}$ as bijections of $\mathcal{B}$ gives actually rise to an action of the Weyl group W_n on $\mathcal{B}$, but we do not need this fact in this article.

Proposition 3.5 *Let $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$ and $i \in [\![0; n-1]\!]$ be such that $w + \alpha_i \notin \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$. Let $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{s}_l; w)$ and $\boldsymbol{\mu}_l := \sigma_i(\boldsymbol{\lambda}_l)$. Then we have the following*:

- (i) *$\boldsymbol{\mu}_l$ is the multi-partition obtained by adding to $\boldsymbol{\lambda}_l$ all its addable i-nodes, and we have $|\boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l| = k_i := (w, \alpha_i)$.*
- (ii) *In $\mathbf{F}_q[\boldsymbol{s}_l]$, we have $\boldsymbol{\mu}_l = f_i^{(k_i)}.\boldsymbol{\lambda}_l$ and $\boldsymbol{\lambda}_l = e_i^{(k_i)}.\boldsymbol{\mu}_l$.*
- (iii) *In $\mathcal{B}$, we have $\boldsymbol{\mu}_l = (\tilde{f}_i^{\,\mathrm{low}})^{k_i}.\boldsymbol{\lambda}_l$ and $\boldsymbol{\lambda}_l = (\tilde{e}_i^{\,\mathrm{low}})^{k_i}.\boldsymbol{\mu}_l$.*

Proof Let $\mathcal{C}$ be the i-chain in $\mathcal{B}$ containing $\boldsymbol{\lambda}_l$ and $\boldsymbol{\mu}_l$. Since $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$, $\boldsymbol{\lambda}_l$ has no removable i-node. This implies, by [19, Thm. 2.4], that $\boldsymbol{\lambda}_l$ is the head of the chain $\mathcal{C}$, and by symmetry $\boldsymbol{\mu}_l$ is the tail of $\mathcal{C}$. Note that since $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$, $\sigma_i.(w + \alpha_i) = (\sigma_i.w) - \alpha_i$ is also not a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$, so $\boldsymbol{\mu}_l$ has no addable i-node. By [19, Thm. 2.4], $\boldsymbol{\mu}_l$ is obtained by adding some i-nodes (let us say k of them) to $\boldsymbol{\lambda}_l$. The integer $k = |\boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l|$ is none other than the length of the chain $\mathcal{C}$. By a well-known formula for crystal graphs relating weights and positions in the i-chains, since $\boldsymbol{\lambda}_l$ is the head of $\mathcal{C}$, we have $k_i = (\mathrm{wt}(\boldsymbol{\lambda}_l), \alpha_i) = k$. Thus $\mathcal{C}$ is a i-chain of length k_i with head $\boldsymbol{\lambda}_l$ and tail $\boldsymbol{\mu}_l$, which proves (iii). The divided powers $e_i^{(k)}, f_i^{(k)} \in U_q(\widehat{\mathfrak{sl}}_n)$ $(k \in \mathbb{N}^*)$ act on $\boldsymbol{\nu}_l \in \mathbf{F}_q[\boldsymbol{s}_l]$ as follows, with notation of Section 2.2.2:

$$e_i^{(k)}.\boldsymbol{\nu}_l = \sum_{\boldsymbol{\kappa}_l \xrightarrow{i:k} \boldsymbol{\nu}_l} q^{-N_i^{<}(\boldsymbol{\kappa}_l;\boldsymbol{\nu}_l;\boldsymbol{s}_l;n)} \boldsymbol{\kappa}_l \qquad \text{and} \qquad f_i^{(k)}.\boldsymbol{\nu}_l = \sum_{\boldsymbol{\nu}_l \xrightarrow{i:k} \boldsymbol{\xi}_l} q^{N_i^{>}(\boldsymbol{\nu}_l;\boldsymbol{\xi}_l;\boldsymbol{s}_l;n)} \boldsymbol{\xi}_l. \tag{$*$}$$

(For $k=1$, this is a part of Theorem 2.3. The general case follows by induction on k; see *e.g.* [5] for a detailed proof.) Since $\boldsymbol{\mu}_l$ has no addable i-node and $\boldsymbol{\lambda}_l$ has no removable i-node, $\boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l$ is the set of the addable i-nodes of $\boldsymbol{\lambda}_l$ and also the set of the removable i-nodes of $\boldsymbol{\mu}_l$, and this set has exactly k_i elements. This proves (i), and this together with $(*)$ proves (ii). □

Example 3.6 Take $n=3$, $l=2$, $\boldsymbol{s}_l=(1,2)$, $i=0$, $\boldsymbol{\lambda}_l=((2,2,1),(3,2))$ and $w=\mathrm{wt}(|\boldsymbol{\lambda}_l,\boldsymbol{s}_l\rangle)$. Then we have $\sigma_i(\boldsymbol{\lambda}_l)=((3,2,2),(3,3,1))$. Note that $w+\alpha_i$ is not a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$, so Proposition 3.5 can be applied in this case. ◇

Remark 3.7 The proof of Proposition 3.5 shows (with the assumptions and notation of this proposition) that $\boldsymbol{\lambda}_l=\sigma_i^{-1}(\boldsymbol{\mu}_l)$ is the multi-partition obtained by removing to $\boldsymbol{\mu}_l$ all its removable i-nodes. ◇

3.2 A first theorem of comparison

Define a symmetric bilinear non-degenerate form $(.\,,.)$ on $\mathbf{F}_q[\boldsymbol{s}_l]$ by

$$(\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l)=q^{\|\boldsymbol{\lambda}_l\|}\delta_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}\qquad(\boldsymbol{\lambda}_l,\ \boldsymbol{\mu}_l\in\Pi^l),\tag{55}$$

where we put

$$\|\boldsymbol{\lambda}_l\|:=(\mathrm{wt}(\boldsymbol{\lambda}_l),\mathrm{wt}(\boldsymbol{\lambda}_l))/2\qquad(\boldsymbol{\lambda}_l\in\Pi^l).\tag{56}$$

This form enjoys the following property:

Lemma 3.8 *For $u,v\in\mathbf{F}_q[\boldsymbol{s}_l]$, $0\le i\le n-1$, we have*

$$(t_i.u\,,v)=(u\,,t_i.v)\qquad\textit{and}\qquad(e_i.u\,,v)=(u\,,f_i.v).$$

Proof Identical to the proof of [12, Prop. 8.1]. □

Let $\{G^*(\boldsymbol{\lambda}_l)\mid\boldsymbol{\lambda}_l\in\Pi^l\}$ denote the adjoint basis of $\{G^+(\boldsymbol{\lambda}_l)\mid\boldsymbol{\lambda}_l\in\Pi^l\}$ with respect to the form $(.\,,.)$. Since the basis $\{G^+(\boldsymbol{\lambda}_l)\mid\boldsymbol{\lambda}_l\in\Pi^l\}$ is a lower global crystal basis of $\mathbf{F}_q[\boldsymbol{s}_l]$ in the sense of [8], it follows by Lemma 3.8 and [8, Prop. 3.2.2] that $\{G^*(\boldsymbol{\lambda}_l)\mid\boldsymbol{\lambda}_l\in\Pi^l\}$ is an upper global crystal basis of $\mathbf{F}_q[\boldsymbol{s}_l]$.

We are now ready to prove the following result, which is a generalization to higher-level Fock spaces of [13, Thm. 20].

Theorem 3.9 *Let $\boldsymbol{s}_l\in\mathbb{Z}^l(s)$, $w\in\mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$ and $i\in[\![0;n-1]\!]$ be such that $w+\alpha_i$ is not a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$. Let $\sigma_i:\Pi^l(\boldsymbol{s}_l;w)\to\Pi^l(\boldsymbol{s}_l;\sigma_i.w)$ be the bijection defined by (54). Then we have, for $\boldsymbol{\lambda}_l,\ \boldsymbol{\mu}_l\in\Pi^l(\boldsymbol{s}_l;w)$,*

$$\text{(i)}\quad\Delta^+_{\sigma_i(\boldsymbol{\lambda}_l),\sigma_i(\boldsymbol{\mu}_l);\,\boldsymbol{s}_l}(q)=\Delta^+_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l;\,\boldsymbol{s}_l}(q)\quad\textit{and}$$

$$\text{(ii)}\quad\Delta^-_{\sigma_i(\boldsymbol{\lambda}_l),\sigma_i(\boldsymbol{\mu}_l);\,\boldsymbol{s}_l}(q)=\Delta^-_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l;\,\boldsymbol{s}_l}(q).$$

As a consequence, the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ *and* $\mathbf{F}_q[\boldsymbol{s}_l]\langle \sigma_i.w\rangle$ *are similar in the sense of Definition* 3.2.

Proof Let us prove (i). Let $\boldsymbol{\mu}_l \in \Pi^l(\boldsymbol{s}_l; w)$. Taking adjoint bases in (49) yields

$$q^{-\|\sigma_i(\boldsymbol{\mu}_l)\|}\sigma_i(\boldsymbol{\mu}_l) = \sum_{\boldsymbol{\nu}_l\in\Pi^l(\boldsymbol{s}_l;\sigma_i.w)} \Delta^+_{\sigma_i(\boldsymbol{\mu}_l),\boldsymbol{\nu}_l;\,\boldsymbol{s}_l}(q)\; G^*(\boldsymbol{\nu}_l).$$

Since $\sigma_i : \Pi^l(\boldsymbol{s}_l; w) \to \Pi^l(\boldsymbol{s}_l; \sigma_i.w)$ is a bijection, we can make in the sum above the reindexing $\boldsymbol{\nu}_l = \sigma_i(\boldsymbol{\lambda}_l)$. If we now apply $e_i^{(k_i)}$ with $k_i := (w, \alpha_i)$ to both hand-sides of this equality, we get

$$q^{-\|\sigma_i(\boldsymbol{\mu}_l)\|}e_i^{(k_i)}.\sigma_i(\boldsymbol{\mu}_l) = \sum_{\boldsymbol{\lambda}_l\in\Pi^l(\boldsymbol{s}_l;w)} \Delta^+_{\sigma_i(\boldsymbol{\mu}_l),\sigma_i(\boldsymbol{\lambda}_l);\,\boldsymbol{s}_l}(q)\; e_i^{(k_i)}.G^*(\sigma_i(\boldsymbol{\lambda}_l)). \qquad (*)$$

Note that

$$\|\sigma_i(\boldsymbol{\mu}_l)\| = \big(\sigma_i.\mathrm{wt}(\boldsymbol{\mu}_l)\,,\,\sigma_i.\mathrm{wt}(\boldsymbol{\mu}_l)\big)/2 = \big(\mathrm{wt}(\boldsymbol{\mu}_l)\,,\,\mathrm{wt}(\boldsymbol{\mu}_l)\big)/2 = \|\boldsymbol{\mu}_l\|. \qquad (**)$$

