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On antipodal Euclidean tight (2e + 1)-designs

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Abstract Neumaier and Seidel (1988) generalized the concept of spherical designs and defined Euclidean designs in $\mathbb{R}^n$. For an integer t, a finite subset X of $\mathbb{R}^n$ given together with a weight function w is a Euclidean t-design if $\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{S_i} f(\boldsymbol{x}) d\sigma_i(\boldsymbol{x}) = \sum_{x \in X} w(\boldsymbol{x}) f(\boldsymbol{x})$ holds for any polynomial $f(\boldsymbol{x})$ of $\deg(f) \le t$, where $\{S_i,\ 1 \le i \le p\}$ is the set of all the concentric spheres centered at the origin that intersect with X, $X_i = X \cap S_i$, and $w : X \to \mathbb{R}_{>0}$. (The case of $X \subset S^{n-1}$ with $w \equiv 1$ on X corresponds to a spherical t-design.) In this paper we study antipodal Euclidean $(2e+1)$-designs. We give some new examples of antipodal Euclidean tight 5-designs. We also give the classification of all antipodal Euclidean tight 3-designs, the classification of antipodal Euclidean tight 5-designs supported by 2 concentric spheres.

Keywords Euclidean design · Spherical design · 2-distance set · Antipodal · Tight design

1. Introduction

Delsarte, Goethals and Seidel defined the concept of spherical designs [5]. In the paper of Neumaier-Seidel [9], they generalized the concept and gave a definition of designs in the Euclidean space $\mathbb{R}^n$, namely, Euclidean designs. Delsarte and Seidel [6] studied more precise properties of Euclidean designs on a union of p concentric spheres centered at the origin. In a joint paper with Eiichi Bannai [3] we slightly generalized the concept and defined Euclidean designs for finite sets which may possibly contain the origin. In that paper [3], we gave a new approach to give the lower bound for the

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cardinalities of Euclidean $2e$-designs which was obtained by Delsarte and Seidel [6]. This approach gave us the way to understand new properties of Euclidean $2e$-designs. In this paper we apply a similar method, to the one given in [3], to antipodal Euclidean $(2e+1)$-designs and obtain a similar lemma as the one proved in [3]. We also give some examples of antipodal tight 5-designs which are not in the list of Euclidean designs given by B. Bajnok (see [1]) recently.

We say that a finite set $X \subset \mathbb{R}^n$ is supported by p concentric spheres if X intersects with exactly p concentric spheres centered at the origin. In this paper we first review the definitions of tightness of the designs and then classify all the antipodal Euclidean tight 3-designs, and then classify antipodal Euclidean tight 5-designs supported by 2 concentric spheres. (From the definition of antipodal Euclidean tight $(2e+1)$-designs it is easy to see that antipodal Euclidean tight 5-designs must be supported by at least 2 concentric spheres.)

Before stating our main results, we give the definitions and notation we use in this paper. We assume $n \geq 2$ throughout this paper. Let X be a finite set in $\mathbb{R}^n$ supported by p concentric spheres $S_1, \ldots, S_p$. In this definition we regard the set consists of only the origin $\mathbf{0}$ as a special case of spheres and assume one of S_i, $1 \leq i \leq p$, may possibly coincide with $\{\mathbf{0}\}$. Let r_i be the radius of S_i for $i = 1, 2, \ldots, p$. We denote the canonical inner product of $\mathbb{R}^n$ by $(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^n x_i y_i$, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Let $\|\boldsymbol{x}\|^2 = (\boldsymbol{x}, \boldsymbol{x})$. Let $X_i = X \cap S_i$ for $i = 1, 2, \ldots p$. Let $d\sigma(\boldsymbol{x})$ be a Haar measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. We consider a Haar measure $d\sigma_i(\boldsymbol{x})$ on each S_i so that $|S_i| = {r_i}^{n-1}|S^{n-1}|$. Here $|S_i|$ and $|S^{n-1}|$ are the surface areas of S_i and the unit sphere S^{n-1} respectively. Let moreover w be a positive real valued function on X, that we call the weight function on X. We define $w(X_i) = \sum_{\boldsymbol{x} \in X_i} w(\boldsymbol{x})$. Here if $r_i = 0$, then we define $\frac{1}{|S_i|}\int_{S_i} f(\boldsymbol{x}) d\sigma_i(\boldsymbol{x}) = f(\mathbf{0})$ for any function $f(\boldsymbol{x})$ defined on $\mathbb{R}^n$. Let $S = \bigcup_{i=1}^p S_i$. Let $\varepsilon_S \in \{0, 1\}$ be defined by

$$\varepsilon_S = 1 \quad \text{if } \mathbf{0} \in S, \qquad \varepsilon_S = 0 \quad \text{if } \mathbf{0} \notin S.$$

We give some more definitions and notations. Let $\mathcal{P}(\mathbb{R}^n) = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be the vector space of polynomials in n variables $x_1, x_2, \ldots, x_n$ over the field of real numbers. Let $\mathrm{Hom}_l(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ which consists of homogeneous polynomials of degree l. Let $\mathcal{P}_l(\mathbb{R}^n) = \bigoplus_{i=0}^l \mathrm{Hom}_i(\mathbb{R}^n)$. Let $\mathrm{Harm}(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ which consists of all the harmonic polynomials. Let $\mathrm{Harm}_l(\mathbb{R}^n) = \mathrm{Harm}(\mathbb{R}^n) \cap \mathrm{Hom}_l(\mathbb{R}^n)$. Let $\mathcal{P}_l^*(\mathbb{R}^n) = \bigoplus_{\substack{i \equiv l \ (2) \\ 0 \leq i \leq l}} \mathrm{Hom}_i(\mathbb{R}^n)$. Let $\mathcal{P}(S)$, $\mathcal{P}_l(S)$, $\mathrm{Hom}_l(S)$, $\mathrm{Harm}(S)$, $\mathrm{Harm}_l(S)$ and $\mathcal{P}_l^*(S)$ be the sets of corresponding polynomials restricted to the union S of concentric spheres. For example $\mathcal{P}(S) = \{f|_S \mid f \in \mathcal{P}(\mathbb{R}^n)\}$.

A finite subset $X \subset \mathbb{R}^n$ is said to be antipodal if $-\boldsymbol{x} \in X$ holds for any $\boldsymbol{x} \in X$. Let X^* be a subset of X satisfying

$$X = X^* \cup (-X^*), \ \ X^* \cap (-X^*) = \emptyset \text{ or } \{\mathbf{0}\},$$

where $-X^* = \{-\boldsymbol{x} \mid \boldsymbol{x} \in X^*\}$. For a finite subset $X \subset \mathbb{R}^n$, we define

$$A(X) = \{\|\boldsymbol{x} - \boldsymbol{y}\| \mid \boldsymbol{x}, \ \boldsymbol{y} \in X, \ \boldsymbol{x} \neq \boldsymbol{y}\}.$$

If $|A(X)| = s$, then we call X an s-distance set. For $\alpha \in A(X)$, we define

$$v_\alpha(\boldsymbol{x}) = |\{\boldsymbol{y} \in X \mid \|\boldsymbol{x} - \boldsymbol{y}\| = \alpha\}|.$$

If the following condition holds, then we call X a distance invariant set:

> “ $v_\alpha(\boldsymbol{x})$ does not depend on the choice of $\boldsymbol{x} \in X$ and depends only on α for any fixed $\alpha \in A(X)$.”

Definition 1.1 (Euclidean design). Let t be a natural number. Let X be a finite set with a positive weight function w on X. We say that X is a Euclidean t-design, if the following condition is satisfied:

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{\boldsymbol{x} \in S_i} f(\boldsymbol{x}) d\sigma_i(\boldsymbol{x}) = \sum_{\boldsymbol{u} \in X} w(\boldsymbol{u}) f(\boldsymbol{u})$$

for any polynomial $f \in \mathcal{P}_t(\mathbb{R}^n)$.

The following theorem is well known [3, 5, 6].

Theorem 1.2.

(1) *Let X be a Euclidean $2e$-design, then*

$$|X| \geq \dim(\mathcal{P}_e(S))$$

holds.

(2) *Let X be an antipodal Euclidean $(2e+1)$-design. Assume $w(-\boldsymbol{x}) = w(\boldsymbol{x})$ holds for any $\boldsymbol{x} \in X$. Then*

$$|X^*| \geq \dim(\mathcal{P}_e^*(S))$$

holds.

Remark 1.

(i) *Theorem* 1.2 *was proved by Delsarte and Seidel in* [6] (*see also* [2, 5, 9]). *They also gave* $\dim(\mathcal{P}_e(S))$ *and* $\dim(\mathcal{P}_e^*(S))$ *explicitly. In* [3], *we gave a different proof for Theorem* 1.2(1). *It is not a good method to prove the lower bound itself. However equations we obtained in the proof are very effective. In the following section, we will give a proof of Theorem* 1.2(2) *using the method given in* [3].

(ii) *For spherical* $(2e+1)$*-designs, that is, Euclidean* $(2e+1)$*-designs satisfying* $p = 1$ *and* $w \equiv 1$, *the inequality given in Theorem* 1.2(2) *was proved without assuming X is antipodal, and if equality holds then X was proved to be antipodal* [5]. *However if $p \geq 2$, then there is no good lower bound without assuming X is antipodal and $w(-\boldsymbol{x}) = w(\boldsymbol{x})$ for $\boldsymbol{x} \in X$.*

Definition 1.3.

(1) (Tight $2e$-design on p concentric spheres).
Let X be a Euclidean $2e$-design supported by p concentric spheres. If

$$|X| = \dim(\mathcal{P}_e(S))$$

holds, then we call X a tight $2e$-design on p concentric spheres.

(2) (Antipodal tight $(2e+1)$-design on p concentric spheres)
Let X be an antipodal Euclidean $(2e+1)$-design supported by p concentric spheres. If $w(-\boldsymbol{x}) = w(\boldsymbol{x})$ for any $\boldsymbol{x} \in X$ and

$$|X^*| = \dim(\mathcal{P}_e^*(S))$$

holds, then we call X an antipodal tight $(2e+1)$-design on p concentric spheres.

Remark 2.

