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Core blocks of Ariki–Koike algebras

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Abstract We examine blocks of the Ariki–Koike algebra, in an attempt to generalise the combinatorial representation theory of the Iwahori–Hecke algebra of type A. We identify a particular type of combinatorial block, which we call a core block, which may be viewed as an analogue of a simple block of the Iwahori–Hecke algebra. We give equivalent characterisations of core blocks and examine their basic combinatorics.

Keywords Ariki–Koike algebra · Multipartition · Weight

1 Introduction

Let $\mathbb{F}$ be a field and q a non-zero element of $\mathbb{F}$. For each $n \geqslant 0$, one defines the *Iwahori–Hecke algebra* $H_n = H_{\mathbb{F},q}(\mathfrak{S}_n)$ of the symmetric group $\mathfrak{S}_n$. This algebra (of which the group algebra $\mathbb{F}\mathfrak{S}_n$ is a special case) arises naturally, and its representation theory has been extensively studied. There are important H_n-modules indexed by partitions of n, and there are many theorems concerning H_n which reduce representation-theoretic notions to statements about the combinatorics of partitions and Young diagrams.

In this paper, we consider the representation theory of the *Ariki–Koike algebra*. This is a deformation of the group algebra of the complex reflection group $C_r \wr \mathfrak{S}_n$, defined using parameters $q, Q_1, \dots, Q_r \in \mathbb{F}$. The development of the representation theory of this algebra is still in its early stages, but already it seems that in many respects the Ariki–Koike algebra behaves in the same way as the Iwahori–Hecke algebra; many of the combinatorial theorems concerning H_n have been generalised to the Ariki–Koike algebra, with the rôle of partitions being played by multipartitions. In fact, much of the difficulty of understanding the Ariki–Koike algebra seems to lie in finding the right generalisations of the combinatorics of partitions to multipartitions—very simple

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combinatorial notions (such as the definition of an e-restricted partition) can have rather nebulous generalisations (such as 'Kleshchev' multipartitions). This paper is intended as a contribution towards understanding the combinatorics of multipartitions, as it relates to the Ariki–Koike algebra. This paper may also be read from the point of view of quantum groups—the decomposition matrices of Ariki–Koike algebras are in certain cases described using canonical bases of higher-level Fock spaces for the quantum groups $U_v(\widehat{\mathfrak{sl}}_e)$, and the combinatorial notions here should be invaluable for studying these Fock spaces. Our results apply also to the cyclotomic q-Schur algebra of Dipper et al., [2], although the statements about Kleshchev multipartitions are of less importance there.

In the representation theory of the Iwahori–Hecke algebra, the weight and core of a partition play an important rôle; they give rise to block invariants which provide information about the representation theory of a block—the weight of a block is an excellent measure of how complicated the representation theory of that block is. In [4], we generalised the notion of weight to multipartitions. We gave a (non-obvious) definition of the weight of a multipartition, and examined its properties. In this paper (which relies heavily on [4], and may be regarded as a sequel), we consider generalising the notion of the core of a partition. Given a multipartition $\boldsymbol{\lambda}$, it seems that we cannot sensibly define another multipartition which we regard as the core of $\boldsymbol{\lambda}$; rather, we define a combinatorial block which we call the core block of $\boldsymbol{\lambda}$.

Core blocks can be quite complicated (in fact, they can have arbitrarily large weight), but we show that they are well-behaved in certain ways. After giving several equivalent definitions of a core block, we show that we may describe the set of multipartitions in a core block in a simple way. We then show that every core block 'occurs at $e=\infty$', by which we mean that for any core block B there is another Ariki–Koike algebra $\check{\mathcal{H}}_n$ defined using parameters $\check{q}, \check{Q}_1, \dots, \check{Q}_r$ with $\check{q}$ *not* a root of unity in $\mathbb{F}$ and a combinatorial block $\check{B}$ of $\check{\mathcal{H}}_n$ which closely resembles B. This resemblance should reflect underlying algebraic structure, but we content ourselves with examining combinatorics, proving that B and $\check{B}$ contain the same multipartitions and the same Kleshchev multipartitions.

Finally, we examine combinatorial blocks which are 'decomposable' in a certain combinatorial sense. The idea is that the representation theory of such blocks should reduce to studying blocks of Ariki–Koike algebras defined for smaller values of r. We show that a decomposable combinatorial block is a core block, and we show that the set of multipartitions and the set of Kleshchev multipartitions in a decomposable combinatorial block may be determined from those of the 'factor' blocks.

For the remainder of this introduction, we describe the background theory and notation we shall need. In Section 2, we prove some purely combinatorial theorems which will be useful in what follows. In Section 3, we look at core blocks.

1.1 Basic definitions

1.1.1 The Ariki–Koike algebra

Let $\mathbb{F}$ be a field, and let $q, Q_1, \dots, Q_r$ be non-zero elements of $\mathbb{F}$. We also assume that $q \neq 1$; there is a corresponding theory for the case $q=1$, but it requires a 'degenerate' Ariki–Koike algebra, which we shall not describe here.

For a non-negative integer n, we define the *Ariki–Koike algebra* $\mathcal{H}_n$ to be the unital associative $\mathbb{F}$-algebra with generators $T_0, \dots, T_{n-1}$ and relations

$$\begin{aligned}
(T_i + q)(T_i - 1) &= 0 && (1 \leqslant i \leqslant n-1)\\
(T_0 - Q_1)\dots(T_0 - Q_r) &= 0 &&\\
T_iT_j &= T_jT_i && (0 \leqslant i, j \leqslant n-1,\ |i-j| > 1)\\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \leqslant i \leqslant n-2)\\
T_0T_1T_0T_1 &= T_1T_0T_1T_0. &&
\end{aligned}$$

We define e to be the multiplicative order of q in $\mathbb{F}$; the assumption that $q \neq 1$ means that $e \in \{2, 3, \dots\} \cup \{\infty\}$. We shall often consider whether two integers are congruent modulo e, and we allow the case $e = \infty$, where 'congruent modulo e' will mean 'equal', and where the set $\mathbb{Z}/e\mathbb{Z}$ should be read as $\mathbb{Z}$. $Q_1, \dots, Q_r$ are referred to as the *cyclotomic parameters* of $\mathcal{H}_n$.

1.1.2 Multipartitions and Specht modules

A *partition* of n is defined to be a decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers whose sum is n. We write $|\lambda| = n$, and we use $\varnothing$ to denote the unique partition of 0. A partition is often written with equal terms grouped and zeroes omitted, so that $(2, 2, 2, 1, 1, 0, 0, \dots)$ becomes $(2^3, 1^2)$. The Young diagram $[\lambda]$ of a partition λ is defined as

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid j \leqslant \lambda_i\},$$

and the elements of $[\lambda]$ are called *nodes*.

A *multipartition of n with r components* is a sequence $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions such that $|\lambda^{(1)}| + \dots + |\lambda^{(r)}| = n$. Again, we write $|\boldsymbol{\lambda}| = n$, and we write the unique multipartition of 0 as $\varnothing$, if r is understood. The *Young diagram* $[\boldsymbol{\lambda}]$ of a multipartition $\boldsymbol{\lambda}$ is the set

$$\left\{(i, j, k) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, r\} \mid j \leqslant \lambda_i^{(k)}\right\},$$

whose elements are also called nodes. We say that the node (i, j, k) is *higher* than the node (i', j', k') if either $k < k'$ or ($k = k'$ and $i < i'$). A node $\mathfrak{n}$ of $[\boldsymbol{\lambda}]$ is called *removable* if $[\boldsymbol{\lambda}] \setminus \{\mathfrak{n}\}$ is the Young diagram of some multipartition, while a triple $\mathfrak{n} = (i, j, k)$ not in $[\boldsymbol{\lambda}]$ is called an addable node of $[\boldsymbol{\lambda}]$ if $[\boldsymbol{\lambda}] \cup \{\mathfrak{n}\}$ is the Young diagram of some multipartition with r components. We emphasise the potentially confusing point that an addable node of $[\boldsymbol{\lambda}]$ is not a node of $[\boldsymbol{\lambda}]$.

To each multipartition $\boldsymbol{\lambda}$ of n, one associates a *Specht module* $S^{\boldsymbol{\lambda}}$. These modules arise from a cellular basis of $\mathcal{H}_n$; each Specht module lies in one block of $\mathcal{H}_n$, and we abuse notation by saying that a multipartition $\boldsymbol{\lambda}$ lies in a block B if $S^{\boldsymbol{\lambda}}$ lies in B. On the other hand, each block contains at least one Specht module, so in order to classify the blocks of $\mathcal{H}_n$, it suffices to describe the corresponding partition of the set of multipartitions.

1.1.3 Rim e-hooks and e-cores

If λ is a partition, then the *rim* of $[\lambda]$ is defined to be the set of nodes (i, j) in $[\lambda]$ for which $(i+1, j+1)$ does not lie in $[\lambda]$. If e is finite, then a *rim e-hook* is defined to be a connected subset R of the rim containing exactly e nodes, such that $[\lambda] \setminus R$ is the Young diagram of a multipartition. If $[\lambda]$ does not have any rim e-hooks, or if $e = \infty$, then we say that λ is an *e-core*. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multipartition and each $\lambda^{(j)}$ is an e-core, then we say that $\boldsymbol{\lambda}$ is a *multicore*.

1.1.4 Residues, blocks and combinatorial blocks

If $\boldsymbol{\lambda}$ is a multipartition and (i, j, k) is a node or an addable node of $[\boldsymbol{\lambda}]$, then we define the *residue* of (i, j, k) to be the element $q^{j-i} Q_k$ of $\mathbb{F}$. For each $f \in \mathbb{F}$, we write $c_f(\boldsymbol{\lambda})$ for the number of nodes of $[\boldsymbol{\lambda}]$ of residue f; now we say that two multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ *lie in the same combinatorial block (of $\mathcal{H}_n$)* if $c_f(\boldsymbol{\lambda}) = c_f(\boldsymbol{\mu})$ for all $f \in \mathbb{F}$. Then we have the following.

Theorem 1.1 ([5] Proposition 5.9 (ii)). *If $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of n, then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the same block of $\mathcal{H}_n$ only if they lie in the same combinatorial block.*

Graham and Lehrer have conjectured a converse to this theorem, namely that two multipartitions lie in the same combinatorial block, then they lie in the same block. This has now been proved by Lyle and Mathas [6, Theorem 2.11], but since their result uses the results of the present paper, we cannot assume it here. Accordingly, this paper is entirely concerned with combinatorial blocks. Of course, when re-reading this paper in the light of the work of Lyle and Mathas, the word 'combinatorial' can be ignored. According to our abuse of terminology, we view a (combinatorial) block as a set of multipartitions, and under this interpretation Theorem 1.1 shows that a combinatorial block B is a disjoint union of blocks $B_1, \dots, B_s$. Of course, $B_1, \dots, B_s$ are more correctly viewed as algebras (in particular, indecomposable direct summands of $\mathcal{H}_n$); occasionally, we shall interpret B as an algebra too, namely the direct sum of the algebras $B_1, \dots, B_s$. We switch between these two interpretations without notice; there should be no risk of confusion.

The author's earlier paper [4] is extensively referenced here, and unfortunately refers to a now discredited preprint containing a purported proof of the converse of Theorem 1.1. This result is not used in a fundamental way in [4], but some of the results need to be re-stated when the classification of blocks is not being assumed. Essentially, this means reading 'combinatorial block' instead of 'block' throughout [4]. A new version of [4] with the appropriate changes appears on the author's web site:

`http://www.maths.qmul.ac.uk/~mf/papers/weight.pdf.`

1.1.5 Kleshchev multipartitions

Residues of nodes are also useful in classifying the simple $\mathcal{H}_n$-modules. Suppose $\boldsymbol{\lambda}$ is a multipartition, and given $f \in \mathbb{F}$ define the *f-signature* of $\boldsymbol{\lambda}$ by examining all the addable and removable nodes of $\boldsymbol{\lambda}$ in turn from higher to lower, and writing a $+$ for each addable node of residue f and a $-$ for each removable node of residue f. Now construct the *reduced f-signature* by successively deleting all adjacent pairs $-+$. If there are any $-$ signs in the reduced f-signature of $\boldsymbol{\lambda}$, the corresponding removable nodes are called *normal* nodes of $[\boldsymbol{\lambda}]$. The highest normal node is called a *good* node of $[\boldsymbol{\lambda}]$.

We say that $\boldsymbol{\lambda}$ is *Kleshchev* if and only if there is a sequence

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(n), \boldsymbol{\lambda}(n-1), \dots, \boldsymbol{\lambda}(0) = \varnothing$$

of multipartitions such that for each i, $[\boldsymbol{\lambda}(i-1)]$ is obtained from $[\boldsymbol{\lambda}(i)]$ by removing a good node. The importance of Kleshchev multipartitions lies in the fact (proved by Ariki [1, Theorem 4.2]) that if $\boldsymbol{\lambda}$ is Kleshchev, then $S^{\boldsymbol{\lambda}}$ has an irreducible cosocle $D^{\boldsymbol{\lambda}}$, and the set $\{D^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \text{ a Kleshchev multipartition}\}$ is a complete set of non-isomorphic simple $\mathcal{H}_n$-modules.

We shall need a slightly stronger statement about which multipartitions are Kleshchev.

Lemma 1.2. *Suppose $\boldsymbol{\lambda}$ is a multipartition and $f \in \mathbb{F}$, and suppose $\overline{\boldsymbol{\lambda}}$ is a multipartition whose Young diagram is obtained by removing all the normal nodes of residue f from* $[\boldsymbol{\lambda}]$. *Then $\boldsymbol{\lambda}$ is Kleshchev if and only if $\overline{\boldsymbol{\lambda}}$ is.*

Proof: Suppose the normal nodes are $\mathfrak{n}_1, \dots, \mathfrak{n}_t$ in descending order. We define $\boldsymbol{\lambda}(0), \dots, \boldsymbol{\lambda}(t)$ by putting $\boldsymbol{\lambda}(0) = \boldsymbol{\lambda}$, and then removing $\mathfrak{n}_i$ from $[\boldsymbol{\lambda}(i-1)]$ to obtain $[\boldsymbol{\lambda}(i)]$, for $i = 1, \dots, t$. Then obviously $\boldsymbol{\lambda}(t) = \overline{\boldsymbol{\lambda}}$, and it is easy to check that $\mathfrak{n}_i$ is a good node of $[\boldsymbol{\lambda}(i-1)]$. Now the result follows from [4, Proposition 1.1]. $\square$

1.1.6 q-connected cyclotomic parameters

We say that the parameters $Q_1, \dots, Q_r$ are *q-connected* if there exist integers a_{ij} such that $Q_j = q^{a_{ij}} Q_i$ for each i, j. Dipper and Mathas [3] showed that if $Q_1, \dots, Q_r$ are not q-connected, then $\mathcal{H}_n$ is Morita equivalent to a direct sum of tensor products of 'smaller' Ariki–Koike algebras. So one typically assumes that $Q_1, \dots, Q_r$ are q-connected. We make this assumption in this paper, too; the relationship between q-connectedness and the combinatorics of multipartitions is discussed to some extent in [4, Section 3.1], and the reader should be able to extend this discussion to cover the content of the current paper.

