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Arbitrary groups as two-point stabilisers of symmetric groups acting on partitions

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Abstract We give a short, direct proof that given any finite group G there exist positive integers k and l and partitions α_1 and α_2 of $\{1, \dots, kl\}$ into l subsets of size k such that $(S_{kl})_{\alpha_1,\alpha_2} \cong G$.

The method used will also show that given any finite group G there exists a regular bipartite graph whose automorphism group is isomorphic to G.

Keywords Symmetric groups acting on partitions · Regular bipartite graphs · Two-point stabilisers

1. Introduction

Let k and l be positive integers such that $k, l \geq 2$ and let $\Omega_{k,l}$ be the set of all partitions of $\{1, \dots, kl\}$ into l distinct subsets of size k. In a previous paper [1] the author showed that there exist $\alpha_1, \alpha_2 \in \Omega_{k,l}$ such that $(S_{kl})_{\alpha_1,\alpha_2} = 1$ if and only if $k \geq 3$ and $l \geq \max\{8, k+3\}$. In light of this result it is natural to ask which finite groups can occur as two-point stabilisers of a symmetric group acting on partitions. This is answered by the following theorem.

Theorem 1.1. *Let G be a finite group. There exist positive integers k, l and partitions $\alpha_1, \alpha_2 \in \Omega_{k,l}$ such that $(S_{kl})_{\alpha_1,\alpha_2} \cong G$.*

In fact Theorem 1.1 is a corollary of a result of Kantor [3] who proved that given a finite group G there exists a symmetric design with automorphism group isomorphic to G. For a given G, the proof gives values of k and l which are both exponential in

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$|G|^2$. The aim of this paper is to give a simple direct proof of Theorem 1.1 in which far smaller values of k and l are necessary. We will prove the following result

Theorem 1.2. *Let G be a non-trivial finite group. Set $k = 4$ and $l = 2(r+1)|G|$ where r is the minimum number of elements needed to generate G. There exist α_1, $\alpha_2 \in \Omega_{k,l}$ such that $(S_{kl})_{\alpha_1,\alpha_2} \cong G$.*

1.1. Fences

In [1] the author defined fences. We recall the definition (in a slightly more general form) and the most important result.

Definition 1.3. Let Ω be a finite set and let k be a positive integer such that $2 \leq k < |\Omega|$. A (k, Ω)*-fence* is a collection Γ of $|\Omega|$ distinct k-subsets of Ω such that each member of Ω occurs in precisely k members of Γ.

Remark 1.4. In design theory terminology a (k, Ω)-fence is the blocks of a 1-$(|\Omega|, k, k)$ design.

Lemma 1.5. *Let Ω be a finite set and set $l = |\Omega|$. Let k be a positive integer such that $2 \leq k < l$ and Γ be a (k, Ω)-fence. There exist partitions α_Γ, $\beta_\Gamma \in \Omega_{k,l}$ such that $(S_{kl})_{\alpha_\Gamma,\beta_\Gamma} \cong \mathrm{Sym}(\Omega)_\Gamma$.*

Proof: See Lemma 2.3 and Lemma 3.3 in [1]. □

So to prove Theorem 1.2 it is enough, given G which can be generated by r elements, to construct a $(4, \Omega)$-fence Γ_G, where Ω is a set of size $2(r+1)|G|$, with stabiliser isomorphic to G.

Remark 1.6. The proof of Lemma 1.5 in [1] shows that given a (k, Ω)-fence we can construct a k-regular bipartite graph $\mathcal{G}$ such that $\mathrm{Aut_b}(\mathcal{G}) \cong \mathrm{Sym}(\Omega)_\Gamma$ where $\mathrm{Aut_b}(\mathcal{G})$ is the group of automorphisms of $\mathcal{G}$ fixing the bipartite blocks of $\mathcal{G}$. Some more work shows that there exists a regular bipartite graph such that $\mathrm{Aut}(\mathcal{G}) \cong \mathrm{Sym}(\Omega)_\Gamma$. See [2] for details.

2. Cyclic groups

The Main Construction (detailed in Section 3) requires at least two distinct elements that generate the group. So we deal with cyclic groups separately.

2.1. The group of size two

Set $\Gamma_{C_2} = \Delta_1 \cup \Delta_2$ where

$$\Delta_1 = \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{3, 4, 5, 6\}, \{4, 5, 6, 7\}, \{5, 6, 7, 8\}\}$$

$$\Delta_2 = \{\{6, 2, 8, 1\}, \{2, 8, 1, 7\}, \{8, 1, 7, 3\}\}.$$

We observe that Γ_{C_2} is a $(4, \{1, \ldots, 8\})$-fence.

