Structure and automorphism groups of Hadamard designs*

Eric Merchant

Abstract Let n be the order of a Hadamard design, and G any finite group. Then there exists many non-isomorphic Hadamard designs of order $2^{12|G|+13}n$ with automorphism group isomorphic to G.

Keywords Hadamard designs · Automorphisms of designs

1. Introduction

One of the fundamental questions concerning any class of combinatorial objects is: given any finite group G, can one find a member of the class whose full automorphism group is isomorphic to G? A survey of such results can be found in [1]. More recently, [7] showed that, for any finite group G, and any $q \geq 3$, for all sufficiently large d there exists a symmetric design with the parameters of $\mathcal{PG}(d, q)$ and with full automorphism group isomorphic to G. This paper establishes a stronger version of this when $q = 2$:

Theorem 1.1. *Given a finite group G and the existence of a Hadamard design of order n, then for all $N > 4|G| + 2$, there exist at least $\frac{(16n-2)!}{2^{10}n^3}$ non-isomorphic Hadamard designs $\mathcal{D}$ of order $2^{3N+4}n$ with $Aut(\mathcal{D}) \cong G$.*

The paper is structured as follows: the first 2 sections give basic definitions and notation, as well as structural properties of a "doubling" construction for Hadamard designs. Much of this was covered in [12], where complete proofs can be found.

Section 4 deals with designs obtained from tensor products. It is well known that the tensor (or Kronecker) product of two Hadamard matrices is a Hadamard matrix.

E. Merchant (✉)
On Time Systems Inc., 1850 Millrace Dr., Suite 1, Eugene OR, 97403
e-mail: eric@otsys.com

Properties of the corresponding product of designs appear not to have been studied. The section concludes with Theorem 4.14, essentially a strong "Krull-Schmidt" theorem which shows that, under suitable conditions, a "tensor factorization" must be unique. This uniqueness provides explicit information about the automorphism group of such a design in terms of the "factor" sub-designs.

Central to the proof of Theorem 1.1 is the construction of rigid Hadamard designs, that is, designs with no non-trivial automorphisms. An infinite family of such designs was previously constructed in [13]. Those designs are of order $2n$, where $4n - 1$ is a prime power, and are constructed by doubling the Paley designs of [15]. Crucial to that construction is the fact that the full automorphism groups of the Paley designs was given in [6]. Section 5 shows that *given any n that is the order of a Hadamard design, there exists a rigid Hadamard design of order* $8n$.

Section 6 surveys known results on GMW designs. These designs have particularly nice automorphism groups, as well as useful structural properties. This combination allows us to construct Hadamard designs with classical parameters having full automorphism group isomorphic to any specified group in Section 7, which concludes with a proof of Theorem 1.1.

2. Notation and definitions

This material is discussed at greater length in [12].

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \in)$ be a $2 - (4n - 1, 2n - 1, n - 1)$ Hadamard design, so $v = |\mathcal{P}| = |\mathcal{B}| = b = 4n - 1$, $k = |B| = 2n - 1$ for any $B \in \mathcal{B}$, and for $B_1 \neq B_2 \in \mathcal{B}$, $|B_1 \cap B_2| = \lambda = n - 1$. Since $\mathcal{D}$ is symmetric, the *dual design* $\mathcal{D}^* = (\mathcal{B}, \mathcal{P}, \ni)$ is a design with the same parameters. The *affine completion* $\mathcal{D}^+$ of $\mathcal{D}$ is the following *Hadamard* 3-*design*, with parameters $3 - (4n, 2n, n - 1)$:

- Point-set: $\mathcal{P} \cup \{p_\infty\}$.
- Blocks: $B \cup \{p_\infty\}$ and $B^c = \mathcal{P} - B$ for all $B \in \mathcal{B}$.

Note that $\text{Aut}(\mathcal{D}^+)_{p_\infty} = \{g \in \text{Aut}(\mathcal{D}^+) | p_\infty^g = p_\infty\} \cong \text{Aut}(\mathcal{D})$. Any point p of the 3-design $\mathcal{D}^+$ can be used to derive a Hadamard 2-design $\mathcal{D}_p^+$ on the remaining points, and block set the blocks on the p. Clearly, $\mathcal{D}_{p_\infty}^+ = \mathcal{D}$, while different points p may induce non-isomorphic 2-designs $\mathcal{D}_p$.

Given a block $B \in \mathcal{B}$, the *induced design* $\mathcal{D}_{(B)}$ on B has as points the points of B, and blocks are the point-sets $C \cap B$ for any $C \neq B$. Similarly, given a block $B \in \mathcal{B}$ the *residual design* $\mathcal{D}^B$ off B consists of the points off B, and blocks are the sets $C \cap B^c$.

Distinct blocks A, B, C of $\mathcal{D}$ form a *coline of size* 3 if $B \cap C \subset A$. We will denote by $B * C$ the third block on the coline with B and C, if such a block exists. Dually, 3 distinct points p, q, r form a *line* of a Hadamard design if all blocks on both p and q are also on $r = p * q$.

A block B is *good* if, for any other block C, $B * C$ exists. Equivalently, B is good if and only if $\mathcal{D}_{(B)}$ is a Hadamard design, which is the case if and only if $\mathcal{D}^B$ is an affine Hadamard 3-design (for the elementary proof, see [2, Theorem XII. 5.3]). Dually, we say p is a *good point* if $p * q$ exists for all other points q.

Kimberley [10] provides a careful study of good blocks of Hadamard 3-designs. Just as with symmetric designs, a block B of such a design is *good* if and only if, for

any block $C \notin \{B, B^c\}$, $(B \triangle C)^c$ is also a block.[1] In that case $(B^c \triangle C)^c = B \triangle C$ is also a block, and therefore B^c is a good block as well. Thus, the good blocks occur in parallel pairs, and we say $\{B, B^c\}$ is a *good parallel class*. Given a Hadamard design $\mathcal{D}$, the good blocks of the 3-design $\mathcal{D}^+$ are the blocks $B \cup \{p_\infty\}$ and B^c, where B is a good block of $\mathcal{D}$. Conversely, for every good parallel class $\{B, B^c\}$ of an affine Hadamard design $\mathcal{A}$, the member of the class containing p will induce a good block of $\mathcal{A}_p$. For, if we have B the member of a good parallel class containing p, then given any other block C which contains p, $(B \triangle C)^c$ is another block *which contains* p, hence another block of $\mathcal{A}_p$. Therefore, although different points p and q may induce non-isomorphic designs $\mathcal{A}_p$ and $\mathcal{A}_q$, the number of good blocks of $\mathcal{A}_p$ and of $\mathcal{A}_q$ will equal the number of good parallel classes of $\mathcal{A}$.