In fact, since the cyclotomic parameters of $\mathcal{H}_n$ may be simultaneously re-scaled without affecting the isomorphism type of $\mathcal{H}_n$, we assume that each Q_j is a power of q. So we assume that we can find an r-tuple of integers $\mathbf{a} = (a_1, \dots, a_r)$ such that $Q_j = q^{a_j}$ for each j; following Yvonne [8], we call such an $\mathbf{a}$ a *multi-charge*. If e is finite then we may change any of the a_j by adding a multiple of e, and we shall still have $Q_j = q^{a_j}$. If $e = \infty$, then we have only one possible choice of multi-charge $\mathbf{a}$.

1.1.7 The abacus

Given the assumption that the cyclotomic parameters of $\mathcal{H}_n$ are all powers of q, we may conveniently represent multipartitions on an *abacus display*. Given a multipartition $\boldsymbol{\lambda}$, choose a multi-charge $\mathbf{a} = (a_1, \dots, a_r)$, and then for each $i \geqslant 1$ and each $j \in \{1, \dots, r\}$ define the *beta-number*

$$\beta_i^j = \lambda_i^{(j)} + a_j - i.$$

It is easy to see that the set $B^j = \{\beta_1^j, \beta_2^j, \dots\}$ is a set containing exactly $a_j + N$ integers greater than or equal to $-N$, for sufficiently large N. On the other hand, any such set is the set of beta-numbers (defined using the integer a_j) of some partition.

Now we take an abacus with e vertical runners, which we label $0, \dots, e-1$ from left to right if $e < \infty$, or $\dots, -1, 0, +1, \dots$ from left to right if $e = \infty$. On runner l, we mark positions corresponding to the integers congruent to l modulo e; if e is finite, then we mark these in increasing order down the runner. Now we place a bead at position β_i^j, for each i. The resulting configuration is called an abacus display for $\lambda^{(j)}$; the abacus displays for $\lambda^{(1)}, \dots, \lambda^{(r)}$ together form an abacus display for $\boldsymbol{\lambda}$.

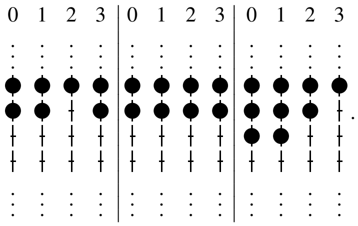
Example. Suppose that $r = 3$, $\mathbf{a} = (-1, 0, 1)$ and $\boldsymbol{\lambda} = ((1), \varnothing, (1^2))$. Then we have

$$B^1 = \{\dots, -5, -4, -3, -1\},$$

$$B^2 = \{\dots, -3, -2, -1\},$$

$$B^3 = \{\dots, -4, -3, -2, 0, 1\}.$$

So an abacus display for $\boldsymbol{\lambda}$ when $e = 4$ is



An abacus display for a partition is useful for visualising the removal of rim e-hooks. If e is finite and we are given an abacus display for $\lambda^{(j)}$, then $[\lambda^{(j)}]$ has a rim e-hook if and only if there is a beta-number $\beta_i^j \in B^j$ such that $\beta_i^j - e \notin B^j$. Furthermore, removing a rim e-hook corresponds to reducing such a beta-number by e. On the abacus, this corresponds to sliding a bead up one position on its runner. So if e is finite, then $\lambda^{(j)}$ is an e-core if and only if every bead in the abacus display has a bead immediately above it.

We now introduce some notation which does not appear in [4]. Suppose e is finite, $\boldsymbol{\lambda}$ is a multicore, and we have chosen a multi-charge $\mathbf{a} = (a_1, \dots, a_r)$. We construct the corresponding abacus display for $\boldsymbol{\lambda}$ as above, and then for each $i \in \mathbb{Z}/e\mathbb{Z}$ and $1 \leqslant j \leqslant r$ we define $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})$ to be the position of the lowest bead on runner i of the abacus for $\lambda^{(j)}$; that is, the largest element of B^j congruent to i modulo e. It is clear that if we choose a different multi-charge $\mathbf{a}' = \mathbf{a} + e\mathbf{x}$ for $\mathbf{x} \in \mathbb{Z}^r$, then we have $\mathfrak{b}^{\mathbf{a}'}_{ij}(\boldsymbol{\lambda}) = \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}) + ex_j$.

We need an alternative notation if $e = \infty$; in this case, we examine the unique abacus display for $\boldsymbol{\lambda}$, and we set $\mathfrak{B}_{ij}(\boldsymbol{\lambda}) = 1$ if $i \in B^j$, and 0 otherwise.

Now for $i \in \mathbb{Z}/e\mathbb{Z}$ and $j, k \in \{1, \dots, r\}$, we define

$$\gamma_i^{jk}(\boldsymbol{\lambda}) = \begin{cases} \dfrac{1}{e}\left(\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}) - \mathfrak{b}^{\mathbf{a}}_{ik}(\boldsymbol{\lambda})\right) & (e < \infty) \\ \mathfrak{B}_{ij}(\boldsymbol{\lambda}) - \mathfrak{B}_{ik}(\boldsymbol{\lambda}) & (e = \infty) \end{cases}.$$

$\gamma_i^{jk}(\boldsymbol{\lambda})$ may then be regarded as the difference in height between the lowest bead on runner i of the abacus display for $\lambda^{(j)}$ and the lowest bead on runner i of the abacus display for $\lambda^{(k)}$. If e is finite, then the integers $\gamma_i^{jk}(\boldsymbol{\lambda})$ depend on the choice of $\mathbf{a}$, but the differences

$$\gamma_{il}^{jk}(\boldsymbol{\lambda}) = \gamma_i^{jk}(\boldsymbol{\lambda}) - \gamma_l^{jk}(\boldsymbol{\lambda})$$

do not; these integers will be very helpful in weight calculations.

1.1.8 The weight and hub of a multipartition

Now we can give the main definition from [4]. Given a multipartition $\boldsymbol{\lambda}$, we define $c_f(\boldsymbol{\lambda})$ for $f \in \mathbb{F}$ as above, and put

$$w(\boldsymbol{\lambda}) = \sum_{j=1}^{r} c_{Q_j}(\boldsymbol{\lambda}) - \frac{1}{2}\sum_{f \in \mathbb{F}} (c_f(\boldsymbol{\lambda}) - c_{qf}(\boldsymbol{\lambda}))^2.$$

$w(\boldsymbol{\lambda})$ is a non-negative integer, called the weight of $\boldsymbol{\lambda}$.

It is also useful to define the *hub* of a multipartition. For each $i \in \mathbb{Z}/e\mathbb{Z}$ and $j \in \{1, \dots, r\}$, define

$$\begin{aligned} \delta_i^j(\boldsymbol{\lambda}) = {} & (\text{the number of removable nodes of } [\lambda^{(j)}] \text{ of residue } q^i) \\ & - (\text{the number of addable nodes of } [\lambda^{(j)}] \text{ of residue } q^i), \end{aligned}$$

and put $\delta_i(\boldsymbol{\lambda}) = \sum_{j=1}^r \delta_i^j(\boldsymbol{\lambda})$. The collection $(\delta_i(\boldsymbol{\lambda}) \mid i \in \mathbb{Z}/e\mathbb{Z})$ of integers is called the hub of $\boldsymbol{\lambda}$.

1.1.9 Notation

Many of the combinatorial notions we have defined, such as the residue of a node, Kleshchev multipartitions and weight, depend upon the parameters $q, Q_1, \dots, Q_r$. Occasionally, we shall be considering Ariki–Koike algebras with different parameters, and we shall use terms such as $(q; Q_1, \dots, Q_r)$-residue, $(q; Q_1, \dots, Q_r)$-Kleshchev and $(q; Q_1, \dots, Q_r)$-weight when there is a danger of ambiguity.

1.2 Background results from [4]

Here we summarise some results from [4], mostly concerning weight calculations. In the published version of [4], two of the results we cite (namely, Propositions 3.2 and 4.6) refer to blocks rather than combinatorial blocks, and as such their proofs are incorrect. However, the statements and proofs given there are correct if blocks are instead interpreted as combinatorial blocks (and if the last sentence of the proof of Proposition 3.2 is ignored). (The proof of Lemma 3.3 also uses arguments from the proof of Proposition 3.2, but is nonetheless valid.)

1.2.1 The weight and hub of a multipartition determine the combinatorial block in which it lies

An important feature of the weight and hub of a multipartition is that they are invariants of the combinatorial block containing $\boldsymbol{\lambda}$, and in fact determine this combinatorial block.

Proposition 1.3 ([4] Proposition 3.2 & Lemma 3.3). *Suppose $\boldsymbol{\lambda}$ is a multipartition of n and $\boldsymbol{\mu}$ is a multipartition of m. Then:*

1. *if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub, then $m \equiv n \pmod e$, and*

$$w(\boldsymbol{\lambda}) - w(\boldsymbol{\mu}) = r\frac{n-m}{e};$$

2. *if $n = m$, then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the same combinatorial block of $\mathcal{H}_n$ if and only if they have the same hub.*

In view of this result, we may define the hub of a combinatorial block B to be the hub of any multipartition $\boldsymbol{\lambda}$ in B, and we write $\delta_i(B) = \delta_i(\boldsymbol{\lambda})$.

1.2.2 Calculating weight from the abacus

In [4], it is shown how to compute the weight of a multipartition efficiently from an abacus display. We summarise the relevant results here.

Proposition 1.4 ([4] Corollary 3.4). *Suppose e is finite, that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions, and that $[\boldsymbol{\mu}]$ is obtained from $[\boldsymbol{\lambda}]$ by removing a rim e-hook. Then $w(\boldsymbol{\mu}) = w(\boldsymbol{\lambda}) - r$.*

Proposition 1.5 ([4] Proposition 3.5). *Suppose $\boldsymbol{\lambda}$ is a multicore, and for each $1 \leqslant j < k \leqslant r$ let $w_{jk}(\boldsymbol{\lambda})$ denote the $(q; Q_j, Q_k)$-weight of $(\lambda^{(j)}, \lambda^{(k)})$. Then*

$$w(\boldsymbol{\lambda}) = \sum_{1 \leqslant j < k \leqslant r} w_{jk}(\boldsymbol{\lambda}).$$

Now suppose that $\boldsymbol{\lambda}$ is a multicore, that $i, l \in \mathbb{Z}/e\mathbb{Z}$ and that $j, k \in \{1, \dots, r\}$. If $e = \infty$, suppose additionally that $\gamma_{il}^{jk}(\boldsymbol{\lambda}) = 2$. Define $s_{il}^{jk}(\boldsymbol{\lambda})$ to be the multicore whose abacus display is obtained by moving a bead from runner i to runner l on the abacus for $\lambda^{(j)}$, and moving a bead from runner l to runner i on the abacus for $\lambda^{(k)}$.

Proposition 1.6. *$s_{il}^{jk}(\boldsymbol{\lambda})$ has the same hub as $\boldsymbol{\lambda}$, and*

$$w\big(s_{il}^{jk}(\boldsymbol{\lambda})\big) = w(\boldsymbol{\lambda}) - r\big(\gamma_{il}^{jk}(\boldsymbol{\lambda}) - 2\big).$$

Proof: Write $\boldsymbol{\mu} = s_{il}^{jk}(\boldsymbol{\lambda})$. Restricting attention to the jth and kth components and calculating using the parameters q, Q_j, Q_k, we see that $\gamma_{il}^{12}((\lambda^{(j)}, \lambda^{(k)})) = \gamma_{il}^{jk}(\boldsymbol{\lambda})$ and $(\mu^{(j)}, \mu^{(k)}) = s_{il}^{12}((\lambda^{(j)}, \lambda^{(k)}))$. So by [4, Lemma 3.7] (which is simply the case $r = 2$ of the present proposition), we have

$$w\big((\mu^{(j)}, \mu^{(k)})\big) = w\big((\lambda^{(j)}, \lambda^{(k)})\big) - 2\big(\gamma_{il}^{jk}(\lambda) - 2\big).$$

Using Proposition 1.3(1) and noting that $(\mu^{(j)}, \mu^{(k)})$ and $(\lambda^{(j)}, \lambda^{(k)})$ have the same hub, we obtain

$$\big|(\mu^{(j)}, \mu^{(k)})\big| = \big|(\lambda^{(j)}, \lambda^{(k)})\big| - e\big(\gamma_{il}^{jk}(\boldsymbol{\lambda}) - 2\big).$$

$\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are identical in all components other than the jth and kth, and so

$$|\boldsymbol{\mu}| = |\boldsymbol{\lambda}| - e\big(\gamma_{il}^{jk}(\boldsymbol{\lambda}) - 2\big),$$

and since $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub, we may apply Proposition 1.3(1) again to get the result. □

Proposition 1.7. *Suppose that $r = 2$, and that $\boldsymbol{\lambda}$ is a multicore.*

1. *If $\gamma_{il}^{12}(\boldsymbol{\lambda}) \leqslant 2$ for all i, l, then $w(\boldsymbol{\lambda})$ is the smaller of the two integers*

$$\big|\{i \,|\, \gamma_{il}^{12}(\boldsymbol{\lambda}) = 2 \text{ for some } l\}\big|$$

and

$$\big|\{l \,|\, \gamma_{il}^{12}(\boldsymbol{\lambda}) = 2 \text{ for some } i\}\big|.$$

2. *$w(\boldsymbol{\lambda}) = 0$ if and only if $\gamma_{il}^{12}(\boldsymbol{\lambda}) \leqslant 1$ for all i, l.*

Proof: (1) is simply [4, Proposition 3.8]. For (2), the result follows from (1) if we have $\gamma_{il}^{12} \leqslant 2$ for all i, l. On the other hand, if we have $\gamma_{il}^{12} \geqslant 3$ for some i, l, then the multipartition $s_{il}^{12}(\boldsymbol{\lambda})$ has strictly smaller weight than $\boldsymbol{\lambda}$, by Proposition 1.6. □

1.2.3 Scopes isometries

Here we introduce maps between combinatorial blocks of Ariki–Koike algebras analogous to those defined by Scopes [7] between blocks of symmetric groups. Suppose $k \in \mathbb{Z}/e\mathbb{Z}$, and let $\phi_k : \mathbb{Z} \to \mathbb{Z}$ be the map given by

$$\phi_k(x) = \begin{cases} x+1 & (x \equiv k-1 \pmod{e}) \\ x-1 & (x \equiv k \pmod{e}) \\ x & (\text{otherwise}) \end{cases} .$$

If e is finite, then ϕ_k descends to give a map from $\mathbb{Z}/e\mathbb{Z}$ to $\mathbb{Z}/e\mathbb{Z}$; we abuse notation by referring to this map as ϕ_k also.