By Proposition 3.5 (ii), we have $e_i^{(k_i)}.\sigma_i(\boldsymbol{\mu}_l) = \boldsymbol{\mu}_l$. Now let $\boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{s}_l; w)$. Since $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$, we have $e_i.\boldsymbol{\lambda}_l = 0$, whence $\tilde{e}_i^{\,\mathrm{up}}.\boldsymbol{\lambda}_l = 0$. Moreover, again by Proposition 3.5, we have $\sigma_i(\boldsymbol{\lambda}_l) = f_i^{(k_i)}.\boldsymbol{\lambda}_l$. Therefore, by Lemma 2.15, we have

$$\boldsymbol{\lambda}_l = (\tilde{e}_i^{\,\mathrm{low}})^{k_i}.(f_i^{(k_i)}.\boldsymbol{\lambda}_l) = (\tilde{e}_i^{\,\mathrm{up}})^{k_i}.(f_i^{(k_i)}.\boldsymbol{\lambda}_l) = (\tilde{e}_i^{\,\mathrm{up}})^{k_i}.(\sigma_i(\boldsymbol{\lambda}_l)),$$

whence $(\tilde{e}_i^{\,\mathrm{up}})^{k_i+1}.\sigma_i(\boldsymbol{\lambda}_l) = 0$. Since $\{G^*(\boldsymbol{\nu}_l) \mid \boldsymbol{\nu}_l \in \Pi^l\}$ is an upper global crystal basis of $\mathbf{F}_q[\boldsymbol{s}_l]$, [8, Lemma 5.1.1 (ii)] then implies

$$e_i^{(k_i)}.G^*(\sigma_i(\boldsymbol{\lambda}_l)) = G^*\big((\tilde{e}_i^{\,\mathrm{up}})^{k_i}.(\sigma_i(\boldsymbol{\lambda}_l))\big) = G^*(\boldsymbol{\lambda}_l). \qquad (***)$$

Combining $(*)$, $(**)$ and $(***)$ we get

$$q^{-\|\boldsymbol{\mu}_l\|}\boldsymbol{\mu}_l = \sum_{\boldsymbol{\lambda}_l\in\Pi^l(\boldsymbol{s}_l;w)} \Delta^+_{\sigma_i(\boldsymbol{\mu}_l),\sigma_i(\boldsymbol{\lambda}_l);\,\boldsymbol{s}_l}(q)\; G^*(\boldsymbol{\lambda}_l).$$

Since this is valid for any $\boldsymbol{\mu}_l \in \Pi^l(\boldsymbol{s}_l; w)$, we get the claimed formula by taking again adjoint bases.

Let us now prove (ii). Let $w' \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$. If we know the basis $\{G^+(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{s}_l; w')\}$, then we can compute the involution of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w'\rangle$ by solving a unitriangular system. Since the canonical basis $\{G^-(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{s}_l; w')\}$ is uniquely determined by the involution of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w'\rangle$, the basis $\{G^-(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{s}_l; w')\}$ is uniquely determined by the basis $\{G^+(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(\boldsymbol{s}_l; w')\}$ (and conversely). Thus (i) implies (ii). (For a different proof of this fact, one can also apply [19, Thm 5.15].) □

Example 3.10 Take $n = 3$, $l = 2$, $\boldsymbol{s}_l = (1, 0)$, $w = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle) - (2\alpha_0 + 3\alpha_1 + \alpha_2)$ and $i = 2$. One can easily check that $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\boldsymbol{s}_l]$. The elements of $\Pi^l(\mathbf{s}_l; w)$ are

$$\big((1),(5)\big),\ \big((4),(2)\big),\ \big((4,2),\emptyset\big),\ \big((1),(2,2,1)\big),\ \big((2,2),(2)\big),$$
$$\big((1,1),(2,1,1)\big),\ \big((1,1,1,1),(2)\big),\ \big((1),(2,1,1,1)\big),$$

and their respective images by the map σ_i are

$$\big((2),(6,1)\big),\ \big((5),(3,1)\big),\ \big((5,3,1),\emptyset\big),\ \big((2),(3,2,2)\big),\ \big((2,2,1),(3,1)\big),$$
$$\big((2,1,1),(3,1,1)\big),\ \big((2,1,1,1),(3,1)\big),\ \big((2),(3,1,1,1,1)\big).$$

With obvious notation, the transition matrices of the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w\rangle$ are

$$\Delta^+_{\mathbf{s}_l}\langle w\rangle(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ q & 1 & . & . & . & . & . & . \\ 0 & q & 1 & . & . & . & . & . \\ q & 0 & 0 & 1 & . & . & . & . \\ q & q^2 & q & 0 & 1 & . & . & . \\ q^2 & 0 & 0 & q & q & 1 & . & . \\ 0 & 0 & q & 0 & q^2 & q & 1 & . \\ 0 & 0 & 0 & q^2 & 0 & q & 0 & 1 \end{pmatrix} \begin{matrix} \big((1),(5)\big) \\ \big((4),(2)\big) \\ \big((4,2),\emptyset\big) \\ \big((1),(2,2,1)\big) \\ \big((2,2),(2)\big) \\ \big((1,1),(2,1,1)\big) \\ \big((1,1,1,1),(2)\big) \\ \big((1),(2,1,1,1)\big) \end{matrix} \quad \text{and}$$

$$\Delta^-_{\mathbf{s}_l}\langle w\rangle(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ q^{-1} & 1 & . & . & . & . & . & . \\ q^{-2} & -q^{-1} & 1 & . & . & . & . & . \\ -q^{-1} & 0 & 0 & 1 & . & . & . & . \\ 0 & 0 & -q^{-1} & 0 & 1 & . & . & . \\ q^{-2} & -q^{-1} & q^{-2} & -q^{-1} & -q^{-1} & 1 & . & . \\ -q^{-3} & q^{-2} & 0 & q^{-2} & 0 & -q^{-1} & 1 & . \\ 0 & q^{-2} & -q^{-3} & 0 & q^{-2} & -q^{-1} & 0 & 1 \end{pmatrix} \begin{matrix} \big((1),(5)\big) \\ \big((4),(2)\big) \\ \big((4,2),\emptyset\big) \\ \big((1),(2,2,1)\big) \\ \big((2,2),(2)\big) \\ \big((1,1),(2,1,1)\big) \\ \big((1,1,1,1),(2)\big) \\ \big((1),(2,1,1,1)\big) \end{matrix}.$$

In the same way, the transition matrices of the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle\sigma_i.w\rangle$ are

$$\Delta^+_{\mathbf{s}_l}\langle\sigma_i.w\rangle(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ q & 1 & . & . & . & . & . & . \\ 0 & q & 1 & . & . & . & . & . \\ q & 0 & 0 & 1 & . & . & . & . \\ q & q^2 & q & 0 & 1 & . & . & . \\ q^2 & 0 & 0 & q & q & 1 & . & . \\ 0 & 0 & q & 0 & q^2 & q & 1 & . \\ 0 & 0 & 0 & q^2 & 0 & q & 0 & 1 \end{pmatrix} \begin{matrix} \big((2),(6,1)\big) \\ \big((5),(3,1)\big) \\ \big((5,3,1),\emptyset\big) \\ \big((2),(3,2,2)\big) \\ \big((2,2,1),(3,1)\big) \\ \big((2,1,1),(3,1,1)\big) \\ \big((2,1,1,1),(3,1)\big) \\ \big((2),(3,1,1,1,1)\big) \end{matrix} \quad \text{and}$$

$$\Delta^-_{s_l}\langle\sigma_i.w\rangle(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ q^{-1} & 1 & . & . & . & . & . & . \\ q^{-2} & -q^{-1} & 1 & . & . & . & . & . \\ -q^{-1} & 0 & 0 & 1 & . & . & . & . \\ 0 & 0 & -q^{-1} & 0 & 1 & . & . & . \\ q^{-2} & -q^{-1} & q^{-2} & -q^{-1} & -q^{-1} & 1 & . & . \\ -q^{-3} & q^{-2} & 0 & q^{-2} & 0 & -q^{-1} & 1 & . \\ 0 & q^{-2} & -q^{-3} & 0 & q^{-2} & -q^{-1} & 0 & 1 \end{pmatrix} \begin{matrix} \big((2),(6,1)\big) \\ \big((5),(3,1)\big) \\ \big((5,3,1),\emptyset\big) \\ \big((2),(3,2,2)\big) \\ \big((2,2,1),(3,1)\big) \\ \big((2,1,1),(3,1,1)\big) \\ \big((2,1,1,1),(3,1)\big) \\ \big((2),(3,1,1,1,1)\big) \end{matrix},$$

in agreement with Theorem 3.9. ◇

4 Comparison of canonical bases of $\mathbf{F}_q[s_l]\langle w\rangle$ for w in a given coset in $\mathcal{P}(\Lambda^s)/\mathbb{Z}\delta$ and different multi-charges s_l

Notation 4.1 For $\boldsymbol{a}_l, \boldsymbol{b}_l \in \mathbb{Z}^l(s)$, introduce the shorter notation

$$d(\boldsymbol{a}_l, \boldsymbol{b}_l) := \Delta(\boldsymbol{a}_l, n) - \Delta(\boldsymbol{b}_l, n) \in \mathbb{Z}. \tag{57}$$

Throughout this section we fix $\boldsymbol{s}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$ and $i \in [\![0; l-1]\!]$. Finally, let $(\boldsymbol{s}_n, \dot{w}) \in \mathbb{Z}^n(s) \times \dot{\mathcal{P}}(\Lambda^s)$ be the pair such that $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle = \mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{w}\rangle$ (see Proposition 2.12 (i)). ◇

4.1 A second theorem of comparison

Recall that $\dot{W}_l = \langle\dot{\sigma}_0, \ldots, \dot{\sigma}_{l-1}\rangle \cong \widetilde{\mathfrak{S}}_l$ is the Weyl group of $U_p(\widehat{\mathfrak{sl}}_l)$.