(i) If $p = 1$, $X \neq \{\mathbf{0}\}$ and $w \equiv 1$ on X, then the definitions given above coincide with the definitions of spherical tight designs [5]. As we mentioned in Remark 1(ii), it is proved that spherical tight $(2e+1)$-designs are antipodal [5].
(ii) We will give a list of the dimensions of subspaces of $\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{P}(S)$ in Section 3.

Definition 1.4.

(1) (Euclidean Tight $2e$-design)
Let X be a Euclidean $2e$-design X. If

$$|X| = \dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$$

holds, then we call X a Euclidean Tight $2e$-design.
(2) (Antipodal Euclidean tight $(2e+1)$-design)
Let X be an antipodal Euclidean $(2e+1)$-design. If

$$|X^*| = \dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n)) \left(= \sum_{i=0}^{[\frac{e}{2}]} \binom{n+e-2i-1}{e-2i} \right)$$

holds, then we call X an antipodal Euclidean tight $(2e+1)$-design.

Remark 3. If X is a Euclidean tight $2e$-design, then we should have $|X| = \dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$. The following example shows the reason why the condition $\dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$ is important. Let X_1 and X_2 be the sets of the vertices of regular triangles in $\mathbb{R}^2$ defined by

$$X_1 = \left\{(1,0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\},\ X_2 = \left\{(-r,0), \left(\frac{r}{2}, \frac{\sqrt{3}}{2}r\right), \left(\frac{r}{2}, -\frac{\sqrt{3}}{2}r\right)\right\}.$$

Let $w(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in X_1$ and $w(\boldsymbol{x}) = \frac{1}{r^3}$ for $\boldsymbol{x} \in X_2$. Then $X = X_1 \cup X_2$ is a Euclidean 4-design. If $r = 1$, then X is on the unit circle S^1 and we have $\dim(\mathcal{P}_2(S)) = 5 <$

$\dim(\mathcal{P}_2(\mathbb{R}^2)) = 6$. Therefore X is not a tight 4-design on $S = S^1$. However, if $r \neq 1$, then X is supported by 2 concentric spheres and $\dim(\mathcal{P}_2(S)) = \dim(\mathcal{P}_2(\mathbb{R}^2)) = 6$ holds. Hence X is a Euclidean tight 4-design. Similarly examples given in the following Theorem 1.6 explain why the equality $\dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n))$ is important in the definition of antipodal Euclidean tight $(2e+1)$-designs.

If X is an antipodal Euclidean 1-design, then $|X^*| \geq \dim(\mathcal{P}_0^*(S)) = \dim(P_0^*(\mathbb{R}^n)) = 1$. Hence $X = \{\mathbf{0}\}$ is an antipodal Euclidean tight 1-design. If we consider the case $\mathbf{0} \notin X$, then any antipodal 2-point set $\{\boldsymbol{u}, -\boldsymbol{u}\}$ is an antipodal Euclidean tight 1-design, which is similar to a spherical tight 1-design.

Theorem 1.5. *Let X be an antipodal Euclidean tight 3-design in $\mathbb{R}^n$, then X is similar to one of the following*:

$$X = \bigcup_{i=1}^{p} X_i \quad \textit{with} \quad X_i = \left\{ \pm r_i \boldsymbol{e}_j \;\middle|\; 1 + \sum_{l=1}^{i-1} N_l \leq j \leq \sum_{l=1}^{i} N_l \right\}$$

and

$$w(\boldsymbol{x}) = \frac{1}{n{r_i}^2} \quad \textit{for } \boldsymbol{x} \in X_i \textit{ and } i = 1, \ldots, p.$$

In above $N_i = |X_i^*|$ for $i = 1, \ldots, p$ and $|X| = 2\sum_i^p N_i = 2n$.

Remark 4. Examples of antipodal 3-designs as in above are given by Bajnok [1].

Theorem 1.6. *Let X be an antipodal Euclidean tight 5-design in $\mathbb{R}^n$ supported by 2 concentric spheres. Then X is similar to one of the following*:

(1) *$\mathbf{0} \in X$ and $X \backslash \{\mathbf{0}\}$ is a tight spherical 5-design.*
(2) *$n = 2$. $X = X_1 \cup X_2$, where*

$$X_1 = \{(\pm 1, 0), (0, \pm 1)\}, \quad X_2 = \left\{ \left(\pm \frac{r}{\sqrt{2}}, \pm \frac{r}{\sqrt{2}} \right) \right\},$$

$w(\boldsymbol{x}) = 1,\ \boldsymbol{x} \in X_1\ w(\boldsymbol{x}) = \frac{1}{r^4},\ \boldsymbol{x} \in X_2, r \neq 1.$
(3) *$n = 3$. $X = X_1 \cup X_2$, where*

$$X_1 = \{\pm \boldsymbol{e}_i \mid i = 1, 2, 3\}, \quad X_2 = \left\{ \frac{r}{\sqrt{3}}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_i \in \{1, -1\}, 1 \leq i \leq 3 \right\},$$

$r \neq 1$, $w(\boldsymbol{x}) = 1$ *for* $\boldsymbol{x} \in X_1$ *and* $w(\boldsymbol{x}) = \frac{9}{8r^4}$ *for* $\boldsymbol{x} \in X_2$.

(4) $n = 5$. $X = X_1 \cup X_2 \subset V = \{(x_1, \ldots, x_6) \in \mathbb{R}^6 \mid \sum_{i=1}^{6} x_i = 0\} \cong \mathbb{R}^5$, *where*

$$X_1 = \{\pm \boldsymbol{u}_i \mid 1 \le i \le 6\},$$
$$X_2 = \left\{ \frac{r}{\sqrt{6}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6) \,\middle|\, \varepsilon_i \in \{1, -1\},\ |\{i | \varepsilon_i = 1\}| = 3 \right\},$$

$r \neq 1$, $w(\boldsymbol{x}) = 1$ *for* $\boldsymbol{x} \in X_1$, $w(\boldsymbol{x}) = \frac{27}{25r^4}$ *for* $\boldsymbol{x} \in X_2$ *and* $\boldsymbol{u}_i = (u_{i,1}, \ldots, u_{i,6}), \in V$ *are defined by*

$$u_{i,j} = \begin{cases} -\dfrac{5}{\sqrt{30}} & \text{if } j = i, \\ \dfrac{1}{\sqrt{30}} & \text{otherwise} \end{cases}$$

(5) $n = 6$. $X = X_1 \cup X_2$, *where*

$$X_1 = \{\pm \boldsymbol{e}_i \mid 1 \le i \le 6\},$$
$$X_2 = \left\{ \frac{r}{\sqrt{6}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6) \,\middle|\, \varepsilon_i \in \{1, -1\},\ |\{i | \varepsilon_i = 1\}| \equiv 0 \ (mod\ 2) \right\},$$

$r \neq 1$, $w(\boldsymbol{x}) = 1$ *for* $\boldsymbol{x} \in X_1$ *and* $w(\boldsymbol{x}) = \frac{9}{8r^4}$ *for* $\boldsymbol{x} \in X_2$.

Remark 5.

(i) Examples (2) and (3) above are given by Bajnok [1]. Examples (4) and (5) are the newly found ones.

(ii) If X is an antipodal Euclidean tight 5-design, then X must be supported by at least 2 concentric spheres. The antipodal Euclidean tight 3- and 5-designs in Theorems 1.5 and 1.6 are non-rigid, because we obtain a distinct antipodal Euclidean tight 3- or 5-designs by changing one of the radii r_i of the spheres which support the given antipodal Euclidean tight 3- or 5-design and the corresponding weight $w(x)$, $x \in X_i$. By a recent result on non-rigid Euclidean designs obtained by Eiichi Bannai and Djoko Suprijanto [4], it seems that if a Euclidean tight $2e$-design or an antipodal Euclidean tight $(2e+1)$-design which is supported by more than $[\frac{e+\varepsilon_S}{2}] + 1$ spheres exists, then there may possibly exist infinitely many Euclidean tight $2e$-designs or antipodal Euclidean tight $(2e+1)$-designs respectively. Actually they showed some of the tight Euclidean designs are strongly non-rigid. That means there are infinitely many tight Euclidean designs which are not transformed to each other by orthogonal transformations, scaling, or adjustment of weight functions.

The following Lemma is one of the key lemmas to prove Theorems 1.5 and 1.6.

Lemma 1.7. *Let* X *be an antipodal tight* $(2e+1)$*-design on* p *concentric spheres. Let* $X_i^* = X^* \cap X_i$. *Then the following conditions hold:*

(1) *On each X_i, the weight function w is constant.*
(2) *Each X_i^* is an at most e-distance set.*
(3) *Each X_i is an at most $(e+1)$-distance set.*
(4) *If the weight function w is constant on $X\backslash\{\mathbf{0}\}$, then $p-\varepsilon_S \le e$.*

As an application of Lemma 1.7, we obtained the following theorem which is very useful.

Theorem 1.8.

(1) Let X be a tight $2e$-design on p concentric spheres. If $e-p+\varepsilon_S \ge 0$, then each X_i is similar to a spherical $(2e-2p+2\varepsilon_S+2)$-design. Moreover, if $p \le [\frac{e+2\varepsilon_S+3}{2}]$, then each X_i is a distance invariant set.
(2) Let X be an antipodal tight $(2e+1)$-design on p concentric spheres. If $e-p+\varepsilon_S \ge 0$, then each X_i is similar to an antipodal spherical $(2e-2p+2\varepsilon_S+3)$-design. Moreover, if $p \le [\frac{e+2\varepsilon_S+3}{2}]$, then each X_i is a distance invariant set.

In Section 2 we give some basic facts about Euclidean designs including the facts already known, state and prove Theorem 2.3. Then we give the proof of Theorem 1.8 using Lemma 1.7. In Section 3, we will prove Lemma 1.7. In Section 4 we will prove Theorem 1.5 and in Section 5 we will prove Theorem 1.6. In Section 6 we will state some remarks.

2. Basic facts

In this section we give basic facts on Euclidean t-designs which we use to prove our results. The following theorem is proved in [9].

Theorem 2.1 ([9]). *Let X be a finite subset which may possibly contains $\mathbf{0}$ and with a weight ω. Then the following* (1) *and* (2) *are equivalent*:

(1) *X is a Euclidean t-design with weight w.*
(2) $\sum_{\boldsymbol{u}\in X} w(\boldsymbol{u})f(\boldsymbol{u})=0$ *for any polynomial* $f \in \|\boldsymbol{x}\|^{2j}Harm_l(\mathbb{R}^n)$ *with* $1 \le l \le t$, $0 \le j \le [\frac{t-l}{2}]$.