Now suppose $\boldsymbol{\lambda}$ is a multipartition, and that we have chosen an abacus display for $\boldsymbol{\lambda}$. For each j, we define a partition $\Phi_k(\lambda^{(j)})$ by replacing each beta-number β with $\phi_k(\beta)$. Equivalently, we simultaneously remove all removable nodes of residue q^k from $[\lambda^{(j)}]$ and add all addable nodes of residue q^k. We define $\Phi_k(\boldsymbol{\lambda})$ to be the multipartition $(\Phi_k(\lambda^{(1)}), \dots, \Phi_k(\lambda^{(r)}))$.

Proposition 1.8 ([4] Proposition 4.6). *If B is a combinatorial block of $\mathcal{H}_n$, then there is a combinatorial block C of $\mathcal{H}_{n-\delta_k(B)}$ such that Φ_k gives a self-inverse bijection from the set of multipartitions in B to the set of multipartitions in C.*

We write $\Phi_k(B)$ for the combinatorial block C described in Proposition 1.8. Now we note that in a special case, Φ_k preserves the Kleshchev property.

Lemma 1.9. *Suppose $\boldsymbol{\lambda}$ is a multipartition, that $k \in \mathbb{Z}/e\mathbb{Z}$, and that $[\boldsymbol{\lambda}]$ has no addable nodes of residue q^k. Then $\boldsymbol{\lambda}$ is Kleshchev if and only if $\Phi_k(\boldsymbol{\lambda})$ is.*

Proof: Since $[\boldsymbol{\lambda}]$ has no addable nodes of residue q^k, every removable node of residue q^k is normal, and $[\Phi_k(\boldsymbol{\lambda})]$ is obtained by removing all these nodes. Now the result follows from Lemma 1.2. □

2 Some combinatorial results

In this section, we prove three combinatorial results which we shall need later. It is possible that these may be of independent interest. It is also possible that they are already known; the author has not searched the literature in great detail.

2.1 A combinatorial lemma concerning the weight lattice of type A_{r-1}

First we prove a result which will be essential for the proof of the main result of Section 3, but we phrase it here in terms of the weight lattice for a root system of type A_{r-1}, where $r \geqslant 1$. (If $r = 1$, then this lattice consists of a single point, but the results all hold without modification.)

We consider the weight lattice L_r of type A_{r-1}: let $\mathbb{Z}^r$ denote the free $\mathbb{Z}$-module with basis $\{e_1, \dots, e_r\}$, and define

$$L_r = \frac{\mathbb{Z}^r}{\mathbb{Z}(e_1 + \cdots + e_r)}.$$

We write elements of L_r simply by writing representative elements in $\mathbb{Z}^r$.

We adopt the following conventions concerning multisets. If T is a set and X a multiset, we say that X is a multisubset of T if every element of X is an element of T. If X contains several copies of some $t \in T$, we write $X \setminus \{t\}$ to mean X with one of these copies removed. Similarly, we write $X \cup \{t\}$ to mean X with a copy of t added (so $\cup$ really means 'disjoint union'). We want to consider finite multisubsets of L_r. First we introduce an equivalence relation on the set of such multisets.

Suppose X is a multisubset of L_r, and that there are $x, y \in X$ and $k, l \in \{1, \dots, r\}$ such that

$$(x_k - x_l) - (y_k - y_l) = 2.$$

Define the multiset Y by

$$Y = X \setminus \{x, y\} \cup \{x - e_k + e_l, y + e_k - e_l\},$$

and say that $X \equiv Y$ whenever X and Y are related in this way. Clearly $\equiv$ is a symmetric relation; we extend it transitively and reflexively to obtain an equivalence relation.

Now say that a multisubset X of L_r is *tight* if

$$(x_k - x_l) - (y_k - y_l) \leqslant 2$$

for all $x, y \in X$ and all $1 \leqslant k, l \leqslant r$, and that X is *ultra-tight* if every multiset in the same $\equiv$-class as X is tight.

Example. Let $r = 3$. Then the set

$$X = \{(0, 0, 0), (2, 0, 0), (0, -2, 0)\}$$

is tight, but it is not ultra-tight, since

$$X \equiv Y = \{(0, 0, 0), (2, -1, 1), (0, -1, -1)\}.$$

Our aim is to classify ultra-tight multisets.

Lemma 2.1. *Given $s \in L_r$, define*

$$N(s) = \{x \in L_r \mid x_k - x_l \leqslant s_k - s_l + 1 \;\forall\, k, l \in \{1, \dots, r\}\}.$$

Then any multisubset of $N(s)$ is ultra-tight.

Proof: Clearly any multisubset of $N(s)$ is tight, so it will suffice to prove that if X is a multisubset of $N(s)$ and $X \equiv Y$, then Y is also a multisubset of $N(s)$. To see this, it is enough to observe the following: if $x, y \in N(s)$ such that $(x_k - x_l) - (y_k - y_l) = 2$ for some k, l, then $x - e_k + e_l$ and $y + e_k - e_l$ also lie in $N(s)$. □

What we want to do is prove a converse to the above lemma, i.e. that every ultra-tight multisubset of L_r is a multisubset of $N(s)$ for some s. In fact, we write the result slightly differently. Given $a_1, \dots, a_{r-1} \in \mathbb{Z}$ and $k, l \in \{1, \dots, r\}$, we write

$$a_{kl} = (a_k + a_{k+1} + \dots + a_{r-1}) - (a_l + a_{l+1} + \dots + a_{r-1}),$$

where we regard $a_k + \dots + a_{r-1}$ as 0 if $k = r$.

Proposition 2.2. *If X is an ultra-tight multisubset of L_r, then there are $a_1, \dots, a_{r-1} \in \mathbb{Z}$ such that*

$$x_k - x_l \leqslant a_{kl} + 1$$

for all $x \in X$ and $k, l \in \{1, \dots, r\}$.

It is clear that this result shows that an ultra-tight multiset X is a multisubset of $N(s)$, where $s_k = (a_k + a_{k+1} + \dots + a_{r-1})$ for each k. The inductive step used to prove Proposition 2.2 is the following.

Proposition 2.3. *Suppose X is an ultra-tight multisubset of L_r and $1 \leqslant t \leqslant r - 1$, and that there exist $a_1, \dots, a_{r-1} \in \mathbb{Z}$ such that for any $x \in X$ we have*

$$x_k - x_l \leqslant a_{kl} + 1$$

whenever $k, l \in \{1, \dots, r - 1\}$ or $k, l \in \{t + 1, \dots, r\}$. Then there exist $\hat{a}_1, \dots, \hat{a}_{r-1} \in \mathbb{Z}$ such that for any $x \in X$ we have

$$x_k - x_l \leqslant \hat{a}_{kl} + 1$$

whenever $k, l \in \{1, \dots, r - 1\}$ or $k, l \in \{t, \dots, r\}$.

Proof: The case where $t = r - 1$ is easy: since X is tight, we can choose $\hat{a}_{r-1}$ such that $\hat{a}_{r-1} - 1 \leqslant x_{r-1} - x_r \leqslant \hat{a}_{r-1} + 1$ for all x, while for $1 \leqslant i \leqslant r - 2$ we set $\hat{a}_i = a_i$. So we suppose $t < r - 1$.

If $a_1, \dots, a_{r-1}$ will not serve as $\hat{a}_1, \dots, \hat{a}_{r-1}$, then there is some $x \in X$ such that either

$$x_t - x_r \geqslant a_{tr} + 2$$

or

$$x_t - x_r \leqslant a_{tr} - 2.$$

We assume the first of these inequalities; the proof in the other case is similar.

Given the inequalities we already have, we find that for any $u \in \{t+1, \dots, r-1\}$ we have

$$\begin{aligned} x_t - x_r &= (x_t - x_u) + (x_u - x_r) \\ &\leqslant a_{tu} + 1 + a_{ur} + 1 \\ &= a_{tr} + 2. \end{aligned}$$

So we must have $x_t - x_u = a_{tu} + 1$ and $x_u - x_r = a_{ur} + 1$, and this implies that

$$\begin{aligned} x_t - x_{t+1} &= a_t + 1, \\ x_{t+1} - x_{t+2} &= a_{t+1}, \\ x_{t+2} - x_{t+3} &= a_{t+2}, \\ &\vdots \\ x_{r-2} - x_{r-1} &= a_{r-2}, \\ x_{r-1} - x_r \quad &= a_{r-1} + 1. \end{aligned}$$

Note also that for any $y \in X$ we have

$$a_{tr} \leqslant y_t - y_r \leqslant a_{tr} + 2; \tag{$\dagger$}$$

the first inequality follows because $x_t - x_r = a_{tr} + 2$ and X is tight, and the second inequality follows because by assumption $y_t - y_{r-1} \leqslant a_{t(r-1)} + 1$ and $y_{r-1} - y_r \leqslant a_{(r-1)r} + 1$.

Now we draw a directed graph G on the set $\{1, \dots, r\}$, with an arrow from k to l if and only if there is some $y \in X$ with

$$y_k - y_l = a_{kl} - 1.$$

We consider two cases.

Case 1. G does not contain a directed path from t to r

Under this assumption, the set $\{1, \dots, r\}$ can be partitioned into two sets T and R such that $t \in T$, $r \in R$, and there is no arrow $k \to l$ for any $k \in T, l \in R$. (For example, we could let T be the set of all k such that there is a directed path from t to k.) We define

$$\hat{a}_k = \begin{cases} a_k + 1 & (k \in T,\ k+1 \in R) \\ a_k - 1 & (k \in R,\ k+1 \in T). \\ a_k & \text{(otherwise)} \end{cases}$$

Then we have

$$\hat{a}_{kl} = \begin{cases} a_{kl} + 1 & (k \in T,\ l \in R) \\ a_{kl} - 1 & (k \in R,\ l \in T), \\ a_{kl} & \text{(otherwise)} \end{cases}$$

and we claim that $y_k - y_l \leqslant \hat{a}_{kl}$ for any $y \in X$ whenever $k, l \in \{1, \dots, r-1\}$ or $k, l \in \{t, \dots, r\}$. Since $t \in T$ and $r \in R$ we have $\hat{a}_{tr} = a_{tr} + 1$, and so by (†) we have

$$\hat{a}_{tr} - 1 \leqslant y_t - y_r \leqslant \hat{a}_{tr} + 1,$$

which deals with cases where $\{k, l\} = \{t, r\}$. For the other cases, the result is immediate if $\hat{a}_{kl} \geqslant a_{kl}$. If $\hat{a}_{kl} = a_{kl} - 1$, then $k \in R$ and $l \in T$. The fact that there are no arrows from T to R implies that $y_l - y_k \geqslant a_{lk}$, so that

$$y_k - y_l \leqslant a_{kl} = \hat{a}_{kl} + 1.$$

Case 2. G does contain a directed path from t to r

Under this assumption, we'll show that X is not ultra-tight, which gives a contradiction. First we note the following.

Claim. *For any $k < r$, we have $x_k - x_r \geqslant a_{kr} + 1$.*

Proof: This comes directly from above if $k \geqslant t$ (and in fact we have $x_t - x_r = a_{tr} + 2$). For $k < t$, we have

$$\begin{aligned} x_k - x_r &= (x_k - x_t) + (x_t - x_r) \\ &\geqslant a_{kt} - 1 + a_{tr} + 2 \\ &= a_{kr} + 1. \end{aligned}$$

Now suppose

$$t = k_1 \to k_2 \to \cdots \to k_s = r$$

is a path from t to r of minimal length in G, and choose $y^1,\dots,y^{s-1}\in X$ such that

$$y^m_{k_m}-y^m_{k_{m+1}}=a_{k_mk_{m+1}}-1$$

for all m. We also define $x^m=x-e_{k_{m+1}}+e_r$ for $m=1,\dots,s-1$. We want to prove the following statements by (downwards) induction, for $m=s-1,\dots,1$:

$A_m:$ there is a multiset Y_m containing $y^1,\dots,y^m$ and x^m such that $Y_m\equiv X$;

$B_m:\big(x^m_{k_m}-x^m_{k_{m+1}}\big)-\big(y^m_{k_m}-y^m_{k_{m+1}}\big)=2;$

$C_m: x_{k_m}-x_r=a_{k_mr}+1.$

If we can prove statement C_1, we shall have a contradiction.
A_{s-1} is immediate—we can take $Y_{s-1}=X$. Now we prove B_{s-1} and C_{s-1}; by the claim, we have

$$\big(x_{k_{s-1}}-x_r\big)-\big(y^{s-1}_{k_{s-1}}-y^{s-1}_r\big)\geqslant\big(a_{k_{s-1}r}+1\big)-\big(a_{k_{s-1}r}-1\big)=2;$$

we must have equality since X is tight, which gives B_{s-1}. Since $y^{s-1}_{k_{s-1}}-y^{s-1}_r=a_{k_{s-1}r}-1$, we have C_{s-1} too.
Now we perform our inductive steps: first, we show that for $m\leqslant s-2$ the statements A_{m+1} and B_{m+1} imply A_m: we construct the set Y_m by taking the multiset Y_{m+1}, removing x^{m+1} and y^{m+1}, and adding x^m and $y^{m+1}+e_{k_{m+1}}-e_{k_{m+2}}$. By B_{m+1}, we have $Y_m\equiv Y_{m+1}\equiv X$.
Next we show that for $m\leqslant s-2$ statements A_m and C_{m+1} imply B_m and C_m. Since the chosen path has minimal length, $k_1,\dots,k_s$ are pairwise distinct; in particular, $(e_r)_{k_m}=(e_r)_{k_{m+1}}=0$. Hence

$$\begin{aligned}x^m_{k_m}-x^m_{k_{m+1}}&=\big(x-e_{k_{m+1}}+e_r\big)_{k_m}-\big(x-e_{k_{m+1}}+e_r\big)_{k_{m+1}}\\&=\big(x_{k_m}-x_r\big)-\big(x_{k_{m+1}}-x_r\big)+1\\&\geqslant\big(a_{k_mr}+1\big)-\big(a_{k_{m+1}r}+1\big)+1\end{aligned}$$

(by the claim and statement C_{m+1})

$$\begin{aligned}&=a_{k_mk_{m+1}}+1\\&=y^m_{k_m}-y^m_{k_{m+1}}+2;\end{aligned}$$

since Y_m is tight, we must have equality, which implies B_m and C_m.
So by induction statement C_1 is true, which is a contradiction. □

Proof of Proposition 2.2. We proceed by induction on r, with the case $r=1$ being trivial. For $r>1$, let X be an ultra-tight multisubset of L_r. The natural projection $\mathbb{Z}^r\to\mathbb{Z}^{r-1}$ induces a map $L_r\to L_{r-1}$, and we write $\overline{X}$ for the image of X under this map. It is clear that $\overline{X}$ is ultra-tight, so by induction we can find integers $a_1, \dots, a_{r-2}$ such that

$$x_k - x_l \leqslant a_{kl} + 1$$

for all $x \in X$ and $k, l \in \{1, \dots, r-1\}$.