Definition 4.2 Keep Notation 4.1. By analogy with Definition 3.4, define a bijection

$$\dot{\sigma}_i : \Pi^n(\boldsymbol{s}_n; \dot{w}) \to \Pi^n(\boldsymbol{s}_n; \dot{\sigma}_i.\dot{w}) \tag{58}$$

which enjoys similar properties as the bijections σ_j from (54). Since $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle = \mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{w}\rangle$, we have a bijection between the standard basis of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ (as a subspace of $\mathbf{F}_q[\boldsymbol{s}_l]$) and the standard basis of $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet\langle\dot{w}\rangle$ (as a subspace of $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$). We thus have a bijection $\Pi^l(\boldsymbol{s}_l; w) \xrightarrow{\sim} \Pi^n(\boldsymbol{s}_n; \dot{w})$. Put $\boldsymbol{t}_l := \dot{\sigma}_i.\boldsymbol{s}_l$. The same argument gives, by Lemma 2.14, a bijection $\Pi^l\big(\boldsymbol{t}_l; w + d(\boldsymbol{s}_l, \boldsymbol{t}_l)\,\delta\big) \xrightarrow{\sim} \Pi^n(\boldsymbol{s}_n; \dot{\sigma}_i.\dot{w})$. We therefore have the following commutative diagram of bijections, in which the dashed arrow will still be denoted by $\dot{\sigma}_i$.

$$\begin{array}{ccc} \Pi^n(\boldsymbol{s}_n; \dot{w}) & \xrightarrow{\dot{\sigma}_i} & \Pi^n(\boldsymbol{s}_n; \dot{\sigma}_i.\dot{w}) \\ \uparrow & & \uparrow \\ \Pi^l(\boldsymbol{s}_l; w) & \overset{\sim}{\dashrightarrow} & \Pi^l\big(\dot{\sigma}_i.\boldsymbol{s}_l\,;\ w + d(\boldsymbol{s}_l, \dot{\sigma}_i.\boldsymbol{s}_l)\,\delta\big). \end{array}$$

◇

Example 4.3 Take $n = 2$, $l = 3$, $\boldsymbol{s}_l = (0, 2, -1)$ and $i = 2$. Note that $d(\boldsymbol{s}_l, \dot{\sigma}_i.\boldsymbol{s}_l) = 0$ in this case. Take $\boldsymbol{\lambda}_l = \big(\emptyset, (2), \emptyset\big) \in \Pi^l(\boldsymbol{s}_l; w)$, where $w := \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle) - (\alpha_0 + \alpha_1)$. By Proposition 2.12 (i), we have $\dot{w} = \dot{\mathrm{wt}}(|\boldsymbol{\emptyset}_n, \boldsymbol{s}_n\rangle^\bullet) - (2\dot{\alpha}_0 + 3\dot{\alpha}_1 + \dot{\alpha}_2)$ with $\boldsymbol{s}_n := (1, 0)$. We have $|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle = |\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle^\bullet$ with $\boldsymbol{\lambda}_n := \big((1), (5)\big)$. One computes

$\boldsymbol{\mu}_n := \dot\sigma_i(\boldsymbol{\lambda}_n) = \big((2),(6,1)\big)$ (see Example 3.10). Let $\boldsymbol{t}_l := \dot\sigma_i.\boldsymbol{s}_l = (0,-1,2)$. Then $\dot\sigma_i(\boldsymbol{\lambda}_l)$ is the l-multi-partition such that $|\dot\sigma_i(\boldsymbol{\lambda}_l), \boldsymbol{t}_l\rangle = |\boldsymbol{\mu}_n, \boldsymbol{s}_n\rangle^\bullet$, namely $\dot\sigma_i(\boldsymbol{\lambda}_l) = \big(\emptyset,\emptyset,(2)\big)$. $\diamond$

Since $\{G^-(\boldsymbol{\lambda}_n, \boldsymbol{s}_n)^\bullet \mid \boldsymbol{\lambda}_n \in \Pi^n\}$ is a lower global crystal basis of $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$, one can prove for the Fock space $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$ an analogue of Theorem 3.9. Rephrasing this result in terms of the indexation $\boldsymbol{\lambda}_l$ leads to the following result.

Theorem 4.4 *Keep Notation 4.1 and assume that $\dot w + \dot\alpha_i$ is not a weight of $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$. Let $\dot\sigma_i : \Pi^l(\boldsymbol{s}_l; w) \to \Pi^l\big(\dot\sigma_i.\boldsymbol{s}_l\,;\, w + d(\boldsymbol{s}_l, \dot\sigma_i.\boldsymbol{s}_l)\,\delta\big)$ be the bijection from Definition 4.2. Then we have, for $\boldsymbol{\lambda}_l,\ \boldsymbol{\mu}_l \in \Pi^l(\boldsymbol{s}_l; w)$, $\epsilon = \pm 1$:*

$$\Delta^{\epsilon}_{\dot\sigma_i(\boldsymbol{\lambda}_l), \dot\sigma_i(\boldsymbol{\mu}_l);\, \dot\sigma_i.\boldsymbol{s}_l}(q) = \Delta^{\epsilon}_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l;\, \boldsymbol{s}_l}(q).$$

Proof Apply the analogue for $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$ of Theorem 3.9 mentioned above, then use Lemma 2.14 and (52). □

Example 4.5 (see Examples 3.10 and 4.3) Take n, l, $\boldsymbol{s}_l$, w and i as in Example 4.3 (namely, $n := 2$, $l := 3$, $\boldsymbol{s}_l := (0,2,-1)$, $w := \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle) - (\alpha_0 + \alpha_1)$ and $i := 2$). Note that $\dot w + \dot\alpha_i \notin \dot{\mathcal{P}}(\mathbf{F}_p[\boldsymbol{s}_n]^\bullet)$ (see Example 3.10). The elements of $\Pi^l(\boldsymbol{s}_l; w)$ are

$$\big(\emptyset,(2),\emptyset\big),\ \big(\emptyset,\emptyset,(1,1)\big),\ \big(\emptyset,(1),(1)\big),\ \big(\emptyset,\emptyset,(2)\big),$$
$$\big((2),\emptyset,\emptyset\big),\ \big((1),\emptyset,(1)\big),\ \big((1,1),\emptyset,\emptyset\big),\ \big(\emptyset,\emptyset,(1,1)\big),$$

and their respective images by the map $\dot\sigma_i$ are

$$\big(\emptyset,\emptyset,(2)\big),\ \big(\emptyset,\emptyset,(1,1)\big),\ \big(\emptyset,(1),(1)\big),\ \big((2),\emptyset,\emptyset\big),$$
$$\big(\emptyset,(2),\emptyset\big),\ \big((1),(1),\emptyset\big),\ \big((1,1),\emptyset,\emptyset\big),\ \big(\emptyset,(1,1),\emptyset\big),$$

On the one hand, the transition matrices of the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w\rangle$ are

$$\Delta^+_{\boldsymbol{s}_l}\langle w\rangle(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ q & 1 & . & . & . & . & . & . \\ q^2 & q & 1 & . & . & . & . & . \\ 0 & 0 & q & 1 & . & . & . & . \\ q & 0 & 0 & 0 & 1 & . & . & . \\ q^2 & q & q^2 & q & q & 1 & . & . \\ q^3 & q^2 & 0 & 0 & q^2 & q & 1 & . \\ 0 & q^2 & q^3 & q^2 & 0 & q & 0 & 1 \end{pmatrix} \begin{matrix} \big(\emptyset,(2),\emptyset\big) \\ \big(\emptyset,\emptyset,(1,1)\big) \\ \big(\emptyset,(1),(1)\big) \\ \big(\emptyset,\emptyset,(2)\big) \\ \big((2),\emptyset,\emptyset\big) \\ \big((1),\emptyset,(1)\big) \\ \big((1,1),\emptyset,\emptyset\big) \\ \big(\emptyset,\emptyset,(1,1)\big) \end{matrix} \qquad \text{and}$$

$$\Delta^-_{s_l}\langle w\rangle(q)=\begin{pmatrix}1&.&.&.&.&.&.&.\\-q^{-1}&1&.&.&.&.&.&.\\0&-q^{-1}&1&.&.&.&.&.\\-q^{-1}&q^{-2}&-q^{-1}&1&.&.&.&.\\-q^{-1}&0&0&0&1&.&.&.\\q^{-2}&0&0&-q^{-1}&-q^{-1}&1&.&.\\0&0&-q^{-1}&q^{-2}&0&-q^{-1}&1&.\\0&0&0&0&q^{-2}&-q^{-1}&0&1\end{pmatrix}\begin{matrix}(\emptyset,(2),\emptyset)\\(\emptyset,\emptyset,(1,1))\\(\emptyset,(1),(1))\\(\emptyset,\emptyset,(2))\\((2),\emptyset,\emptyset)\\((1),\emptyset,(1))\\((1,1),\emptyset,\emptyset)\\(\emptyset,\emptyset,(1,1))\end{matrix}.$$

On the other hand, the transition matrices of the canonical bases of $\mathbf{F}_q[\dot\sigma_i.\boldsymbol{s}_l]\langle w\rangle$ are

$$\Delta^+_{\dot\sigma_i.s_l}\langle w\rangle(q)=\begin{pmatrix}1&.&.&.&.&.&.&.\\q&1&.&.&.&.&.&.\\q^2&q&1&.&.&.&.&.\\0&0&q&1&.&.&.&.\\q&0&0&0&1&.&.&.\\q^2&q&q^2&q&q&1&.&.\\q^3&q^2&0&0&q^2&q&1&.\\0&q^2&q^3&q^2&0&q&0&1\end{pmatrix}\begin{matrix}(\emptyset,\emptyset,(2))\\(\emptyset,\emptyset,(1,1))\\(\emptyset,(1),(1))\\((2),\emptyset,\emptyset)\\(\emptyset,(2),\emptyset)\\((1),(1),\emptyset)\\((1,1),\emptyset,\emptyset)\\(\emptyset,(1,1),\emptyset)\end{matrix}\qquad\text{and}$$

$$\Delta^-_{\dot\sigma_i.s_l}\langle w\rangle(q)=\begin{pmatrix}1&.&.&.&.&.&.&.\\-q^{-1}&1&.&.&.&.&.&.\\0&-q^{-1}&1&.&.&.&.&.\\-q^{-1}&q^{-2}&-q^{-1}&1&.&.&.&.\\-q^{-1}&0&0&0&1&.&.&.\\q^{-2}&0&0&-q^{-1}&-q^{-1}&1&.&.\\0&0&-q^{-1}&q^{-2}&0&-q^{-1}&1&.\\0&0&0&0&q^{-2}&-q^{-1}&0&1\end{pmatrix}\begin{matrix}(\emptyset,\emptyset,(2))\\(\emptyset,\emptyset,(1,1))\\(\emptyset,(1),(1))\\((2),\emptyset,\emptyset)\\(\emptyset,(2),\emptyset)\\((1),(1),\emptyset)\\((1,1),\emptyset,\emptyset)\\(\emptyset,(1,1),\emptyset)\end{matrix},$$

in agreement with Theorem 4.4 (note that $d(\boldsymbol{s}_l,\dot\sigma_i.\boldsymbol{s}_l)=0$ in this case). ◇

4.2 A sufficient condition for Theorem 4.4

Keep again Notation 4.1. We can apply Theorem 4.4 if $\dot w+\dot\alpha_i$ is not a weight of $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$. By [7, Prop. 3.6 (iv)], this condition holds if and only if $\dot e_i.(|\boldsymbol{\lambda}_l,\boldsymbol{s}_l\rangle)=0$ for all $\boldsymbol{\lambda}_l\in\Pi^l(\boldsymbol{s}_l;w)$. We therefore have to check, for all $\boldsymbol{\lambda}_l\in\Pi^l(\boldsymbol{s}_l;w)$, whether $\boldsymbol{\lambda}_n$ has a removable i-node, where $\boldsymbol{\lambda}_n\in\Pi^n$ is related to $\boldsymbol{\lambda}_l$ by $|\boldsymbol{\lambda}_l,\boldsymbol{s}_l\rangle=|\boldsymbol{\lambda}_n,\boldsymbol{s}_n\rangle^\bullet$. It is not very convenient to make such tests in practice when the cardinality of $\Pi^l(\boldsymbol{s}_l;w)$ is large. We shall therefore give a sufficient condition on $\boldsymbol{s}_l$ and w that ensures, without further computation, that $\dot w+\dot\alpha_i$ is not a weight of $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$.