Corollary 2.2. *Let X be an antipodal set with weight w satisfying $w(-\boldsymbol{x})=w(\boldsymbol{x})$ for $\boldsymbol{x}\in X$. Then the following* (1) *and* (2) *are equivalent:*

(1) *X is a Euclidean t-design with weight w.*
(2) $\sum_{\boldsymbol{u}\in X^*} w(\boldsymbol{u})\|\boldsymbol{u}\|^{2j}\varphi(\boldsymbol{u})=0$ *for any polynomial* $\varphi \in Harm_{2l}(\mathbb{R}^n)$ *with* $1\le l\le [\frac{t}{2}]$, $0 \le j \le [\frac{t}{2}]-l$.

Applying Theorem 2.1 and Corollary 2.2 we can prove the following theorem.

Theorem 2.3.

(1) Let X be a t-design supported by p concentric spheres. Assume $p \le [\frac{t+1}{2}] + \varepsilon_S$. Then

$$\sum_{\boldsymbol{u} \in X_i} w(\boldsymbol{u})\varphi(\boldsymbol{u}) = 0$$

holds for any i satisfying $r_i > 0$ and $\varphi \in \mathrm{Harm}_l(\mathbb{R}^n)$ with $1 \le l \le t - 2p + 2\varepsilon_S + 2$. In particular, if $w(u)$ is constant on X_i, then X_i is similar to a spherical $(t - 2p + 2\varepsilon_S + 2)$-design.

(2) Let X be an antipodal Euclidean t-design supported by p concentric spheres. Assume $w(-\boldsymbol{x}) = w(\boldsymbol{x})$ for any $\boldsymbol{x} \in X$ and $p \le [\frac{t}{2}] + \varepsilon_S$. Then

$$\sum_{\boldsymbol{u} \in X_i} w(\boldsymbol{u})\varphi(\boldsymbol{u}) = 0$$

holds for any i satisfying $r_i > 0$ and $\varphi \in \mathrm{Harm}_{2l}(\mathbb{R}^n)$ with $1 \le l \le t - 2p + 2\varepsilon_S + 2$. In particular, if $w(\boldsymbol{u})$ is constant on X_i, then X_i is similar to an antipodal spherical $(2([\frac{t}{2}] - p + \varepsilon_S) + 3)$-design.

Proof: We may assume $r_i > 0$ for $i = 1, \ldots, p - \varepsilon_S$.

(1) Theorem 2.1 implies that for any j and l satisfying $1 \le l \le t$ and $0 \le j \le [\frac{t-l}{2}]$ the following condition holds:

$$\sum_{\boldsymbol{u} \in X} w(\boldsymbol{u})\|\boldsymbol{u}\|^{2j}\varphi(\boldsymbol{u}) = \sum_{i=1}^{p-\varepsilon_S} {r_i}^{2j} \sum_{\boldsymbol{u} \in X_i} w(\boldsymbol{u})\varphi(\boldsymbol{u}) = 0, \tag{2.1}$$

for any $\varphi \in \mathrm{Harm}_l(\mathbb{R}^n)$. Let $p' = p - \varepsilon_S$. Therefore $1 \le l \le t - 2p' + 2$ implies $[\frac{t-l}{2}] \ge p' - 1$ and the coefficient matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ {r_1}^2 & {r_2}^2 & \ldots & {r_{p'}}^2 \\ \vdots & \vdots & & \vdots \\ {r_1}^{2[\frac{t-l}{2}]} & {r_2}^{2[\frac{t-l}{2}]} & \ldots & {r_{p'}}^{2[\frac{t-l}{2}]} \end{pmatrix}$$

of the Eq. (2.1) is of rank p'. Hence

$$\sum_{\boldsymbol{u} \in X_i} w(\boldsymbol{u})\varphi(\boldsymbol{u}) = 0$$

holds for any i and $\varphi \in \mathrm{Harm}_{2l}(\mathbb{R}^n)$ with $1 \le l \le t - 2p' + 2 = t - 2p + 2\varepsilon_S + 2$. Then Theorem 2.1 implies (1).

(2) Let X be an antipodal Euclidean t-design. Then Corollary 2.2 implies that for any l and j satisfying $1 \leq l \leq [\frac{t}{2}]$ and $0 \leq j \leq [\frac{t}{2}] - l$ the following condition holds:

$$\sum_{i=1}^{p-\varepsilon_S} {r_i}^{2j} \sum_{\boldsymbol{u} \in X_i^*} w(\boldsymbol{u})\varphi(\boldsymbol{u}) = 0 \tag{2.2}$$

for any i and $\varphi \in \mathrm{Harm}_{2l}(\mathbb{R}^n)$. Therefore $[\frac{t}{2}] - l \geq p' - 1$ implies that the coefficient matrix of the above Eq. (2.2) is of full rank and

$$\sum_{\boldsymbol{u} \in X_i^*} w(\boldsymbol{u})\varphi(\boldsymbol{u}) = 0$$

holds for any i and $\varphi \in \mathrm{Harm}_{2l}(\mathbb{R}^n)$ with $1 \leq l \leq [\frac{t}{2}] - p + \varepsilon_S + 1$. Since X_i is antipodal if w is constant on X_i, Corollary 2.2 implies that X_i is similar to an antipodal spherical $(2([\frac{t}{2}] - p + \varepsilon_S) + 3)$-design.

□

Proof of Theorem 1.8

(1) Let X be a tight $2e$-design on p-concentric spheres. In [2], we proved a lemma similar to Lemma 1.7, which shows that each X_i is at most an e-distance set and the weight function is constant on X_i for any tight $2e$-design X on p-concentric spheres. On the other hand Theorem 2.3 implies that X_i is similar to a $(2e - 2p + 2\varepsilon_S + 2)$-design. Since $(2e - 2p + 2\varepsilon_S + 2) \geq e - 1$ holds by the assumption of Theorem 1.8, Theorem 7.4 in [5] implies that X_i is distance invariant (Theorem 7.4 in [5] shows that every spherical t-design which is at most a $(t + 1)$-distance set is distance invariant).

(2) Let X be an antipodal tight $(2e + 1)$-design on p concentric spheres. Then Lemma 1.7 implies that X_i is at most an $(e + 1)$-distance set and the weight is constant on X_i. On the other hand Theorem 2.3 implies that X_i is similar to an antipodal spherical $(2e - 2p + 2\varepsilon_S + 3)$-design. The assumption of Theorem 1.8 give $(2e - 2p + 2\varepsilon_S + 3) \geq (e + 1) - 1$. Hence Theorem 7.4 in [5] shows that X_i is distance invariant.

3. Proof of Lemma 1.7

Let S be a union of p concentric spheres centered at the origin. Let $e \geq 1$. Then the following are well known [2, 6, 7]:

(i) $\dim(\mathcal{P}_e(\mathbb{R}^n)) = \binom{n+e}{e}$.

(ii) $\dim(\mathcal{P}_e^*(\mathbb{R}^n)) = \sum_{i=0}^{[\frac{e}{2}]} \binom{n+e-2i-1}{e-2i}$

(ii) $\dim(\mathcal{P}_e(S)) = \varepsilon_S + \sum_{i=0}^{2(p-\varepsilon_S)-1} \binom{n+e-i-1}{e-i} < \binom{n+e}{e}$ for $p \leq [\frac{e+\varepsilon_S}{2}]$.

(iii) $\dim(\mathcal{P}_e(S)) = \sum_{i=0}^{e} \binom{n+e-i-1}{e-i} = \binom{n+e}{e}$ for $p \geq [\frac{e+\varepsilon_S}{2}] + 1$.

(iv) $\dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n)) = \sum_{i=0}^{[\frac{e}{2}]} \binom{n+e-2i-1}{e-2i}$ for $p \geq [\frac{e}{2}] + 1$.

(v) $\dim(\mathcal{P}_e^*(S)) = \sum_{i=0}^{p-1} \binom{n+e-2i-1}{e-2i} < \dim(\mathcal{P}_e^*(\mathbb{R}^n))$

for $p \leq [\frac{e}{2}]$ and e is odd or e is even and $\mathbf{0} \notin S$.

(vi) $\dim(\mathcal{P}_e^*(S)) = 1 + \sum_{i=0}^{p-2} \binom{n+e-2i-1}{e-2i} < \dim(\mathcal{P}_e^*(\mathbb{R}^n))$

for $p \leq [\frac{e}{2}]$, e is even and $\mathbf{0} \in S$.

Let X be an antipodal finite subset in $\mathbb{R}^n$ supported by p concentric spheres. Let S be the union of the p concentric spheres. First we define a basis of $\mathcal{P}_e^*(S)$. Let $\{\varphi_{l,i}(\boldsymbol{x}) \mid 1 \leq i \leq h_l\}$ be an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^n)$ with respect to the inner product

$$\langle f, g \rangle = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(\boldsymbol{x}) g(\boldsymbol{x}) d\sigma(\boldsymbol{x}),$$

where $h_l = \dim(\mathrm{Harm}_l(\mathbb{R}^n))$. Let

$$\mathcal{H}_0 = \left\{ g_{0,j}(\|\boldsymbol{x}\|^2) \mid 0 \leq j \leq \min\left\{ p-1, \left[\frac{e}{2}\right] \right\} \right\}$$

and

$$\mathcal{H}_l = \left\{ g_{l,j}(\|\boldsymbol{x}\|^2) \varphi_{l,i}(\boldsymbol{x}) \mid 1 \leq i \leq h_l, 0 \leq j \leq \min\left\{ p - \varepsilon_S - 1, \left[\frac{e-l}{2}\right] \right\} \right\}$$

for $1 \leq l \leq e$, where $g_{l,\nu}(\|\boldsymbol{x}\|^2)$ $(0 \leq \nu \leq p-1)$ is a polynomial of degree 2ν, which is linear combinations of $1, \|\boldsymbol{x}\|^2, \ldots, \|\boldsymbol{x}\|^{2j}$ and satisfying the following condition:

$$\sum_{\boldsymbol{x} \in X^*} w(\boldsymbol{x}) \|\boldsymbol{x}\|^{2l} g_{l,j}(\|\boldsymbol{x}\|^2) g_{l,j'}(\|\boldsymbol{x}\|^2) = \delta_{j,j'}.$$

Such polynomials always exist because $\{1, \|\boldsymbol{x}\|^2, \ldots, \|\boldsymbol{x}\|^{2(p-1)}\}$ is a linearly independent subset of $\mathcal{P}(X^*) = \{f|_{X^*} \mid f \in \mathcal{P}(\mathbb{R}^n)\}$ and for each l

$$\langle f, g \rangle_l = \sum_{x \in X^*} w(\boldsymbol{x}) \|\boldsymbol{x}\|^{2l} f(\boldsymbol{x}) g(\boldsymbol{x})$$

defines a positive definite inner product of $\mathcal{P}(X^*)$.