We let a_{r-1} be an arbitrary integer, and then we apply Proposition 2.3 for $t = r-1, \dots, 1$ in turn. □

2.2 Results concerning integer matrices

Now we prove two simple results concerning manipulation of integer matrices. Suppose A and B are both $e \times r$ matrices with integer entries; for consistency with later sections, we index the rows with the integers $0, \dots, e-1$ and the columns with the integers $1, \dots, r$.

First we suppose that all the entries of A and B are 0 or 1. We write $A \leftrightarrow B$ if there are indices k, l, m, n with $k \neq l$ and $m \neq n$ such that

$$a_{ij} - b_{ij} = \delta_{ik}\delta_{jm} + \delta_{il}\delta_{jn} - \delta_{ik}\delta_{jn} - \delta_{il}\delta_{jm}$$

for all i, j, where we employ the Kronecker delta. That is, A and B differ by the addition of a 2×2 submatrix of the form $\left(\begin{smallmatrix} +1 & -1 \\ -1 & +1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & +1 \\ +1 & -1 \end{smallmatrix}\right)$.

Clearly if $A \leftrightarrow B$, then A and B have the same row sums and column sums. Our result is a converse to this statement.

Proposition 2.4. *Suppose A and B are $e \times r$ matrices with all entries equal to 0 or 1 and with the same row and column sums, that is,*

$$a_{i1} + \cdots + a_{ir} = b_{i1} + \cdots + b_{ir}$$

for all $i \in \{0, \dots, e-1\}$ and

$$a_{0j} + \cdots + a_{(e-1)j} = b_{0j} + \cdots + b_{(e-1)j}$$

for all $j \in \{1, \dots, r\}$. Then there is a sequence $A = A_0, \dots, A_s = B$ of matrices with all entries equal to 0 or 1 such that $A_0 \leftrightarrow A_1 \leftrightarrow \cdots \leftrightarrow A_s$.

Proof: It suffices to assume that $A \neq B$ and to find a matrix A' such that either

1. $A \leftrightarrow A'$ and there are fewer positions where A' and B differ than positions where A and B differ, or
2. $A' \leftrightarrow B$ and there are fewer positions where A and A' differ than positions where A and B differ.

Put $C = A - B$. Then the entries of C are all equal to -1, 0 or $+1$, and the row and column sums of C are all zero. For $t > 0$, we define a *chain of length t* to be a sequence $(i_0, j_0), \dots, (i_{t-1}, j_{t-1})$ such that

$$c_{i_k j_k} = +1, \qquad c_{i_k j_{k+1}} = -1$$

for all $k = 0, \dots, t-1$, where we interpret i_t as i_0. We can certainly find a chain of some length, by the following procedure:

- since C is non-zero, we can find g_0, h_0 such that $c_{g_0 h_0} = +1$;
- suppose we have g_k, h_k with $c_{g_k h_k} = +1$; since the g_kth row sum of C is zero, we can find h_{k+1} such that $c_{g_k h_{k+1}} = -1$;
- suppose we have g_k, h_{k+1} with $c_{g_k h_{k+1}} = -1$; since the h_{k+1}th column sum of C is zero, we can find g_{k+1} such that $c_{g_{k+1} h_{k+1}} = +1$.

This enables us to construct a sequence $g_0, h_0, g_1, h_1, \dots$ such that $c_{g_k h_k} = +1$ and $c_{g_k h_{k+1}} = -1$ for each k; since there are only finitely many entries in C, the sequence $(g_0, h_0), (g_1, h_1), \dots$ must repeat at some point, say $(g_v, h_v) = (g_u, h_u)$ with $v > u$. Defining $t = v - u$ and $i_k = g_{u+k}$, $j_k = h_{u+k}$ for $k = 0, \dots, t-1$ gives a chain.

Take a chain $(i_0, j_0), \dots, (i_{t-1}, j_{t-1})$ of length t with $t > 0$ minimal, and consider the position (i_0, j_{t-1}).

- If $a_{i_0, j_{t-1}} = 0$, then we define

  $$a'_{ij} = a_{ij} + \delta_{ii_0}\delta_{jj_{t-1}} + \delta_{ii_{t-1}}\delta_{jj_0} - \delta_{ii_0}\delta_{jj_0} - \delta_{ii_{t-1}}\delta_{jj_{t-1}}.$$

  Certainly $i_0 \neq i_{t-1}$ and $j_0 \neq j_{t-1}$, so the matrix $A' = (a'_{ij})$ satisfies the conditions of (1) above.
- If $b_{i_0, j_{t-1}} = 1$, then we define

  $$a'_{ij} = b_{ij} - \delta_{ii_0}\delta_{jj_{t-1}} - \delta_{ii_{t-1}}\delta_{jj_0} + \delta_{ii_0}\delta_{jj_0} + \delta_{ii_{t-1}}\delta_{jj_{t-1}}.$$

  Now $A' = (a'_{ij})$ satisfies the conditions of (2) above.
- Otherwise, we have $c_{i_0, j_{t-1}} = +1$; but this implies that $t \geqslant 3$ and that

  $$((i_0, j_{t-1}), (i_1, j_1), (i_2, j_2), \dots, (i_{t-2}, j_{t-2}))$$

  is a chain of length $t-1$; contradiction.

$\square$

Now we prove our second result concerning integer matrices.

Proposition 2.5. *Suppose A and B are $e \times r$ matrices satisfying the following conditions:*

- *there exist integers $\alpha_0, \dots, \alpha_{e-1}$ and $\beta_0, \dots, \beta_{e-1}$ with*

$$\alpha_i \equiv \beta_i \equiv i \pmod e$$

for each i and with

$$a_{ij} \in \{\alpha_i, \alpha_i + e\}, \qquad b_{ij} \in \{\beta_i, \beta_i + e\}$$

for all i, j;

- *there is a constant K such that*

$$b_{i1} + \cdots + b_{ir} = a_{i1} + \cdots + a_{ir} + K$$

for all i;

- *for any j,*

$$b_{0j} + \cdots + b_{(e-1)j} \equiv a_{0j} + \cdots + a_{(e-1)j} \pmod{e^2}.$$

Then there is an $e \times r$ integer matrix C with entries constant down each column and such that:

-

$$b_{ij} + c_{ij} \in \{\alpha_i, \alpha_i + e\}$$

for each i, j;

-

$$(b_{i1} + c_{i1}) + \cdots + (b_{ir} + c_{ir}) = a_{i1} + \cdots + a_{ir}$$

for $i = 0, \ldots, e-1$;

-

$$(b_{0j} + c_{0j}) + \cdots + \left(b_{(e-1)j} + c_{(e-1)j}\right) = a_{0j} + \cdots + a_{(e-1)j}$$

for $j = 1, \ldots, r$.

Proof: By reducing β_i by e if necessary, we assume that for each i there is at least one j with $b_{ij} = \beta_i + e$. Then we have

$$r\beta_i + e \leqslant b_{i1} + \cdots + b_{ir} \leqslant r\beta_i + re$$

as well as

$$r\alpha_i \leqslant a_{i1} + \cdots + a_{ir} \leqslant r\alpha_i + re$$

so that

$$\frac{K - re}{r} \leqslant \beta_i - \alpha_i \leqslant \frac{K + (r-1)e}{r}$$

for each i. Since $\beta_i - \alpha_i$ is divisible by e, this means that the integers $\beta_i - \alpha_i$ can only take two different values (which differ by e) as i varies. By adding a constant multiple of e to all entries of B and to each β_i, we may assume that $\beta_i - \alpha_i = 0$ or e for all i, and equals 0 for at least one value of i.

Now we examine the column sums of A and B. We write a_{*j} for the sum $a_{0j} + \cdots + a_{(e-1)j}$ and similarly b_{*j}. We have

$$b_{*j} - a_{*j} \equiv 0 \pmod{e^2}$$

and

$$\sum_{i=0}^{e-1}(\beta_i - \alpha_i) - e^2 \leqslant b_{*j} - a_{*j} \leqslant \sum_{i=0}^{e-1}(\beta_i - \alpha_i) + e^2$$

and by the above assumptions we have $0 \leqslant \sum_{i=0}^{e-1}(\beta_i - \alpha_i) \leqslant e(e-1)$, so that $b_{*j} - a_{*j}$ can only equal 0 or $\pm e^2$. But if $b_{*j} - a_{*j} = -e^2$, then we must have $b_{ij} = \beta_i$ for all i, and we may increase each b_{ij} by e without affecting earlier hypotheses to get $b_{*j} - a_{*j} = 0$. So we assume that $b_{*j} - a_{*j}$ equals 0 or e^2 for each j. By re-ordering rows and columns, we may assume that $\beta_i - \alpha_i = e$ for $i = 0, \dots, l-1$ only, and that $b_{*j} - a_{*j} = e^2$ for $j = 1, \dots, k$ only. Note that we then have $K = ek$. We would like to define

$$c_{ij} = \begin{cases} -e & (j \leqslant k) \\ 0 & (j > k) \end{cases};$$

this would give the correct row and column sums for $B + C$, and would give $b_{ij} + c_{ij} = \alpha_i$ or $\alpha_i + e$, except possibly when $i \geqslant l$ and $j \leqslant k$. So it suffices to show that we have $b_{ij} = \beta_i + e$ when $i \geqslant l$ and $j \leqslant k$. For any $1 \leqslant j \leqslant k$, we have

$$\begin{aligned}
\big(b_{0j} + \cdots + b_{(l-1)j}\big) - \big(a_{0j} + \cdots + a_{(l-1)j}\big) &= e^2 + \big(a_{lj} + \cdots + a_{(e-1)j}\big) \\
&\quad - \big(b_{lj} + \cdots + b_{(e-1)j}\big) \\
&\geqslant e^2 + \big(\alpha_l + \cdots + \alpha_{e-1}\big) \\
&\quad - \big((\beta_l + e) + \cdots + (\beta_{e-1} + e)\big) \\
&= el. \qquad (*)
\end{aligned}$$

with equality only if $b_{ij} = \beta_i + e$ for $i = l, \dots, e-1$. On the other hand, for any $0 \leqslant i \leqslant l-1$ we have

$$\begin{aligned}
(b_{i1} + \cdots + b_{ik}) - (a_{i1} + \cdots + a_{ik}) &= ek + \big(a_{i(k+1)} + \cdots + a_{ir}\big) \\
&\quad - \big(b_{i(k+1)} + \cdots + b_{ir}\big) \\
&\leqslant ek + (r-k)(\alpha_i + e) - (r-k)\beta_i \\
&= ek. \qquad (**)
\end{aligned}$$

And so (summing (∗) over j and (∗∗) over i) we get

$$ekl \leqslant \sum_{i=0}^{l-1} \sum_{j=1}^{k} (b_{ij} - a_{ij}) \leqslant ekl.$$

So equality holds in (∗) and (∗∗), and in particular we have $b_{ij} = \beta_i + e$ for $i \geqslant l$ and $j \leqslant k$, as required. □

3 Core blocks

In this section, we introduce core blocks of Ariki–Koike algebras, giving several equivalent definitions. For the rest of this paper $q, Q_1, \dots, Q_r$ are fixed, and we assume that there are integers $a_1, \dots, a_r$ such that $Q_i = q^{a_i}$ for each i. Let $\mathcal{H}_n$ be the Ariki–Koike algebra with these parameters.

3.1 The definition of a core block

In order to introduce core blocks, we need to consider separately the case $e = \infty$; in this case, every combinatorial block of $\mathcal{H}_n$ will be a core block. For the case where e is finite, the definition is given by the equivalent statements in the following theorem. It is straightforward to check that these statements, appropriately re-phrased, all hold for every combinatorial block of $\mathcal{H}_n$ when $e = \infty$, with property (4) following from Proposition 1.3.

Theorem 3.1. *Suppose that e is finite, and that $\boldsymbol{\lambda}$ is a multipartition lying in a combinatorial block B of $\mathcal{H}_n$. The following are equivalent.*

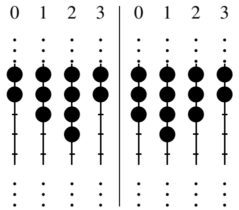
1. *$\boldsymbol{\lambda}$ is a multicore, and there exist a multi-charge $\mathbf{a} = (a_1, \dots, a_r)$ and integers $\alpha_0, \dots, \alpha_{e-1}$ such that for each i, j, $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})$ equals either α_i or $\alpha_i + e$.*
2. *$\boldsymbol{\lambda}$ is a multicore, and there exist a multi-charge $\mathbf{a} = (a_1, \dots, a_r)$ and integers $s_1, \dots, s_r$ such that*
$$\frac{\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}) - \mathfrak{b}^{\mathbf{a}}_{ik}(\boldsymbol{\lambda})}{e} \leqslant s_j - s_k + 1$$
for all $i \in \{0, \dots, e-1\}$, $j, k \in \{1, \dots, r\}$.
3. *$\boldsymbol{\lambda}$ is a multicore, and for any multi-charge $\mathbf{a} = (a_1, \dots, a_r)$ there exist integers $s_1, \dots, s_r$ such that*
$$\frac{\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}) - \mathfrak{b}^{\mathbf{a}}_{ik}(\boldsymbol{\lambda})}{e} \leqslant s_j - s_k + 1$$
for all $i \in \{0, \dots, e-1\}$, $j, k \in \{1, \dots, r\}$.
4. *There is no combinatorial block of any $\mathcal{H}_m$ with the same hub as B and smaller weight.*
5. *Every multipartition in B is a multicore.*

Now we can make the main definition of this paper.