A *translation* of an affine Hadamard design $\mathcal{A}$ is a non-trivial automorphism f with the property that $B^f = B$ or B^c for all blocks B. Clearly, such an automorphism is fixed-point-free (since for some B, $B^f = B^c$) and of order 2 (as $B^{f^2} = B$ for all blocks). These were characterized in [12, Lemma 4.4]:

Fact 2.1. *Let p and q be distinct points of an affine Hadamard design $\mathcal{A}$. There exists a translation $f \in \operatorname{Aut}(\mathcal{A})$ with $p^f = q$ if and only if q is a good point of $\mathcal{A}_p$.*

An incident point/block pair $p \in B$ is called a *flag*; if p and B are both good, we will call this a *good flag*. A non-incident good point/block pair will be called a *good anti-flag*.

3. Doubling Hadamard designs

In this section we define the "doubling" construction first described in [18], and subsequently used by many authors (cf., e.g. [4, 7, 10, 11, 13]). We will then focus on the lines and colines that arise with this construction.

Let $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1)$ be a $2 - (4n - 1, 2n - 1, n - 1)$ Hadamard design, and $\mathcal{A}_2 = (\mathcal{P}_2, \mathcal{B}_2)$ a Hadamard $3 - (4n, 2n, n - 1)$ design. Then $\mathcal{A}_2$ has $4n - 1 = |\mathcal{B}_1|$ parallel classes of blocks. Let σ be any bijection from the parallel classes of $\mathcal{A}_2$ to $\mathcal{B}_1$. Denote by $\overline{B_2}$ the parallel class containing B_2. Then we define the $2 - (8n - 1, 4n - 1, 2n - 1)$ Hadamard design $\mathcal{D}_1 \sigma \mathcal{A}_2$ as follows:

Point-set: $\mathcal{P}_1 \cup \mathcal{P}_2$
Blocks: $B_\infty = \mathcal{P}_1$ and $\overline{B_2}^\sigma \cup B_2$ for all $B \in \mathcal{B}_2$.

We first characterize the good blocks that can arise from this construction. A block B of a Hadamard 2-design is good if and only if, for any other block C, $(B \triangle C)^c$ is also a block. Therefore B_∞ is a good block of $\mathcal{D}_1 \sigma \mathcal{A}_2$, as $((\overline{B_2}^\sigma \cup B_2) \triangle B_\infty)^c = \overline{B_2}^\sigma \cup B_2^c$ is a block (clearly $\overline{B_2} = \overline{B_2^c}$). It is then easy to see that

$$(\mathcal{D}_1 \sigma \mathcal{D}_2)_{(B_\infty)} \cong \mathcal{D}_1 \tag{3.1}$$

---

[1] Throughout, $\triangle$ denotes the symmetric difference of sets, and X^c denotes the complement of X.

and

$$(\mathcal{D}_1 \sigma \mathcal{D}_2)^{B_\infty} \cong \mathcal{A}_2. \tag{3.2}$$

The existence of other good blocks of $\mathcal{D}_1 \sigma \mathcal{A}_2$ is a function both of the constituent designs themselves, as well as σ. Given two blocks of the form $\overline{B_2}^\sigma \cup B_2$ and $\overline{C_2}^\sigma \cup C_2$, with $C_2 \notin \{B_2, B_2^c\}$, we have

$$\left((\overline{B_2}^\sigma \cup B_2) \triangle (\overline{C_2}^\sigma \cup C_2) \right)^c = \left(\mathcal{P}_1 - (\overline{B_2}^\sigma \triangle \overline{C_2}^\sigma) \right) \cup (\mathcal{P}_2 - (B_2 \triangle C_2)). \tag{3.3}$$

Thus, $\overline{B_2}^\sigma \cup B_2$ is good if and only if the following 2 conditions are met:

1. Both B_2 and $\overline{B_2}^\sigma$ are good, so that the sets $\mathcal{P}_1 - (\overline{B_2}^\sigma \triangle \overline{C_2}^\sigma)$ and $\mathcal{P}_2 - (B_2 \triangle C_2)$ are blocks of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, for all blocks C_2.
2. $\overline{(\mathcal{P}_2 - (B_2 \triangle C_2))}^\sigma = \mathcal{P}_1 - (\overline{B_2}^\sigma \triangle \overline{C_2}^\sigma)$ for all blocks C_2, so that in (3.3), the set on the right is indeed a block of $\mathcal{D}_1 \sigma \mathcal{A}_2$.[2]

Given a $\mathcal{D}_1$ and $\mathcal{A}_2$ of order n, a natural question is: how many non-isomorphic designs can be obtained via the doubling procedure? This question was the main focus of [12], which showed the following (Lemma 3.1):

Fact 3.4. *Let $\mathcal{D}_1 \sigma \mathcal{A}_2$ and $\mathcal{D}_1 \tau \mathcal{A}_2$ have B_∞ as their only good block. Then $\mathcal{D}_1 \sigma \mathcal{A}_2 \cong \mathcal{D} \tau \mathcal{A}_2$ if and only if $\sigma \in \overline{G_2} \tau G_1$, where $G_1 = Aut(\mathcal{D}_1)$ in its permutation representation on blocks, and $\overline{G_2} = Aut(\mathcal{A}_2)$ in its representation on parallel classes.*

Note 3.5. *One special case of the doubling procedure deserves mention. Since every block B of a Hadamard design $\mathcal{D}$ determines a unique parallel class $\{B \cup \{p_\infty\}, B^c\}$, there is a natural mapping from parallel classes of $\mathcal{D}^+$ to blocks of $\mathcal{D}$, which we denote σ_0, so $\{B \cup \{p_\infty\}, B^c\}^{\sigma_0} = B$.*

The resulting design $\mathcal{D} \sigma_0 \mathcal{D}^+$ is well-known, and can also be constructed by taking the Hadamard matrix associated with $\mathcal{D}$ (the ± 1 incidence matrix with an additional row and column of all 1s) and using its Kronecker product with the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. See Section 4 for more details of that construction.

