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Derivation modules of orthogonal duals of hyperplane arrangements

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Abstract Let A be an $n \times d$ matrix having full rank n. An orthogonal dual $A^{\perp}$ of A is a $(d - n) \times d$ matrix of rank $(d - n)$ such that every row of $A^{\perp}$ is orthogonal (under the usual dot product) to every row of A. We define the orthogonal dual for arrangements by identifying an essential (central) arrangement of d hyperplanes in n-dimensional space with the $n \times d$ matrix of coefficients of the homogeneous linear forms for which the hyperplanes are kernels. When $n \geq 5$, we show that if the matroid (or the lattice of intersection) of an n-dimensional essential arrangement $\mathcal{A}$ contains a modular copoint whose complement spans, then the derivation module of the orthogonally dual arrangement $\mathcal{A}^{\perp}$ has projective dimension at least $\lceil n(n + 2)/4 \rceil - 3$.

Keywords Hyperplane arrangement · Module of derivations · Projective dimension · Matroid · Orthogonal duality

1. Introduction

An important conjecture in the theory of hyperplane arrangements is *Terao's conjecture* [14]: whether the derivation module $D(\mathcal{A})$ of a central arrangement $\mathcal{A}$ is free depends only on the "combinatorics," that is to say, the matroid of $\mathcal{A}$. Since being free is equivalent to having zero projective dimension, a natural extension of Terao's conjecture is that the projective dimension $\mathrm{pdim}(D(\mathcal{A}))$ of $D(\mathcal{A})$ depends

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only on the matroid of $\mathcal{A}$. No counterexamples to this extended conjecture are known, although in [21] Ziegler gives two arrangements with the same matroid but non-isomorphic derivation modules. The common matroid in Ziegler's example has rank 3 and is the truncation of the orthogonal dual $M^{\perp}(K_{3,3})$ of the cycle matroid of the complete bipartite graph $K_{3,3}$. This matroid has two inequivalent representations, one from projecting a representation of $M^{\perp}(K_{3,3})$ from a point in general position, the other, a less special one which cannot be "erected". However, the derivation modules of these two arrangements both have projective dimension 1. Ziegler's examples provide one motivation to study derivation modules of orthogonal duals of arrangements. Another motivation comes from computer experiments suggesting that derivation modules of duals of free arrangements tend to have high projective dimension.

The notion of orthogonal duality is pervasive in combinatorics. H. Whitney first defined duality for matroids in his 1932 paper [16] to extend the notion of a dual or face graph of a planar graph to arbitrary graphs. He proved the theorem (equivalent to Kuratowski's theorem for planarity) that a graph is planar if and only if its matroid dual is a graphic matroid. Another example (suggested by a referee) is the concept of *association* introduced by Coble in [2]; for an application to generic arrangements see [5]. Duality also occurs in linear programming, combinatorial optimization, and coding theory. It is closely related to Alexandrov and other kinds of duality in algebraic topology. See, for example, [3, 4, 10].

Let X be a subspace in the lattice $L_{\mathcal{A}}$ of intersection of the arrangement $\mathcal{A}$. The *closed subarrangement* $\mathcal{A}_X$ is the subset of all hyperplanes in $\mathcal{A}$ containing X. When X is 1-dimensional, $\mathcal{A}_X$ is a *copoint*. A closed subarrangement $\mathcal{A}_X$ (or its associated subspace X) is *modular* if

$$\mathrm{rank}(X \vee Y) + \mathrm{rank}(X \cap Y) = \mathrm{rank}(X) + \mathrm{rank}(Y)$$

for every subspace Y in $L_{\mathcal{A}}$. Chains of modular flats occur (by definition) in supersolvable arrangements. In addition, it is easy to show by induction and the addition-deletion lemma (see [14] or [9], Chap. 4) that if an arrangement $\mathcal{A}$ has a modular copoint $\mathcal{A}_X$ which is free, then $\mathcal{A}$ itself is free.