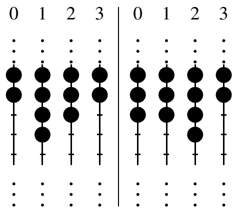
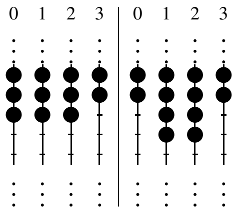
Definition. Suppose B is a combinatorial block of $\mathcal{H}_n$. Then we say that B is a *core block* if and only if either

- e is finite and the equivalent conditions of Theorem 3.1 are satisfied for any multipartition $\boldsymbol{\lambda}$ in B, or
- $e = \infty$.

Example. Suppose $r = 2$, $e = 4$, $Q_1 = q^3$, $Q_2 = 1$, and consider the combinatorial block B of $\mathcal{H}_8$ containing the bipartition $((4, 1^2), (2))$. Choosing the multi-charge $(3, 4)$, we get an abacus display



for this bipartition. So we may take $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (-4, 1, 2, -1)$, and we find that B is a core block. The other bipartitions in B are $((3, 1^2), (3))$ and $(\varnothing, (3^2, 1^2))$, with abacus displays


,

.

In order to prove Theorem 3.1, we need some preliminary results. First we observe that from the integers $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})$ we may recover the multi-charge $\mathbf{a}$ and the hub of $\boldsymbol{\lambda}$. The proof of the following lemma is straightforward—for (1), recall that for sufficiently large N, the number of beta-numbers for $\lambda^{(j)}$ which are greater than or equal to $-N$ is $a_j + N$.

Lemma 3.2. *Suppose e is finite, that $\boldsymbol{\lambda}$ is a multicore and that $\mathbf{a} = (a_1, \dots, a_r)$ is a multi-charge. Then*

1.

$$a_j = \frac{\sum_{i=0}^{e-1} \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})}{e} + \frac{e+1}{2}$$

for $j = 1, \dots, r$;

2.

$$\delta_i(\boldsymbol{\lambda}) = \frac{\sum_{j=1}^r \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}) - \sum_{j=1}^r \mathfrak{b}^{\mathbf{a}}_{(i-1)j}(\boldsymbol{\lambda}) - r}{e}$$

for $i = 0, \dots, e-1$ (reading the subscript $i-1$ modulo e).

Proposition 3.3. *Suppose e is finite, and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multicores with the same hub. Suppose that:*

- *there exist a multi-charge $\mathbf{a} = (a_1, \dots, a_r)$ and integers $\alpha_0, \dots, \alpha_{e-1}$ such that $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}) \in \{\alpha_i, \alpha_i + e\}$, for each i, j;*
- *there exist a multi-charge $\mathbf{b} = (b_1, \dots, b_r)$ and integers $\beta_0, \dots, \beta_{e-1}$ such that $\mathfrak{b}^{\mathbf{b}}_{ij}(\boldsymbol{\mu}) \in \{\beta_i, \beta_i + e\}$, for each i, j.*

Then $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu}) \in \{\alpha_i, \alpha_i + e\}$, for each i, j.

Proof: Let A be the matrix with entries $a_{ij} = \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})$, and let B be the matrix with entries $b_{ij} = \mathfrak{b}^{\mathbf{b}}_{ij}(\boldsymbol{\mu})$. We wish to use Proposition 2.5, so we need to verify the hypotheses of that proposition concerning the row and column sums of A and B.

Take $j \in \{1, \dots, r\}$. The fact that a_j and b_j are congruent modulo e means that

$$\frac{\sum_{i=0}^{e-1} a_{ij}}{e} + \frac{e+1}{2} \equiv \frac{\sum_{i=0}^{e-1} b_{ij}}{e} + \frac{e+1}{2} \pmod{e}$$

by Lemma 3.2(1), so that

$$\sum_{i=0}^{e-1} a_{ij} \equiv \sum_{i=0}^{e-1} b_{ij} \pmod{e^2}.$$

Now we look at row sums. Using Lemma 3.2(2), the fact that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub implies that

$$\sum_{j=1}^r a_{ij} - \sum_{j=1}^r a_{(i-1)j} = \sum_{j=1}^r b_{ij} - \sum_{j=1}^r b_{(i-1)j}$$

for all i, so that there is a constant K such that

$$\sum_{j=1}^r b_{ij} = \sum_{j=1}^r a_{ij} + K$$

for all i.

By Proposition 2.5 there is a matrix C with entries constant down each column such that any entry in row i of $B + C$ is equal to α_i or $\alpha_i + e$, and such that $B + C$ has the same row sums and the same column sums as A. Clearly the entries of C are all divisible by e, i.e. there is a vector $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{Z}^r$ such that the entries of any row of C are $(ec_1, \dots, ec_r)$. So (by the comment following the definition of the integers $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})$) the (i,j) entry of $B+C$ is $\mathfrak{b}^{\mathbf{b}+e\mathbf{c}}_{ij}(\boldsymbol{\mu})$, for each i,j. But the fact that the column sums of $B+C$ equal the column sums of A means that we have $b_j + ec_j = a_j$ for each j, by Lemma 3.2(1). Hence $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu}) = (B+C)_{ij} = \alpha_i$ or $\alpha_i + e$, for any i,j. □

Proposition 3.4. *Suppose e is finite, and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions satisfying the hypotheses of Proposition 3.3. Then $w(\boldsymbol{\lambda}) = w(\boldsymbol{\mu})$.*

Proof: Let $\mathbf{a} = (a_1, \dots, a_r)$ and $\alpha_0, \dots, \alpha_{e-1}$ be as in Proposition 3.3, and write $a_{ij} = \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})$ and $b_{ij} = \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu})$. Then a_{ij} and b_{ij} each equal either α_i or $\alpha_i + e$, for each i,j. Furthermore, the row and column sums of $A = (a_{ij})$ equal the corresponding row and column sums of $B = (b_{ij})$. Define

$$\hat{a}_{ij} = \frac{a_{ij} - \alpha_i}{e}, \quad \hat{b}_{ij} = \frac{b_{ij} - \alpha_i}{e}.$$

Then the matrices $\hat{A} = (\hat{a}_{ij})$ and $\hat{B} = (\hat{b}_{ij})$ are 0–1 matrices with corresponding row and column sums equal. So by Proposition 2.4 there is a sequence $\hat{A} = A^0 \leftrightarrow A^1 \leftrightarrow \cdots \leftrightarrow A^t = \hat{B}$. If we let $\hat{a}^k_{ij}$ be the i,j entry of A^k and define $a^k_{ij} = e\hat{a}^k_{ij} + \alpha_i$, then we have $a^k_{ij} = \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}_k)$ for some multipartition $\boldsymbol{\lambda}_k$ (with $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}$, $\boldsymbol{\lambda}_t = \boldsymbol{\mu}$). Moreover, the relation $A^{k-1} \leftrightarrow A^k$ means that $\boldsymbol{\lambda}_k = s^{jm}_{il}(\boldsymbol{\lambda}_{k-1})$ for some i,l,j,m with $\gamma^{jm}_{il}(\boldsymbol{\lambda}_{k-1}) = 2$. By Proposition 1.6, this means that $w(\boldsymbol{\lambda}_k) = w(\boldsymbol{\lambda}_{k-1})$. □

Proof of Theorem 3.1. Trivially (3)⇒(2), and it is easy to see that (2)⇒(3): suppose the sequence $(s_1, \dots, s_r)$ satisfies the given condition for the multi-charge $\mathbf{a} = (a_1, \dots, a_r)$. Then for any other multi-charge $\mathbf{a}' = (a_1 + ex_1, \dots, a_r + ex_r)$, the sequence $(s_1 + x_1, \dots, s_r + x_r)$ will work.

The preceding paragraph also shows that we may choose the multi-charge $\mathbf{a}$ and the integers $s_1, \dots, s_r$ in (2) in such a way that $s_1 = \cdots = s_r$. If we do this, and then set $\alpha_i = \min\{\mathfrak{b}^{\mathbf{a}}_{ij} \mid j \in \{1, \dots, r\}\}$ for each i, then the inequalities for (1) will follow. So (2)⇒(1). To show that (1)⇒(2), we use the chosen multi-charge and put $s_1 = \cdots = s_r = 0$.

It is also straightforward to show that (4) and (5) are equivalent. Suppose first that (4) is false, so there is a multipartition $\boldsymbol{\mu}$ in a combinatorial block C with the same hub as B but smaller weight. By Proposition 1.3, we have $w(C) = w(B) - ar$ for some positive integer a. If we add a rim ae-hook to the Young diagram of $\boldsymbol{\mu}$, then we shall have a multipartition in B, by Proposition 1.4. This is not a multicore, and so (5) is false. So (5)⇒(4). To show that (4)⇒(5) is even easier: if there is a multipartition $\boldsymbol{\mu}$ in B which is not a multicore, then we may remove a rim e-hook from the Young diagram of $\boldsymbol{\mu}$ to get a multipartition with the same hub and smaller weight.

The hard part, then, is to show that (1), (2) and (3) are equivalent to (4) and (5). First we show that (4)⇒(2), for which we use Proposition 2.2. Certainly (4) implies that $\boldsymbol{\lambda}$ is a multicore, because otherwise we could remove a rim e-hook from $[\boldsymbol{\lambda}]$ to get a multipartition of lower weight. Choose a multi-charge $\mathbf{a} = (a_1, \dots, a_r)$, and then for $i = 0, \dots, e-1$ define $x^{(i)} = (\mathfrak{b}^{\mathbf{a}}_{i1}(\boldsymbol{\lambda}), \dots, \mathfrak{b}^{\mathbf{a}}_{ir}(\boldsymbol{\lambda}))$. Let $X(\boldsymbol{\lambda})$ be the multiset $\{x^{(0)}, \dots, x^{(e-1)}\}$, regarded as a multisubset of the weight lattice L_r. It is straightforward to see that the condition that $X(\boldsymbol{\lambda})$ is tight is exactly the condition $\gamma_{il}^{jk}(\boldsymbol{\lambda}) \leqslant 2$ for all i, l, j, k. Moreover, if Y is a multiset such that $Y \equiv X(\boldsymbol{\lambda})$, then $Y = X(\boldsymbol{\mu})$ for some multipartition $\boldsymbol{\mu} \in B$: for the condition

$$(x_k - x_l) - (y_k - y_l) = 2$$

for some $x, y \in X$ is exactly the condition $\gamma_{ij}^{kl}(\boldsymbol{\lambda}) = 2$ for some i, j; and replacing x and y with $x - e_k + e_l$ and $y + e_k - e_l$ corresponds to replacing $\boldsymbol{\lambda}$ with $s_{ij}^{kl}(\boldsymbol{\lambda})$, which has the same weight as $\boldsymbol{\lambda}$ and so lies in the same combinatorial block. Now it easy to see that condition (4) implies that $X(\boldsymbol{\lambda})$ is ultra-tight: for if there is $Y \equiv X(\boldsymbol{\lambda})$ which is not tight, then we have $Y = X(\boldsymbol{\mu})$ for some $\boldsymbol{\mu}$ in B, and $\gamma_{ij}^{kl}(\boldsymbol{\mu}) \geqslant 3$ for some i, j, k, l. But then the multipartition $s_{ij}^{kl}(\boldsymbol{\mu})$ has the same hub as $\boldsymbol{\mu}$ but smaller weight, contradicting (4). So if we assume (4), then $X(\boldsymbol{\lambda})$ is ultra-tight, and so by Proposition 2.2 and the comment following it, we find that $X(\boldsymbol{\lambda})$ is a multisubset of $N(s)$ for some s, and this gives condition (2).

Finally, we show that (1)⇒(4). Suppose (1) holds for $\boldsymbol{\lambda}$, with $\mathfrak{b}_{ij}^{\mathbf{a}}(\boldsymbol{\lambda}) = \alpha_i$ or $\alpha_i + e$ for each i, j. Suppose also that (4) is false, and take a multipartition $\boldsymbol{\mu}$ in a combinatorial block C of minimal weight having the same hub as $\boldsymbol{\lambda}$. Then condition (4) holds for $\boldsymbol{\mu}$ and C, and so (since (4)⇒(2)⇒(1)) we can find $\mathbf{b} = (b_1, \dots, b_r)$ and integers $\beta_0, \dots, \beta_{e-1}$ such that $\mathfrak{b}_{ij}^{\mathbf{b}}(\boldsymbol{\mu}) = \beta_i$ or $\beta_i + e$ for each i, j. By Proposition 3.4 we have $w(\boldsymbol{\mu}) = w(\boldsymbol{\lambda})$, which is a contradiction. □

3.2 The multipartitions in a core block

Theorem 3.1 gives us several equivalent conditions for a multipartition to lie in a core block. As a corollary, we can give a simple description of all the multipartitions lying in a given core block; this will be useful later. We give separate statements for the cases $e < \infty$ and $e = \infty$; this is an artefact of our notation, and the results are really the same in substance.

Proposition 3.5. *Suppose that e is finite, that $\boldsymbol{\lambda}$ is a multipartition lying in a core block B, and that $\mathbf{a}$ and $\alpha_0, \dots, \alpha_{e-1}$ are chosen so that $\mathfrak{b}_{ij}^{\mathbf{a}}(\boldsymbol{\lambda}) \in \{\alpha_i, \alpha_i + e\}$ for each i, j. Then the multipartitions lying in B are precisely those multicores $\boldsymbol{\mu}$ for which:*

- $\mathfrak{b}_{ij}^{\mathbf{a}}(\boldsymbol{\mu}) \in \{\alpha_i, \alpha_i + e\}$ *for each i, j;*
- *for each i,*

$$\sum_{j=1}^{r} \mathfrak{b}_{ij}^{\mathbf{a}}(\boldsymbol{\mu}) = \sum_{j=1}^{r} \mathfrak{b}_{ij}^{\mathbf{a}}(\boldsymbol{\lambda}),$$

i.e. the number of $\mathfrak{b}_{ij}^{\mathbf{a}}(\boldsymbol{\mu})$ equal to $\alpha_i + e$ equals the number of $\mathfrak{b}_{ij}^{\mathbf{a}}(\boldsymbol{\lambda})$ equal to $\alpha_i + e$.

Proof: First suppose that $\boldsymbol{\mu}$ does satisfy the given conditions. The second condition implies that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub, by Lemma 3.2(2). Now Proposition 3.4 shows that they have the same weight, and so they lie in the same combinatorial block.