Lemma 2.1. *The stabiliser of* Γ_{C_2} *in* S_8 *has size two.*

Proof: Consider a graph $\mathcal{K}_{C_2}$ with vertex set Γ_{C_2} in which two sets are joined by an edge if they intersect in three points. Elements of $(S_8)_{\Gamma_{C_2}}$ act as automorphisms of $\mathcal{K}_{C_2}$. The connected components of $\mathcal{K}_{C_2}$ are a line of length five and a line of length three corresponding to Δ_1 and Δ_2. So if $g \in (S_8)_{\Gamma_{C_2}}$ then g either fixes or reflects the line corresponding to Δ_1 hence g is either the identity or the involution $(1, 8)(2, 7)(3, 6)(4, 5)$. □

2.2. Other cyclic groups

Let n be an integer such that $n \geq 3$. Set $\Omega_{C_n} = \{0, \ldots, 4n - 1\}$ and $\Gamma_{C_n} = \bigcup_{i=0}^{n-1} \Delta_i$ where

$$\begin{aligned}\Delta_i = \{&\{4i, \ldots, 4i + 3\}, \{4i + 1, \ldots, 4i + 4\}, \{4i + 2, \ldots, 4i + 5\},\\ &\{4i + 3, \ldots, 4i + 10\}\}\end{aligned}$$

with addition modulo $4n$. We observe that Γ_{C_n} is a $(4, \Omega_{C_n})$-fence.

Lemma 2.2. *The stabiliser of* Γ_{C_n} *in* $\mathrm{Sym}(\Omega_{C_n})$ *is isomorphic to* C_n.

Proof: As above $\mathrm{Sym}(\Omega_{C_n})_{\Gamma_{C_n}} \leq \mathrm{Aut}(\mathcal{K}_{C_n})$ where $\mathcal{K}_{C_n}$ is a graph with vertices Γ_{C_n} in which two sets are joined if they intersect in three points. The connected components of $\mathcal{K}_{C_n}$ are n lines of length four corresponding to the sets $\Delta_0, \ldots, \Delta_{n-1}$. So $g \in \mathrm{Sym}(\Omega_{C_n})_{\Gamma_{C_n}}$ permutes the sets $\Delta_0, \ldots, \Delta_{n-1}$.

We observe that the permutation $\sigma \in \mathrm{Sym}(\Omega_{C_n})$ given by $i\sigma = i + 4 \bmod 4n$ stabilises Γ_{C_n} and sends Δ_j to Δ_{j+1} where $j \in [0, n - 1]$ and addition is modulo n. So $\mathrm{Sym}(\Omega_{C_n})_{\Gamma_{C_n}}$ is transitive on the connected components of $\mathcal{K}_{\Gamma_{C_n}}$ and contains a subgroup isomorphic to C_n. To show that $\mathrm{Sym}(\Omega_{C_n})_{\Gamma_{C_n}} \cong C_n$ it is enough to show that the stabiliser of Δ_0 in $(\mathrm{Sym}(\Omega_{C_n})_{\Gamma_{C_n}})$ is trivial.

Let $i \in [0, n - 1]$ and suppose $g \in \mathrm{Sym}(\Omega_{C_n})_{\Gamma_{C_n}}$ such that $\Delta_i g = \Delta_i$. So g either fixes or reflects the corresponding connected component. Hence g acts on $\mathrm{supp}(\Delta_i) = \{4i, \ldots, 4i + 5\} \cup \{4i + 10\}$ either as the identity or the involution $(4i, 4i + 10)(4i + 1, 4i + 5)(4i + 2, 4i + 4)$ where all addition is modulo $4n$. We observe that

- the elements $4i$ and $4i + 1$ are only contained in members of Δ_i and Δ_{i-1};
- the element $4i + 5$ is only contained in members of Δ_i and Δ_{i+1};
- the element $4i + 10$ is only contained in members of Δ_i and Δ_{i+2}

where addition in indices is modulo n and all other addition is modulo $4n$. Hence g can not act as an involution on $\mathrm{supp}(\Delta_i)$ otherwise it must send Δ_{i-1} to both Δ_{i+1} and Δ_{i+2} simultaneously. Therefore g is the identity on $\mathrm{supp}(\Delta_i)$ and must stabilise Δ_{i+1}. So the stabiliser of Δ_0 in $(\mathrm{Sym}(\Omega_{C_n})_{\Gamma_{C_n}})$ is trivial by induction. □

3. The main construction

Let G be a finite group which is not cyclic and r be the minimal number of elements required to generate G. Since G is not cyclic $r \geq 2$. Pick distinct $f(1), \ldots, f(r) \in G \setminus \{1\}$ such that $G = \langle f(i) \mid 1 \leq i \leq r \rangle$. Set $f(0) = f(1)$.