Notation 4.6 Let $\boldsymbol{s}_l\in\mathbb{Z}^l(s)$, $w\in\mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$ and $i\in[\![0;l-1]\!]$. By (32), the integer $M_i(\boldsymbol{\lambda}_l;\boldsymbol{s}_l;n)$ only depends on $\boldsymbol{s}_l$ and w, but not on $\boldsymbol{\lambda}_l\in\Pi^l(\boldsymbol{s}_l;w)$. From now on, this number will be denoted by $M_i(w;\boldsymbol{s}_l)$. ◇

Lemma 4.7 *Keep Notation 4.1. Then for all $\dot\sigma \in \dot W_l$, $0 \le i \le n-1$, $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{s}_l])$, we have*

$$M_i\big(w + d(\boldsymbol{s}_l, \dot\sigma.\boldsymbol{s}_l)\,\delta\,;\,\dot\sigma.\boldsymbol{s}_l\big) = M_i(w;\boldsymbol{s}_l).$$

Proof By (32) and the definition of the integers $M_j\big(w + d(\boldsymbol{s}_l, \dot\sigma.\boldsymbol{s}_l)\,\delta\,;\,\dot\sigma.\boldsymbol{s}_l\big)$ and $d(\boldsymbol{s}_l, \dot\sigma.\boldsymbol{s}_l)$, we have

$$\begin{aligned} &w + d(\boldsymbol{s}_l, \dot\sigma.\boldsymbol{s}_l)\,\delta \\ &\quad = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \dot\sigma.\boldsymbol{s}_l\rangle) - \sum_{j=0}^{n-1} M_j\big(w + d(\boldsymbol{s}_l, \dot\sigma.\boldsymbol{s}_l)\,\delta\,;\,\dot\sigma.\boldsymbol{s}_l\big)\,\alpha_j \\ &\quad = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle) + d(\boldsymbol{s}_l, \dot\sigma.\boldsymbol{s}_l)\,\delta - \sum_{j=0}^{n-1} M_j\big(w + d(\boldsymbol{s}_l, \dot\sigma.\boldsymbol{s}_l)\,\delta\,;\,\dot\sigma.\boldsymbol{s}_l\big)\,\alpha_j, \end{aligned}$$

whence $w = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{s}_l\rangle) - \sum_{j=0}^{n-1} M_j\big(w + d(\boldsymbol{s}_l, \dot\sigma.\boldsymbol{s}_l)\,\delta\,;\,\dot\sigma.\boldsymbol{s}_l\big)\,\alpha_j$. The lemma follows. □

Lemma 4.8 *Keep Notation 4.1. Assume that*

$$s_i - s_{i+1} \ge n\big(M_0(w\,;\boldsymbol{s}_l) + 1\big)$$

(*where we put $s_0 := n + s_l$ if $i = 0$*). *Then $\dot w + \dot\alpha_i$ is not a weight of $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$.*

Proof Assume on the contrary that $\dot w + \dot\alpha_i$ is a weight of $\mathbf{F}_p[\boldsymbol{s}_n]^\bullet$. Let $(\boldsymbol{\lambda}_l, \boldsymbol{t}_l) \in \Pi^l \times \mathbb{Z}^l(s)$ be such that $\dot{\mathrm{wt}}(|\boldsymbol{\lambda}_l, \boldsymbol{t}_l\rangle) = \dot w + \dot\alpha_i$. Write $\dot w = d\dot\delta + \sum_{j=1}^l b_j \dot\Lambda_{j-1}$ with $b_1, \dots, b_l, d \in \mathbb{Z}$. Then by (2), we have $\dot w + \dot\alpha_i = (d + \delta_{i,0})\dot\delta + \sum_{j=1}^l (b_j + a_{i,j-1})\dot\Lambda_{j-1}$, where $(a_{i',j'})_{0\le i',j'\le l-1}$ is the Cartan matrix of $\widehat{\mathfrak{sl}}_l$. Since the vectors $\dot\Lambda_0, \dots, \dot\Lambda_{l-1}$ and $\dot\delta$ are linearly independent, we get by (33) the following relations:

$$-\big(\Delta(\boldsymbol{t}_l, n) + M_0(\boldsymbol{\lambda}_l; \boldsymbol{t}_l; n)\big) = -\big(\Delta(\boldsymbol{s}_l, n) + M_0(w\,;\boldsymbol{s}_l)\big) + \delta_{i,0}, \tag{$*$}$$

$$\boldsymbol{t}_l = \begin{cases} (s_1, \dots, s_{i-1}, s_i + 1, s_{i+1} - 1, s_{i+2}, \dots, s_l) & \text{if } i \ge 1, \\ (s_1 - 1, s_2, \dots, s_{l-1}, s_l + 1) & \text{if } i = 0. \end{cases} \tag{$**$}$$

For $a \in \mathbb{Z}$, denote temporarily by $\overline{a} \in [\![0; n-1]\!]$ the residue of a modulo n. Using $(**)$ and the fact that $s_0 - \overline{s_0} = 1 + (s_l - \overline{s_l})$, we get

$$\Delta(\boldsymbol{t}_l, n) = \Delta(\boldsymbol{s}_l, n) + \frac{1}{n}(s_i - s_{i+1}) + \frac{1}{n}(\overline{s_{i+1}} - \overline{s_i}) + \delta_{\overline{s_{i+1}},0} - \delta_{i,0}. \tag{$***$}$$

Moreover, by assumption we have $M_0(w\,;\boldsymbol{s}_l)-\frac{1}{n}(s_i-s_{i+1})\le -1$. This together with $(*)$ and $(***)$ imply

$$M_0(\boldsymbol{\lambda}_l;\boldsymbol{t}_l;n)=-\frac{1}{n}(\overline{s_{i+1}}-\overline{s_i})-\delta_{\overline{s_{i+1}},0}+M_0(w\,;\boldsymbol{s}_l)-\frac{1}{n}(s_i-s_{i+1})$$

$$\le -\frac{1}{n}(\overline{s_{i+1}}-\overline{s_i})-\delta_{\overline{s_{i+1}},0}-1<0,$$

which is absurd since $M_0(\boldsymbol{\lambda}_l;\boldsymbol{t}_l;n)$ is the number of 0-nodes of the multi-partition $\boldsymbol{\lambda}_l$. □

Remark 4.9 The lower bound $s_i-s_{i+1}\ge n\big(M_0(w\,;\boldsymbol{s}_l)+1\big)$ from Lemma 4.8 is certainly not the best to ensure that $\dot{w}+\dot{\alpha}_i\notin\dot{\mathcal{P}}(\mathbf{F}_p[\boldsymbol{s}_n]^\bullet)$. We actually conjecture that the latter statement holds if

$$s_i-s_{i+1}\ge M_0(w\,;\boldsymbol{s}_l)+\cdots+M_{n-1}(w\,;\boldsymbol{s}_l)$$

(this lower bound is in general better). ◇

4.3 A graph containing multi-charges conjugated under the action of $\dot{W}_l$

Definition 4.10 Fix $\boldsymbol{r}_l\in A_{l,n}(s)$ and $M\in\mathbb{N}^*$. (Recall that the set $A_{l,n}(s)$ defined by (38) is a fundamental domain for the action of $\dot{W}_l$.) For $\boldsymbol{s}_l=(s_1,\dots,s_l)\in\dot{W}_l.\boldsymbol{r}_l$, $\boldsymbol{t}_l\in\dot{W}_l.\boldsymbol{r}_l$, $0\le i\le l-1$, write $\boldsymbol{s}_l\overset{i}{\longrightarrow}\boldsymbol{t}_l$ if $\boldsymbol{t}_l=\dot{\sigma}_i.\boldsymbol{s}_l$ and $s_i-s_{i+1}\ge M$ (if $i=0$, we put $s_0:=n+s_l$). Let $\Gamma(M)$ be the graph containing $\dot{W}_l.\boldsymbol{r}_l$ as set of vertices and the arrows $\boldsymbol{s}_l\overset{i}{\longrightarrow}\boldsymbol{t}_l$ ($\boldsymbol{s}_l,\boldsymbol{t}_l\in\dot{W}_l.\boldsymbol{r}_l$, $0\le i\le l-1$). ◇

Remark 4.11 Note that $\Gamma(1)$ is connected. More generally, we claim that $\Gamma(M)$ has finitely many connected components. To see this, introduce the following notation. For $\boldsymbol{s}_l\in\dot{W}_l.\boldsymbol{r}_l$, let $\dot{\sigma}(\boldsymbol{s}_l)\in\dot{W}_l$ be the element of minimal length such that $\boldsymbol{s}_l=\dot{\sigma}(\boldsymbol{s}_l).\boldsymbol{r}_l$, and let $\ell(\boldsymbol{s}_l)$ denote the length of $\dot{\sigma}(\boldsymbol{s}_l)$. Now, for each connected component C in $\Gamma(M)$, choose $\boldsymbol{s}_l(C)\in C$ such that $\ell(\boldsymbol{s}_l(C))$ is minimal in the set $\{\ell(\boldsymbol{t}_l)\mid\boldsymbol{t}_l\in C\}$. (We think that this determines $\boldsymbol{s}_l(C)$ in a unique way, but we do not have the proof for this fact.) One can then show easily that $\boldsymbol{s}_l(C)$ lies in the finite set

$$\{(t_1,\dots,t_l)\in\mathbb{Z}^l(s)\mid\forall\,0\le i\le l-1,\ t_{i+1}-t_i\le M-1\},$$

which proves the claim. ◇

We now give a relation between (the connected components of) $\Gamma(M)$ and the comparison of canonical bases. We shall give an important application of this result in Section 5.

Proposition 4.12 *Let $\boldsymbol{r}_l\in A_{l,n}(s)$ and $w\in\mathcal{P}(\mathbf{F}_q[\boldsymbol{r}_l])$. Put $M:=n(M_0(w\,;\boldsymbol{r}_l)+1)$. Let $\boldsymbol{s}_l$ and $\boldsymbol{t}_l$ be two multi-charges in the same connected component of $\Gamma(M)$ (in particular, $\boldsymbol{s}_l$ and $\boldsymbol{t}_l$ are in the $\dot{W}_l$-orbit of $\boldsymbol{r}_l$). Then the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w+d(\boldsymbol{r}_l,\boldsymbol{s}_l)\,\delta\rangle$ and $\mathbf{F}_q[\boldsymbol{t}_l]\langle w+d(\boldsymbol{r}_l,\boldsymbol{t}_l)\,\delta\rangle$ are similar in the sense of Definition 3.2.*

Proof We may assume that $\boldsymbol{s}_l \xrightarrow{i} \boldsymbol{t}_l$ with $i \in [\![0; l-1]\!]$. With obvious notation, we have by Lemma 4.7 : $s_i - s_{i+1} \geq n(M_0(w\,;\boldsymbol{r}_l)+1) = n\big(M_0(w + d(\boldsymbol{r}_l, \boldsymbol{s}_l)\,\delta\,;\boldsymbol{s}_l)+1\big)$. We can therefore apply Lemma 4.8 and then Theorem 4.4 to conclude. □

With the notation above, Proposition 4.12 and Remark 4.11 show that there are only finitely many similarity classes of canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w + d(\boldsymbol{r}_l, \boldsymbol{s}_l)\,\delta\rangle$, where $\boldsymbol{s}_l$ ranges over the $\dot{W}_l$-orbit of $\boldsymbol{r}_l$ and $(\boldsymbol{r}_l, w)$ is fixed.