A subarrangement of an essential arrangement *spans* if it is essential. The *ceiling* $\lceil x \rceil$ of a real number x is the the smallest integer greater than or equal to x. The *floor* $\lfloor x \rfloor$ of x is the largest integer less than or equal to x.

Our main result is the following theorem.

Theorem 1.1. *Let $\mathcal{A}$ be an essential arrangement over an arbitrary field with a modular copoint X such that its complement $\mathcal{A} \backslash \mathcal{A}_X$ spans. Suppose that the dimension n of $\mathcal{A}$ is at least 5. Then the projective dimension of the derivation module of the orthogonal dual $\mathcal{A}^{\perp}$ is bounded below by $\lceil n(n+2)/4 \rceil - 3$.*

The proof of Theorem 1.1 is combinatorial. We show that an arrangement $\mathcal{A}$ satisfying the main hypotheses in 1.1 contains a spanning subarrangement with the same matroid as the braid arrangement A_{n+1}. This implies that the dual $\mathcal{A}^{\perp}$ contains a closed circuit with at least $\lceil n(n+2)/4 \rceil$ hyperplanes. The proof is completed by combining a result of Terao on projective dimension of closed subarrangements with results of Rose and Terao, and Yuzvinsky on the projective dimension of generic arrangements.

2. Projective dimension of $D(\mathcal{A})$ and closed subarrangements

In this section, we discuss the two theorems from hyperplane arrangements we need. Both theorems hold over arbitrary fields.

A *generic arrangement* is an arrangement of at least $n+1$ hyperplanes in n-dimensional space for which every subset of n hyperplanes is independent. In particular, matroids of generic arrangements are uniform matroids. The following theorem is due to Rose and Terao [11] and Yuzvinsky [19].

Theorem 2.1. *If $\mathcal{A}$ is a generic arrangement in $\mathbb{k}^n$, then* $\operatorname{pdim}(D(\mathcal{A})) = n-2$.

We shall also use the following theorem of Terao [15] (see also [1]).

Theorem 2.2. *If $\mathcal{A}_X$ is a closed subarrangement of $\mathcal{A}$, then*

$$\operatorname{pdim}(D(\mathcal{A})) \geq \operatorname{pdim}(D(\mathcal{A}_X)).$$

Terao's proof is unpublished. Yuzvinsky gives a proof in [20]. For the reader's convenience, we give another proof, which is a more elementary version of the proof in [20] (but requires the hypothesis that the field $\mathbb{k}$ has characteristic zero). Let S be the symmetric algebra $\operatorname{Sym}(V^*)$ of the dual space V^*. The algebra S is isomorphic to the polynomial algebra $\mathbb{k}[x_1, x_2, \ldots, x_n]$, where $\{x_i\}$ is a dual basis for V. Let $\mathcal{A} = \{H_i : 1 \leq i \leq d\}$ and Q be the polynomial $Q = \prod_{i=1}^{d} l_i$, where for each i, l_i is a homogeneous linear form such that the kernel $V(l_i)$ of l_i is the hyperplane H_i. The *derivation module* $D(\mathcal{A})$ is the S-module of all S-derivations θ such that for all i, $\theta(l_i)$ is in the principal ideal $\langle l_i \rangle \subseteq S$. If char $\mathbb{k} = 0$, this is equivalent to the single condition $\theta(Q) \in \langle Q \rangle$. The Euler derivation $\sum x_i \partial/\partial x_i$ generates a free summand $S(-1)$ of $D(\mathcal{A})$ and

$$D(\mathcal{A}) = S(-1) \oplus D_0(\mathcal{A}),$$

where $D_0(\mathcal{A})$ is the kernel of the Jacobian matrix J_Q, the $n \times 1$ matrix with $(i, 1)$-entry equal to $\partial Q/\partial x_i$ (see, for example, [19]). In particular, the projective dimension of $D(\mathcal{A})$ is one less than the projective dimension of the ideal $\langle J_Q \rangle$ generated by the entries of the matrix J_Q.