3.1. Definitions

In order to define a fence Γ_G with stabiliser G we require a G-space. Let $G_0, G_1, \ldots, G_r, \widehat{G_0}, \widehat{G_1}, \ldots, \widehat{G_r}$ be disjoint copies of G and set

$$\Omega_G = \bigcup_{i=0}^{r} (G_i \cup \widehat{G_i}).$$

We will write g_i (respectively $\widehat{g_i}$) for the element of G_i (respectively $\widehat{G_i}$) corresponding to $g \in G$. The group G acts on Ω_G by right multiplication:

$$g_i h = (gh)_i$$
$$\widehat{g_i} h = \widehat{(gh)_i} \qquad \text{for } g, h \in G \text{ and } i \in [0, r]$$

For each $i \in [0, r]$ we define

$$\mathcal{R}_i = R_i G \qquad \text{with} \qquad R_i = \{1_i, \widehat{1_i}, 1_{i+1}, \widehat{1_{i+1}}\}$$
$$\mathcal{T}_i = T_i G \qquad \text{with} \qquad T_i = \{\widehat{1_i}, 1_{i+1}, \widehat{1_{i+1}}, f(i)_{i+2}\}$$

where all addition is modulo $(r + 1)$. We define

$$\Gamma_G = \bigcup_{i=0}^{r} (\mathcal{R}_i \cup \mathcal{T}_i).$$

and observe that Γ_G is a $(4, \Omega_G)$-fence and a union of G-orbits, so $G \leq \mathrm{Sym}(\Omega_G)_{\Gamma_G}$.

3.2. Intersections

The proofs in the next section rely on the following detailed analysis of intersections between members of Γ_G.

Lemma 3.1. *Let $g \in G$ and $i \in [0, r]$. If $U \in \Gamma_G$ then*

$$|R_i g \cap U| = \begin{cases} 4 & \text{if } U = R_i g \\ 3 & \text{if } U = T_i g \\ 2 & \text{if } U \in \{T_{i-1}g, R_{i+1}g, R_{i-1}g\} \\ 1 & \text{if } U \in \{T_{i-1}f(i-1)^{-1}g, T_{i+1}g, T_{i-2}f(i-2)^{-1}g\} \\ 0 & \text{otherwise} \end{cases}$$

$$|T_i g \cap U| = \begin{cases} 4 & \textit{if } U = T_i g \\ 3 & \textit{if } U = R_i g \\ 2 & \textit{if } U = R_{i+1} g \\ 1 & \textit{if } U \in \{R_{i+1} f(i) g, T_{i-1} g, T_{i-1} f(i-1)^{-1} g, \\ & \qquad\qquad T_{i+1} g, T_{i+1} f(i) g, R_{i-1} g, R_{i+2} f(i) g\} \\ 0 & \textit{otherwise} \end{cases}$$

Proof: Each $\omega \in \Omega_G$ is contained in exactly 4 members of Γ_G and $|U| = 4$ so $\sum_{V \in \Gamma_G} |U \cap V| = 16$. Hence if there exists a positive integer s and distinct $V_1, \ldots, V_s \in \Gamma_G$ such that $\sum_{i=1}^{s} |U \cap V_i| = 16$ then $U \cap V = \emptyset$ for all $V \in \Gamma_G \setminus \{V_1, \ldots, V_s\}$. □

Corollary 3.2. *Let $g \in G$ and $i \in [0, r]$. If $h \in \mathrm{Sym}(\Omega_G)_{\Gamma_G}$ then there exist $g' \in G$ and an integer i' such that $0 \le i' \le r$ and $(R_i g)h = R_{i'} g'$.*

Proof: By Lemma 3.1, if $k \in G$ and $j \in [0, r]$ then exactly three members of Γ_G intersect $R_j k$ in two points whereas only one member of Γ_G intersects $T_j k$ in two points. □

Corollary 3.3. *Let g, $g' \in G$ and i, $i' \in [0, r]$.*
1. *If $h \in \mathrm{Sym}(\Omega_G)_{\Gamma_G}$ such that $(R_i g)h = R_{i'} g'$ then $(T_i g)h = T_{i'} g'$.*
2. *If $h \in \mathrm{Sym}(\Omega_G)_{\Gamma_G}$ such that $(T_i g)h = T_{i'} g'$ then $(R_{i+1} g)h = R_{i'+1} g'$.*

Proof: By Lemma 3.1, if $k \in G$ and $j \in [0, r]$ then $T_j k$ is the unique member of Γ_G which shares exactly three points with $R_j k$ and $R_{j+1} k$ is the unique member of Γ_G which shares two points with $T_j k$. □

3.3. The proof

We set $\mathcal{S} = \{R_i, T_i \mid 0 \le i \le r\}$ and observe that $\mathcal{S}G$ is a partition of Γ_G into $|G|$ sets of size $2(r+1)$.