The existence of good points in these designs was studied in previous work. We mention the two salient facts [12, Theorem 4.3, 4.5]:

Fact 3.6. *Let $\mathcal{E}$ be a Hadamard design containing a good anti-flag $p \notin B$. Then $\mathcal{E} \cong \mathcal{D} \sigma_0 \mathcal{D}^+$, where $\mathcal{E}_{(B)} \cong \mathcal{D}$.*

Fact 3.7. *Let $p \in B$ be a good flag of a Hadamard design $\mathcal{D}$. Then there exists an elation[3] of $\mathcal{D}$ inducing a translation of the Hadamard 3-design $\mathcal{D}^B$.*

---

[2] See [14, Lemma 1] for the corresponding characterization in the affine Hadamard design case.

[3] An *elation* is an automorphism which, given an incident point/block pair, fixes all points on the block and all blocks on the point.

Much of this paper is dedicated to understanding properties of designs resulting from the "doubling" procedure. Since any such design has at least one good block B_∞, a good point of such a design automatically implies the existence of a good flag or a good anti-flag. For that reason, the above facts are used throughout the paper. Another application was given in [12, Theorem 1.3]:

Theorem 3.8. *Given a* $2 - (4n - 1, 2n - 1, n - 1)$ *Hadamard design with* $n > 2$, *there exists a Hadamard design with the same parameters having no good blocks and at most one good point.*

In the case where B_∞ is the only good block of $\mathcal{D}_1 \sigma \mathcal{A}_2$, the automorphism group can be nicely described. Let $G_1 = \text{Aut}(\mathcal{D}_1)$ in its permutation representation on blocks, $G_2 = \text{Aut}(\mathcal{A}_2)$ in its permutation representation on blocks, and $\overline{G_2} = \text{Aut}(\mathcal{A}_2)$ in its permutation representation on *parallel classes* of blocks.

Lemma 3.9. *Let* $\mathcal{D}_1 \sigma \mathcal{A}_2$ *have* B_∞ *as its only good block, and* $g \in Aut(\mathcal{D}_1 \sigma \mathcal{A}_2)$. *Then* g *induces a* $g_1 \in G_1$ *on* B_∞ *and a* $g_2 \in G_2$ *off* B_∞ *satisfying* $g_1 = \overline{g_2}^\sigma$.
Conversely, given g_1 *and* g_2 *satisfying* $g_1 = \overline{g_2}^\sigma$, *there exists a* $g \in Aut(\mathcal{D}_1 \sigma \mathcal{A}_2)$ *inducing* g_1 *and* g_2 *on and off* B_∞, *respectively.*[4]

Proof: $\Rightarrow$ Since B_∞ is the unique good block, $B_\infty^g = B_\infty$. Therefore, g induces an automorphism $g_1 \in G_1$ acting on B_∞, and a $g_2 \in G_2$ acting off B_∞. Then, for any other block:

$$(\overline{B_2}^\sigma \cup B_2)^g = \overline{B_1}^{\sigma g_1} \cup B_2^{g_2} = \overline{B_2}^{\overline{g_2}\sigma} \cup B_2$$

which implies

$$\sigma g_1 = \overline{g_2}\sigma$$

$\Leftarrow$ Suppose $g_1 = \overline{g_2}^\sigma$ for $g_1 \in G_1, \overline{g_2} \in \overline{G_2}$. Fix $g_2 \in G_2$, an arbitrary pre-image of $\overline{g_2}$. We then define an automorphism g acting as g_1 on B_∞ and as g_2 off B_∞ as follows:

$$(\overline{B_2}^\sigma \cup B_2)^g = \overline{B_2}^{\sigma g_1} \cup B_2^{g_2} = \overline{B_2}^{\sigma \sigma^{-1}\overline{g_2}\sigma} \cup B_2^{g_2} = \overline{B_2}^{\overline{g_2}\sigma} \cup B_2^{g_2}.$$

Which is a block of $\mathcal{D}_1 \sigma \mathcal{A}_2$, proving that g induces an automorphism. $\square$

Corollary 3.10. *Let* $\mathcal{D}_1 \sigma \mathcal{A}_2$ *have* B_∞ *as its only good block. If* $Aut(\mathcal{A}_2)$ *contains no translations, then*:

$$Aut(\mathcal{D}_1 \sigma \mathcal{A}_2) \cong G_1 \cap \overline{G_2}^\sigma.$$

---

[4] Similar ideas appear in [7, 13].

Proof: Since G_2 contains no translations, the homomorphism $G_2 \to \overline{G_2}$ is injective, so every g_2 induces a unique $\overline{g_2}$. Thus, Lemma 3.9 provides the bijection between $\mathrm{Aut}(\mathcal{D}_1 \sigma \mathcal{A}_2)$ and $G_1 \cap \overline{G_2}^\sigma$. $\square$

Lemma 3.11. *Let p be a point of $\mathcal{D}_1$ and q, r be points of $\mathcal{A}_2$. Then $\{p, q, r\}$ is a line of $\mathcal{D}_1 \sigma \mathcal{A}_2$ if and only if, for the $2n - 1$ blocks B of $\mathcal{A}_2$ with $q, r \in B$, we have $p \in \overline{B}^\sigma$.*

Proof: $\Rightarrow$ If $\{p, q, r\}$ is a line of $\mathcal{D}_1 \sigma \mathcal{A}_2$, then every time $p, q \in \overline{B}^\sigma \cup B$, we have $r \in \overline{B}^\sigma \cup B$, so $q, r \in B$ and $p \in \overline{B}^\sigma$.