Then $\mathcal{H}_l$ is a basis of $\bigoplus_{j=0}^{\min\{p-\varepsilon_S-1,[\frac{e-l}{2}]\}} \|\boldsymbol{x}\|^{2j}\mathrm{Harm}_l(S)$. Let

$$\mathcal{H}^* = \bigcup_{l=0}^{[\frac{e}{2}]} \mathcal{H}_{e-2l}.$$

Since

$$\mathcal{P}_e^*(S) = \sum_{i=0}^{[\frac{e}{2}]} \mathrm{Hom}_{e-2i}(S) = \sum_{i=0}^{[\frac{e}{2}]} \sum_{j=0}^{[\frac{e-2i}{2}]} \|\boldsymbol{x}\|^{2j}\mathrm{Harm}_{e-2i-2j}(S)$$

and $\mathcal{H}^*$ is a linearly independent subset of $P_e^*(S)$, $\mathcal{H}^*$ is a basis of $\mathcal{P}_e^*(S)$.

Let M be a matrix whose columns and rows are indexed by $X^* \times \mathcal{H}^*$ whose $(\boldsymbol{x}, f)$-entry is given by $\sqrt{w(\boldsymbol{x})}f(\boldsymbol{x})$ for $\boldsymbol{x} \in X^*$ and $f \in \mathcal{H}^*$.

Proposition 3.1. *Let X be an antipodal $(2e+1)$-design. Let M be the matrix defined as above for X^*. Then ${}^tM\,M = I$ holds. Hence we have $|X^*| \geq \dim(\mathcal{P}_e^*(S))$.*

Proof: First we prove the case when $\mathbf{0} \notin X$. Since $e - 2l_1 + 2j_1 + e - 2l_2 + 2j_2 \leq 2e$, the $(g_{e-2l_1,j_1}\varphi_{e-2l_1,i_1}, g_{e-2l_2,j_2}\varphi_{e-2l_2,i_2})$-entry of ${}^tM\,M$ is given by

$$\begin{aligned}
&\sum_{\boldsymbol{x}\in X^*} w(\boldsymbol{x}) g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2)\varphi_{e-2l_1,i_1}(\boldsymbol{x}) g_{e-2l_2,j_2}(\|\boldsymbol{x}\|^2)\varphi_{e-2l_2,i_2}(\boldsymbol{x}) \\
&= \frac{1}{2}\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|}\int_{S_i} g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2) g_{e-2l_2,j_2}(\|\boldsymbol{x}\|^2)\varphi_{e-2l_1,i_1}(\boldsymbol{x})\varphi_{e-2l_2,i_2}(\boldsymbol{x})d\sigma_i(\boldsymbol{x}) \\
&= \frac{1}{2}\sum_{i=1}^{p} \frac{w(X_i) g_{e-2l_1,j_1}\left(r_i{}^2\right) g_{e-2l_2,j_2}\left(r_i{}^2\right) r_i^{2(e-l_1-l_2)}}{|S^{n-1}|}\int_{S^{n-1}} \varphi_{e-2l_1,i_1}(\boldsymbol{x})\varphi_{e-2l_2,i_2}(\boldsymbol{x})d\sigma(\boldsymbol{x}) \\
&= \frac{1}{2}\delta_{l_1,l_2}\delta_{i_1,i_2}\sum_{i=1}^{p} w(X_i) r_i^{2(e-2l_1)} g_{e-2l_1,j_1}\left(r_i{}^2\right) g_{e-2l_1,j_2}\left(r_i{}^2\right) \\
&= \frac{1}{2}\delta_{l_1,l_2}\delta_{i_1,i_2}\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2(e-2l_1)} g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2) g_{e-2l_1,j_2}(\|\boldsymbol{x}\|^2) \\
&= \delta_{l_1,l_2}\delta_{i_1,i_2}\sum_{\boldsymbol{x}\in X^*} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2(e-2l_1)} g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2) g_{e-2l_1,j_2}(\|\boldsymbol{x}\|^2) = \delta_{l_1,l_2}\delta_{i_1,i_2}\delta_{j_1,j_2}.
\end{aligned}$$

Next we consider the case when $\mathbf{0} \in X$. Let $S_p = \{\mathbf{0}\}$.

Since $X\backslash\{\mathbf{0}\}$ is also a $(2e+1)$-design we have the following:

(i) If $e - 2l_1 > 0$ or $e - 2l_2 > 0$ holds, then

$$\begin{aligned}
&\sum_{\boldsymbol{x}\in X^*} w(\boldsymbol{x}) g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2)\varphi_{e-2l_1,i_1}(\boldsymbol{x}) g_{e-2l_2,j_2}(\|\boldsymbol{x}\|^2)\varphi_{e-2l_2,i_2}(\boldsymbol{x}) \\
&= w(\mathbf{0}) g_{e-2l_1,j_1}(\mathbf{0}) g_{e-2l_2,j_2}(\mathbf{0})\varphi_{e-2l_1,i_1}(\mathbf{0})\varphi_{e-2l_2,i_2}(\mathbf{0})
\end{aligned}$$

$$+\frac{1}{2}\sum_{\boldsymbol{x}\in X\setminus\{\boldsymbol{0}\}} w(\boldsymbol{x})g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2)\varphi_{e-2l_1,i_1}(\boldsymbol{x})g_{e-2l_2,j_2}(\|\boldsymbol{x}\|^2)\varphi_{e-2l_2,i_2}(\boldsymbol{x}).$$

$$=\frac{1}{2}\sum_{i=1}^{p-1}\frac{w(X_i)}{|S_i|}\int_{S_i} g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2)g_{e-2l_2,j_2}(\|\boldsymbol{x}\|^2)\varphi_{e-2l_1,i_1}(\boldsymbol{x})\varphi_{e-2l_2,i_2}(\boldsymbol{x})d\sigma_i(\boldsymbol{x})$$

$$=\frac{1}{2}\sum_{i=1}^{p-1}\frac{w(X_i)g_{e-2l_1,j_1}\left({r_i}^2\right)g_{e-2l_2,j_2}\left({r_i}^2\right)r_i^{2(e-l_1-l_2)}}{|S^{n-1}|}\int_{S^{n-1}}\varphi_{e-2l_1,i_1}(\boldsymbol{x})\varphi_{e-2l_2,i_2}(\boldsymbol{x})d\sigma(\boldsymbol{x})$$

$$=\frac{1}{2}\delta_{l_1,l_2}\delta_{i_1,i_2}\sum_{i=1}^{p} w(X_i)r_i^{2(e-2l_1)}g_{e-2l_1,j_1}\left({r_i}^2\right)g_{e-2l_1,j_2}\left({r_i}^2\right)$$

$$=\frac{1}{2}\delta_{l_1,l_2}\delta_{i_1,i_2}\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2(e-2l_1)}g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2)g_{e-2l_1,j_2}(\|\boldsymbol{x}\|^2)$$

$$=\delta_{l_1,l_2}\delta_{i_1,i_2}\sum_{\boldsymbol{x}\in X^*} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2(e-2l_1)}g_{e-2l_1,j_1}(\|\boldsymbol{x}\|^2)g_{e-2l_1,j_2}(\|\boldsymbol{x}\|^2)=\delta_{l_1,l_2}\delta_{i_1,i_2}\delta_{j_1,j_2}.$$

(ii) If $e-2l_1=e-2l_2=0$, then

$$\sum_{\boldsymbol{x}\in X^*} w(\boldsymbol{x})g_{0,j_1}(\|\boldsymbol{x}\|^2)g_{0,j_2}(\|\boldsymbol{x}\|^2)=\delta_{j_1,j_2}.$$

This completes the proof. □

Proof of Lemma 1.7

Since $|X^*|=\dim(\mathcal{P}_e^*(S))$, M is a square matrix. Hence Proposition 3.1 implies that M is a regular matrix and ${}^tM\ M=M\ {}^tM=I$.
If $\boldsymbol{x},\boldsymbol{y}\neq\boldsymbol{0}$, then the $(\boldsymbol{x},\boldsymbol{y})$ entry of $M\ {}^tM$ is given by the following:

$$\sum_{l,j,i}\sqrt{w(\boldsymbol{x})w(\boldsymbol{y})}g_{e-2l,j}(\|\boldsymbol{x}\|^2)g_{e-2l,j}(\|\boldsymbol{y}\|^2)\varphi_{e-2l,i}(\boldsymbol{x})\varphi_{e-2l,i}(\boldsymbol{y})$$

$$=\sum_{l,j}\sqrt{w(\boldsymbol{x})w(\boldsymbol{y})}(\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^{(e-2l)}g_{e-2l,j}(\|\boldsymbol{x}\|^2)g_{e-2l,j}(\|\boldsymbol{y}\|^2)\sum_i\varphi_{e-2l,i}\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right)\varphi_{e-2l,i}\left(\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right)$$

$$=\sum_{l,j}\sqrt{w(\boldsymbol{x})w(\boldsymbol{y})}(\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^{(e-2l)}g_{e-2l,j}(\|\boldsymbol{x}\|^2)g_{e-2l,j}(\|\boldsymbol{y}\|^2)Q_{e-2l}\left(\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|},\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right)\right),$$

where Q_l is the Gegenbauer polynomial of degree l (see [2, 3, 5, 7] for the explicit definition). This implies

$$w(\boldsymbol{x})\sum_{l,j}\|\boldsymbol{x}\|^{2(e-2l)}g_{e-2l,j}(\|\boldsymbol{x}\|^2)^2Q_{e-2l}(1)=1, \tag{3.1}$$

