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# On the evaluation at $(j, j^2)$ of the Tutte polynomial of a ternary matroid

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**Abstract** F. Jaeger has shown that up to a ± sign the evaluation at $(j, j^2)$ of the Tutte polynomial of a ternary matroid can be expressed in terms of the dimension of the bicycle space of a representation over $GF(3)$. We give a short algebraic proof of this result, which moreover yields the exact value of ±, a problem left open in Jaeger's paper. It follows that the computation of $t(j, j^2)$ is of polynomial complexity for a ternary matroid.

**Keywords** Matroid · Ternary matroid · Tutte polynomial · Graph · Knot theory · Jones polynomial · Computational complexity

In the seminal paper [4] on the complexity of Tutte polynomials, it is shown that the point $(j, j^2)$ and its conjugate $(j^2, j)$ are two out of eight 'easy' special points, where 'easy' is intended from a computational point of view. Each of these eight points have remarkable combinatorial interpretations. A result of F. Jaeger [3] relates $t(j, j^2)$ and $t(j^2, j)$ to ternary matroids. Specifically, let $E$ be a finite set, $V$ be a subspace of the vector space $GF(3)^E$, and $M(V)$ be the matroid on $E$ whose circuits are the inclusion-minimal supports of non-zero vectors of $V$. Then $t(M(V); j, j^2) = \pm j^{|E|+\dim V}(i\sqrt{3})^{\dim(V\cap V^\perp)}$. Graphs, via graphic matroids, are a special case of ternary matroids. We refer the reader to the introduction of [3] (see also [4] Section 6) for

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the relevance of these properties to the Jones polynomial in Knot Theory. We also mention the related paper [5], where the problem of the complexity of the computation of $t(M;x,y)$, for $x, y$ algebraic numbers and $M$ vectorial over a given finite field, is addressed in full generality.

The main step of the proof in [3] is to establish that $\sum_{u\in V} j^{|s(u)|} = \pm(i\sqrt{3})^{\dim V+\dim(V\cap V^\perp)}$, where $s(u)$ denotes the support of $u$. The proof of this last property in Jaeger's paper uses deletion/contraction of elements of $E$, and is about four pages long. Our purpose in the present note is to provide a short algebraic proof. Moreover, we obtain the exact value of $\pm$, a question left open in Jaeger's paper. As a consequence $t(M;j,j^2)$ is of polynomial complexity for a ternary matroid $M$.

Let $K$ be a field and $E$ be a finite set. The *canonical bilinear form* on the space $K^E$ is defined by $<u,v>=\sum_{e\in E} u(e)v(e)$ for $u,v\in K^E$. In $K^E$, the subspace *orthogonal* to a subspace $V$ is defined by $V^\perp=\{u'\in K^E \mid <u,u'>=0 \text{ for all } u\in V\}$. A vector $u\in K^E$ is *isotropic* if $<u,u>=0$.

We will use two classical results about orthogonal bases. The orthogonalization algorithm of Lemma 1.1, which allows isotropic vectors in the orthogonal basis, is different from the current Gram-Schmidt orthogonalization algorithm valid for real spaces. We include proofs for completeness.

**Lemma 1.1.** *Let $V$ be a finite dimensional vector space over a field of characteristic $\neq 2$ endowed with a bilinear form. Then $V$ has an orthogonal basis.*

*More specifically an orthogonal basis of $V$ can be constructed from any given basis in polynomial time.*

**Proof:** Let $(u_k)_{1\le k\le d}$ be a basis of $V$. If there is an index $1\le \ell\le d$ such that $<u_\ell,u_\ell>\neq 0$, then reindex in such a way that $\ell=1$ and set $u'_1=u_1$. Otherwise, if there is an index $2\le\ell\le d$ such that $<u_1+u_\ell,u_1+u_\ell>\neq 0$, then set $u'_1=u_1+u_\ell$. In both cases, update $u_k$ as $u_k-<u'_1,u_k><u'_1,u'_1>^{-1}u'_1$ for $2\le k\le d$. We have $<u'_1,u_k>=0$ for $2\le k\le d$.

Otherwise we have $<u_k,u_k>=0$ for $1\le k\le d$, and $<u_1+u_k,u_1+u_k>=0$ for $2\le k\le d$. From $<u_1+u_k,u_1+u_k>=0$, we get $<u_1,u_1>+2<u_1,u_k>+<u_k,u_k>=2<u_1,u_k>=0$, hence $<u_1,u_k>=0$ in characteristic $\neq 2$. We set $u'_1=u_1$.