Let X be a subspace in the intersection lattice of $\mathcal{A}$. Order the hyperplanes of $\mathcal{A}$ so that the closed subarrangement $\mathcal{A}_X$ equals $\{H_1, H_2, \ldots, H_s\}$. Choose coordinates so that X is the subspace $V(x_1, x_2, \ldots, x_k)$ defined by the equations $x_1 = 0, x_2 = 0, \ldots, x_k = 0$, and hence, a hyperplane H_i in $\mathcal{A}_X$ may be written as the kernel $V(l_i)$ with l_i a homogeneous linear form in $\mathbb{k}[x_1, \ldots, x_k]$.

Let P be the prime ideal $\langle x_1, \ldots, x_k \rangle$ in S. By our choice of coordinates, if the hyperplane $V(l_i)$ does not contain the subspace X, then l_i equals $\gamma_i + \delta_i$ where γ_i is a linear form in $\mathbb{k}[x_1, \ldots, x_k]$, δ_i is a linear form in $\mathbb{k}[x_{k+1}, \ldots, x_n]$, and $\delta_i \not\equiv 0$.

Write $Q = LK$, where $L = \prod_{i=1}^{s} l_i$ and $K = \prod_{i=s+1}^{|\mathcal{A}|} l_i$. Computing the $(i, 1)$-entry of J_Q by the product rule, we have

$$\frac{\partial Q}{\partial x_i} = L\,\frac{\partial K}{\partial x_i} + K\,\frac{\partial L}{\partial x_i}.$$

By our choice of coordinates, $\partial L/\partial x_i = 0$ when $i > k$. Hence, the Jacobian matrix J_Q simplifies to the transpose of the matrix

$$\left[L\,\frac{\partial K}{\partial x_1} + K\,\frac{\partial L}{\partial x_1}, \ldots, L\,\frac{\partial K}{\partial x_k} + K\,\frac{\partial L}{\partial x_k}, L\,\frac{\partial K}{\partial x_{k+1}}, \ldots, L\,\frac{\partial K}{\partial x_n}\right].$$

We localize at the prime ideal P. In the local ring S_P, every element not in P is a unit. Since each l_i with $i > s$ has the non-zero form δ_i in $\mathbb{k}[x_{k+1}, \ldots, x_n]$, the product K contains at least one monomial in $\mathbb{k}[x_{k+1}, \ldots, x_n]$. Hence K is a unit in S_P. Similarly, $\partial K/\partial x_i$ is nonzero for some $i \in \{s + 1, \ldots, n\}$ *and* still contains a nonzero monomial in $\mathbb{k}[x_{k+1}, \ldots, x_n]$. In particular, as an element in S_P, L equals $K^{-1}Q$ and since $Q \in \langle J_Q \rangle$ by Euler's identity, L is in the ideal $\langle J_Q \rangle_P$ generated by the entries of J_Q in S_P. We conclude that

$$\langle J_Q \rangle_P = \left\langle K\frac{\partial L}{\partial x_1}, \ldots, K\frac{\partial L}{\partial x_k}, L \right\rangle_P.$$

Since K is a unit, it can be removed. Further, we can use Euler's relation to write the last generator L as a linear combination of the first k generators. We thus obtain

$$\begin{aligned}\langle J_Q \rangle_P &= \left\langle \frac{\partial L}{\partial x_1}, \ldots, \frac{\partial L}{\partial x_k} \right\rangle_P \\ &= \langle J_L \rangle_P.\end{aligned}$$

Since localization is an exact functor (see, for example, [6]), localizing a minimal free resolution of $\langle J_Q \rangle$ yields a free resolution (possibly non-minimal) of $\langle J_Q \rangle_P$, which equals $\langle J_L \rangle_P$. The free resolution obtained for $\langle J_L \rangle_P$ is also a free resolution for $\langle J_L \rangle$ because L is in P. We conclude that

$$\mathrm{pdim}(\langle J_L \rangle) \leq \mathrm{pdim}(\langle J_Q \rangle).$$

This completes the proof of Theorem 2.2.