$\Leftarrow$ Given such a σ, if $q, r \in \overline{B}^\sigma \cup B$ then $p \in \overline{B}^\sigma \cup B$. Thus, $\{p, q, r\}$ form a line of $\mathcal{D}_1 \sigma \mathcal{A}_2$. $\square$

Corollary 3.12. *Let $\mathcal{D}_1$ be a Hadamard design of order $n > 2$, and $\mathcal{A}_2$ an affine Hadamard design of order n. Then there exists a σ such that $\mathcal{D}_1 \sigma \mathcal{A}_2$ has at least 1 point on no lines of size 3.*

Proof: By Lemma 3.11, given a point p of $\mathcal{D}_1$ and points q, r of $\mathcal{A}_2$, there are $(2n)!(2n - 1)!$ choices of σ for which $\mathcal{D}_1 \sigma \mathcal{D}_2$ has $\{p, q, r\}$ as a line. Fixing q, but letting p and r vary, we see that there are $(4n - 1)! - (4n - 1)^2(2n)!(2n - 1)! > 0$ choices of σ for which $\mathcal{D}_1 \sigma \mathcal{A}_2$ has no lines of size 3 through q. $\square$

4. The tensor product

If $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2)$ are Hadamard designs, we denote by $\mathcal{D}_1 \otimes \mathcal{D}_2$ the Hadamard design on $\mathcal{P}_1 \cup \mathcal{P}_2 \cup (\mathcal{P}_1 \times \mathcal{P}_2)$, with blocks of the following form.

$$B_1^* = B_1 \cup \mathcal{P}_2 \cup (B_1 \times \mathcal{P}_2) \tag{4.1}$$

$$B_2^* = \mathcal{P}_1 \cup B_2 \cup (\mathcal{P}_1 \times B_2) \tag{4.2}$$

$$(B_1, B_2) = (B_1^* \triangle B_2^*)^c. \tag{4.3}$$

We will need the line structure of the design (colines are described similarly).

Observation 4.4. The lines of size 3 in the design $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2$ are of the following types, where $\{x_i, y_i, z_i\}$ is a line of $\mathcal{D}_i$ of size 3 and p_i is an arbitrary point of $\mathcal{D}_i$:

i. $\{x_i, y_i, z_i\}$
ii. $\{(x_1, x_2), (y_1, y_2), (z_1, z_2)\}$
iii. $\{(x_1, p_2), (y_1, p_2), z_1\}$ or $\{(p_1, x_2), (p_1, y_2), z_2\}$
iv. $\{p_1, p_2, (p_1, p_2)\}$

Two special cases are worth mentioning. First, the trivial case of an "empty" Hadamard design $\mathcal{E}$ with no points or blocks. Clearly, $\mathcal{D} \otimes \mathcal{E} = \mathcal{D}$ (just as taking a Kronecker product with the matrix $[1]$ leaves the original matrix unchanged) and this case shall hereafter be ignored. More interesting is the tensor product with the

design consisting of one point and block which are non-incident. As mentioned in Section 3 above, this corresponds to the doubling construction of $\mathcal{D}\sigma_0\mathcal{D}^+$.

Corollary 4.5. *If $\mathcal{D} \cong \mathcal{D}_1 \otimes \mathcal{D}_2$, then every point of $\mathcal{D}$ is on at least one line of size 3.*

Proof: Lines of the form $\{p_1, p_2, (p_1, p_2)\}$ cover all points. $\square$

For any Hadamard design $\mathcal{D}$ and point p, define:

$$p^\perp = \{p\} \cup \{q \mid p * q \text{ exists}\},$$

and for any $X \subseteq \mathcal{P}$:

$$X^\perp = \bigcap_{p \in X} p^\perp.$$

Lemma 4.6. *Let p_1 be a point of $\mathcal{D}_1 \otimes \mathcal{D}_2$, and let $\perp_1$ denote the perp structure of $\mathcal{D}_1$. Then*

$$p_1^\perp = p_1^{\perp_1} \cup \mathcal{P}_2 \cup \left(p_1^{\perp_1} \times \mathcal{P}_2\right).$$

Proof: Lines of type i give $p_1^\perp \cap \mathcal{P}_1 = p_1^{\perp_1}$. All points of $\mathcal{P}_2$ are contained in $p_1^\perp$ courtesy of lines of type iii. Finally, $p_1^\perp \cap (\mathcal{P}_1 \times \mathcal{P}_2) = p_1^{\perp_1} \times \mathcal{P}_2$ as lines of type iv connect p_1 and a point of the form (q_1, p_2) if and only if $q_1 \in p_1^{\perp_1}$. $\square$

Of course, we can similarly characterize $p_2^\perp$.

Lemma 4.7. *Let (p_1, p_2) be a point of $\mathcal{D}_1 \otimes \mathcal{D}_2$. Then*

$$(p_1, p_2)^\perp = p_1^\perp \cap p_2^\perp = p_1^{\perp_1} \cup p_2^{\perp_2} \cup \left(p_1^{\perp_1} \times p_2^{\perp_2}\right).$$

Proof: $p_1^{\perp_1} \subseteq (p_1, p_2)^\perp$ as lines of type iii will exist whenever the corresponding line exists in $\mathcal{D}_1$. Also, $p_1 \in (p_1, p_2)^\perp$ via the unique line of type iv through (p_1, p_2). Similarly for $p_2^{\perp_2} \subseteq (p_1, p_2)^\perp$. Finally, lines of type ii show that $(p_1, p_2)^\perp \cap (\mathcal{P}_1 \times \mathcal{P}_2) = p_1^{\perp_1} \times p_2^{\perp_2}$.