Lemma 3.4. *Let $g \in G$. If $h \in \mathrm{Sym}(\Omega_G)_{\Gamma_G}$ then there exist an integer j and $g' \in G$ such that $(R_i g)h = R_{i+j} g'$ and $(T_i g)h = T_{i+j} g'$ for all integers $i \in [0, r]$. In particular, if $h \in \mathrm{Sym}(\Omega_G)_{\Gamma_G}$ then there exists $g' \in G$ such that $(\mathcal{S}g)h = \mathcal{S}g'$.*

Proof: Corollary 3.2 shows that there exists an integer j and $g' \in G$ such that $(R_0 g)h = R_j g'$. The rest follows by induction using Corollary 3.3. □

Lemma 3.5. *Let $g \in G$. If $h \in \mathrm{Sym}(\Omega_G)_{\Gamma_G}$ such that $(\mathcal{S}g)h = \mathcal{S}g$ then*
1. *$(R_i g)h = R_i g$;*
2. *$(T_i g)h = T_i g$;*
3. *$(\mathcal{S}(f(i)g))h = \mathcal{S}(f(i)g)$*

for all $i \in [0, r]$.

Proof: By Lemma 3.4 there exists an integer j such that $(R_i g)h = R_{i+j}g$ and $(T_i g)h = T_{i+j}g$ for all integers $i \in [0, r]$. The element $(f(s)g)_{s+2}$ is the unique member of $T_s g \setminus R_s g$ for all $s \in [0, r]$ so $(f(i)g)_{i+2}h = (f(i + j)g)_{i+j+2}$ for all i such that $0 \leq i \leq r$. Given s such that $0 \leq s \leq r$ the three members of $\Gamma_G \setminus \{T_s g\}$ which contain $(f(s)g)_{s+2}$ are all contained in $\mathcal{S}(f(s)g)$ hence $(\mathcal{S}(f(i)g))h = \mathcal{S}(f(i + j)g)$ by Lemma 3.4. So

$$\mathcal{S}(f(j)g) = (\mathcal{S}(f(0)g))h = (\mathcal{S}(f(1)g))h = \mathcal{S}(f(j + 1)g),$$

therefore $j = 0$. □

Lemma 3.6. *Let $g \in G$ and $i \in [0, r]$. The element g_i is the unique member of $R_i g \cap T_{i-2} f(i - 2)^{-1} g$ and $\widehat{g_i}$ is the unique member of $T_{i-1} g \cap T_i g$.*

Theorem 3.7. $\mathrm{Sym}(\Omega_G)_{\Gamma_G} \cong G$.

Proof: By Lemma 3.4 the orbit of $\mathcal{S}$ under $\mathrm{Sym}(\Omega_G)_{\Gamma_G}$ is the same as the orbit under G. Hence

$$|\mathrm{Sym}(\Omega_G)_{\Gamma_G}| = |G| \cdot |(\mathrm{Sym}(\Omega_G)_{\Gamma_G})_{\mathcal{S}}|$$

and since $G \leq \mathrm{Sym}(\Omega_G)_{\Gamma_G}$ it is enough to show that $(\mathrm{Sym}(\Omega_G)_{\Gamma_G})_{\mathcal{S}} = 1$.

Let $h \in \mathrm{Sym}(\Omega_G)_{\Gamma_G}$ such that $\mathcal{S}h = \mathcal{S}$. For $g \in G$ we will write $|g|$ for the length of the shortest word in $\{f(1), \ldots, f(r)\}$ which is equal to g. Induction on $|g|$ using Lemma 3.5 shows that $(\mathcal{S}g)h = \mathcal{S}g$ for all $g \in G$. Hence $Uh = U$ for all $U \in \Gamma_G$ by Lemma 3.5. So $\omega h = \omega$ for all $\omega \in \Omega_G$ since there exists $V, W \in \Gamma_G$ such that $V \cap W = \{\omega\}$ by Lemma 3.6. □

Theorem 3.7 completes the proof of Theorem 1.2.

Remark 3.8. The stabilser of the partitions $\alpha_{\Gamma_G}, \beta_{\Gamma_G} \in \Omega_{k,l}$ constructed from Γ_G by Lemma 1.5 can be shown to lie in A_{kl}. Hence $(A_{kl})_{\alpha_{\Gamma_G}, \beta_{\Gamma_G}} \cong G$. See [2] for details.

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