for any $\boldsymbol{x} \in X^*$ satisfying $\boldsymbol{x} \neq \boldsymbol{0}$ and

$$\sum_{l,j} (\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^{(e-2l)} g_{e-2l,j}(\|\boldsymbol{x}\|^2) g_{e-2l,j}(\|\boldsymbol{y}\|^2) Q_{e-2l}\left(\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right)\right) = 0, \tag{3.2}$$

for any $(\boldsymbol{x}, \boldsymbol{y}) \in X^* \times X^*$ satisfying $\boldsymbol{x}, \boldsymbol{y} \neq 0$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Then Eq. (3.1) implies that $w(\boldsymbol{x})$ only depends on $\|\boldsymbol{x}\|^2$. Hence $w(\boldsymbol{x})$ is constant on each X_i (Lemma 1.7(1)). Equation (3.2) implies that for any $\boldsymbol{x},\ \boldsymbol{y} \in X_i^*$ the inner product $(\boldsymbol{x}, \boldsymbol{y})$ is a root of the same polynomial of degree at most e. Hence, each X_i^* is an at most e-distance set (Lemma 1.7(2)). If w is constant on $X\backslash\{\boldsymbol{0}\}$, then all the $r_i \neq 0$ are roots of the same Eq. (3.1) of degree at most e. This implies Lemma 1.7(4). Next we prove Lemma 1.7(3). Let $r_i > 0$ and $I(X_i) = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x}, \boldsymbol{y} \in X_i,\ \boldsymbol{x} \neq \boldsymbol{y}\}$. Since X_i is on a sphere, we have $|I(X_i)| = |A(X_i)|$. For any $\boldsymbol{x},\ \boldsymbol{y} \in X_i^*$, Eq. (3.2) is a polynomial of $(\boldsymbol{x}, \boldsymbol{y})$ with degree at most e and the coefficients are functions of r_i. This means Eq. (3.2) depends only on i and does not depend on the choice of X_i^*. Let $\boldsymbol{x}, \boldsymbol{y} \in X_i$. Assume $\boldsymbol{x} \neq \pm\boldsymbol{y}$. Then there is a subset $Z \subset X_i$ satisfying $\boldsymbol{x}, \boldsymbol{y} \in Z$, $Z \cup (-Z) = X_i$ and $Z \cap (-Z) = \{\boldsymbol{0}\}$ or $\emptyset$. Then $(\boldsymbol{x}, \boldsymbol{y})$ is a root of Eq. (3.2). Hence $I(X_i) \subset \{-{r_i}^2\} \cup$ (set of all the real roots of the Eq. (3.2)). This implies $|I(X_i)| \leq e + 1$ and X_i is at most an $(e+1)$-distance set.

4. Antipodal Euclidean tight 3-design

In this section we will prove Theorem 1.5.

Let X be an antipodal tight Euclidean 3-design. Then $|X^*| = n$. If $\boldsymbol{0} \in X$, then $p \geq 2$ and $Y = X\backslash\{\boldsymbol{0}\}$ is also an antipodal Euclidean 3-design. Then Y is supported by $p - 1(\geq 1)$ concentric spheres and $|Y^*| = |X^*\backslash\{\boldsymbol{0}\}| = n - 1$. This is a contradiction. Hence $\boldsymbol{0} \notin X$ holds. Thus $r_i > 0$ for $i = 1, \ldots, p$. In this case $\mathcal{H}^* = \mathcal{H}_1$ holds. Since w is constant on X_i for each i, let $w(\boldsymbol{x}) = w_i$ for $\boldsymbol{x} \in X_i$. Let $R_i = {r_i}^2$. With these notation we have

$$g_{1,0} = \frac{1}{\sqrt{\sum_{i=1}^{p} N_i w_i R_i}}.$$

By scaling we may assume

$$\sum_{j=1}^{p} N_j w_j R_j = 1.$$

Then $g_{1,0} = 1$ and Eq. (3.1) implies

$$w_i R_i Q_1(1) = 1, \quad \text{hence} \quad w_i = \frac{1}{nR_i}$$

for $i = 1, \ldots, p$, and Eq. (3.2) implies $(\boldsymbol{x}, \boldsymbol{y}) = 0$ holds for any $\boldsymbol{x},\ \boldsymbol{y} \in X^*$ and $\boldsymbol{x} \neq \boldsymbol{y}$. We may assume

$$X_1^* = \{r_1 \boldsymbol{e}_i \mid 1 \leq i \leq N_1\}.$$

Let $\boldsymbol{x} \in X_2^*$. Then $(\boldsymbol{x}, \boldsymbol{e}_i) = 0$ holds for any $i = 1, \ldots, N_1$. By a transformation in the orthogonal group $O(n)$, we may assume $\boldsymbol{x} = r_2 \boldsymbol{e}_{N_1+1}$. By continuing this argument we may assume

$$X_i^* = \left\{ r_i \boldsymbol{e}_j \mid 1 + \sum_{l=1}^{i-1} N_l \leq j \leq \sum_{l=1}^{i} N_l \right\}.$$

Hence

$$X_i = \left\{ \pm r_i \boldsymbol{e}_j \mid 1 + \sum_{l=1}^{i-1} N_l \leq j \leq \sum_{l=1}^{i} N_l \right\}$$

and

$$w(\boldsymbol{x}) = \frac{1}{n {r_i}^2}$$

for $\boldsymbol{x} \in X_i$ and $i = 1, \ldots, p$. This completes the proof.

5. Antipodal Euclidean tight 5-design with $p = 2$

Let X be an antipodal Euclidean tight 5-design. Then $p \geq 2$ and $|X^*| = \dim(\mathcal{P}_2^*(\mathbb{R}^n)) = \binom{n+1}{2} + 1$. In this section we consider the case $p = 2$. Assume $\boldsymbol{0} \in X$. Then $X \backslash \{\boldsymbol{0}\}$ is similar to an antipodal spherical 5-design. Hence $X \backslash \{\boldsymbol{0}\}$ is a spherical tight 5-design. In the following we assume $\boldsymbol{0} \notin X$.

By assumption we have $|X^*| = \frac{n^2+n+2}{2}$. We use the same notation given before. By Lemma 1.7, $w(\boldsymbol{x})$ is constant on each X_i^*, $i = 1, 2$. Let $w_i = w(\boldsymbol{x})$ for $\boldsymbol{x} \in X_i^*$ and $i = 1, 2$. By a similar transformation of $\mathbb{R}^n$ and a multiplication with a positive real number to the weight w we may assume $|X_1| \leq |X_2|$, $r_1 = w_1 = 1$. Let $w = w_2$ and $r = r_2 \neq 1$. Let $R = r^2$ and $N_i = |X_i^*|,\ i = 1, 2$.

Theorem 1.8 implies that X_i is an antipodal spherical 3-design and distance invariant. Hence Theorem 5.12 in [5] implies $|X_i| \geq 2n$ and the graph defined on X_i by any one of the distances in $A(X_i)$ is regular. Therefore we have

$$n \leq N_1 \leq \frac{n^2 + n + 2}{4} \leq N_2 \leq \frac{n^2 - n + 2}{2}.$$

The polynomials $g_{l,j}(\|\boldsymbol{x}\|^2)$ are given by the following formula:

$$g_{0,0}(\|\boldsymbol{x}\|^2) \equiv \frac{2}{\sqrt{4(1-w)N_1 + 2w(n^2+n+2)}},$$

$$g_{0,1}(\|\boldsymbol{x}\|^2) = \sqrt{\frac{2N_1 + w(n^2+n+2-2N_1)}{N_1 w(R-1)^2(n^2+n+2-2N_1)}}$$
$$\times \frac{(2N_1 + w(n^2+n+2-2N_1))\|\boldsymbol{x}\|^2 - 2N_1 - wR(n^2+n+2-2N_1)}{2N_1 + w(n^2+n+2-2N_1)},$$

$$g_{2,0}(\|\boldsymbol{x}\|^2) \equiv \frac{2}{\sqrt{4N_1 + 2wR^2(n^2+n+2-2N_1)}}$$

Then Eq. (3.1) for $\boldsymbol{x} \in X_1^*$ and X_2^* implies the following Eqs. (5.1) and (5.2) respectively. Also Eq. (3.2) for $\boldsymbol{x},\ \boldsymbol{y} \in X_i^*, \boldsymbol{x} \neq \boldsymbol{y}$ implies the following (5.3) and (5.4) respectively. Equation (3.2) for $\boldsymbol{x} \in X_1^*,\ \boldsymbol{y} \in X_2^*$ implies the following Eq. (5.5).

$$\frac{wR^2(n^2+n+2-2N_1) + n(n+1)N_1}{(2N_1 + wR^2(n^2+n+2-2N_1))N_1} = 1, \tag{5.1}$$

$$\frac{wR^2 n(n+1)(n^2+n+2-2N_1) + 4N_1}{(n^2+n+2-2N_1)(2N_1 + wR^2(n^2+n+2-2N_1))} = 1, \tag{5.2}$$

$$\frac{nN_1(n+2)A^2 - 4nN_1(n+2)A + 4wR^2(n^2+n+2-2N_1) + 4nN_1(1+n)}{4N_1(2N_1 + wR^2(n^2+n+2-2N_1))} = 0,$$
$$\text{where } A = \|\boldsymbol{x}-\boldsymbol{y}\|^2 \neq 0,\ \boldsymbol{x},\ \boldsymbol{y} \in X_1^*), \tag{5.3}$$

$$(wn(n+2)(n^2+n+2-2N_1)B^2 - 4wRn(n+2)(n^2+n+2-2N_1)B$$
$$+ 4wR^2 n(n+1)(n^2+n+2-2N_1) + 16N_1)$$
$$\times \frac{1}{4w(n^2+n+2-2N_1)(2N_1 + wR^2(n^2+n+2-2N_1))} = 0,$$
$$\text{where } B = \|\boldsymbol{x}-\boldsymbol{y}\|^2 \neq 0,\ \boldsymbol{x},\ \boldsymbol{y} \in X_2^*, \tag{5.4}$$

$$\frac{(n+2)(nC^2 - 2n(1+R)C + R^2 n + 2Rn - 4R + n)}{4(2N_1 + wR^2(n^2+n+2-2N_1))} = 0,$$
$$\text{where } C = \|\boldsymbol{x}-\boldsymbol{y}\|^2,\ \boldsymbol{x} \in X_1^*,\ \boldsymbol{y} \in X_2^*. \tag{5.5}$$