We remark that Theorem 2.2 fails if one does not assume that the subarrangement is closed. An easy example is the braid arrangement A_4. It is free but contains three generic subarrangements of four lines, none of them closed.

Combining the results on generic arrangements and Theorem 2.2, we obtain a simple but useful combinatorial corollary.

Corollary 2.3. *Let $\mathcal{A}$ be an arrangement whose matroid contains a generic flat of rank r. Then* $\mathrm{pdim}(D(\mathcal{A})) \geq r - 2$. *In particular, if the matroid of $\mathcal{A}$ contains a closed circuit of size m, then* $\mathrm{pdim}(D(\mathcal{A})) \geq m - 3$.

This corollary extends the folk-lore lemma that an arrangement cannot be free if it contains a closed subarrangement consisting of four hyperplanes in general position in 3-dimensional space.

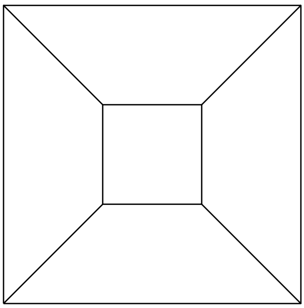
Let G be a graph (without loops or multiple edges) with vertex set $\{1, 2, \ldots, n\}$ and edge set E. The *graphic* arrangement $\mathcal{A}_G$ is the collection $\{V(x_i - x_j) \mid \{i, j\} \in E\}$. For example, the braid arrangement A_n is the arrangement associated to the complete graph K_n, the graph containing all possible edges. Graphic arrangements are never essential; an arrangement from a connected graph can be made essential by suppressing a variable. For example, A_n can be made into the essential arrangement $\hat{A}_n$ consisting of the hyperplanes $V(x_i)$ and $V(x_i - x_j)$, where $1 \leq i < j \leq n - 1$.

The only generic flats in graphic arrangements are closed circuits. Closed circuits correspond to induced cycles. Thus, Corollary 2.3 also extends the reverse implication of a theorem (combining results in [13] and [14]) that a graphic arrangement is free if and only if its graph is *chordal,* or equivalently, its graph has no induced cycles of length greater than 3. In particular, we have:

Corollary 2.4. *If a graph G contains an induced cycle of length m, then* $\mathrm{pdim}(D(\mathcal{A}_G) \geq m - 3$.

We close this section with some illustrations of Corollary 2.4 and several related problems.

Example 2.5. Consider the graph G (with 8 vertices) given by the 1-skeleton of the cube:

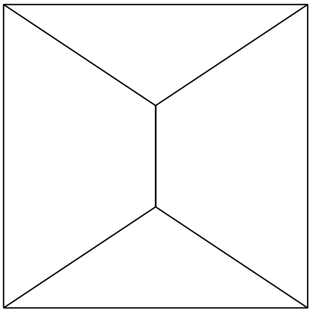


A free resolution for $D(\mathcal{A}_G)$ is:

$$0 \longrightarrow S^3(-7) \longrightarrow S^{13}(-6) \longrightarrow \begin{matrix} S(-4) \\ \oplus \\ S^{19}(-5) \end{matrix} \longrightarrow \begin{matrix} S(-1) \\ \oplus \\ S(-2) \\ \oplus \\ S^9(-3) \\ \oplus \\ S^6(-4) \end{matrix} \longrightarrow D(A_G) \longrightarrow 0$$

The diagram gives the degree (but not the explicit expressions) of the generators of the free modules. For example, from the diagram, one sees that $D(\mathcal{A}_G)$ can be generated by 17 generators, one of degree 1 (the Euler derivation), one of degree 2, nine of degree 3, and six of degree 4. These generators have relations which can be generated by 20 relations. The indexing of a free resolution starts at zero, and so $D(\mathcal{A}_G)$ has projective dimension 3. Since G has an induced cycle of length 6, this is the lower bound predicted by Corollary 2.4.