Conversely, suppose $\boldsymbol{\mu}$ lies in B. Then $\boldsymbol{\mu}$ lies in a core block and has the same hub as $\boldsymbol{\lambda}$, so the hypotheses of Proposition 3.3 are satisfied. So we have $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu}) = \alpha_i$ or $\alpha_i + e$ for each i, j; furthermore, the fact that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub means that there is a constant K such that

$$\sum_{j=1}^{r} \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu}) = \sum_{j=1}^{r} \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}) + K$$

for each i. By Lemma 3.2(1) we have

$$\sum_{j=1}^{r} \sum_{i=0}^{e-1} \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu}) = \sum_{j=1}^{r} \sum_{i=0}^{e-1} \mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda}),$$

which gives $K = 0$. □

Proposition 3.6. *Suppose $e = \infty$, and that $\boldsymbol{\lambda}$ is a multipartition lying in a combinatorial block B of $\mathcal{H}_n$. Then the multipartitions in B are precisely those multipartitions $\boldsymbol{\mu}$ for which*

$$\sum_{j=1}^{r} \mathfrak{B}_{ij}(\boldsymbol{\mu}) = \sum_{j=1}^{r} \mathfrak{B}_{ij}(\boldsymbol{\lambda})$$

for all $i \in \mathbb{Z}$.

Proof: Note that the sums $\mathfrak{B}_{i*}(\boldsymbol{\lambda}) = \mathfrak{B}_{i1}(\boldsymbol{\lambda}) + \cdots + \mathfrak{B}_{ir}(\boldsymbol{\lambda})$ determine the hub of $\boldsymbol{\lambda}$: we have

$$\delta_i(\boldsymbol{\lambda}) = \mathfrak{B}_{i*}(\boldsymbol{\lambda}) - \mathfrak{B}_{(i-1)*}(\boldsymbol{\lambda}).$$

Also, the hub of $\boldsymbol{\lambda}$ determines the $\mathfrak{B}_{i*}(\boldsymbol{\lambda})$: from the above equation, the $\delta_i(\boldsymbol{\lambda})$ determine these sums up to addition of a constant, and we have $\mathfrak{B}_{i*}(\boldsymbol{\lambda}) = 0$ for sufficiently large i.

So $\boldsymbol{\mu}$ satisfies the condition given in the proposition if and only if $\boldsymbol{\mu}$ has the same hub as $\boldsymbol{\lambda}$; but this happens if and only if $\boldsymbol{\mu}$ lies in the same combinatorial block as $\boldsymbol{\lambda}$, by Proposition 1.3. □

3.3 Elementary moves

In this section, we study the relationship between two multipartitions lying in the same combinatorial block of $\mathcal{H}_n$. In the case of Iwahori–Hecke algebras of type A, we know that, given two partitions λ and μ lying in the same combinatorial block, we may get from one to the other by a sequence of simple ‘moves’, i.e. addition and removal of rim e-hooks in the Young diagram. Furthermore, by first removing and then adding rim hooks, we may guarantee that the intermediate partitions have weight equal to or less than the common weight of λ and μ. We want to prove a similar result for multipartitions: that one may get from a multipartition to any other multipartition in the same combinatorial block by a sequence of 'elementary moves', without going via any multipartition of higher weight. Given that there are combinatorial blocks containing more than one partition when $e = \infty$, it is clear that addition and removal of rim e-hooks will not suffice; we must use the functions s_{ij}^{kl} as well.

Recall that the hub and weight of a combinatorial block B of $\mathcal{H}_n$ determine B. Moreover, condition (4) of Theorem 3.1 implies that, of the combinatorial blocks with a given hub, only the one with the smallest weight is a core block. So if $\boldsymbol{\lambda}$ is a partition with this hub, then we may speak of this core block as the *core block of* $\boldsymbol{\lambda}$. It seems that this is as close as we can get to a generalisation of the core of a partition.

Given multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we write $\boldsymbol{\lambda} \leftrightsquigarrow \boldsymbol{\mu}$ (and say that $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by an *elementary move*) if one of the following holds:

- $[\boldsymbol{\mu}]$ is obtained from $[\boldsymbol{\lambda}]$ by adding or removing a rim e-hook;
- $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are both multicores, and $\boldsymbol{\mu} = s_{ij}^{kl}(\boldsymbol{\lambda})$ for some i, j, k, l.

Proposition 3.7. *Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions lying in the same combinatorial block of $\mathcal{H}_n$. Then there is a sequence $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_t = \boldsymbol{\mu}$ of multipartitions such that $\boldsymbol{\lambda}_{i-1} \leftrightsquigarrow \boldsymbol{\lambda}_i$ for each i, and $w(\boldsymbol{\lambda}_i) \leqslant w(\boldsymbol{\lambda})$ for each i.*

Proof: We prove two statements:

1. if $\boldsymbol{\lambda}$ is a multipartition not lying in a core block of $\mathcal{H}_n$, then there is a sequence $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_t$ of multipartitions such that $\boldsymbol{\lambda}_{i-1} \leftrightsquigarrow \boldsymbol{\lambda}_i$ for each i, $w(\boldsymbol{\lambda}_i) \leqslant w(\boldsymbol{\lambda})$ for each i and $w(\boldsymbol{\lambda}_t) < w(\boldsymbol{\lambda})$;
2. if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions lying in a core block B, then there is a sequence $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_t = \boldsymbol{\mu}$ of multipartitions in B such that $\boldsymbol{\lambda}_{i-1} \leftrightsquigarrow \boldsymbol{\lambda}_i$ for each i.

It is clear that these two statements will imply the theorem: using (1) repeatedly, we can get from $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ to two multipartitions in the core block of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ using elementary moves, without passing multipartitions of higher weight. We can then pass between these two multipartitions using (2).

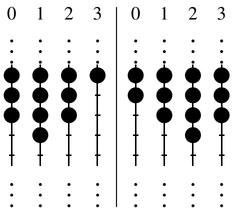
First we prove (1); the assertion that $\boldsymbol{\lambda}$ does not lie in a core block means that $e < \infty$. If $\boldsymbol{\lambda}$ is not a multicore, then we may take $t = 1$, removing a rim e-hook from the Young diagram of $\boldsymbol{\lambda}$ to get $\boldsymbol{\lambda}_1$. So we suppose that $\boldsymbol{\lambda}$ is a multicore, and we use Proposition 2.2. As in the proof that (4)⇒(2) in Theorem 3.1, we choose a multi-charge $\mathbf{a}$, and for each $i = 0, \ldots, e-1$ we set $x^{(i)} = (\mathfrak{b}_{i1}^{\mathbf{a}}, \ldots, \mathfrak{b}_{ir}^{\mathbf{a}})$. We regard the multiset $X(\boldsymbol{\lambda}) = \{x^{(0)}, \ldots, x^{(e-1)}\}$ as a multisubset of the weight lattice L_r, and we assert that $X(\boldsymbol{\lambda})$ is not ultra-tight. Indeed, if $X(\boldsymbol{\lambda})$ is ultra-tight, then Proposition 2.2 implies that $X(\boldsymbol{\lambda})$ is a multisubset of $N(s)$ for some s; but then condition (2) of Theorem 3.1 holds, contradicting the fact that $\boldsymbol{\lambda}$ does not lie in a core block. So there is some $Y \equiv X(\boldsymbol{\lambda})$ which is not tight. We have $Y = X(\boldsymbol{\mu})$ for some $\boldsymbol{\mu}$ in the same combinatorial block as $\boldsymbol{\lambda}$, and the definition of the relation $\equiv$ means that we can get from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu}$ via a sequence $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 \leftrightsquigarrow \cdots \leftrightsquigarrow \boldsymbol{\lambda}_{t-1} = \boldsymbol{\mu}$ of elementary moves, with all the $\boldsymbol{\lambda}_i$ in the same combinatorial block as $\boldsymbol{\lambda}$. The fact that $X(\boldsymbol{\mu})$ is not tight means that $\gamma_{ij}^{kl}(\boldsymbol{\mu}) \geqslant 3$ for some i, j, k, l. Now we define $\boldsymbol{\lambda}_t = s_{ij}^{kl}(\boldsymbol{\mu})$, and we have $w(\boldsymbol{\lambda}_t) < w(\boldsymbol{\lambda})$.

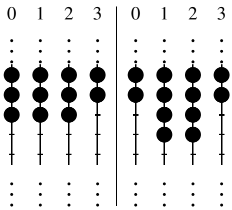
Now we prove (2), supposing first that e is finite. Obviously $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are both multicores, and by Proposition 3.5 we may choose $\mathbf{a}$ and integers $\alpha_0, \ldots, \alpha_{e-1}$ so that

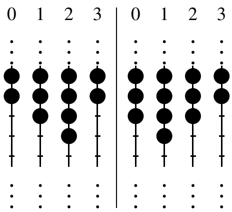
$\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})$ equals α_i or $\alpha_i + e$ and $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu})$ equals α_i or $\alpha_i + e$ for each i, j. Now we let $\boldsymbol{\lambda}_0, \dots, \boldsymbol{\lambda}_t$ be as in the proof of Proposition 3.4.

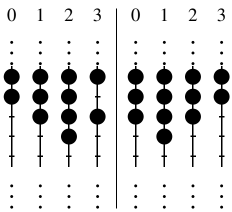
If $e = \infty$, then (2) follows by a very similar application of Proposition 2.4, using the integers $\mathfrak{B}_{ij}(\boldsymbol{\lambda})$. □

Example. Suppose $r = 2$, $e = 4$, $Q_1 = q^3$, $Q_2 = 1$, $\boldsymbol{\lambda} = ((3, 1^3), (3, 1^3))$ and $\boldsymbol{\mu} = ((4, 2^3), (2))$. Then it is easily checked that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the same combinatorial block of $\mathcal{H}_{12}$. The core block of $\boldsymbol{\lambda}$ and of $\boldsymbol{\mu}$ is the combinatorial block B of $\mathcal{H}_8$ described in the last example. We have $\boldsymbol{\lambda} \leftrightsquigarrow \boldsymbol{\lambda}_1 \leftrightsquigarrow \boldsymbol{\lambda}_2 \leftrightsquigarrow \boldsymbol{\mu}$, where

$\boldsymbol{\lambda} = ((3, 1^3), (3, 1^3)) =$  ,

$\boldsymbol{\lambda}_1 = (\varnothing, (3^2, 1^2)) =$  ,

$\boldsymbol{\lambda}_2 = ((4, 1^2), (2)) =$  ,

$\boldsymbol{\mu} = ((4, 2^3), (2)) =$  .

3.4 Every core block occurs for $e = \infty$

Suppose B is a core block of $\mathcal{H}_n$, with $e < \infty$, and suppose that $\mathbb{F}$ contains elements of infinite multiplicative order. The aim of this section is to show that there is an Ariki–Koike algebra $\check{\mathcal{H}}_n$ over the same field, with parameters $\check{q}, \check{Q}_1, \dots, \check{Q}_r$, with $\check{q}$ being of infinite order, such that B is also a combinatorial block of $\check{\mathcal{H}}_n$. By this we mean that there is a combinatorial block $\check{B}$ of $\check{\mathcal{H}}_n$ containing exactly the same multipartitions as B, that the $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-weight of $\check{B}$ equals the $(q; Q_1, \dots, Q_r)$-weight of B and that a multipartition $\boldsymbol{\lambda}$ in B is $(q; Q_1, \dots, Q_r)$-Kleshchev if and only if it is $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-Kleshchev. It is tempting to speculate that B and $\check{B}$ have similar structure—perhaps even that they are isomorphic, with compatible isomorphisms between corresponding Specht modules—but we leave such issues for a future paper, and restrict attention to combinatorics here.

Suppose, then, that e is finite and that B is a core block of $\mathcal{H}_n$. We choose and fix a multi-charge $\mathbf{a} = (a_1, \dots, a_r)$ and integers $\alpha_0, \dots, \alpha_{e-1}$ such that $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\lambda})$ equals α_i or $\alpha_i + e$ for each multipartition $\boldsymbol{\lambda}$ in B and each i, j. Let $\check{q}$ be any element of $\mathbb{F}$ of infinite order, and let $\check{Q}_j = \check{q}^{a_j}$ for $j = 1, \dots, r$. Now let $\check{\mathcal{H}}_n$ be the Ariki–Koike algebra over $\mathbb{F}$ with parameters $\check{q}, \check{Q}_1, \dots, \check{Q}_r$.

Proposition 3.8. *There is a combinatorial block $\check{B}$ of $\check{\mathcal{H}}_n$ such that a multipartition $\boldsymbol{\lambda}$ lies in $\check{B}$ if and only if it lies in B.*

Proof: It is clear from the definition of residues that if two multipartitions lie in the same combinatorial block of $\check{\mathcal{H}}_n$, then they lie in the same combinatorial block of $\mathcal{H}_n$. So we need to show that all the multipartitions in B lie in the same combinatorial block of $\check{\mathcal{H}}_n$. Given $\boldsymbol{\lambda}$ in B, define

$$b_k = \frac{\sum_{j=1}^{r} \mathfrak{b}^{\mathbf{a}}_{kj}(\boldsymbol{\lambda}) - r\alpha_k}{e};$$

that is, b_k is the number of j for which $\mathfrak{b}^{\mathbf{a}}_{kj}(\boldsymbol{\lambda})$ equals $\alpha_k + e$. Now construct the abacus display for $\boldsymbol{\lambda}$ with $e = \infty$ using the multi-charge $\mathbf{a}$, and for each integer l let d_l be the number of $j \in \{1, \dots, r\}$ for which $\mathfrak{B}_{kj}(\boldsymbol{\lambda}) = 1$. It is straightforward the express the d_l in terms of the b_k; for any integer k, we write $\overline{k}$ for its image in $\mathbb{Z}/e\mathbb{Z}$. Since $\mathfrak{b}^{\mathbf{a}}_{kj}(\boldsymbol{\lambda})$ equals α_k or $\alpha_k + e$ for each k, j, we have

$$d_l = \begin{cases} r & (l < \alpha_{\overline{l}} + e) \\ b_{\overline{l}} & (l = \alpha_{\overline{l}} + e) \\ 0 & (l > \alpha_{\overline{l}} + e) \end{cases}.$$

Now the $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-hub of $\boldsymbol{\lambda}$ is given by $\delta_i(\boldsymbol{\lambda}) = d_i - d_{i-1}$; by Proposition 3.5, the integers b_k do not depend on the choice of $\boldsymbol{\lambda}$ in B, and so the $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-hub $(\delta_i(\boldsymbol{\lambda}))_{i\in\mathbb{Z}}$ does not depend on the choice of $\boldsymbol{\lambda}$ either. So by Proposition 1.3(2), all the multipartitions in B lie in the same combinatorial block of $\check{\mathcal{H}}_n$. □

Given combinatorial blocks B and $\check{B}$ as above, we say that $\check{B}$ is a *lift* of B. Clearly, choosing a lift corresponds to choosing an appropriate multi-charge. It is a straightforward exercise, using Propositions 1.5 and 1.7, to show that B and $\check{B}$ have the same weight. In fact, more is true: if $\boldsymbol{\lambda}$ is any multipartition in B and if $1 \leqslant j_1 < \cdots < j_s \leqslant r$, then the $(q; Q_{j_1}, \dots, Q_{j_s})$-weight of $(\lambda^{(j_1)}, \dots, \lambda^{(j_s)})$ equals its $(\check{q}; \check{Q}_{j_1}, \dots, \check{Q}_{j_s})$-weight.