Then Eqs. (5.1) and (5.2) are both equivalent to the following equation.

$$w = \frac{N_1(n^2+n-2N_1)}{R^2(N_1-1)(n^2+n+2-2N_1)}. \tag{5.6}$$

By substituting this value w in Eqs. (5.3), (5.4) and (5.5) we obtain following Eqs. (5.7), (5.8) and (5.9) respectively.

$$n(N_1-1)A^2-4n(N_1-1)A+4N_1(n-1)=0, \tag{5.7}$$

$$\begin{aligned}&(n^2+n-2N_1)nB^2-4Rn(n^2+n-2N_1)B\\&\quad+4R^2(n-1)(n^2+n+2-2N_1)=0,\end{aligned} \tag{5.8}$$

$$nC^2-2n(R+1)C+R^2n+2Rn-4R+n=0. \tag{5.9}$$

By solving the above equations we obtain

$$A=\frac{2n(N_1-1)\pm2\sqrt{n(N_1-1)(N_1-n)}}{n(N_1-1)} \tag{5.10}$$

$$B=\frac{2R\big(n(n^2+n-2N_1)\pm\sqrt{n(n^2-n+2-2N_1)(n^2+n-2N_1)}\big)}{n(n^2+n-2N_1)} \tag{5.11}$$

$$C=\frac{n(R+1)\pm2\sqrt{nR}}{n} \tag{5.12}$$

Since $n\le N_1\le\frac{n^2+n+2}{4}\le\frac{n^2-n+2}{2}\le\frac{n^2+n}{2}$, we have positive solutions A, B and C.

Let A_1 and A_2 be the two solutions of Eq. (5.7) and B_1 and B_2 be the two solutions of Eq. (5.8). Assume $A_1\le A_2$ and $B_1\le B_2$. If $N_1=n$, then $A_1=A_2$ holds. Also if $n=2$, then $N_1=N_2=2$ and $B_1=B_2$ holds. If $n\ge5$, then $N_2>n+1$ holds and X_2^* is a 2-distance set. For any $A_1<A_2$ and $B_1<B_2$ we define positive real numbers k_A and k_B by

$$\frac{A_1}{A_2}=\frac{k_A-1}{k_A}, \tag{5.13}$$

$$\frac{B_1}{B_2}=\frac{k_B-1}{k_B}. \tag{5.14}$$

By definition $k_A,k_B>1$ holds. Then if $N_1>2n+3$, then Theorem 2 in [8] implies that k_A and k_B are integers. If $n\ge9$, then $N_2\ge\frac{n^2+n+2}{4}>2n+3$ holds. Therefore, k_B is an integer for $n\ge9$. Equations. (5.13), (5.14), (5.10) and (5.11) imply the following equations.

$$(2k_A-1)^2=\left(\frac{A_2+A_1}{A_2-A_1}\right)^2=\frac{n(N_1-1)}{N_1-n} \tag{5.15}$$

$$(2k_B-1)^2=\left(\frac{B_2+B_1}{B_2-B_1}\right)^2=\frac{n(n^2+n-2N_1)}{n^2-n+2-2N_1}. \tag{5.16}$$

Next, we assume $n \geq 9$ and study the behavior of the functions $G_A(x)$ and $G_B(x)$ defined by

$$G_A(x) = \frac{n(x-1)}{x-n}, \tag{5.17}$$

$$G_B(x) = \frac{n(n^2+n-2x)}{n^2-n+2-2x} \tag{5.18}$$

for $n \leq x \leq \frac{n^2+n+2}{4}$.

Proposition 5.1. *Let $n \geq 9$. Then*

$$n+2 < G_B(x) < n+6$$

holds for $n \leq x \leq \frac{n^2+n+2}{4}$.

Proof: We have $\frac{dG_B(x)}{dx} = \frac{4n(n-1)}{(n^2-n+2-2x)^2} > 0$, $G_B(n) = \frac{n^2}{n-2} > n+2$ and $G_B(\frac{n^2+n+2}{4}) = \frac{n(n+2)}{n-2} < n+6$. □

Proposition 5.2. *If there exists an antipodal Euclidean tight 5-design in $\mathbb{R}^n$ supported by 2 concentric spheres centered at the origin and $\mathbf{0} \notin X$, then $n \leq 8$ holds.*

Proof: Assume $n \geq 14$. By Proposition 5.1 we have

$$(2k_B-1)^2 = G_B(N_1) \in \{n+3, n+4, n+5\}.$$

(i) If $(2k_B-1)^2 = G_B(N_1) = n+3$, then $n = 4k_B^2 - 4k_B - 2$ and

$$N_1 = \frac{n^2-n+6}{6} = 2 + \frac{2}{3}k_B(k_B-1)\left(4k_B^2 - 4k_B - 5\right)$$

is an integer. Since $n \geq 14$, we have $\frac{n^2-n+6}{6} > 2n+3$. Hence X_1^* is a 2-distance set and

$$(2k_A-1)^2 = G_A(N_1) = G_A\left(\frac{n^2-n+6}{6}\right) = \frac{n^2}{n-6} = n+6+\frac{36}{n-6}$$

is an integer. Therefor $n \in \{15,\ 18,\ 24,\ 42\}$. If $n \in \{15, 42\}$, then k_B cannot be an integer. If $n \in \{18, 24\}$, then k_A cannot be an integer.

(ii) If $(2k_B-1)^2 = G_B(N_1) = n+4$, then $n = 4{k_B}^2 - 4k_B - 3$ and

$$N_1 = \frac{n^2-n+4}{4} = 4{k_B}^4 - 8{k_B}^3 - 3{k_B}^2 + 7k_B + 4.$$

Since $n \geq 14$, $N_1 > 2n+3$. Hence X_1^* is a 2-distance set and k_A is an integer. On the other hand we have

$$(2k_A - 1)^2 = G_A(N_1) = G_A\left(\frac{n^2-n+4}{4}\right) = \frac{n^2}{n-4} = n+4+\frac{16}{n-4}.$$

Hence $n = 20$. Then k_B cannot be an integer.

(iii) If $(2k_B-1)^2 = G_B(N_1) = n+5$, then $n = 4{k_B}^2 - 4k_B - 4$ and $N_1 = \frac{3n^2-3n+10}{10}$. Since $n \geq 14$, $\frac{3n^2-3n+10}{10} > 2n+3$. Hence X_1^* is a 2-distance set and k_A is an integer. On the other hand we have

$$(2k_A - 1)^2 = G_A(N_1) = G_A\left(\frac{3n^2-3n+10}{10}\right) = n+3+\frac{n+30}{3n-10}.$$

Since $G_A(N_1)$ is an integer we have $n \leq 20$. However k_A cannot be an integer for any n satisfying $14 \leq n \leq 20$.

Next we assume $9 \leq n \leq 13$. Then $N_2 \geq \frac{n^2+n+2}{4} > 2n+3$ holds. Hence X_2^* is a 2-distance set and k_B is an integer. Proposition 5.1 implies that $G_B(N_1) = n+3,\ n+4$ or $n+5$.

(a) If $G_B(N_1) = n+3$, then $n = 4{k_B}^2 - 4k_B - 2$ as we have seen in the proof of (i). There is no integer k_B which satisfies $n = 4{k_B}^2 - 4k_B - 2$ for $9 \leq n \leq 13$
(b) If $G_B(N_1) = n+4$, then $n = 4{k_B}^2 - 4k_B - 3$ as we have seen in the proof of (ii). There is no integer k_B which satisfies $n = 4{k_B}^2 - 4k_B - 3$ for $9 \leq n \leq 13$.
(c) If $G_B(N_1) = n+5$, then $n = 4{k_B}^2 - 4k_B - 4$ as we have seen in the proof of (iii). There is no integer k_B which satisfies $n = 4{k_B}^2 - 4k_B - 4$ for $9 \leq n \leq 13$. This completes the proof of Proposition 5.2

□

In the following we discuss the case $2 \leq n \leq 8$ and prove Theorem 1.6. To eliminate the possibilities of existence of antipodal tight Euclidean 5-designs, we used a computer for calculation. Theorem 1.8 implies that each layer X_i of an antipodal tight Euclidean 5-design supported by 2 concentric spheres is a distance invariant set.

Let $I(X_i) = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x}, \boldsymbol{y} \in X_i,\ \boldsymbol{x} \neq \boldsymbol{y}\}$. If X_i is a 3-distance set, then $I(X_i) = \{-{r_i}^2, \alpha {r_i}^2, -\alpha {r_i}^2\}$ with a positive real number α. Let $\boldsymbol{u}_1 \in X_i$. Since X_i is antipodal and $(\boldsymbol{x}, \boldsymbol{u}_1) = \alpha {r_i}^2$ holds if and only if $(-\boldsymbol{x}, \boldsymbol{u}_1) = -\alpha {r_i}^2$, we have

$$X_i = \left\{\boldsymbol{x} \in X_i \mid (\boldsymbol{u}_1, \boldsymbol{x}) = \alpha {r_i}^2\right\} \cup \left\{\boldsymbol{x} \in X_i \mid (\boldsymbol{u}_1, \boldsymbol{x}) = -\alpha {r_i}^2\right\} \cup \{\boldsymbol{u}_1, -\boldsymbol{u}_1\}.$$

Hence if X_1 (X_2 resp.) is a 3-distance set, then the graph defined on X_i by the distance $\sqrt{A_1}$, or $\sqrt{A_2}$ ($\sqrt{B_1}$, or $\sqrt{B_2}$ resp.) is of valency $N_1 - 1$ ($N_2 - 1$ resp.).

If X_i is a 2-distance set then every distinct vectors in X_i^* are perpendicular to each other.

Case $n = 8$. We have $|X^*| = 37$, $8 \leq N_1 \leq 18 < 19 \leq N_2$.