5 Comparison of canonical bases for dominant multi-charges

Definition 5.1 Let $M \in \mathbb{N}$. We say that $(x_1, \dots, x_N) \in \mathbb{R}^N$ is *M-dominant* if for all $1 \leq i \leq N-1$, we have

$$x_i - x_{i+1} \geq M. \qquad \diamond$$

Throughout Section 5, we keep the following notation.

5.1 Notation

* Recall that $\mathbb{R}^l(s) = \{(x_1, \dots, x_l) \in \mathbb{R}^l \mid x_1 + \cdots + x_l = s\}$. The subset of $\mathbb{R}^l(s)$ formed by the M-dominant elements will be denoted by $\mathcal{C}_M$.

* The group $\dot{W}_l \cong \widetilde{\mathfrak{S}}_l$ is a semidirect product of the finite symmetric group $\mathfrak{S}_l$ and an Abelian group $\dot{\mathcal{Q}}$ which is free of rank $l-1$. More precisely, $\dot{\mathcal{Q}}$ is spanned by $\dot{\tau}_1, \dots, \dot{\tau}_{l-1}$, where $\dot{\tau}_i$ $(1 \leq i \leq l-1)$ acts on $\mathbb{Z}^l(s)$ by

$$\dot{\tau}_i.(s_1, \dots, s_l) = (s_1, \dots, s_{i-1}, s_i + n, s_{i+1} - n, s_{i+2}, \dots, s_l) \quad \big((s_1, \dots, s_l) \in \mathbb{Z}^l(s)\big) \tag{59}$$

(since $\dot{W}_l$ acts faithfully on $\mathbb{Z}^l(s)$, this determines $\dot{\tau}_i$ completely).

* For $\boldsymbol{a}_l = (a_1, \dots, a_l) \in \mathbb{Z}^l$, put

$$\mathcal{L}_{\boldsymbol{a}_l} := \{(s_1, \dots, s_l) \in \mathbb{Z}^l(s) \mid \forall\, 1 \leq i \leq l,\ s_i \equiv a_i \pmod{n}\}. \tag{60}$$

Note that $\dot{\mathcal{Q}}$ acts transitively on $\mathcal{L}_{\boldsymbol{a}_l}$; in particular, two elements of $\mathcal{L}_{\boldsymbol{a}_l}$ lie in a same $\dot{W}_l$-orbit.

* For $\boldsymbol{s}_l, \boldsymbol{t}_l \in \mathbb{Z}^l(s)$ and $M \in \mathbb{N}$, write

$$\boldsymbol{s}_l \underset{M}{\equiv} \boldsymbol{t}_l \tag{61}$$

if $\mathcal{L}_{\boldsymbol{s}_l} = \mathcal{L}_{\boldsymbol{t}_l}$, and there exist $\boldsymbol{s}_l^{(0)}, \dots, \boldsymbol{s}_l^{(r)} \in \mathcal{L}_{\boldsymbol{s}_l} \cap \mathcal{C}_M$ such that $\boldsymbol{s}_l^{(0)} = \boldsymbol{s}_l$, $\boldsymbol{s}_l^{(r)} = \boldsymbol{t}_l$ and for all $1 \leq i \leq r$, we have $\boldsymbol{s}_l^{(i)} \in \{\dot{\tau}_j^{\pm 1}.\boldsymbol{s}_l^{(i-1)} \mid 1 \leq j \leq l-1\}$. In other words, put into a non-oriented graph all the elements of $\mathbb{Z}^l(s)$ and draw an edge between two vertices if they are $\dot{W}_l$-conjugated to each other by a generator of $\dot{\mathcal{Q}}$ or its inverse. Then $\boldsymbol{s}_l \underset{M}{\equiv} \boldsymbol{t}_l$ if and only if there exists a path in this graph connecting $\boldsymbol{s}_l$ and $\boldsymbol{t}_l$ through M-dominant vertices (including $\boldsymbol{s}_l$ and $\boldsymbol{t}_l$).

* Let $\boldsymbol{r}_l = (r_1, \ldots, r_l) \in A_{l,n}(s)$ and $w \in \mathcal{P}(\mathbf{F}_q[\boldsymbol{r}_l])$. Recall the definition of the integers $d(\boldsymbol{r}_l, \boldsymbol{s}_l)$ ($\boldsymbol{s}_l \in \mathbb{Z}^l(s)$) from (57) and $M_i(w; \boldsymbol{r}_l)$ ($0 \le i \le n-1$) from Notation 4.6.

5.2 A third theorem of comparison

The goal of Section 5 is to prove the following theorem.

Theorem 5.2

Keep Notation 5.1. Then there exists $N \in \mathbb{N}$ (which only depends on n, l and $M_0(w; \boldsymbol{r}_l)$) such that for all N-dominant multi-charges $\boldsymbol{s}_l, \boldsymbol{t}_l \in \dot{W}_l.\boldsymbol{r}_l$ with $\mathcal{L}_{\boldsymbol{s}_l} = \mathcal{L}_{\boldsymbol{t}_l}$, the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w + d(\boldsymbol{r}_l, \boldsymbol{s}_l)\,\delta\rangle$ and $\mathbf{F}_q[\boldsymbol{t}_l]\langle w + d(\boldsymbol{r}_l, \boldsymbol{t}_l)\,\delta\rangle$ are similar. □

Remark 5.3 The proof of Theorem 5.1 will provide an integer N of the form

$$N = nM_0(w; \boldsymbol{r}_l) + c,$$

where c is a constant that can be explicitly calculated. This c *a priori* depends on the multi-charge $\boldsymbol{r}_l \in A_{l,n}(s)$ that we have fixed in Notation 5.1. However, by taking a maximum over the *finite* set $A_{l,n}(s)$, we can make c independent of $\boldsymbol{r}_l$ (see the proof of Proposition 5.6). More precisely, by Remark 5.13 we can take $c \ge n(l^2 + l + 3)$. However, the corresponding value of N is probably not optimal. Indeed, according to Remark 4.9 and explicit calculus of canonical bases, we conjecture that Theorem 5.2 holds if we replace this N by

$$N' := M_0(w; \boldsymbol{r}_l) + \cdots + M_{n-1}(w; \boldsymbol{r}_l)$$

(the latter lower bound is in general better). ◇

Example 5.4 Take $n = 3$, $l = 2$, $\boldsymbol{r}_l = (1, 0)$ and $w = \mathrm{wt}(|\boldsymbol{\emptyset}_l, \boldsymbol{r}_l\rangle) - (\alpha_0 + \alpha_1 + \alpha_2)$. With notation from Theorem 5.2 and Remark 5.3, we can take $N = 30$ and $N' = 3$. Note that all the multi-charges $\boldsymbol{s}_l^{(k)} := (3k+1, -3k)$ ($k \in \mathbb{Z}$) are in the $\dot{W}_l$-orbit of $\boldsymbol{r}_l$ and they have the same pair of residues modulo n. Put $w_k := w + d(\boldsymbol{r}_l, \boldsymbol{s}_l^{(k)})\,\delta$ ($k \in \mathbb{Z}$). By Theorem 5.2, the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l^{(k)}]\langle w_k\rangle$, $k \ge 5$ are pairwise similar. By Remark 5.3, the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l^{(k)}]\langle w_k\rangle$, $k \ge 1$ should be pairwise similar, which can be actually checked for $1 \le k \le 5$ by explicit calculus. Namely, the transition matrices $\Delta_k^{\epsilon}(q)$ of the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l^{(k)}]\langle w_k\rangle$ ($k \ge 1$, $\epsilon = \pm 1$) are

$$\Delta_k^{+}(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ q & 1 & . & . & . & . & . & . \\ q^2 & q & 1 & . & . & . & . & . \\ 0 & q & 0 & 1 & . & . & . & . \\ 0 & q^2 & q & q & 1 & . & . & . \\ 0 & 0 & q & 0 & 0 & 1 & . & . \\ 0 & 0 & q^2 & 0 & q & q & 1 & . \\ 0 & 0 & 0 & q & q^2 & 0 & q & 1 \end{pmatrix} \begin{matrix} \big((3), \emptyset\big) \\ \big((2,1), \emptyset\big) \\ \big((2), (1)\big) \\ \big((1,1,1), \emptyset\big) \\ \big((1), (1,1)\big) \\ \big(\emptyset, (3)\big) \\ \big(\emptyset, (2,1)\big) \\ \big(\emptyset, (1,1,1)\big) \end{matrix} \qquad \text{and}$$

$$\Delta_k^-(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ -q^{-1} & 1 & . & . & . & . & . & . \\ 0 & -q^{-1} & 1 & . & . & . & . & . \\ q^{-2} & -q^{-1} & 0 & 1 & . & . & . & . \\ 0 & q^{-2} & -q^{-1} & -q^{-1} & 1 & . & . & . \\ -q^{-1} & q^{-2} & -q^{-1} & 0 & 0 & 1 & . & . \\ q^{-2} & -q^{-3} & q^{-2} & q^{-2} & -q^{-1} & -q^{-1} & 1 & . \\ -q^{-3} & 0 & 0 & 0 & 0 & q^{-2} & -q^{-1} & 1 \end{pmatrix} \begin{matrix} ((3),\emptyset) \\ ((2,1),\emptyset) \\ ((2),(1)) \\ ((1,1,1),\emptyset) \\ ((1),(1,1)) \\ (\emptyset,(3)) \\ (\emptyset,(2,1)) \\ (\emptyset,(1,1,1)) \end{matrix} .$$

One can check moreover that

$$\Delta_0^+(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ 0 & 1 & . & . & . & . & . & . \\ 0 & q & 1 & . & . & . & . & . \\ q & q & 0 & 1 & . & . & . & . \\ 0 & q^2 & q & q & 1 & . & . & . \\ 0 & 0 & q^2 & 0 & q & 1 & . & . \\ q^2 & 0 & 0 & q & 0 & 0 & 1 & . \\ 0 & 0 & 0 & q^2 & q & 0 & q & 1 \end{pmatrix} \begin{matrix} ((3),\emptyset) \\ ((2,1),\emptyset) \\ ((2),(1)) \\ ((1,1,1),\emptyset) \\ ((1),(1,1)) \\ (\emptyset,(3)) \\ (\emptyset,(2,1)) \\ (\emptyset,(1,1,1)) \end{matrix} .$$

Note that $\Delta_1^+(q)$ has 22 nonzero entries, whereas $\Delta_0^+(q)$ has only 21 nonzero entries. As a consequence, the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l^{(k)}]\langle w_k\rangle$ for $k=0$ and $k=1$ are not similar. ◇

The proof of Theorem 5.2 relies on the two following propositions.