$\square$

Lemma 4.8. *The good points of $\mathcal{D}_1 \otimes \mathcal{D}_2$ are exactly those of the form p_1, p_2, or (p_1, p_2) where p_i is a good point of $\mathcal{D}_i$.*

Proof: Lemma 4.6 shows that p_1 is good if and only if $p_1^{\perp_1} = \mathcal{P}_1$, i.e., if and only if p_1 is a good point of $\mathcal{D}_1$. Similarly with a point p_2. For a point of the form (p_1, p_2) to be good, Lemma 4.7 shows that both p_1 and p_2 will necessarily be good as well. $\square$

Corollary 4.9. *Let $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2$ where $\mathcal{D}_2$ has no good points. If p is a point of $\mathcal{D}$, then $p^\perp \supseteq \mathcal{P}_2$ if and only if $p \in \mathcal{P}_1$.*

Lemma 4.10. *Let $\mathcal{D}_1 \otimes \mathcal{D}_2$ have a point-transitive automorphism group. Then $\mathcal{D}_1 \otimes \mathcal{D}_2$ is isomorphic to a projective space over $\mathbb{F}_2$.*

Proof: For any $p_1 \in \mathcal{P}_1$ and $p_2 \in \mathcal{P}_2$, there exists an automorphism $g \in \text{Aut}(\mathcal{D}_1 \otimes \mathcal{D}_2)$ so $p_1^g = (p_1, p_2)$. Therefore, $|p_1^\perp| = |(p_1, p_2)^\perp|$, which implies by Lemma 4.7 $p_1^\perp \subseteq p_2^\perp$. Thus, $\mathcal{P}_2 \subseteq p_2^\perp$, so p_2 is a good point of $\mathcal{D}_2$ and hence of $\mathcal{D}_1 \otimes \mathcal{D}_2$ by Lemma 4.8. Since $\text{Aut}(\mathcal{D}_1 \otimes \mathcal{D}_2)$ is transitive, all points of $\mathcal{D}_1 \otimes \mathcal{D}_2$ are good. This implies that $\mathcal{D}_1 \otimes \mathcal{D}_2$ is a projective space by the Dembowski-Wagner Theorem [2, Theorem XII.2.10]. $\square$

Theorem 4.11. *Given a Hadamard design $\mathcal{D}$, we have $\mathcal{D} \cong \mathcal{D}_1 \otimes \mathcal{D}_2$ where $\mathcal{D}_i = (\mathcal{P}_i, \mathcal{B}_i)$ if and only if the point set and the block set of $\mathcal{D}$ can be partitioned as $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup (\mathcal{P}_1 \times \mathcal{P}_2)$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup (\mathcal{B}_1 \times \mathcal{B}_2)$ satisfying the following*:

1. *The incidence structures $(\mathcal{P}_i, \mathcal{B}_i)$ are both induced 2-designs of $\mathcal{D}$.[5]*
2. *For all $p_1 \in \mathcal{P}_1$ and $B_2 \in \mathcal{B}_2$ we have $p_1 \in B_2$, and for all $p_2 \in \mathcal{P}_2$ and $B_1 \in \mathcal{B}_1$ we have $p_2 \in B_1$*
3. *For every $(p_1, p_2) \in \mathcal{P}_1 \times \mathcal{P}_2$, $(p_1, p_2) = p_1 * p_2$. Dually, every block $(B_1, B_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, $(B_1, B_2) = B_1 * B_2$.*

Proof: The forward implication is immediate, by the definition of $\mathcal{D}_1 \otimes \mathcal{D}_2$. For the converse, first note that if such sub-designs of $\mathcal{D}$ exist, they must themselves be symmetric designs. The existence of the partitions shows:

$$v_1 + v_2 + v_1 v_2 = v = b = b_1 + b_2 + b_1 b_2.$$

Combined with Fisher's Inequality: $v_i \le b_i$ for any 2-design gives $v_i = b_i$, implying that both induced sub-designs are symmetric. To see that both designs are Hadamard, consider any block $B_1 \in \mathcal{B}_1$. Since $\{p_1, p_2, (p_1, p_2)\}$ forms a line by condition *3*, we have $(p_1, p_2) \in B_1 \Leftrightarrow p_1 \in B_1$. Thus, since $\mathcal{D}$ is Hadamard, we have:

$$|B_1| = k_1 + v_2 + k_1 v_2 = \frac{1}{2}(v + 1) - 1 = \frac{1}{2}(v_1 + v_2 + v_1 v_2 + 1) - 1.$$

Which gives:

$$k_1 + 1 = \frac{1}{2}(v_1 + 1),$$

which shows that $\mathcal{D}_1$ is Hadamard. Similarly, so is $\mathcal{D}_2$.

---

[5] By conventional abuse of notation, $(\mathcal{P}_i, \mathcal{B}_i)$ is the design consisting of the point set $\mathcal{P}_i$ and blocks being the intersection of $\mathcal{B}_i$ with the point set. Note that $(\mathcal{P}_i, \mathcal{B}_i)$ may be empty or tirvial, as indicated above.

Also, condition *2* and the lines in *3* show that the blocks of $\mathcal{B}_1$ consist of the same points required by Eq. (4.1). Similarly for Eq. (4.2), and the colines of condition *3* ensure that the blocks of the form (B_1, B_2) satisfy Eq. (4.3). $\qquad\square$

If $\mathcal{D}$ contains subdesigns $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfying the conditions of Theorem 4.11, then we say $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2$ is an *internal* tensor product. The associativity of this product follows from the corresponding statement about matrices:

Lemma 4.12. *For Hadamard designs $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$,*

$$(\mathcal{D}_1 \otimes \mathcal{D}_2) \otimes \mathcal{D}_3 = \mathcal{D}_1 \otimes (\mathcal{D}_2 \otimes \mathcal{D}_3).$$

Proof: The design on the left has point-set:

$$(\mathcal{P}_1 \cup \mathcal{P}_2 \cup (\mathcal{P}_1 \times \mathcal{P}_2)) \cup \mathcal{P}_3 \cup ((\mathcal{P}_1 \cup \mathcal{P}_2 \cup (\mathcal{P}_1 \times \mathcal{P}_2)) \times \mathcal{P}_3)$$

and blocks similarly. The subdesigns $(\mathcal{P}_1, \mathcal{B}_1)$ and $(\mathcal{P}_2 \cup \mathcal{P}_3 \cup (\mathcal{P}_2 \times \mathcal{P}_3), \mathcal{B}_2 \cup \mathcal{B}_3 \cup (\mathcal{B}_2 \times \mathcal{B}_3))$ satisfy the conditions of Theorem 4.11. $\qquad\square$