(i) If $N_1 \leq 17$, then $N_2 \geq 20 > 2 \cdot 8 + 3$. Hence L-R-S's theorem implies that $G_B(N_1)$ is the square of an odd integer. Since $n = 8$ implies $G_B(N_1) = 8 + \frac{56}{29-N_1}$,

$G_B(N_1)$ cannot be an integer for any N_1 with $8 \le N_1 \le 17$. Hence this case does not occur.

(ii) If $N_1 = 18$ then $N_2 = 19$. In this case we have $(\boldsymbol{x}, \boldsymbol{y}) = \pm\frac{\sqrt{11}}{12}R$ or $-R$ for any $\boldsymbol{x}, \boldsymbol{y} \in X_2$. Fix $\boldsymbol{u}_1 \in X_2$. As we mentioned above we may assume

$$X_2^* = \{\boldsymbol{u}_1\} \cup \left\{\boldsymbol{x} \in X_2 \mid (\boldsymbol{x}, \boldsymbol{u}_1) = \frac{\sqrt{11}}{12}R\right\}$$

Since $N_2 = 19$, X_2^* is a 2-distance set. Let $\boldsymbol{u}_2, \ldots, \boldsymbol{u}_5 \in X_2^*$ satisfying $(\boldsymbol{u}_1, \boldsymbol{u}_i) = \frac{\sqrt{11}}{12}R$. Then $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_5\}$ is a five point 1 or 2-distance set whose distances are among $\{\sqrt{(2-\frac{\sqrt{11}}{6})R}, \sqrt{(2+\frac{\sqrt{11}}{6})R}\}$. There are eleven possible configurations between $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_5\}$. For each of the configurations we did computer computation to find out there is no 2-distance set having more than 9 points and containing $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_5\}$.

Case $n = 7$. We have $|X^*| = 29$, $7 \le N_1 \le 14 < 15 \le N_2$.

(i) If $N_1 \le 11$, then $N_2 \ge 18 > 2 \cdot 7 + 3$. On the other hand $G_B(x) = 7 + \frac{42}{22-x}$. Hence $G_B(x)$ cannot be an integer for any N_1 with $7 \le N_1 \le 11$. Hence, this case does not occur.
The following are the remaining cases for $n = 7$.

(ii) $N_1 = 12$, then $N_2 = 17$

(iii) $N_1 = 13$, then $N_2 = 16$.

(iv) $N_1 = 14$, then $N_2 = 15$.

In each case X_i is a 3-distance set. Let $\boldsymbol{u}_1 \in X_1$ and $X_1^* = \{\boldsymbol{u}_1\} \cup \{\boldsymbol{u} \in X_1 \mid \|\boldsymbol{u} - \boldsymbol{u}_1\| = A_1\}$. Then X_1^* is a 2-distance set with $A(X_1^*) = \{\sqrt{A_1}, \sqrt{A_2}\}$. We apply the same method explained in the case $n = 8$ and found out there is no 2-distance satisfying the required conditions.

Case $n = 6$. We have $|X^*| = 22$, $6 \le N_1 \le 11 \le N_2$.

(i) If $N_1 = 6$, then $N_2 = 16 > 2 \cdot 6 + 3$, $k_B = 3$, $A_1 = A_2 = 2$, $B_1 = \frac{4R}{3}$, $B_2 = \frac{8R}{3}$, $C_1 = R + 1 - \sqrt{\frac{2}{3}}\sqrt{R}$, and $C_2 = R + 1 + \sqrt{\frac{2}{3}}\sqrt{R}$. Hence X_1^* is a six point 1-distance set on S^5 and every distinct $\boldsymbol{x}, \boldsymbol{y} \in X_1^*$ satisfy $(\boldsymbol{x}, \boldsymbol{y}) = 0$. Therefore we may assume $X_1^* = \{\boldsymbol{e}_i \mid 1 \le i \le 6\}$, where $\boldsymbol{e}_i$ is the unit vector whose i-th entry is 1. Let

$$Y = \{v \in S_2 \mid \|v - \boldsymbol{e}_i\|^2 = C_1 \text{ or } C_2, \quad 1 \le i \le 6\}$$

and $\boldsymbol{v} = (v_1, v_2, \ldots, v_6) \in Y$. Then we obtain $v_i = \varepsilon_i \frac{\sqrt{R}}{\sqrt{6}}$ for $i = 1, \ldots, 6$, where $\varepsilon_i = 1$ or -1. Therefore

$$Y = \left\{ \frac{\sqrt{R}}{\sqrt{6}}(\varepsilon_1, \ldots, \varepsilon_6) \;\middle|\; \varepsilon_i = 1 \text{ or } -1, 1 \le i \le 6 \right\}$$

holds. Since X_2^* is a subset of Y and Y is an antipodal set, $X_2 \subset Y$. Moreover we may assume that X_2^* contains the vector in Y whose entries are all positive. If it is not so then apply an isometry defined by changing the sign of the canonical vectors. Since Y and $X_1 = X_1^* \cup (-X_1^*)$ are invariant under such an isometry, the image of X_2^* satisfies the condition. For $\boldsymbol{u} = (u_1, \ldots, u_6)$, $\boldsymbol{v} = (v_1, \ldots, v_6) \in Y$, $\|\boldsymbol{u} - \boldsymbol{v}\|^2 = \frac{4R}{3}$ holds if and only if $|\{i \mid u_i \neq v_i\}| = 2$ holds. Also $\|\boldsymbol{u} - \boldsymbol{v}\|^2 = \frac{8R}{3}$ holds if and only if $|\{i \mid u_i \neq v_i\}| = 4$ holds.
Therefore we obtain

$$X_2^* = \{\boldsymbol{u} \in Y \mid \quad |\{i|u_i > 0\}| \in \{0,\ 2,\ 4,\ 6\}\ \}.$$

Then $|X_2^*| = 16$ and $\|\boldsymbol{u} - \boldsymbol{v}\|^2 \in \{0,\ \frac{4R}{3},\ \frac{8R}{3}\}$ holds. Then $X = X_1^* \cup (-X_1^*) \cup X_2^* \cup (-X_2^*)$ is an antipodal Euclidean tight 5-design in $\mathbb{R}^6$.

(ii) If $N_1 = 7$, then $A_1 = \sqrt{\frac{5}{3}}$, $A_2 = \sqrt{\frac{7}{3}}$, $B_1 = (2 - \sqrt{\frac{3}{7}})R$, $B_2 = (2 + \sqrt{\frac{3}{7}})R$, and $C_1 = R + 1 - \sqrt{\frac{2}{3}}\sqrt{R}$, and $C_2 = R + 1 + \sqrt{\frac{2}{3}}\sqrt{R}$. Note that the length of the edges of a regular simplex on S^5 is $\sqrt{\frac{7}{3}}$. Moreover if $\boldsymbol{x},\ \boldsymbol{y} \in X_1$, then $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \frac{7}{3}$ holds if and only if $\|\boldsymbol{x} + \boldsymbol{y}\| = \frac{5}{3}$. Hence we can find out easily that the only possibility is when X_1 contains the regular simplex on S^5. Then we may assume that X_1^* is a regular simplex on S^5. We consider X_1^* in the intersection of S^6 and the subspace $V = \{(x_1, \ldots, x_7) \mid \sum_{i=1}^7 x_i = 0\} (\cong \mathbb{R}^6)$. Then we may assume that X_1^* consists of the following 7 points $\boldsymbol{u}_i = (u_{i,1}, \ldots, u_{i,7})$, $1 \leq i \leq 7$, defined by

$$u_{i,j} = \begin{cases} -\dfrac{6}{\sqrt{42}} & \text{if } j = i, \\ \dfrac{1}{\sqrt{42}} & \text{otherwise.} \end{cases}$$

Then X_2^* is a subset of the set Y defined by

$$Y = \{\boldsymbol{u} \in V \mid \|\boldsymbol{u}\|^2 = R, \|\boldsymbol{u} - \boldsymbol{u}_i\|^2 = C_1,\ \text{or } C_2,\ \ 1 \leq i \leq 7\}.$$

Then

$$Y = \left\{ \sqrt{\frac{R}{7}}\,(\varepsilon_1, \ldots, \varepsilon_7) \,\middle|\, \varepsilon_i = 1,\ \text{or } -1,\ 1 \leq i \leq 7 \right\}$$

holds. This contradicts the fact $Y \subset V$. Hence this case does not occur.

(iii) If $11 \geq N_1 \geq 8$, then X_1 is a 3-distance set. We apply the same method explained in the case $n = 8$ and find out that these cases do not occur.

Case $n = 5$. We have $|X^*| = 16$, $5 \leq N_1 \leq 8 \leq N_2$.

(i) If $N_1 = 6$, then $N_2 = 10$, $A_1 = \frac{8}{5}$, $A_2 = \frac{12}{5}$, $B_1 = \frac{4R}{3}$, $B_2 = \frac{8R}{3}$, $C_1 = R + 1 - 2\sqrt{\frac{R}{5}}$, and $C_2 = R + 1 + 2\sqrt{\frac{R}{5}}$.
Note that the length of the edges of a regular simplex on S^4 is $\sqrt{\frac{12}{5}}$. By a similar argument given in case $n = 6$ (ii), we may assume that X_1^* is a regular simplex on S^4. We consider X_1^* in the intersection of S^5 and the subspace

$V = \{(x_1, \ldots, x_6) \mid \sum_{i=1}^{6} x_i = 0\} (\cong \mathbb{R}^5)$. Then we may assume that X_1^* consists of the following 6 points $\boldsymbol{u}_i = (u_{i,1}, \ldots, u_{i,6})$, $1 \le i \le 6$, defined by

$$u_{i,j} = \begin{cases} -\dfrac{5}{\sqrt{30}} & \text{if } j = i, \\ \dfrac{1}{\sqrt{30}} & \text{otherwise.} \end{cases}$$

Then our X_2^* is a subset of V and the intersection of the sphere S_2. Let $\boldsymbol{v} = (v_1, \ldots, v_6) \in V \cap S_2$. Then $\|\boldsymbol{v} - \boldsymbol{u}_i\|^2 = R + 1 + 2\varepsilon\sqrt{\frac{R}{5}}$ implies $v_i = \varepsilon\sqrt{\frac{R}{6}}$. Let

$$Y = \{\boldsymbol{v} \in V \mid \|\boldsymbol{v}\|^2 = R,\ \|\boldsymbol{v} - \boldsymbol{u}_i\|^2 \in \{C_1, C_2\},\ 1 \le i \le 6\}.$$

Then we have

$$Y = \left\{ \sqrt{\frac{R}{6}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6) \,\middle|\, \varepsilon_i = 1 \text{ or } -1,\ |\{i \mid \varepsilon_i = 1\}| = 3 \right\}.$$

Since Y is antipodal, $|Y| = 20 = |X_2|$ and $X_2 = X_2^* \cup (-X_2^*) \subset Y$ we have $X_2 = Y$. Clearly X_2 is a 3-distance set with distance $\sqrt{\frac{8}{5}}\,r$, $\sqrt{\frac{12}{5}}\,r$, $2r$ and $X = X_1 \cup X_2$ is an antipodal Euclidean tight 5-design in $\mathbb{R}^5$.