Proposition 5.5 *Keep Notation 5.1. Let $\boldsymbol{s}_l \in \dot{W}_l.\boldsymbol{r}_l$ be an M-dominant multi-charge with $M := n(M_0(w;\boldsymbol{r}_l)+2)$. Let $1 \le i \le l-1$ and $\boldsymbol{t}_l := \dot{\tau}_i.\boldsymbol{s}_l$. Then the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l]\langle w + d(\boldsymbol{r}_l,\boldsymbol{s}_l)\,\delta\rangle$ and $\mathbf{F}_q[\boldsymbol{t}_l]\langle w + d(\boldsymbol{r}_l,\boldsymbol{t}_l)\,\delta\rangle$ are similar.*

Proof We give the proof for $2 \le i \le l-3$ (the proof for $i=1$, $i=l-2$ and $i=l-1$ is similar). One easily checks that

$$\dot{\tau}_i = \dot{\sigma}_{i-1}\dot{\sigma}_{i-2}\cdots\dot{\sigma}_1\dot{\sigma}_0\dot{\sigma}_{l-1}\cdots\dot{\sigma}_{i+2}\dot{\sigma}_{i+1}\dot{\sigma}_{i+2}\cdots\dot{\sigma}_{l-1}\dot{\sigma}_0\dot{\sigma}_1\cdots\dot{\sigma}_i .$$

Let $0 \le k \le 2l-2$. Denote by $\dot{\tau}_i[k]$ the right factor of length k in this word (we thus have $\dot{\tau}_i[0] = \mathrm{id}$, $\dot{\tau}_i[1] = \dot{\sigma}_i$, $\dot{\tau}_i[2] = \dot{\sigma}_{i-1}\dot{\sigma}_i$ and so on). Put

$$\boldsymbol{s}_l^{(k)} = (s_1^{(k)},\dots,s_l^{(k)}) := \dot{\tau}_i[k].\boldsymbol{s}_l .$$

For $1 \le k \le 2l-2$, let $i_k \in [\![0;l-1]\!]$ be such that $\dot{\tau}_i[k] = \dot{\sigma}_{i_k}\dot{\tau}_i[k-1]$. Let now $0 \le k \le 2l-3$. By computing the action of $\dot{\tau}_i[k]$ on $\boldsymbol{s}_l$, one can show the following:

(i) $s_{i_{k+1}}^{(k)}, s_{i_{k+1}+1}^{(k)} \in \{s_1,\dots,s_l, s_l+n, s_i+n, s_{i+1}-n\}$.

(ii) If $s_{i_{k+1}}^{(k)} = s_a + \epsilon n$ and $s_{i_{k+1}+1}^{(k)} = s_{a'} + \epsilon' n$ with $a, a' \in [\![1;l]\!]$, $\epsilon, \epsilon' \in \{-1,0,1\}$, then $a \ne a'$ and $\epsilon\epsilon' = 0$.

Note moreover that by assumption on $\boldsymbol{s}_l$, we have for all $a, b \in [\![1; l]\!]$ such that $a \neq b$, $\left|s_b - s_a\right| \geq M\left|b - a\right| \geq M$. This together with (i), (ii) imply that

$$\left|s^{(k)}_{i_{k+1}} - s^{(k)}_{i_{k+1}+1}\right| \geq M - n.$$

As a consequence, $\Gamma(M-n)$ contains the arrow $\boldsymbol{s}_l^{(k)} \xrightarrow{i_k} \boldsymbol{s}_l^{(k+1)}$ or $\boldsymbol{s}_l^{(k+1)} \xrightarrow{i_k} \boldsymbol{s}_l^{(k)}$. It follows that $\boldsymbol{s}_l$ and $\boldsymbol{t}_l$ are in the same connected component of $\Gamma(M-n) = \Gamma\big(n(M_0(w; \boldsymbol{r}_l) + 1)\big)$. We can therefore apply Proposition 4.12 to conclude. □

Proposition 5.6 *Keep Notation 5.1. Let $M \in \mathbb{N}$. Then there exists $c \in \mathbb{Z}$ (which only depends on l and n, but not on M nor $\boldsymbol{r}_l$) such that for all $(M+c)$-dominant multi-charges $\boldsymbol{s}_l, \boldsymbol{t}_l \in \dot{W}_l.\boldsymbol{r}_l$ with $\mathcal{L}_{\boldsymbol{s}_l} = \mathcal{L}_{\boldsymbol{t}_l}$, we have $\boldsymbol{s}_l \underset{M}{\equiv} \boldsymbol{t}_l$.*

Proof We shall prove this proposition in Section 5.3. □

Proof of Theorem 5.2 from Propositions 5.5 and 5.6. Let $M := n(M_0(w; \boldsymbol{r}_l) + 2)$. Let $c \in \mathbb{Z}$ be the integer given by Proposition 5.6 and put $N := M + c$. Let $\boldsymbol{s}_l, \boldsymbol{t}_l \in \dot{W}_l.\boldsymbol{r}_l$ be two N-dominant multi-charges such that $\mathcal{L}_{\boldsymbol{s}_l} = \mathcal{L}_{\boldsymbol{t}_l}$. Put $\mathcal{L} := \mathcal{L}_{\boldsymbol{s}_l} = \mathcal{L}_{\boldsymbol{t}_l}$. By Proposition 5.6, there exist $\boldsymbol{s}_l^{(0)}, \dots, \boldsymbol{s}_l^{(r)} \in \mathcal{L} \cap \mathcal{C}_M$ such that $\boldsymbol{s}_l^{(0)} = \boldsymbol{s}_l$, $\boldsymbol{s}_l^{(r)} = \boldsymbol{t}_l$ and for all $1 \leq i \leq r$, we have $\boldsymbol{s}_l^{(i)} \in \{\dot{\tau}_j^{\pm 1}.\boldsymbol{s}_l^{(i-1)} \mid 1 \leq j \leq l-1\}$. Let $1 \leq i \leq r$. Since $\mathcal{L}_{\boldsymbol{s}_l^{(i-1)}} = \mathcal{L}_{\boldsymbol{s}_l^{(i)}} = \mathcal{L}$, we have $\boldsymbol{s}_l^{(i-1)}, \boldsymbol{s}_l^{(i)} \in \dot{W}_l.\boldsymbol{r}_l$. By Proposition 5.5, the canonical bases of $\mathbf{F}_q[\boldsymbol{s}_l^{(i-1)}]\langle w + d(\boldsymbol{r}_l, \boldsymbol{s}_l^{(i-1)})\,\delta\rangle$ and $\mathbf{F}_q[\boldsymbol{s}_l^{(i)}]\langle w + d(\boldsymbol{r}_l, \boldsymbol{s}_l^{(i)})\,\delta\rangle$ are similar. Theorem 5.2 follows. □

5.3 Proof of Proposition 5.6

The idea of the proof is the following. Let $\boldsymbol{s}_l$ and $\boldsymbol{t}_l$ be two multi-charges as in Proposition 5.6 and put $\mathcal{L} := \mathcal{L}_{\boldsymbol{s}_l} = \mathcal{L}_{\boldsymbol{t}_l}$. First, we introduce a suitable change of coordinates φ that maps $\mathcal{L}$ to $\mathbb{Z}^{l-1}$ and such that $\boldsymbol{a}_l \underset{M}{\equiv} \boldsymbol{b}_l$ if and only if $\varphi(\boldsymbol{a}_l)$ is $\mathbb{Z}$-connected to $\varphi(\boldsymbol{b}_l)$, that is there exists a piecewise affine path connecting $\varphi(\boldsymbol{a}_l)$ to $\varphi(\boldsymbol{b}_l)$ with edges parallel to the axes of coordinates of $\mathbb{Z}^{l-1}$ (see Lemma 5.8). Roughly speaking, the aim is to replace the lattice $\mathcal{L}$ by $\mathbb{Z}^{l-1}$ and the action of $\dot{\mathcal{Q}}$ by the obvious action of $\mathbb{Z}^{l-1}$ by translations. Doing this, we replace the set of M-dominance $\mathcal{C}_M$ by a certain cone (that is, an intersection of half-spaces), temporarily denoted by C_M. Note that two arbitrary points in $C_M \cap \mathbb{Z}^{l-1}$ are not necessarily $\mathbb{Z}$-connected in C_M. However, and this is the second step of the proof, we shall construct an integer c such that $C_{M+c} \subset C_M$ and any two points in $C_{M+c} \cap \mathbb{Z}^{l-1}$ are $\mathbb{Z}$-connected in C_M (see Proposition 5.12).

Notation 5.7 In addition to Notation 5.1, we shall use for the proof the following notation.

* For $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ ($N \in \mathbb{N}^*$), put

$$\lfloor \boldsymbol{x} \rfloor := (\lfloor x_1 \rfloor, \dots, \lfloor x_N \rfloor) \in \mathbb{Z}^N \tag{62}$$

and define $\lceil \boldsymbol{x} \rceil$ in a similar way.

* Let

$$A := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \tag{63}$$

denote the Cartan matrix of $\mathfrak{sl}_l$. In particular, A has $l-1$ rows and $l-1$ columns.

* For $1 \leq j \leq l-1$, let $\epsilon_j := (\delta_{i,j})_{1\leq i\leq l-1}$ be the j-th vector of the natural basis of $\mathbb{R}^{l-1}$. Put also $\mathbf{1} := \epsilon_1 + \cdots + \epsilon_{l-1}$.

* Define a partial ordering on the set of matrices (or vectors) of a given size with entries in $\mathbb{R}$ by writing $\boldsymbol{x} = (x_i)_{i\in I} \leq \boldsymbol{y} = (y_i)_{i\in I}$ if $x_i \leq y_i$ for all $i \in I$. By definition, the maximum of $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\max(\boldsymbol{x}, \boldsymbol{y}) := (\max(x_i, y_i))_{i\in I}$. One may similarly define the maximum of a greater number of matrices (or vectors) of a given size provided the max in the right hand-side above still exists. Now, for $\boldsymbol{b} \in \mathbb{R}^{l-1}$ define the cones

$$C_{\boldsymbol{b}} := \{\boldsymbol{x} \in \mathbb{R}^{l-1} \mid A.\boldsymbol{x} \geq \boldsymbol{b}\} \qquad \text{and} \qquad C'_{\boldsymbol{b}} := \{\boldsymbol{x} \in \mathbb{R}^{l-1} \mid A.\boldsymbol{x} \leq \boldsymbol{b}\}. \tag{64}$$

The unique vector $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{b}) \in \mathbb{R}^{l-1}$ such that $A.\boldsymbol{\omega} = \boldsymbol{b}$ is called the *vertex* of $C_{\boldsymbol{b}}$ (or $C'_{\boldsymbol{b}}$).