Theorem 4.13. *If $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2 = \mathcal{D}'_1 \otimes \mathcal{D}'_2$ and $\mathcal{D}$ contains no good points, then:*

$$\mathcal{D} = (\mathcal{D}_1 \cap \mathcal{D}'_1) \otimes (\mathcal{D}_1 \cap \mathcal{D}'_2) \otimes (\mathcal{D}_2 \cap \mathcal{D}'_1) \otimes (\mathcal{D}_2 \cap \mathcal{D}'_2),$$

where $\mathcal{D}_i \cap \mathcal{D}'_j$ is the induced subdesign on $(\mathcal{P}_i \cap \mathcal{P}'_j, \mathcal{B}_i \cap \mathcal{B}'_j)$.[6]

Proof: It suffices to show that $\mathcal{D}_1 = (\mathcal{D}_1 \cap \mathcal{D}'_1) \otimes (\mathcal{D}_1 \cap \mathcal{D}'_2)$; $\mathcal{D}_2$ will decompose similarly. We have:

$$\mathcal{P}_1 = (\mathcal{P}_1 \cap \mathcal{P}'_1) \cup (\mathcal{P}_1 \cap \mathcal{P}'_2) \cup (\mathcal{P}_1 \cap (\mathcal{P}'_1 \times \mathcal{P}'_2)),$$

as a partition of $\mathcal{P}_1$. We first show that this partition satisfies Property 3 of Theorem 4.11. Let $p_1 = (p'_1, p'_2)$, so $p_1 = p'_1 * p'_2$. We need to show that $p'_1, p'_2 \in \mathcal{P}_1$. By Lemmas 4.6 and 4.7, we have:

$$\mathcal{P}_2 \subseteq p_1^\perp = (p'_1, p'_2)^\perp = {p_1'}^\perp \cap {p_2'}^\perp.$$

Thus, $\mathcal{P}_2 \subseteq p_1'^\perp$, so ${p_1'}^\perp \in \mathcal{P}_1$ by Corollary 4.9. Clearly, the same argument shows that $p'_2 \in \mathcal{P}_1$, so we have verified Property 3 for the points of $\mathcal{P}_1$. For the dual condition, suppose $B_1 = (B'_1, B'_2) \in \mathcal{B}_1$. We need to show $B_1'^*, B_2'^* \in \mathcal{B}_1$. It suffices to show that $\mathcal{P}_2 \subset B_1'^*$ as the blocks of $\mathcal{B}_1$ are the only blocks which contain all points of $\mathcal{P}_2$, and we already have $\mathcal{P}_2 \subset B_1^* = (B'_1, B'_2)$. Let $q_2 \in \mathcal{P}_2$, and we consider the following 3 cases:

---

[6] Note that the $\mathcal{D}_i \cap \mathcal{D}_j$ may be trivial, or even empty.

1. $q_2 = q_1' \in \mathcal{P}_2 \cap \mathcal{P}_1'$. But, $q_1' \in (B_1', B_2')$ implies $q_1' \in B_1'^*$.
2. $q_2 = q_2' \in \mathcal{P}_2 \cap \mathcal{P}_2'$. Then $q_2' \in B_1'^*$ is immediate.
3. $q_2 = (q_1', q_2') \in \mathcal{P}_2 \cap (\mathcal{P}_1' \times \mathcal{P}_2')$. By the argument above, this means $q_1', q_2' \in \mathcal{P}_2 \subset (B_1', B_2')$, so $q_1' \in B_1'^*$, implying $(q_1', q_2') \in B_1'^*$.

The same argument shows $B_2' \in \mathcal{B}_1$, completing the proof of Property 3.

Property 2 of Theorem 4.11 is immediate, as any point of $\mathcal{P}_1 \cap \mathcal{P}_1'$ is contained in every block of $\mathcal{B}_1 \cap \mathcal{B}_2'$ simply by the incidence relations in $\mathcal{D}_1' \otimes \mathcal{D}_2'$.

For Property 1 of Theorem 4.11, first note $\mathcal{D}_1$ is a 2-design. Any two points $p, q \in \mathcal{P}_1 \cap \mathcal{P}_1'$ lie on λ_1 blocks of $\mathcal{B}_1$. But, $\mathcal{B}_1 = (\mathcal{B}_1 \cap \mathcal{B}_1') \cup (\mathcal{B}_1 \cap \mathcal{B}_2') \cup (\mathcal{B}_1 \cap (\mathcal{B}_1' \times \mathcal{B}_2'))$. Suppose p and q lie on λ_{pq} blocks of $\mathcal{B}_1 \cap \mathcal{B}_1'$. Then they lie on every block of $\mathcal{B}_1 \cap \mathcal{B}_2'$ by Property 2 (say this set has size β), and Property 3 gives us that the blocks of $\mathcal{B}_1 \cap (\mathcal{B}_1' \times \mathcal{B}_2')$ are on unique colines with a block from each of the first two sets, so p and q must lie on $\lambda_{pq}\beta$ blocks of that type. Therefore:

$$\lambda_1 = \lambda_{pq} + \beta + \lambda_{pq}\beta$$

Since λ_1 and β are both independent of the choice of p and q, λ_{pq} must be too. In other words, $(\mathcal{P}_1 \cap \mathcal{P}_1', \mathcal{B}_1 \cap \mathcal{B}_1')$ is an induced 2-design of $\mathcal{D}_1$. The same argument gives $(\mathcal{P}_1 \cap \mathcal{P}_2', \mathcal{B}_1 \cap \mathcal{B}_2')$ a 2-design, completing the proof.   □

In the above theorem, "no good points" can be replaced with the hypothesis "no good blocks" by duality.

If $\mathcal{D} \neq \mathcal{D}_1 \otimes \mathcal{D}_2$ for any non-empty $\mathcal{D}_1, \mathcal{D}_2$, we say $\mathcal{D}$ is *tensor-indecomposable*. The following corollary demonstrates that, in the absence of good points, a design decomposes into unique tensor factors. This is in contrast to the classical case of $\mathcal{PG}(d, 2)$, which can be decomposed into complementary subspaces in many ways.