(ii) If $N_1 = 5$, then $N_2 = 11$, $A_1 = A_2 = 2$, $B_1 = \frac{2}{5}(5 - \sqrt{3})R$, $B_2 = \frac{2}{5}(5 + \sqrt{3})R$, $C_1 = R + 1 - 2\sqrt{\frac{R}{5}}$ and $C_1 = R + 1 + 2\sqrt{\frac{R}{5}}$. Then we may assume $X_1^* = \{\boldsymbol{e}_i \mid i = 1, \ldots, 5\}$. Let Y be the set defined by

$$Y = \{\boldsymbol{u} \in S_2 \mid \|\boldsymbol{u} - \boldsymbol{e}_i\| = C_1, \text{ or } C_2,\ 1 \le i \le 5\}.$$

Then we have

$$Y = \left\{ \sqrt{\frac{R}{5}}\,(\varepsilon_1, \ldots, \varepsilon_5) \,\middle|\, \varepsilon_i = 1 \text{ or } -1, 1 \le i \le 5 \right\}.$$

Then X_2^* must be contained in Y. However the distance between every distinct two points in Y is not equal to B_1 or B_2. This is a contradiction.

The following are the remaining cases for $n = 5$:

(iii) $N_1 = 7, N_2 = 9$.

(iv) $N_1 = N_2 = 8$.

For each case we apply the same method as before and eliminate the possibilities of X_1^*.

Case $n = 4$. We have $|X| = 22$, $|X^*| = 11$, $4 \le N_1 \le 5 \le N_2$.

(i) If $N_1 = 4$, then $A_1 = A_2 = 2$, $B_1 = (2 - \frac{\sqrt{2}}{2})R$, $B_2 = (2 + \frac{\sqrt{2}}{2})R$, $C_1 = R + 1 - \sqrt{R}$ and $C_2 = R + 1 + \sqrt{R}$. Hence we may assume

$$X_1^* = \{\boldsymbol{e}_1 = (1, 0, 0, 0), \boldsymbol{e}_2 = (0, 1, 0, 0), \boldsymbol{e}_3 = (0, 0, 1, 0), \boldsymbol{e}_4 = (0, 0, 0, 1)\}.$$

Let

$$Y = \{\boldsymbol{u} \in S_2 \mid \|\boldsymbol{u} - \boldsymbol{e}_i\|^2 \in \{C_1, C_2\},\ 1 \leq i \leq 4\}.$$

Then X_2^* must be contained in Y. On the other hand we have

$$Y = \left\{ \frac{r}{2}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \,\middle|\, \varepsilon_i = 1 \text{ or } -1, 1 \leq i \leq 4 \right\}.$$

However if $\boldsymbol{u},\ \boldsymbol{v} \in Y$, then $\|\boldsymbol{u} - \boldsymbol{v}\|^2 \in \{0, 2R, 3R, 4R\}$. This is a contradiction.

(ii) Let $N_1 = 5$ and $N_2 = 6$. Then $A_1 = \frac{3}{2},\ A_2 = \frac{5}{2},\ B_1 = (2 - \sqrt{\frac{2}{5}})R,\ B_2 = (2 + \sqrt{\frac{2}{5}})R$, $C_1 = R + 1 - \sqrt{R}$ and $C_2 = R + 1 + \sqrt{R}$.

Note that the length of the edges of a regular simplex on S^3 is $\sqrt{\frac{5}{2}}$. By a similar argument given as before we may assume that X_1^* is a regular simplex on S^3 .

We consider X_1^* in the intersection of S^4 and the subspace V in $\mathbb{R}^5$ defined by $V = \{(x_1, \ldots, x_5) | \sum_{i=1}^{5} x_i = 0\}$. Then we may assume that X_1^* consists of the following 5 points $\boldsymbol{u}_i = (u_{i,1}, \ldots, u_{i,5}),\ 1 \leq i \leq 5$, defined by

$$\begin{cases} -\dfrac{2}{\sqrt{5}} & \text{if } j = i, \\ \dfrac{1}{2\sqrt{5}} & \text{otherwise.} \end{cases}$$

Then X_2^* must be contained in the set Y defined by

$$Y = \{\boldsymbol{u} \in V \mid \|\boldsymbol{u}\|^2 = R, \|\boldsymbol{u} - \boldsymbol{u}_i\|^2 = C_1, \text{ or } C_2, \text{ for } i = 1, \ldots, 5\}.$$

Then

$$Y = \left\{ \sqrt{\frac{R}{5}}\,(\varepsilon_1, \ldots, \varepsilon_5) \,\middle|\, \varepsilon_i = 1, \text{ or } -1 \text{ for } i = 1, \ldots, 5 \right\}$$

holds. This contradicts the fact $Y \subset V$. Hence this case does not occur.

Case $n = 3$. We have $|X| = 14$, $|X^*| = 7$, $N_1 = 3$ and $N_2 = 4$.

In this case $A_1 = A_2 = 2$, $B_1 = \frac{4R}{3}$, $B_2 = \frac{8R}{3}$, $C_1 = R + 1 - 2\sqrt{\frac{R}{3}}$ and $C_2 = R + 1 + 2\sqrt{\frac{R}{3}}$ hold. We may assume

$$X_1^* = \{\boldsymbol{e}_1 = (1, 0, 0), \boldsymbol{e}_2 = (0, 1, 0), \boldsymbol{e}_3 = (0, 0, 1)\}.$$

and $X_1 = X_1^* \cup (-X_1^*)$. Let

$$Y = \{\boldsymbol{u} \in S^2 \mid \|\boldsymbol{u} - \boldsymbol{e}_i\|^2 = C_1, \text{ or } C_2, \text{ for } 1 \leq i \leq 3\}.$$

Then we have

$$Y = \left\{ \frac{r}{\sqrt{3}}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \;\middle|\; \varepsilon_i = 1 \text{ or } -1, \text{ for } 1 \le i \le 3 \right\},$$

Then $X_2^* \subset Y$. Since Y is antipodal and $|Y| = 8$, we obtain $X_2 = X_2^* \cup (-X_2^*) = Y$. Clearly for any $\boldsymbol{u},\ \boldsymbol{v} \in Y$, $\|\boldsymbol{u} - \boldsymbol{v}\|^2 = \frac{4R}{3}, \frac{8R}{3}$ or $4R$ and $X = X_1 \cup Y$ is an antipodal Euclidean tight 5-design.

Case $n = 2$. We have $|X^*| = 4$, $N_1 = N_2 = 2$.

In this case let $X_1^* = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $X_2^* = \{\boldsymbol{u}_3, \boldsymbol{u}_4\}$. Since $\|\boldsymbol{u}_1 - \boldsymbol{u}_2\| = \sqrt{2}$, $\|\boldsymbol{u}_3 - \boldsymbol{u}_4\| = \sqrt{2R}$ and $\{C_1, C_2\} = \{R + 1 + \sqrt{2R}, R + 1 - \sqrt{2R}\}$, X is isometric to the following set consisting of 8 points.

$$X = X_1 \cup X_2, \quad X_1 = \{(\pm 1, 0), (0, \pm 1)\}, \quad X_2 = \left\{ \left(\pm \frac{r}{\sqrt{2}}, \pm \frac{r}{\sqrt{2}} \right) \right\}.$$

This is an antipodal Euclidean tight 5-design.

6. Concluding remark

One of the reason why the author chooses the order of the words in the name "antipodal Euclidean tight $(2e + 1)$-design" is because the known examples of Euclidean tight $2e$-designs are not Euclidean $(2e + 1)$-designs and she believes that the following conjecture holds.

Conjecture . Let X be a Euclidean $(2e + 1)$-design supported by p concentric spheres. Then the following holds:

$$|X^*| \ge \dim(\mathcal{P}_e^*(S)).$$

Moreover if equality holds above, then the weight function is constant on each X_i, $1 \le i \le p$, and X is antipodal.

If this conjecture is true then we can drop the word "antipodal" and define "tight $(2e + 1)$-design on p concentric spheres" and "Euclidean tight $(2e + 1)$-design."

To have a Euclidean tight $2e$-design or an antipodal Euclidean tight $(2e + 1)$-design we need to have $p \ge [\frac{e+\varepsilon_S}{2}] + 1$ or $p \ge [\frac{e}{2}] + 1$ respectively. Theorem 1.8 implies that if $p \le [\frac{e+2\varepsilon_S+3}{2}]$, then every layer X_i of a Euclidean tight $2e$-design or an antipodal Euclidean tight $(2e + 1)$-design is also a Euclidean $(2e - 2p + 2\varepsilon_S + 2)$-design or an antipodal Euclidean $(2e - 2p + 2\varepsilon_S + 3)$-design respectively which is distance invariant. These facts indicate that it is very important to study Euclidean tight $2e$-designs or antipodal Euclidean tight $(2e + 1)$-designs with $p = [\frac{e+\varepsilon_S}{2}] + 1$ or $p = [\frac{e}{2}] + 1$ respectively. (For example, if we assume $\varepsilon_S = 0$ for simplification, then $p = [\frac{e+\varepsilon_S}{2}] + 1 = [\frac{e}{2}] + 1$ implies $2e - 2p + 2\varepsilon_S + 2 = e$ or $e + 1$ according to e being even or odd. Therefore, in this case, each X_i of a Euclidean tight $2e$-design becomes a Euclidean e- or $(e+1)$-design and also at most an e-distance set. Consequently X_i is a distance invariant. If such a design exists it will be very interesting from a combinatorial view point.)

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