Now we turn our attention to Kleshchev multipartitions.

Proposition 3.9. *Suppose B is a core block of $\mathcal{H}_n$, that $\check{B}$ is a lift of B and that $\boldsymbol{\lambda}$ is a multipartition in B. Then $\boldsymbol{\lambda}$ is $(q; Q_1, \dots, Q_r)$-Kleshchev if and only if it is $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-Kleshchev.*

Before we can prove Proposition 3.9, we need to examine the relationship between lifts and Scopes pairs. Suppose B is a core block of $\mathcal{H}_n$, and that $\check{B}$ is a lift of B. Given $k \in \mathbb{Z}/e\mathbb{Z}$, define $C = \Phi_k(B)$ as in Section 1.2.3, and define

$$\check{C} = \Bigg(\prod_{l \in \mathbb{Z} \mid \bar{l}=k} \Phi_l\Bigg)(\check{B}).$$

We remark that this is a valid definition: any two of the values $l \in \mathbb{Z}$ for which $\bar{l} = k$ differ by at least e, and so the corresponding operators Φ_l commute; moreover, only finitely many of these operators have a non-trivial effect on $\check{B}$.

Lemma 3.10. *Suppose $B, \check{B}, C, \check{C}$ are as above. Then C is a core block of $\mathcal{H}_{n-\delta_k(B)}$, and $\check{C}$ is a lift of C.*

Proof: It is clear that the set of multipartitions in $\check{C}$ equals the set of multipartitions in C: these are precisely the multipartitions which may be obtained from some multipartition in B by simultaneously adding all addable nodes of $(q; Q_1, \dots, Q_r)$-residue q^k and removing all removable nodes of $(q; Q_1, \dots, Q_r)$-residue q^k. So in order to prove that C is a core block with $\check{C}$ as a lift, it suffices to show that there are integers $\beta_0, \dots, \beta_{e-1}$ such that for each multipartition $\boldsymbol{\mu}$ in C we have $\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu}) \in \{\beta_i, \beta_i + e\}$ for each i, j.

Each $\boldsymbol{\mu}$ is of the form $\Phi_k(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda}$ in B, and so we have

$$\begin{aligned}\mathfrak{b}^{\mathbf{a}}_{ij}(\boldsymbol{\mu}) &= \mathfrak{b}^{\mathbf{a}}_{ij}\big(\Phi_k(\boldsymbol{\lambda})\big)\\ &= \phi_k\big(\mathfrak{b}^{\mathbf{a}}_{\phi_k(i)j}(\boldsymbol{\lambda})\big)\\ &\in \big\{\phi_k\big(\alpha_{\phi_k(i)}\big), \phi_k\big(\alpha_{\phi_k(i)} + e\big)\big\}\\ &= \big\{\phi_k\big(\alpha_{\phi_k(i)}\big), \phi_k\big(\alpha_{\phi_k(i)}\big) + e\big\}.\end{aligned}$$

□

Now we set up the inductive step of the proof of Proposition 3.9.

Lemma 3.11. *Suppose $B, \check{B}, C, \check{C}$ are as above, and that the integers $\alpha_0, \dots, \alpha_{e-1}$ can be chosen in such a way that $\alpha_k > \alpha_{k-1} + 1$. If $\boldsymbol{\lambda}$ is a multipartition in B, then:*

- *$\boldsymbol{\lambda}$ is $(q; Q_1, \dots, Q_r)$-Kleshchev if and only if $\Phi_k(\boldsymbol{\lambda})$ is $(q; Q_1, \dots, Q_r)$-Kleshchev;*
- *$\boldsymbol{\lambda}$ is $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-Kleshchev if and only if $\Phi_k(\boldsymbol{\lambda})$ is $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-Kleshchev.*

Proof: The fact that $\alpha_k > \alpha_{k-1} + 1$ means that $\mathfrak{b}^{\mathbf{a}}_{kj}(\boldsymbol{\lambda}) \geqslant \mathfrak{b}^{\mathbf{a}}_{(k-1)j}(\boldsymbol{\lambda}) + 1$ for each j, which implies that $\boldsymbol{\lambda}$ has no addable nodes of $(q; Q_1, \dots, Q_r)$-residue q^k, and therefore no addable nodes of $(\check{q}; \check{Q}_1, \ldots, \check{Q}_r)$-residue $\check{q}^l$ for any l with $\bar{l} = k$. The result now follows by Lemma 1.9. $\square$

The initial case of Proposition 3.9 deals with a particular type of core block, which we now describe. If B and $\check{B}$ are as above, then we define $\mathfrak{B}_{i*}(\check{B})$ to be the sum $\mathfrak{B}_{i1}(\boldsymbol{\lambda}) + \cdots + \mathfrak{B}_{ir}(\boldsymbol{\lambda})$ for any $\boldsymbol{\lambda}$ in $\check{B}$; by Proposition 3.6, these integers do not depend on the choice of $\boldsymbol{\lambda}$. Now we say that $\check{B}$ is *e-flat* if, for every i, l with $i - l \geqslant e$ we have either $\mathfrak{B}_{i*}(\check{B}) = 0$ or $\mathfrak{B}_{l*}(\check{B}) = r$.

Lemma 3.12. *Suppose $\check{B}$ is e-flat, and that $\boldsymbol{\lambda}$ is a multipartition in $\check{B}$. Then there is some $i \in \mathbb{Z}$ such that the $(\check{q}; \check{Q}_1, \ldots, \check{Q}_r)$-residue of any node in $[\boldsymbol{\lambda}]$ lies in $\{\check{q}^{i+1}, \check{q}^{i+2}, \ldots, \check{q}^{i+e-1}\}$, and the $(\check{q}; \check{Q}_1, \ldots, \check{Q}_r)$-residue of any addable node of $[\boldsymbol{\lambda}]$ lies in $\{\check{q}^{i}, \check{q}^{i+1}, \ldots, \check{q}^{i+e}\}$.*

Proof: The e-flat condition means that there is some $i \in \mathbb{Z}$ (independent of $\boldsymbol{\lambda}$) such that

$$\mathfrak{B}_{kj}(\boldsymbol{\lambda}) = \begin{cases} 1 & (\text{if } k < i) \\ 0 & (\text{if } k \geqslant i + e) \end{cases}$$

for any j, k. Furthermore, the definition of the abacus display guarantees that among $\mathfrak{B}_{ij}, \mathfrak{B}_{(i+1)j}, \ldots, \mathfrak{B}_{(i+e-1)j}$ there are exactly $(a_j - i)$ 1s and $(e - a_j + i)$ 0s. Hence $[\lambda^{(j)}] \subseteq [\nu^{(j)}]$, where $\nu^{(j)}$ is the partition with abacus display

$$\begin{array}{cccccccccccccccccccc} \cdots & i-3 & i-2 & i-1 & i & i+1 & \cdots & e-a_j+2i-2 & e-a_j+2i-1 & e-a_j+2i & e-a_j+2i+1 & \cdots & i+e-2 & i+e-1 & i+e & i+e+1 & i+e+2 & \cdots \\ \cdots & \bullet & \bullet & \bullet & | & | & \cdots & | & | & \bullet & \bullet & \cdots & \bullet & \bullet & | & | & | & \cdots \end{array},$$

i.e. the partition $((e - a_j + i)^{a_j - i})$. So every node of $[\lambda^{(j)}]$ is a node of $[\nu^{(j)}]$, and every addable node of $[\lambda^{(j)}]$ is a node or an addable node of $[\nu^{(j)}]$, and it is straightforward to check the conditions on the residues of the nodes and addable nodes of $[\nu^{(j)}]$. $\square$

Corollary 3.13. *Suppose B and $\check{B}$ are as above, with $\check{B}$ e-flat, and that $\boldsymbol{\lambda}$ is a multipartition in $\check{B}$. Suppose that $\boldsymbol{\mu}$ is a multipartition with $[\boldsymbol{\mu}] \subseteq [\boldsymbol{\lambda}]$, and that $\mathfrak{n}$ is a removable node of $[\boldsymbol{\mu}]$. Then $\mathfrak{n}$ is $(q; Q_1, \ldots, Q_r)$-good if and only if it is $(\check{q}; \check{Q}_1, \ldots, \check{Q}_r)$-good.*

Proof: The $(\check{q}; \check{Q}_1, \ldots, \check{Q}_r)$-residue of $\mathfrak{n}$ is $\check{q}^{i+d}$ for some $1 \leqslant d \leqslant e - 1$, and so its $(q; Q_1, \ldots, Q_r)$-residue is q^{i+d}. The lemma implies that a node or an addable node of $[\boldsymbol{\mu}]$ has $(\check{q}; \check{Q}_1, \ldots, \check{Q}_r)$-residue $\check{q}^{i+d}$ if and only if it has $(q; Q_1, \ldots, Q_r)$-residue q^{i+d}. So the $\check{q}^{i+d}$-signature of $[\boldsymbol{\mu}]$ coincides with the q^{i+d}-signature. $\square$

Proof of Proposition 3.9. We proceed by induction on n. First we suppose that $\check{B}$ is e-flat. Then the result follows easily using Corollary 3.13: if we are given

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(n), \boldsymbol{\lambda}(n-1), \dots, \boldsymbol{\lambda}(0) = \varnothing$$

where $\boldsymbol{\lambda}(i-1)$ is obtained from $\boldsymbol{\lambda}(i)$ by removing a node $\mathfrak{n}(i)$, then $\mathfrak{n}(i)$ is a $(q; Q_1, \dots, Q_r)$-good node of $\boldsymbol{\lambda}(i)$ if and only if it is a $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-good node. Hence $\boldsymbol{\lambda}$ is $(q; Q_1, \dots, Q_r)$-Kleshchev if and only if it is $(\check{q}; \check{Q}_1, \dots, \check{Q}_r)$-Kleshchev.

Next we suppose that $\check{B}$ is not e-flat. So there exist i, l with $i - l \geqslant e$ such that $\mathfrak{B}_{l*}(B) < r$ and $\mathfrak{B}_{i*}(B) > 0$. Since B is a core block, i is less than or equal to $\alpha_{\bar{\imath}} + e$, and l is at least $\alpha_{\bar{l}} + e$. The first consequence of this is that i and l cannot be congruent modulo e, so the interval $\{\bar{l}+1, \bar{l}+2, \dots, \bar{\imath}\}$ in $\mathbb{Z}/e\mathbb{Z}$ contains fewer than e elements; the second consequence is that

$$\alpha_{\bar{\imath}} - \alpha_{\bar{l}} \geqslant (i-e) - (l-e) \geqslant e.$$

Hence (since $\alpha_k - \alpha_{k-1} \equiv 1 \pmod{e}$ for every k) there must be some $k \in \{\bar{l}+1, \bar{l}+2, \dots, \bar{\imath}\}$ such that α_k exceeds α_{k-1} by at least $e+1$. Now we define $C = \Phi_k(B)$ as above, and the result follows by induction, using Lemma 3.10 and Lemma 3.11. □

3.5 Decomposable blocks

In this section, we examine a special type of combinatorial block of $\mathcal{H}_n$ which may be 'decomposed' into smaller combinatorial blocks. That is, for a combinatorial block B satisfying certain conditions, we can decompose B as a 'product' of B^J and B^K, where (J, K) is a partition of the set $\{1, \dots, r\}$, B^J is a combinatorial block of an Ariki–Koike algebra defined using the parameters $(Q_j \mid j \in J)$, and B^K is a combinatorial block of an Ariki–Koike algebra defined using $(Q_k \mid k \in K)$. Our results are purely combinatorial, but it seems likely that there is algebraic structure underlying them—a bold conjecture might be that B is Morita equivalent to the tensor product $B^J \otimes B^K$. This would in some sense be a generalisation of the main result of Dipper and Mathas [3].

Suppose B is a combinatorial block of $\mathcal{H}_n$, and $\boldsymbol{\lambda}$ is a multipartition in B. For any distinct j, k in $\{1, \dots, r\}$, we examine the bipartition $(\lambda^{(j)}, \lambda^{(k)})$, and calculate its weight using the parameters q, Q_j, Q_k. If J is a non-empty proper subset of $\{1, \dots, r\}$ and $K = \{1, \dots, r\} \setminus J$, then we say that $\boldsymbol{\lambda}$ is (J, K)-*decomposable* if $w((\lambda^{(j)}, \lambda^{(k)})) = 0$ for all $j \in J, k \in K$.

Proposition 3.14. *Suppose $\boldsymbol{\lambda}$ is (J, K)-decomposable. Then every multipartition in B is (J, K)-decomposable, and B is a core block of $\mathcal{H}_n$.*

This reduces to the following lemma.

Lemma 3.15. *Suppose $\boldsymbol{\lambda}$ is (J, K)-decomposable, and $\boldsymbol{\mu}$ is a multipartition in the same combinatorial block which is obtained from $\boldsymbol{\lambda}$ by an elementary move. Then $\boldsymbol{\mu}$ is (J, K)-decomposable.*

Proof: First we note that $\boldsymbol{\lambda}$ must be a multicore; for if $\lambda^{(j)}$ is not an e-core, and (say) $j \in J$, then $w((\lambda^{(j)}, \lambda^{(k)})) \geqslant 2$ for any $k \in K$. $\boldsymbol{\mu}$ is of the form $s^{hh'}_{gg'}(\boldsymbol{\lambda})$ for some $g, g' \in \mathbb{Z}/e\mathbb{Z}$ and $h, h' \in \{1, \dots, r\}$. The fact that $\boldsymbol{\mu}$ has the same weight as $\boldsymbol{\lambda}$ means that $\gamma^{hh'}_{gg'}(\boldsymbol{\lambda}) = 2$. This implies that h, h' both lie in J or both lie in K; we assume without loss that they both lie in J. We must prove that $\gamma^{jk}_{il}(\boldsymbol{\mu}) \leqslant 1$ for all $i, l \in \mathbb{Z}/e\mathbb{Z}$, $j \in J$ and $k \in K$. The definition of $\boldsymbol{\mu}$ shows that

$$\gamma^{jk}_{i}(\boldsymbol{\mu}) = \begin{cases} \gamma^{jk}_{i}(\boldsymbol{\lambda}) - 1 & (i = g,\ j = h) \\ \gamma^{jk}_{i}(\boldsymbol{\lambda}) + 1 & (i = g,\ j = h') \\ \gamma^{jk}_{i}(\boldsymbol{\lambda}) + 1 & (i = g',\ j = h) \\ \gamma^{jk}_{i}(\boldsymbol{\lambda}) - 1 & (i = g',\ j = h') \\ \gamma^{jk}_{i}(\boldsymbol{\lambda}) & (\text{otherwise}). \end{cases}$$

So (since by assumption $\gamma^{jk}_{il}(\boldsymbol{\lambda}) \leqslant 1$) we may assume that $j = h$ or $j = h'$. In fact, we assume $j = h$; the proof in the other case is similar. We may also assume that $i = g'$ or $l = g$, or both.