Corollary 4.14. *Let*

$$\mathcal{D} = \bigotimes_{i=1}^{n} \mathcal{D}_i = \bigotimes_{i=1}^{m} \mathcal{D}_i'$$

where the $\mathcal{D}_i$ and $\mathcal{D}_i'$ are tensor-indecomposable, and $\mathcal{D}$ has no good points.
Then there exists a bijection $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ satisfying $\mathcal{D}_{if} \cong \mathcal{D}_i'$.

Proof: We have:

$$\mathcal{D} = \mathcal{D}_1 \otimes \bigotimes_{i=2}^{n} \mathcal{D}_i = \mathcal{D}_1' \otimes \bigotimes_{i=2}^{m} \mathcal{D}_i'.$$

If $\mathcal{D}_1 = \mathcal{D}_1'$, we are done by induction. Otherwise, Theorem 4.13 gives:

$$\mathcal{D}_1 = (\mathcal{D}_1 \cap \mathcal{D}_1') \otimes \left(\mathcal{D}_1 \cap \bigotimes_{i=2}^{m} \mathcal{D}_i' \right).$$

Since $\mathcal{D}_1$ is tensor-indecomposable, we must have $\mathcal{D}_1 = \mathcal{D}_1 \cap \otimes_{i=2}^{m} \mathcal{D}_i'$. Another application of Theorem 4.13 yields:

$$\bigotimes_{i=2}^{m} \mathcal{D}_i' = \left(\mathcal{D}_1 \cap \bigotimes_{i=2}^{m} \mathcal{D}_i' \right) \otimes \left(\bigotimes_{i=2}^{n} \mathcal{D}_i \cap \bigotimes_{i=2}^{m} \mathcal{D}_i' \right) = \mathcal{D}_1 \otimes \left(\bigotimes_{i=2}^{n} \mathcal{D}_i \cap \bigotimes_{i=2}^{m} \mathcal{D}_i' \right).$$

By induction, this decomposition is unique, so $m - 1 = n - 1$, and for some i, we have $\mathcal{D}_i' = \mathcal{D}_1$. $\qquad \square$

Similarly, determining a factorization determines the automorphism group in terms of the automorphism groups of the factors involved.

Corollary 4.15. *Let $\mathcal{D}$ have no good points and*

$$\mathcal{D} = \bigotimes_{i=1}^{n_1} \mathcal{D}_1 \otimes \bigotimes_{i=1}^{n_2} \mathcal{D}_2 \otimes \cdots \otimes \bigotimes_{i=1}^{n_m} \mathcal{D}_m$$

for tensor-indecomposable designs $\mathcal{D}_i$. Then:

$$Aut(\mathcal{D}) \cong Aut(\mathcal{D}_1) \wr S_{n_1} \times Aut(\mathcal{D}_2) \wr S_{n_2} \times \cdots \times Aut(\mathcal{D}_m) \wr S_{n_m}$$

Proof: Any automorphism of $\mathcal{D}$ maps a decomposition to a decomposition, but Theorem 4.14 ensures that this decomposition is unique. Thus, the above group is the only possibility. $\qquad \square$

The corollary of the following Lemma gives a necessary condition for designs of the form $\mathcal{D}_1 \sigma \mathcal{A}_2$ to be tensor-decomposable. Recall the definition of a block B^* from Eq. (4.1).

Lemma 4.16. *Let $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2$ and let B be a good block of $\mathcal{D}_1$. Then $\mathcal{D}_{(B^*)} = \mathcal{D}_{1(B)} \otimes \mathcal{D}_2$.*

Proof: The point-set of $\mathcal{D}_{(B^*)}$ are: $B \cup \mathcal{P}_2 \cup B \times \mathcal{P}_2$, which is also the point set of $\mathcal{D}_{1(B)} \otimes \mathcal{D}_2$. For blocks, $(B_1, B_2) \cap B^* = (B_1 \cap B, B_2)$. $\qquad \square$

Corollary 4.17. *Let $\mathcal{D} = \mathcal{D}_1 \sigma \mathcal{A}_2$ have a unique good block and no good points. If $\mathcal{D}_1$ is tensor-indecomposable, so is $\mathcal{D}$.*

Proof: Since $\mathcal{D}$ has no good points, $\sigma \neq \sigma_0$, so $\mathcal{D}$ is not trivially decomposable. Suppose $\mathcal{D} = \mathcal{D}_3 \otimes \mathcal{D}_4$. By the dual of Lemma 4.8, we may assume the unique good block of $\mathcal{D}$ is induced by a good block B of $\mathcal{D}_3$. Thus, the good block is a block of the form B^*, so by the Lemma above, $\mathcal{D}_{(B^*)} = \mathcal{D}_{3(B)} \otimes \mathcal{D}_4$. However, by (3.1) we have $\mathcal{D}_{(B^*)} \cong \mathcal{D}_1$, which is tensor-indecomposable by assumption. $\qquad \square$

5. Constructing rigid designs

In this section we construct Hadamard designs with no non-trivial automorphisms. First, we mention a well-known fact regarding automorphisms of Hadamard designs.

Fact 5.1. *Let $\mathcal{D}$ be a $2 - (4n - 1, 2n - 1, n - 1)$ Hadamard design and $1 \neq g \in \text{Aut}(\mathcal{D})$. Then g fixes at most $2n$ points and at most $2n$ blocks.*

Proof: Choose a block B such that $B^g = C \neq B$. Therefore $h : B \cap C^c \to B^c \cap C$, and $|B \cap C^c| = |B^c \cap C| = n$, so the support of g contains at least $2n$ points. Thus, g *fixes at most* $4n - 1 - 2n = 2n$ *points*. Dually, g fixes at most $2n$ blocks. □

Theorem 5.2. *If a Hadamard design of order $n > 2$ exists, there exist at least $\frac{(16n-2)!}{2^{10}n^3}$ pairwise non-isomorphic, tensor-indecomposable Hadamard designs of order $8n$ with exactly one good block, no good points, and no non-trivial automorphisms.*

Proof: By Theorem 3.8, we may assume $\mathcal{D}$ has no good blocks. Set $\mathcal{D}_1 = \mathcal{D}\sigma(\mathcal{D}^*)^+$, where σ is chosen as in Corollary 3.12 so there exists a point q of $\mathcal{D}_1$ which is on no lines of size 3. We list some properties of $\mathcal{D}_1$.