Example 2.6. Let G be the *triangular prism*:



A free resolution for $D(\mathcal{A}_G)$ is:

$$0 \longrightarrow S(-5) \longrightarrow S^5(-4) \longrightarrow \begin{matrix} S(-1) \\ \oplus \\ S(-2) \\ \oplus \\ S^7(-3) \end{matrix} \longrightarrow D(A_G) \longrightarrow 0$$

The maximum length of an induced cycle in G is 4, but the projective dimension of $D(\mathcal{A}_G)$ is 2. Hence, $\mathrm{pdim}(D(\mathcal{A}_G))$ can be strictly greater than the bound given in Corollary 2.4.

Example 2.6 raises several questions. Is there a characterization of graphs G for which $\mathrm{pdim}(D(\mathcal{A}_G)) = m - 3$, where m is the maximum size of an induced circuit? Are there reasonable formulas involving graph parameters for $\mathrm{pdim}(D(\mathcal{A}_G))$? In analogy to excluded minors in matroid theory (see, for example, [8], Section 8), define an arrangement $\mathcal{A}$ to be *k-minimal* if $\mathrm{pdim}(D(\mathcal{A})) = k$ and for every proper closed subarrangement $\mathcal{A}_X \subset A$, $\mathrm{pdim}(D(\mathcal{A}_X)) < k$. The graphic arrangement of the triangular prism is 2-minimal and rank-m generic arrangements are $(m-2)$-minimal. It seems an interesting problem to classify k-minimal arrangements.

3. Orthogonal duals of arrangements

Let $\mathcal{A}$ be a hyperplane arrangement in n-dimensional space. We construct an $n \times |\mathcal{A}|$ matrix A as follows: each hyperplane H in $\mathcal{A}$ labels a column equal to $(c_1, c_2, \ldots, c_n)^t$, where $H = V(c_1x_1 + c_2x_2 + \cdots + c_nx_n)$. Conversely, given a matrix A, we construct an arrangement by *simplifying,* that is, removing all zero columns, constructing a multiset of hyperplanes corresponding to the kernels of the linear forms defined by the columns, giving a multiarrangement, and disregarding the multiplicities to obtain an arrangement.

If $\mathcal{A}$ is essential, the hyperplanes in $\mathcal{A}$ intersect in the zero subspace, and the matrix $\mathcal{A}$ has full rank n. The correspondence between essential arrangements $\mathcal{A}$ and $n \times |\mathcal{A}|$ matrices A, with no zero columns and no two columns a non-zero multiple of each other, is bijective up to left multiplication by elements of $GL(n)$, and right multiplication by a product of a permutation matrix and a non-singular diagonal matrix. The matrix A is a representation for the matroid $M(\mathcal{A})$ of the arrangement $\mathcal{A}$.

Suppose that A is an $n \times d$ matrix having full rank n. An (*orthogonal*) *dual* of A is an $(d-n) \times d$ matrix B having full rank $d-n$ such that any row of A is orthogonal (under the usual dot product) to any row of B. The matrix B exists and is determined up to left multiplication by a non-singular matrix. In addition, if $A^\perp$ is a dual of A, then it is also a dual of any matrix obtained from A by left multiplication by a non-singular matrix. Thus, duality is an operation defined between equivalence classes of matrices. In particular, there is an easy way to construct a dual of A. Put A into the form $[I|C]$, where I is the $n \times n$ identity matrix. Then a dual of A is $[-C^t|I]$, where I is the $(d-n) \times (d-n)$ identity matrix.