In all three cases, $\{u'_1,u_2,\ldots,u_d\}$ is a basis of $V$ such that $u'_1$ is orthogonal to the space generated by $u_k$ for $2\le k\le d$. Lemma 1.1 follows by induction. □

**Lemma 1.2.** *The isotropic vectors of any orthogonal basis of $V$ constitute a basis of $V\cap V^\perp$.*

**Proof:** Let $(u_k)_{1\le k\le d}$ be an orthogonal basis of $V$, and $u=\sum_{1\le k\le d} a_k u_k\in V\cap V^\perp$. For $1\le\ell\le d$, we have $0=<u,u_\ell>=\sum_{1\le k\le d} a_k<u_k,u_\ell>=a_\ell<u_\ell,u_\ell>$. Hence if $<u_\ell,u_\ell>\neq 0$, we have $a_\ell=0$. It follows that $u$ is generated by the isotropic vectors of the basis. These vectors being independent, they constitute a basis of $V\cap V^\perp$. □

Our basic result is the following strengthening of Jaeger's proposition.

**Proposition 1.** *Let $E$ be a finite set, and $V$ be a subspace of $GF(3)^E$. We have*

$$\sum_{u\in V} j^{|s(u)|} = (-1)^{d+d_1}(i\sqrt{3})^{d+d_0}$$

*where $d = \dim V$, $d_0 = \dim V \cap V^\perp$ and $d_1$ is the number of vectors with support of size congruent to 1 modulo 3 in any orthogonal basis of $V$ with respect to the canonical bilinear form.*

**Proof:** We have $GF(3) \cong Z/3Z$, in other words the elements of $GF(3)$ can be assimilated to integer residues modulo 3. We observe that for $u \in GF(3)^E$ we have $|s(u)|$ modulo $3 =< u, u >$, where $< u, v >= \sum_{e\in E} u(e)v(e)$ is the canonical bilinear form. It follows that $j^{|s(u)|} = j^{<u,u>}$.

By Lemma 1.1, there is an orthogonal basis $(u_k)_{1\le k\le d}$ of $V$. We have

$$\begin{aligned}
\sum_{u\in V} j^{|s(u)|} &= \sum_{u\in V} j^{<u,u>} \\
&= \sum_{(a_1,a_2,\ldots,a_d)\in GF(3)^d} j^{<\sum_{1\le k\le d} a_k u_k, \sum_{1\le k\le d} a_k u_k>} \\
&= \sum_{(a_1,a_2,\ldots,a_d)\in GF(3)^d} j^{\sum_{1\le k\le d} a_k^2 <u_k,u_k>} \\
&= \sum_{(a_1,a_2,\ldots,a_d)\in GF(3)^d} \prod_{1\le k\le d} j^{a_k^2 <u_k,u_k>} \\
&= \prod_{1\le k\le d} \sum_{a_k\in GF(3)} j^{a_k^2 <u_k,u_k>} \\
&= \prod_{1\le k\le d} (1 + 2j^{<u_k,u_k>}) \\
&= 3^{d_0}(1+2j)^{d_1}(1+2j^2)^{d_2}
\end{aligned}$$

where $d_0$ resp. $d_1$, $d_2$ is the number of vectors $u_k$ $1 \le k \le d$ such that $< u_k, u_k >= 0$ resp. $= 1 = 2$. We have $1 + 2j = i\sqrt{3}$, $1 + 2j^2 = -i\sqrt{3}$, $d = d_0 + d_1 + d_2$, and $d_0 = \dim V \cap V^\perp$ by Lemma 1.2. Proposition 1 follows. □

It follows from Proposition 1 and Lemma 1.1 that

**Corollary 2.** *Let $E$ be a finite set, and $V$ be a subspace of $GF(3)^E$.*

*The parity of the number of vectors with support of cardinality congruent to 1 resp. 2 modulo 3 in an orthogonal basis of $V$ does not depend on the particular orthogonal basis.*

By Corollary 2 the residue modulo 2 of the number of vectors with support of cardinality congruent to 1 resp. 2 modulo 3 in an orthogonal basis of a subspace $V$ of $GF(3)^E$ is a 0-1 invariant of $V$. We will denote it by $\bar{d}_1(V)$ resp. $\bar{d}_2(V)$. It follows from Lemma 1.1 that $\bar{d}_1(V)$ can be computed in polynomial time from any given basis of $V$.

We recall that by a theorem of Greene [2], given a subspace $V$ of $GF(q)^E$, $q$ a prime power, we have $\sum_{u\in V} z^{|s(u)|} = z^{|E|-d}(1-z)^d t(M; 1/z, 1+qz/(1-z))$, where $d = \dim V$.