Now we have

$$2 = \gamma^{hh'}_{gg'}(\boldsymbol{\lambda}) = \gamma^{hk}_{gg'}(\boldsymbol{\lambda}) + \gamma^{kh'}_{gg'}(\boldsymbol{\lambda})$$

with $\gamma^{hk}_{gg'}(\boldsymbol{\lambda}), \gamma^{kh'}_{gg'}(\boldsymbol{\lambda}) \leqslant 1$. So in fact $\gamma^{hk}_{gg'}(\boldsymbol{\lambda}) = \gamma^{kh'}_{gg'}(\boldsymbol{\lambda}) = 1$. Hence

$$\gamma^{hk}_{g'g}(\boldsymbol{\mu}) = \gamma^{hk}_{g'g}(\boldsymbol{\lambda}) + 2 = 1.$$

Also, if $l \neq g, g'$, then

$$\begin{aligned} \gamma^{hk}_{g'l}(\boldsymbol{\mu}) &= \gamma^{hk}_{g'l}(\boldsymbol{\lambda}) + 1 \\ &= \gamma^{hk}_{g'g}(\boldsymbol{\lambda}) + \gamma^{hk}_{gl}(\boldsymbol{\lambda}) + 1 \\ &\leqslant -1 \qquad + 1 \qquad\quad + 1 \\ &= 1, \end{aligned}$$

and if $i \neq g, g'$ then

$$\begin{aligned} \gamma^{hk}_{ig}(\boldsymbol{\mu}) &= \gamma^{hk}_{ig}(\boldsymbol{\lambda}) + 1 \\ &= \gamma^{hk}_{g'g}(\boldsymbol{\lambda}) + \gamma^{hk}_{ig'}(\boldsymbol{\lambda}) + 1 \\ &\leqslant -1 \qquad + 1 \qquad\quad + 1 \\ &= 1. \end{aligned}$$

□

Proof of Proposition 3.14. Suppose $\boldsymbol{\mu}$ is another multipartition in B. By Proposition 3.7, we can get from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu}$ via a sequence $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 \leftrightsquigarrow \cdots \leftrightsquigarrow \boldsymbol{\lambda}_t = \boldsymbol{\mu}$ of elementary moves, with $w(\boldsymbol{\lambda}_1) \leqslant w(\boldsymbol{\lambda})$. We prove that $\boldsymbol{\mu}$ is (J, K)-decomposable by induction on t, with the case $t = 0$ being vacuous.

As noted in the proof of Lemma 3.15, $\boldsymbol{\lambda}$ must be a multicore, so that $\boldsymbol{\lambda}_1$ is a multicore of the form $s^{hh'}_{gg'}(\boldsymbol{\lambda})$ for some $g, g' \in \mathbb{Z}/e\mathbb{Z}$ and $h, h' \in \{1, \dots, r\}$. The fact that $\boldsymbol{\lambda}$ is (J, K)-decomposable means that $\gamma^{jk}_{gg'}(\boldsymbol{\lambda}) \leqslant 1$ whenever $j \in J$ and $k \in K$, and this implies that $\gamma^{hh'}_{gg'}(\boldsymbol{\lambda}) \leqslant 2$ for any $h, h' \in \{1, \dots, r\}$. So by Proposition 1.6, we have $w(\boldsymbol{\lambda}_1) \geqslant w(\boldsymbol{\lambda})$, and so $w(\boldsymbol{\lambda}_1) = w(\boldsymbol{\lambda})$. $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_1$ have the same hub, and hence $\boldsymbol{\lambda}_1$ lies in B. By Lemma 3.15 $\boldsymbol{\lambda}_1$ is (J, K)-decomposable, and so by induction (replacing $\boldsymbol{\lambda}$ with $\boldsymbol{\lambda}_1$) $\boldsymbol{\mu}$ is (J, K)-decomposable.

So every multipartition in B is (J, K)-decomposable. This implies in particular that every multipartition in B is a multicore, and so B is a core block. □

In view of Proposition 3.14, we may say that a (core) block B is (J, K)-decomposable, meaning that any multipartition in B is (J, K)-decomposable. By the comments concerning weight following the proof of Proposition 3.8, if B is (J, K)-decomposable and $e < \infty$, then a lift of B is also (J, K)-decomposable. The aim of the rest of this section is to describe the set of multipartitions in a decomposable combinatorial block, and to describe which of them are Kleshchev.

Suppose that $\boldsymbol{\lambda}$ is (J, K)-decomposable, and suppose that $J = \{j_1 < \dots < j_s\}$, $K = \{k_1 < \dots < k_t\}$. Let $\boldsymbol{\lambda}^J$ be the multipartition $(\lambda^{(j_1)}, \dots, \lambda^{(j_s)})$ with s components. For any p, let $\mathcal{H}^J_p$ be the Ariki–Koike algebra with parameters $q, Q_{j_1}, \dots, Q_{j_s}$. Define $\boldsymbol{\lambda}^K$ and $\mathcal{H}^K_p$ similarly. We abuse notation by regarding a node of $[\boldsymbol{\lambda}]$ as a node of $[\boldsymbol{\lambda}^J]$ or $[\boldsymbol{\lambda}^K]$ in the obvious way, and when we speak of the residue of a node of $[\boldsymbol{\lambda}^J]$, we calculate this using the parameters $q, Q_{j_1}, \dots, Q_{j_s}$ (so that the residue will be the same as that of the corresponding node of $[\boldsymbol{\lambda}]$).

Proposition 3.16. *If B is a (J, K)-decomposable combinatorial block, containing multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, then:*

- $|\boldsymbol{\lambda}^J| = |\boldsymbol{\mu}^J|$;
- *$\boldsymbol{\lambda}^J$ and $\boldsymbol{\mu}^J$ lie in the same combinatorial block of $\mathcal{H}^J_{|\boldsymbol{\lambda}^J|}$;*
- *$\boldsymbol{\lambda}^K$ and $\boldsymbol{\mu}^K$ lie in the same combinatorial block of $\mathcal{H}^K_{|\boldsymbol{\lambda}^K|}$.*

Proof: Using Proposition 3.7 and the fact that B is a core block, it suffices to consider the case where we can get from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu}$ by an elementary move. So assume we have $\boldsymbol{\mu} = s^{jj'}_{il}(\boldsymbol{\lambda})$ for some $j, j' \in \{1, \dots, r\}$ and $i, l \in \mathbb{Z}/e\mathbb{Z}$ with $\gamma^{jj'}_{il}(\boldsymbol{\lambda}) = 2$. This implies that j and k lie both in J or both in K; we assume without loss that they both lie in J, say $j = j_c$, $j' = j_d$. Then we have:

- $\boldsymbol{\mu}^J = s^{cd}_{il}(\boldsymbol{\lambda}^J)$, with $\gamma^{cd}_{il}(\boldsymbol{\lambda}^J) = 2$;
- $\boldsymbol{\mu}^K = \boldsymbol{\lambda}^K$.

The result follows. □

We see from Proposition 3.16 that a (J, K)-decomposable combinatorial block B of $\mathcal{H}_n$ defines an integer $p \in \{0, \dots, n\}$ and a pair (B^J, B^K), where B^J is a combinatorial block of $\mathcal{H}^J_p$ and B^K is a combinatorial block of $\mathcal{H}^K_{n-p}$; each multipartition $\boldsymbol{\lambda}$ in B corresponds to a pair of multipartitions $\boldsymbol{\lambda}^J$ in B^J and $\boldsymbol{\lambda}^K$ in B^K. We say that B *decomposes as the product of B^J and B^K*.

We now consider which multipartitions in B are Kleshchev.

Proposition 3.17. *Suppose B is a (J,K)-decomposable combinatorial block of $\mathcal{H}_n$, and that $\boldsymbol{\lambda}$ is a multipartition in B. Then $\boldsymbol{\lambda}$ is Kleshchev if and only if both $\boldsymbol{\lambda}^J$ and $\boldsymbol{\lambda}^K$ are Kleshchev.*

Proof: Given $i \in \mathbb{Z}/e\mathbb{Z}$, define $\overline{\boldsymbol{\lambda}}$ by removing all the normal nodes of residue q^i from $[\boldsymbol{\lambda}]$, and define $\overline{\boldsymbol{\lambda}^J}$ and $\overline{\boldsymbol{\lambda}^K}$ similarly. Then it suffices to prove that

$$\overline{\boldsymbol{\lambda}}\,\text{is}\,(J,K)\text{-decomposable, with}\quad \overline{\boldsymbol{\lambda}}^J = \overline{\boldsymbol{\lambda}^J}\quad\text{and}\quad \overline{\boldsymbol{\lambda}}^K = \overline{\boldsymbol{\lambda}^K}. \qquad (\ddagger)$$

For if $\boldsymbol{\lambda}$ is Kleshchev, then (assuming $n > 0$) there is some i such that $[\boldsymbol{\lambda}]$ has at least one normal node of residue q^i, and $\overline{\boldsymbol{\lambda}}$ is Kleshchev by Lemma 1.2. By induction and $(\ddagger)$, $\overline{\boldsymbol{\lambda}^J}$ and $\overline{\boldsymbol{\lambda}^K}$ are Kleshchev, and so $\boldsymbol{\lambda}^J$ and $\boldsymbol{\lambda}^K$ are Kleshchev. Conversely, if $\boldsymbol{\lambda}^J$ and $\boldsymbol{\lambda}^K$ are Kleshchev, then (assuming $n > 0$ and without loss of generality that $|\boldsymbol{\lambda}^J| > 0$) there is some i such that $[\boldsymbol{\lambda}^J]$ has at least one normal node of residue q^i, and $\overline{\boldsymbol{\lambda}^J}$ and $\overline{\boldsymbol{\lambda}^K}$ are Kleshchev. By induction this gives $\overline{\boldsymbol{\lambda}}$ Kleshchev, so that $\boldsymbol{\lambda}$ is Kleshchev.

So we prove $(\ddagger)$. If $\boldsymbol{\lambda}$ has no removable nodes of residue q^i, then $(\ddagger)$ is trivial, so we assume that there is at least one removable node. The fact that $\gamma^{jk}_{i(i-1)}(\boldsymbol{\lambda}) \leqslant 1$ for $j \in J$ and $k \in K$ means that if $[\boldsymbol{\lambda}^J]$ has removable nodes of residue q^i, then $[\boldsymbol{\lambda}^K]$ has no addable nodes of this residue; similarly, if $[\boldsymbol{\lambda}^K]$ has removable nodes of residue q^i, then $[\boldsymbol{\lambda}^J]$ has no addable nodes of this residue. So we are (without loss of generality) in one of two situations:

1. $[\boldsymbol{\lambda}]$ has no addable nodes of residue q^i;
2. $[\boldsymbol{\lambda}^K]$ has neither addable nor removable nodes of residue q^i.

In case 1, $\overline{\boldsymbol{\lambda}}$ is obtained simply by removing all removable nodes of residue q^i from $[\boldsymbol{\lambda}]$, and similarly for $\overline{\boldsymbol{\lambda}^J}$ and $\overline{\boldsymbol{\lambda}^K}$. So it is clear that $\overline{\boldsymbol{\lambda}}^J = \overline{\boldsymbol{\lambda}^J}$ and $\overline{\boldsymbol{\lambda}}^K = \overline{\boldsymbol{\lambda}^K}$, and we just need to show that $\overline{\boldsymbol{\lambda}}$ is (J,K)-decomposable. Since $[\boldsymbol{\lambda}]$ has no addable nodes of residue q^i, we have $\overline{\lambda}^{(j)} = \Phi_i(\lambda^{(j)})$ for each $j \in \{1,\dots,r\}$. And so for $j \in J, k \in K$, $l,m \in \mathbb{Z}/e\mathbb{Z}$ we have

$$\gamma^{jk}_{lm}(\overline{\boldsymbol{\lambda}}) = \gamma^{jk}_{\phi_i(l)\phi_i(m)}(\boldsymbol{\lambda}) \leqslant 1,$$

as required.

In case 2, the q^i-signature of $\boldsymbol{\lambda}$ coincides with that of $\boldsymbol{\lambda}^J$; in particular, the normal nodes of $[\boldsymbol{\lambda}]$ of residue q^i are precisely the normal nodes of $[\boldsymbol{\lambda}^J]$ of residue q^i. So again it is clear that $\overline{\boldsymbol{\lambda}}^J = \overline{\boldsymbol{\lambda}^J}$ and $\overline{\boldsymbol{\lambda}}^K = \overline{\boldsymbol{\lambda}^K}$, and we need to show that $\overline{\boldsymbol{\lambda}}$ is (J,K)-decomposable. Assuming we are not in case 1, $[\lambda^J]$ has at least one addable node of residue q^i. The fact that $\gamma^{jj'}_{i(i-1)} \leqslant 2$ for all j, j' implies that each $[\lambda^{(j)}]$ has at most one removable node of residue q^i. Hence for each $j \in J$, we have either $\overline{\lambda}^{(j)} = \lambda^{(j)}$ or $\overline{\lambda}^{(j)} = \Phi_i(\lambda^{(j)})$. The fact that $[\boldsymbol{\lambda}^K]$ has neither addable nor removable nodes of residue q^i means that $\overline{\lambda}^{(k)} = \lambda^{(k)} = \Phi_i(\lambda^{(k)})$ for $k \in K$. Hence for $j \in J, k \in K, l, m \in \mathbb{Z}/e\mathbb{Z}$ we have either

$$\gamma^{jk}_{lm}(\overline{\boldsymbol{\lambda}}) = \gamma^{jk}_{lm}(\boldsymbol{\lambda})$$

or

$$\gamma^{jk}_{lm}(\overline{\boldsymbol{\lambda}}) = \gamma^{jk}_{\phi_i(l)\phi_i(m)}(\boldsymbol{\lambda});$$

in either case, $\gamma^{jk}_{lm}(\overline{\boldsymbol{\lambda}}) \leqslant 1$. □

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