If $\mathcal{A}$ is an essential arrangement with matrix A, we define its (*orthogonal*) *dual* $\mathcal{A}^\perp$ to be the arrangement obtained from a dual of the matrix A. Note that because we discard zero columns and ignore multiplicities, $\mathcal{A}$ is not reconstructible from $\mathcal{A}^\perp$ in general.

We will also need several elementary facts from the theory of matroid duality (see [3, 4, 10, 17]). There are many ways to define the orthogonal dual of a matroid. For us, the best definition is the circuit-cocircuit definition. Recall that a *circuit* is a minimal dependent set and a *cocircuit* is the complement of a copoint. The (*orthogonal*) *dual* $M^\perp$ of M is the matroid on the same ground set whose circuits are exactly the cocircuits of M. Duality interchanges contraction and deletion, that is, for a subset B of the set of elements, $(M^\perp)/B$ equals $(M \backslash B)^\perp$. It is true (but not obvious) that the matroid of the dual arrangement $\mathcal{A}^\perp$ is the simplification of the dual of the matroid of $\mathcal{A}$. Despite its age, the neatest and most accessible proof of this remains Whitney's original proof in [17].

A *loop* is an element e such that the set $\{e\}$ is a circuit. An *isthmus* is an element e such that $\{e\}$ is a cocircuit, so that M is the direct sum $(M \backslash \{e\}) \oplus \{e\}$. For graphs, an isthmus is an edge whose removal increases the number of connected components. Duality interchanges loops and isthmuses.

We shall call closure in the dual matroid $M^\perp$ *$\perp$-closure.*

Lemma 3.1. *Let M be a matroid on the set E and $B \subseteq E$. Then e is in the $\perp$-closure of B if and only if e is in B or e is an isthmus in the deletion $M\backslash B$. In particular, a cocircuit B is $\perp$-closed if its complementary copoint X has no isthmuses.*

Proof: The lemma follows from dualizing the statement: a point e is in the $\perp$-closure if and only if e is a loop in the contraction $M^\perp/B$. □

The *cycle matroid* $M(G)$ of a graph is the matroid on the edge set whose circuits are the cycles of the graph. A *cutset* in a graph G is an edge-subset whose removal increases the number of connected components of G. The circuits of the dual matroid $G^\perp$ are precisely the minimal cutsets of G. For graphs, an isthmus is an edge which is a cutset by itself. Thus, Lemma 3.1 gives an easy way to determine whether a minimal cutset is $\perp$-closed. We remark that the set of all edges incident to a vertex v is a minimal cutset. Such "vertex cutsets" usually contain few edges compared to other minimal cutsets.

The complete graph K_n is the graph on n vertices with all possible edges. The maximum size of a minimal cutset in complete graphs is given in the next lemma.

Theorem 3.2. *The largest cocircuit in the cycle matroid $M(K_{n+1})$ has size $\lceil n(n+2)/4 \rceil$. When $n \geq 5$, the largest cocircuits are $\perp$-closed.*

Proof: Because K_{n+1} contains all possible edges, minimal cutsets are in bijection with partitions of the vertex set into two non-empty subsets and these cutsets disconnect K_{n+1} into two disjoint smaller complete graphs. By Lemma 3.1, every minimal cutset in K_{n+1} gives a $\perp$-closed cocircuit with the exception of the cutsets which divide K_{n+1} into a K_{n-1} and a single edge K_2. The largest minimal cutsets are those which divide K_{n+1} into two connected components of almost equal size. We conclude that the largest cocircuit in $M(K_{n+1})$ has size k^2 if $n+1 = 2k$ and $k(k+1)$ if $n+1 = 2k+1$. To finish, it is easy to check that $\lceil n(n+2)/4 \rceil$ equals k^2 or $k(k+1)$ depending on the parity of $n+1$. □

Theorem 3.3. *Let M be a rank-n matroid on the set S with a modular copoint X. Suppose that the cocircuit $S\backslash X$ spans. If $n \geq 5$, then there exists a $\perp$-closed cocircuit in M of size at least $\lceil n(n+2)/4 \rceil$.*

Proof: We shall use the following lemma.