**Theorem 3.** *Let $M$ be a ternary matroid on a finite set $E$. We have*

$$t(M; j, j^2) = (-1)^{d_2} j^{|E|+d} (i\sqrt{3})^{d_0}$$

*where $d = \dim V$, $d_0 = \dim V \cap V^{\perp}$, and $d_2$ is the number of vectors with support of cardinality congruent to 2 modulo 3 in any orthogonal basis of a subspace $V$ of $GF(3)^E$ such that $M = M(V)$.*

**Proof:** As in Jaeger's paper, we derive Theorem 3 from Proposition 1 by means of Greene's theorem. Specializing this formula to $z = j$ and $q = 3$, and applying Proposition 1, we get

$$t(M; j^2, j) = (-1)^{d+d_1} j^{-|E|-d} (i\sqrt{3})^{d_0}$$

Since $t(M; j, j^2)$ is the complex conjugate of $t(M; j^2, j)$, Theorem 2 follows. □

A short proof of Greene's theorem is given in [3] Proposition 7 (see also [1] for another short proof).

Theorem 3 provides the exact value of $\pm$ in Jaeger's formula for $t(M; j, j^2)$ when $M$ is a ternary matroid. This answers the question in [3] p. 25 asking for an interpretation of the parameter $\epsilon(M)$, defined by $t(M; j, j^2) = \epsilon(M) j^{|E|+d}(i\sqrt{3})^{d_0}$. By Corollary 2, $\bar{d}_1 = \bar{d}_1(V) = \bar{d}_1(M)$ is a 0-1 invariant of polynomial complexity of a ternary matroid $M$. By Theorem 3, we have

$$\epsilon(M) = (-1)^{d+\bar{d}_1} = (-1)^{d_0+\bar{d}_2}$$

As well-known, if $V$ is defined by a basis, the dimension $d_0 = \dim V \cap V^{\perp}$ is of polynomial complexity (also a corollary of Lemmas 1.1 and 1.2). Hence

**Corollary 4.** *The evaluation $t(M; j, j^2)$ of the Tutte polynomial of a ternary matroid $M = M(V)$, with $V$ defined by a basis, is of polynomial complexity.*

Corollary 4 strengthens the previously known polynomial complexity of the modulus $|t(M; j, j^2)|$, used in [4, 5].

As noted by Jaeger (see [3] Proposition 9) $\epsilon(M)$ and $\epsilon(M^*)$ are related.

**Corollary 5.** *Let $M$ be a ternary matroid on a set $E$. We have $\bar{d}_1(M^*) \equiv \bar{d}_1(M) + d_0(M) + |E|$ modulo 2, where $M^*$ denotes the dual matroid of $M$.*

Corollary 5 follows from the relation $\epsilon(M) = (-1)^{d+\bar{d}_1}$, combined with [3] Proposition 9.(i). It can also be easily derived directly from Theorem 2.

Finally, we mention that the initial motivation of the present note was the computation of $\sum_{u\in V} j^{|s(w+u)|}$, where $w$ is any vector of $GF(3)^E$.

**Corollary 6.** *Let $w \in GF(3)^E$.*

- *If $w \in V + V^{\perp}$, say $w = w' + w''$ with $w' \in V$ and $w'' \in V^{\perp}$, then, with notation of Proposition 1, we have*

$$\sum_{u\in V} j^{|s(w+u)|} = (-1)^{d+d_1}(i\sqrt{3})^{d+d_0} j^{|s(w'')|}$$

- *If $w \notin V + V^{\perp}$, we have*

$$\sum_{u\in V} j^{|s(w+u)|} = 0.$$

**Proof:** If $w = w' + w''$ with $w' \in V$ and $w'' \in V^{\perp}$, we have

$$\begin{aligned}\sum_{u\in V} j^{|s(w+u)|} &= \sum_{u\in V} j^{|s(w'+w''+u)|}\\ &= \sum_{u\in V} j^{|s(w''+u)|} = \sum_{u\in V} j^{<w''+u,w''+u>}\\ &= \sum_{u\in V} j^{<w'',w''>+<u,u>} = j^{<w'',w''>}\sum_{u\in V} j^{<u,u>}\end{aligned}$$

Then we obtain Corollary 4 by applying Proposition 1.

If $w \notin V + V^{\perp}$, then there is $v \in V \cap V^{\perp} = (V + V^{\perp})^{\perp}$ such that $< w, v > \neq 0$. Let $V'$ be a supplement of $< v >$ in $V$. We have

$$\begin{aligned}\sum_{u\in V} j^{|s(w+u)|} &= \sum_{u\in V'}\sum_{a\in GF(3)} j^{|s(w+u+av)|}\\ &= \sum_{u\in V'}\sum_{a\in GF(3)} j^{<w+u+av,w+u+av>}\\ &= \sum_{u\in V'}\sum_{a\in GF(3)} j^{<w+u,w+u>+a<w,v>}\\ &= \sum_{u\in V'} j^{<w+u,w+u>}\left(\sum_{a\in GF(3)} j^{a<w,v>}\right)\\ &= \sum_{u\in V'} j^{<w+u,w+u>}(1+j+j^2) = 0\end{aligned}$$

□

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