* For $M \in \mathbb{N}$, let $\boldsymbol{b}(M) = \boldsymbol{b}(M; n, l, \boldsymbol{r}_l) = (b_1^{(M)}, \dots, b_{l-1}^{(M)}) \in \mathbb{R}^{l-1}$ denote the vector defined by

$$b_i^{(M)} := (M + r_{i+1} - r_i)/n \qquad (1 \leq i \leq l-1). \tag{65}$$

* Define a map $\varphi : (s_1, \dots, s_l) \in \mathbb{R}^l(s) \mapsto (x_1, \dots, x_{l-1}) = \varphi(s_1, \dots, s_l) \in \mathbb{R}^{l-1}$ by

$$x_i := \frac{1}{n}\sum_{j=1}^{i}(s_j - r_j) \qquad (1 \leq i \leq l-1). \tag{66}$$

Conversely, let $\psi : (x_1, \dots, x_{l-1}) \in \mathbb{R}^{l-1} \mapsto (s_1, \dots, s_l) = \psi(x_1, \dots, x_{l-1}) \in \mathbb{R}^l(s)$ be the map defined by

$$s_i := n(x_i - x_{i-1}) + r_i \qquad (1 \leq i \leq l), \tag{67}$$

where we put $x_0 = x_l := 0$. (Note that φ and ψ depend on the multi-charge $\boldsymbol{r}_l \in A_{l,n}(s)$ that we have fixed in Notation 5.1.) ◇

Lemma 5.8 *Keep Notation 5.7. Then we have the following.*

(i) *The maps* $\varphi : \mathbb{R}^l(s) \to \mathbb{R}^{l-1}$ *and* $\psi : \mathbb{R}^{l-1} \to \mathbb{R}^l(s)$ *are bijections inverse to each other.*

(ii) *We have* $\psi(\mathbb{Z}^{l-1}) = \mathcal{L}_{\boldsymbol{r}_l}$, *and for* $1 \le i \le l-1$, $(x_1, \dots, x_{l-1}) \in \mathbb{Z}^{l-1}$, *we have*

$$\psi(x_1, \dots, x_i + 1, \dots, x_{l-1}) = \dot{\tau}_i.\psi(x_1, \dots, x_{l-1}).$$

(iii) *For* $M \in \mathbb{N}$, *we have* $\varphi(\mathcal{C}_M) = C_{\boldsymbol{b}(M)}$.

Proof The proof of (i) and (ii) is straightforward. With obvious notation, we have the equivalence

$$s_i - s_{i+1} \ge M \Longleftrightarrow -x_{i-1} + 2x_i - x_{i+1} \ge (M + r_{i+1} - r_i)/n\,;$$

Statement (iii) follows. □

Definition 5.9 Let $D \subset \mathbb{R}^{l-1}$. We say that $\boldsymbol{x}, \boldsymbol{y} \in D$ are $\mathbb{Z}$*-connected in* D if there exist vectors $\boldsymbol{x}^{(0)}, \dots, \boldsymbol{x}^{(N)} \in D \cap \mathbb{Z}^{l-1}$ such that $\boldsymbol{x}^{(0)} = \boldsymbol{x}$, $\boldsymbol{x}^{(N)} = \boldsymbol{y}$ and for all $0 \le i \le N-1$, we have $\boldsymbol{x}^{(i+1)} - \boldsymbol{x}^{(i)} \in \{\pm\epsilon_j \mid 1 \le j \le l-1\}$; in particular, we have $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^{l-1}$. In this case, write $\boldsymbol{x} \underset{D}{\text{---}} \boldsymbol{y}$. ◇

In order to prove Proposition 5.6, we have to deal with $\mathbb{Z}$-connected points in cones $C_{\boldsymbol{b}}$ ($\boldsymbol{b} \in \mathbb{R}^{l-1}$). Note that two points in $C_{\boldsymbol{b}} \cap \mathbb{Z}^{l-1}$ are not necessary $\mathbb{Z}$-connected in $C_{\boldsymbol{b}}$. For example, take $l = 3$ (so $l - 1 = 2$) and $\boldsymbol{b} = (0, 0)$. Then $\boldsymbol{x} := (1, 1)$ and $\boldsymbol{y} := (0, 0)$ are two points in $C_{\boldsymbol{b}}$ which are not $\mathbb{Z}$-connected in $C_{\boldsymbol{b}}$, because none of the points $(0, \pm 1)$ and $(\pm 1, 0)$ lies in $C_{\boldsymbol{b}}$. However, given $\boldsymbol{b} \in \mathbb{R}^{l-1}$, one can construct $\boldsymbol{c} \le \boldsymbol{b}$ such that any two points in $C_{\boldsymbol{b}}$ with integer coordinates are $\mathbb{Z}$-connected in $C_{\boldsymbol{c}}$. This leads to the introduction of the following set. For $\boldsymbol{b} \in \mathbb{R}^{l-1}$, put

$$\mathcal{A}(\boldsymbol{b}) := \{\boldsymbol{c} \in \mathbb{R}^{l-1} \mid \boldsymbol{c} \le \boldsymbol{b} \text{ and } \forall\, \boldsymbol{x}, \boldsymbol{y} \in C_{\boldsymbol{b}},\ \boldsymbol{x} \underset{C_{\boldsymbol{c}}}{\text{---}} \boldsymbol{y}\}. \tag{68}$$

Proposition 5.10 *Let* $\boldsymbol{b} = (b_1, \dots, b_{l-1}) \in \mathbb{R}^{l-1}$. *Then we have the following.*

(i) *The set* $\mathcal{A}(\boldsymbol{b})$ *is nonempty.*
(ii) *For all* $\boldsymbol{c} \in \mathcal{A}(\boldsymbol{b})$, *we have* $\boldsymbol{c}' \le \boldsymbol{c} \Rightarrow \boldsymbol{c}' \in \mathcal{A}(\boldsymbol{b})$.
(iii) *The map* $\boldsymbol{b} \mapsto \mathcal{A}(\boldsymbol{b})$ *is increasing.*
(iv) *For all* $\boldsymbol{b}' \in \mathbb{R}^{l-1}$ *such that* $A^{-1}.(\boldsymbol{b} - \boldsymbol{b}') \in \mathbb{Z}^{l-1}$, *we have* $\mathcal{A}(\boldsymbol{b}) = \mathcal{A}(\boldsymbol{b}') + (\boldsymbol{b} - \boldsymbol{b}')$.

Proof We prove (i) and leave the other statements to the reader. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{l-1})$ be the vertex of $C_{\boldsymbol{b}}$, $\boldsymbol{b}' := \boldsymbol{b} + 2.\boldsymbol{1}$ and $\boldsymbol{\omega}'$ be the vertex of $C_{\boldsymbol{b}'}$. Let $\boldsymbol{\omega}'' = (\omega''_1, \dots, \omega''_{l-1})$ be equal to $\max(\boldsymbol{\omega}', \lceil \boldsymbol{\omega} \rceil)$ and let $\omega''_0 = \omega''_l := 0$. Now let $\boldsymbol{c} = (c_1, \dots, c_{l-1}) \in \mathbb{R}^{l-1}$ be the vector defined by

$$c_i := -\omega''_{i-1} + 2\omega_i - \omega''_{i+1} \qquad (1 \le i \le l-1).$$

We shall show that $\boldsymbol{c} \in \mathcal{A}(\boldsymbol{b})$. It is well-known that $A^{-1} \ge 0$, whence $\boldsymbol{x} \in C_{\boldsymbol{b}} \Rightarrow \boldsymbol{x} \ge \boldsymbol{\omega}$ and $\boldsymbol{y} \in C'_{\boldsymbol{b}'} \Rightarrow \boldsymbol{y} \le \boldsymbol{\omega}'$. Since $\boldsymbol{b} \le \boldsymbol{b}'$, we get $\boldsymbol{\omega} \in C'_{\boldsymbol{b}'}$ and therefore $\boldsymbol{\omega} \le \boldsymbol{\omega}' \le \boldsymbol{\omega}''$.

Consider the set

$$\mathbf{P} := \{\boldsymbol{x} \in \mathbb{R}^{l-1} \mid \boldsymbol{\omega} \leq \boldsymbol{x} \leq \boldsymbol{\omega}''\}.$$

The argument above shows that $C_{\boldsymbol{b}} \cap C'_{\boldsymbol{b}'} \subset \mathbf{P}$, and by construction we also have $\lceil \boldsymbol{\omega} \rceil \in \mathbf{P}$. By definition of $\boldsymbol{c}$, we have $\boldsymbol{\omega} \in \mathbf{P} \subset C_{\boldsymbol{c}}$, whence $\boldsymbol{c} \leq \boldsymbol{b}$. Let us now show that any $\boldsymbol{x} \in C_{\boldsymbol{b}} \cap \mathbb{Z}^{l-1}$ is $\mathbb{Z}$-connected to $\lceil \boldsymbol{\omega} \rceil$ in $C_{\boldsymbol{c}}$. Note that for $\boldsymbol{x} = (x_1, \ldots, x_{l-1}) \in C_{\boldsymbol{b}} \cap \mathbb{Z}^{l-1}$, we have $\boldsymbol{x} \geq \boldsymbol{\omega}$ (because $\boldsymbol{x} \in C_{\boldsymbol{b}}$) and therefore $\boldsymbol{x} \geq \lceil \boldsymbol{\omega} \rceil$. We can thus argue by induction on

$$N(\boldsymbol{x}) := \sum_{i=1}^{l-1} (x_i - \lceil \omega_i \rceil) \in \mathbb{N}.$$