Lemma 3.4. *Under the hypotheses in the theorem, M contains a spanning submatroid isomorphic to $M(K_{n+1})$.*

Proof: This is a combination of Lemma 5.3 in [7] and Lemma 5.14 in [8]. For the sake of completeness, we will give a proof in the language of arrangements and linear forms. Choose coordinates so that the linear forms x_i, $1 \leq i \leq n$ are in the cocircuit $S\backslash X$ and the copoint X is the subarrangement of all linear forms whose kernel contains the point $(1, 1, \ldots, 1)$. By modularity,

$$\operatorname{rank}((x_i \vee x_j) \wedge X) = \operatorname{rank}(X) + \operatorname{rank}(x_i \vee x_j) - n = 1$$

for every pair x_i and x_j of linear forms. Hence, there is a linear combination of x_i and x_j whose kernel contains $(1, 1, \ldots, 1)$. This form is $x_i - x_j$, so the arrangement contains the subarrangement $\{x_i,\ x_i - x_j \mid 1 \le i < j \le n\}$, which is isomorphic to the graphic arrangement of K_{n+1}. □

Let K be a spanning submatroid in M isomorphic to $M(K_{n+1})$. Take a copoint X' in the submatroid K and let X be the closure of X' in M. A point in $K \backslash X'$ is still not in X. Hence,

$$|S \backslash X| \ge |K \backslash X'|.$$

Choosing X' in $M|K$ so that $K \backslash X'$ has size $\lceil n(n+2)/4 \rceil$, we obtain a cocircuit in M having size at least $\lceil n(n+2)/4 \rceil$. If $n \ge 5$, the copoint X' in K contains no isthmuses. Since X and X' have the same rank, a direct summand of X induces a direct summand of X'. As X' contains no isthmuses, X also contains no isthmuses and the cocircuit $S \backslash X$ is $\perp$-closed. □

Corollary 2.3 and Theorem 3.3 imply Theorem 1.1.

Since the matroid of the braid arrangement A_n is $M(K_n)$, Theorem 2.2 and Theorem 3.2 imply that the projective dimension of the dual of the "essential" braid arrangement $\hat{A}_n$ is at least $\lceil (n-1)(n+1)/4 \rceil - 3$. Lower bounds for the other families of real reflection arrangements can be obtained using the method in the proof of Theorem 3.2.

Theorem 3.5. *When $n \ge 5$,*

$$\operatorname{pdim}\left(B_n^{\perp}\right) \ge \left\lfloor \frac{2}{3}n^2 + \frac{1}{3}n - \frac{1}{24} \right\rfloor - 3.$$

When $n \ge 6$,

$$\operatorname{pdim}\left(D_n^{\perp}\right) \ge \left\lfloor \frac{2}{3}n^2 - \frac{1}{3}n + \frac{1}{24} \right\rfloor - 3.$$

Proof: Consider the copoint isomorphic to the direct sum $A_k \oplus B_{n-k}$ in B_n spanned by the $n-1$ linear forms

$$x_1 - x_2, x_2 - x_3, \ldots, x_{k-1} - x_k, x_{k+1}, x_{k+2}, \ldots, x_n.$$

The cocircuit complementary to X has size

$$\binom{k}{2} + k + 2k(n-k). \tag{1}$$

We obtain a cocircuit of maximum size when k is the integer closest to $2n/3 + \frac{1}{6}$ and this maximum size is obtained by substitution into formula (1) and rounding down. Since A_2 and B_1 contain a single form, the cocircuits of maximum size are $\perp$-closed if $n \geq 5$. The argument for D_n is similar. $\square$

The argument for B_n can also be applied to the complex reflection arrangements $G(n, 1, l)$ to give a rough lower bound of $(l^2/(2l+2))n^2$ for $\mathrm{pdim}(G(n, 1, l)^{\perp})$ when $n \geq 5$.

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