If $N(\boldsymbol{x}) = 0$, then we have $\boldsymbol{x} = \lceil \boldsymbol{\omega} \rceil \in \mathbf{P} \subset C_{\boldsymbol{c}}$ and we are done. Assume now that $\boldsymbol{x} \in C_{\boldsymbol{b}} \cap \mathbb{Z}^{l-1}$, $N(\boldsymbol{x}) > 0$, and consider two cases. Assume first that $\boldsymbol{x} \in C'_{\boldsymbol{b}'}$. Then we have $\boldsymbol{x} \in C_{\boldsymbol{b}} \cap C'_{\boldsymbol{b}'} \subset \mathbf{P}$; moreover, we have $\lceil \boldsymbol{\omega} \rceil \in \mathbf{P}$, so $\boldsymbol{x} \underset{\mathbf{P}}{\text{---}} \lceil \boldsymbol{\omega} \rceil$ because any two points in $\mathbf{P} \cap \mathbb{Z}^{l-1}$ are $\mathbb{Z}$-connected. Since $\mathbf{P} \subset C_{\boldsymbol{c}}$, we can conclude in this case. Assume now that $\boldsymbol{x} \notin C'_{\boldsymbol{b}'}$. Let $1 \leq i \leq l-1$ be such that $-x_{i-1} + 2x_i - x_{i+1} > b_i + 2$ (where we put $x_0 = x_l := 0$). Consider the vector

$$\boldsymbol{y} = (y_1, \ldots, y_{l-1}) := (x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_{l-1}) \in \mathbb{Z}^{l-1}$$

and put $y_0 = y_l := 0$. Note that for $1 \leq j \leq l-1$, we have

$$-y_{j-1} + 2y_j - y_{j+1} \geq -x_{j-1} + 2x_j - x_{j+1} - 2\delta_{i,j}.$$

Using this fact together with the definition of i and the assumption $\boldsymbol{x} \in C_{\boldsymbol{b}}$, we get $\boldsymbol{y} \in C_{\boldsymbol{b}}$. Moreover, since $N(\boldsymbol{y}) = N(\boldsymbol{x}) - 1$, we have by induction $\boldsymbol{y} \underset{C_{\boldsymbol{c}}}{\text{---}} \lceil \boldsymbol{\omega} \rceil$. Note also that $\boldsymbol{x} \underset{C_{\boldsymbol{c}}}{\text{---}} \boldsymbol{y}$ because $\boldsymbol{c} \leq \boldsymbol{b}$, hence $\boldsymbol{x} \underset{C_{\boldsymbol{c}}}{\text{---}} \lceil \boldsymbol{\omega} \rceil$. □

Keep Notation 5.7. We now construct an integer c such that for all $M \in \mathbb{N}$, we have $\boldsymbol{b}(M) \in \mathcal{A}(\boldsymbol{b}(M + c))$. Let $\boldsymbol{b} \in \mathbb{R}^{l-1}$. By Proposition 5.10, we can define

$$m_M = m_M(n, l, \boldsymbol{r}_l) := \max_{\boldsymbol{c}} \min_{1 \leq i \leq l-1} (c_i) \in \mathbb{Z}, \tag{69}$$

where $\boldsymbol{c} = (c_1, \ldots, c_{l-1})$ ranges over the set $\mathcal{A}(\boldsymbol{b}(M; n, l, \boldsymbol{r}_l)) \cap \mathbb{Z}^{l-1}$.

Lemma 5.11 *The sequence $(m_M)_{M \in \mathbb{N}}$ defined by (69) is increasing. Moreover, we have $m_{nlM} = lM + m_0$ for all $M \in \mathbb{N}$.*

Proof The first statement follows from the fact that the maps $M \to \boldsymbol{b}(M)$ and $\boldsymbol{b} \mapsto \mathcal{A}(\boldsymbol{b})$ are increasing (the latter by Proposition 5.10). Let $M \in \mathbb{N}$, $\boldsymbol{b} := \boldsymbol{b}(nlM)$ and $\boldsymbol{b}' := \boldsymbol{b}(0)$. Since $\det(A) = l$, the Cramer formula shows that $A^{-1}.(\boldsymbol{b} - \boldsymbol{b}') \in \mathbb{Z}^{l-1}$. Applying Proposition 5.10 (iv) to the pair $(\boldsymbol{b}, \boldsymbol{b}')$ yields $\mathcal{A}(\boldsymbol{b}(nlM)) = \mathcal{A}(\boldsymbol{b}(0)) + (lM, \ldots, lM)$. The second statement follows. □

Put $m_0^{(\min)} := \min_{\boldsymbol{a}_l \in A_{l,n}(s)} m_0(n, l, \boldsymbol{a}_l) \in \mathbb{R}$. This is well-defined because $A_{l,n}(s)$ is a finite set.

Proposition 5.12 *With the notation above, put*

$$c := n + \lceil -m_0^{(\min)} n \rceil + nl \in \mathbb{Z}.$$

Then c does not depend on $\boldsymbol{r}_l \in A_{l,n}(s)$ and for all $M \in \mathbb{N}$, we have $\boldsymbol{b}(M) \in \mathcal{A}\big(\boldsymbol{b}(M + c)\big)$.

Proof The first statement is clear. Let $\boldsymbol{c} = (c_1, \dots, c_{l-1}) \in \mathcal{A}(\boldsymbol{b}(M+c)) \cap \mathbb{Z}^{l-1}$ be such that $\min_{1 \le i \le l-1}(c_i) = m_{M+c}$. By proposition 5.10 (ii), it is enough to show that $\boldsymbol{b}(M) \le \boldsymbol{c}$. We have

$$M + c \ge nl\left(\frac{M + n - m_0 n}{nl} + 1\right) \ge nl\left\lceil \frac{M + n - m_0 n}{nl}\right\rceil = nl\left\lceil \frac{\left(\frac{M}{n} + 1\right) - m_0}{l}\right\rceil.$$

By Lemma 5.11, we get

$$m_{M+c} \ge l\left\lceil \frac{\left(\frac{M}{n} + 1\right) - m_0}{l}\right\rceil + m_0 \ge l\left(\frac{\left(\frac{M}{n} + 1\right) - m_0}{l}\right) + m_0 \ge \left\lceil \frac{M}{n}\right\rceil.$$

Let $1 \le i \le l - 1$. Since $\boldsymbol{r}_l \in A_{l,n}(s)$, we have $r_{i+1} - r_i \le 0$, whence

$$c_i \ge m_{M+c} \ge \left\lceil \frac{M}{n}\right\rceil \ge \frac{M}{n} + \frac{r_{i+1} - r_i}{n} = b_i^{(M)}.$$

The result follows. □

Remark 5.13 The author proved in [21, Lemma 4.43] that $m_0^{(\min)} \ge -l^2$, which gives an explicit upper bound for the smallest $c \in \mathbb{Z}$ satisfying the conclusions of Propositions 5.12 and 5.6. ◇

Proof of Proposition 5.6 Let $c \in \mathbb{Z}$ be the integer defined in Proposition 5.12. Let $\boldsymbol{s}_l, \boldsymbol{t}_l \in \dot{W}_l.\boldsymbol{r}_l \cap \mathcal{C}_{M+c}$ be such that $\mathcal{L}_{\boldsymbol{s}_l} = \mathcal{L}_{\boldsymbol{t}_l}$. By Lemma 5.8, we have $\varphi(\boldsymbol{s}_l), \varphi(\boldsymbol{t}_l) \in C_{\boldsymbol{b}(M+c)} \cap \mathbb{Z}^{l-1}$. Proposition 5.12 now implies that $\boldsymbol{b}(M)$ is in $\mathcal{A}\big(\boldsymbol{b}(M+c)\big)$, whence $\varphi(\boldsymbol{s}_l) \underset{C_{\boldsymbol{b}(M)}}{\text{———}} \varphi(\boldsymbol{t}_l)$. Applying again Lemma 5.8 yields $\boldsymbol{s}_l \underset{M}{\equiv} \boldsymbol{t}_l$. □

Acknowledgement I thank Bernard Leclerc for his advice and his reading of the preliminary versions of this article.

References

1. Ariki, S. (1996). On the decomposition numbers of the Hecke algebra of $G(m,1,n)$. *J. Math. Kyoto Univ.*, *36*(4), 789–808.
2. Dipper, R., James, G., & Mathas, A. (1998). Cyclotomic q-Schur algebras. *Math. Z.*, *229*, 385–416.
3. Foda, O., Leclerc, B., Okado, M., Thibon, J.-Y., & Welsh, T. (1999). Branching functions of $A_{n-1}^{(1)}$ and Jantzen-Seitz problem for Ariki-Koike algebras. *Adv. Math.*, *141*(2), 322–365.
4. Hayashi, T. (1990). q-analogues of Clifford and Weyl algebras—spinor and oscillator representations of quantum enveloping algebras. *Commun. Math. Phys.*, *127*(1), 129–144.

5. Jacon, N. (2004). On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity. *J. Math. Kyoto Univ.*, *44*(4), 729–767.
6. Jimbo, M., Misra, K., Miwa, T., & Okado, M. (1991). Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}_n)$ at $q=0$. *Commun. Math. Phys.*, *136*(3), 543–566.
7. Kac, V. G. (1990). *Infinite dimensional Lie algebras* (3rd edn.). Cambridge: Cambridge University Press.
8. Kashiwara, M. (1993). Global crystal bases of quantum groups. *Duke Math. J.*, *69*, 455–485.
9. Kashiwara, M. (1993). The crystal base and Littelmann's refined Demazure character formula. *Duke Math. J.*, *71*, 839–958.
10. Kashiwara, M., Miwa, T., & Stern, E. (1995). Decomposition of q-deformed Fock spaces. *Sel. Math.*, *1*, 787–805.
11. Kashiwara, M., & Tanisaki, T. (2002). Parabolic Kazhdan-Lusztig polynomials and Schubert varieties. *J. Algebra*, *249*, 306–325.
12. Lascoux, A., Leclerc, B., & Thibon, J.-Y. (1996). Hecke algebras at roots of unity and crystal bases of quantum affine algebras. *Commun. Math. Phys.*, *181*(1), 205–263.
13. Leclerc, B., & Miyachi, H. (2002). Some closed formulas for canonical bases of Fock spaces. *Represent. Theory*, *6*, 290–312.
14. Leclerc, B., & Thibon, J.-Y. (1996). Canonical bases of q-deformed Fock spaces. *Intern. Math. Res. Not.*, *9*, 447–456.
15. Leclerc, B., & Thibon, J.-Y. (2000). Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials. *Comb. Methods Represent. Theory Adv. Stud. Pure Math.*, *28*, 155–220.
16. Macdonald, I. G. (1990). *Symmetric functions and Hall polynomials* (2nd edn.). *Oxford Science Publications*. London: Oxford University Press.
17. Misra, K. C., & Miwa, T. (1990). Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}_n)$. *Commun. Math. Phys.*, *134*(1), 79–88.
18. Scopes, J. (1991). Cartan matrices and Morita equivalence for blocks of the symmetric groups. *J. Algebra*, *142*(2), 441–455.
19. Uglov, D. (2000). Canonical bases of higher-level q-deformed Fock spaces and Kazhdan-Lusztig polynomials. In M. Kashiwara, T. Miwa (Eds.), *Physical combinatorics*. *Progress in math.* (Vol. *191*). Birkhäuser, Basel, math.QA/9905196 (1999).
20. Varagnolo, M., & Vasserot, E. (1999). On the decomposition matrices of the quantized Schur algebra. *Duke Math. J.*, *100*, 267–297.
21. Yvonne, X. (2005). *Bases canoniques d'espaces de Fock de niveau supérieur*. Thèse de l'Université de Caen.
22. Yvonne, X. (2006). A conjecture for q-decomposition matrices of cyclotomic v-Schur algebras. *J. Algebra*, *304*, 419–456.
23. Yvonne, X. (2007). An algorithm for computing the canonical bases of higher-level q-deformed Fock spaces. *J. Algebra*, *309*, 760–785.