1.1. $\mathcal{D}_1$ *has a unique good block, B_∞.* For, $\mathcal{D}$ has no good blocks, so Eq. (3.3) rules out any other good blocks.
1.2. $\mathcal{D}_1$ *has no good points.* For, a good point would be on a line of size 3 with q, which is ruled out.
1.3. *Any automorphism inducing the identity on B_∞ is the identity.* For, any such automorphism would fix the parallel classes of $D_1^{B_\infty} = (\mathcal{D}^*)^+$, hence induce a translation of that affine design. However, $\mathcal{D}$ has no good blocks, so $\mathcal{D}^*$ has no good points. Therefore, Lemma 2.1 implies $(\mathcal{D}^*)^+$ has no translations.
1.4. $\mathcal{D}_1$ *is tensor-indecomposable.* For, the point q is on no lines of size 3, hence use Corollary 4.5.

We now define $\mathcal{D}_2 = \mathcal{D}_1\tau\mathcal{D}_1^+$ for a carefully chosen τ. Let $\hat{B}$ denote the unique good block of $\mathcal{D}_1$ (this notation is used so as not to confuse it with the new block B_∞ of $\mathcal{D}_2$). Fix a point of $p \in \hat{B}$, so that p is on $2n - 1$ blocks of $\mathcal{D}_{1(\hat{B})} = \mathcal{D}$. These blocks are of the form $B_i \cap \hat{B}$, where $B_i \neq \hat{B}$ is a block of $\mathcal{D}_1$ containing p. So, let $B_1, B_2, \ldots, B_{2n-1}$ be $2n - 1$ blocks inducing those blocks of $\mathcal{D}_{1(\hat{B})}$ (note that we have 2 choices for each block). Let C be an arbitrary block not containing p. Set $\tau = \sigma_0 t$ where σ_0 is as in note 3.5 and t is the following element of the symmetric group on the blocks of $\mathcal{D}_1$.

$$t = (\hat{B}, B_1, B_2, \ldots, B_{2n-2}, C).$$

Properties of $\mathcal{D}_2$:

2.1. B_∞ *is the unique good block of $\mathcal{D}_2$.* Recall that Eq. (3.3) shows that any other good block must arise as the union of a good block of $\mathcal{D}_1^+$ and a good block of $\mathcal{D}_1$. Since $\hat{B} \in \text{supp}(t)$, this does not happen, as although $\hat{B}$ is good, $\hat{B}^t = B_1$ is not.

2.2. $\mathcal{D}_2$ *has no good points on* B_∞. Otherwise, the elation given by Theorem 3.7 would induce a translation of $\mathcal{D}_2^{B_\infty} = \mathcal{D}_1^+$, which is impossible by Lemma 2.1 since $\mathcal{D}_1$ has no good points, by Property 1.2 of $\mathcal{D}_1$.

2.3. $\mathcal{D}_2$ *has no good points off* B_∞. For, such a good point would imply that $\mathcal{D}_2 \cong \mathcal{D}_1 \sigma_0 \mathcal{D}_1^+$ by Theorem 3.6. Setting $G_1 = \mathrm{Aut}(\mathcal{D}_1)$ in its permutation representation on the blocks of $\mathcal{D}_1$ and $\overline{G_1} = \mathrm{Aut}(\mathcal{D}_1^+)$ in its representation on the parallel classes of blocks of $\mathcal{D}_1^+$, the assumed isomorphism and Fact 3.4 give $\tau \in \overline{G_1}\sigma_0 G_1$.

　　On the other hand, $\overline{G_1}$ fixes $\overline{\hat{B}}$, since it is the unique good parallel class of $\mathcal{D}_1^+$, and G_1 fixes $\hat{B}$, since it is the unique good block of $\mathcal{D}_1$. Thus, any element of $\overline{G_1}\sigma_0 G_1$ sends $\overline{\hat{B}}$ to $\hat{B}$. However, $\overline{\hat{B}}^{\tau} = B_1$, so no isomorphism is possible.

2.4. $\mathcal{D}_2$ *is tensor-indecomposable* by Corollary 4.17, as $\mathcal{D}_1$ is indecomposable, by Property 1.4.

2.5. $\mathcal{D}_2$ *has a point moved by any non-trivial automorphism.*

Proof: We'll show p_∞ behaves as stated: if $g \in \mathrm{Aut}(\mathcal{D}_2)$ fixes p_∞, we will show $g = 1$.

　　By Corollary 3.10, g induces a $g_1 \in G_1$ and a $g_2 \in \mathrm{Aut}(\mathcal{D}^+)$ which induces a $\overline{g_2} \in \overline{G_1}$ on parallel classes (G_1 and $\overline{G_1}$ as in Property 2.3 above), with $g_1 = \overline{g_2}^{\tau}$. Since g fixes p_∞, so does g_2. Thus, g_2 fixes (as a set) the blocks on p_∞, and restricted to those blocks g_2 induces an automorphism g_2^* of $(\mathcal{D}_1^+)_{p_\infty} = \mathcal{D}_1$. Since one member of each parallel class contains p_∞, the action of g_2^* on those blocks can be deduced from the action of $\overline{g_2}$ on parallel classes. In fact, since σ_0 associates to each parallel class of $\mathcal{D}_1^+$ the member of the parallel class containing p_∞, we see $g_2^* = \overline{g_2}^{\sigma_0}$ in its representation on blocks of $\mathcal{D}_1$.

　　Since $g_1 = \overline{g_2}^{\tau} = \overline{g_2}^{\sigma_0 t} = (g_2^*)^t$, we have:

$$g_1(g_2^*)^{-1} = \overline{g_2}^{\tau}(g_2^*)^{-1} = t^{-1}\overline{g_2}^{\sigma_0}t(g_2^*)^{-1} = t^{-1}t^{(g_2^*)^{-1}}.$$

A priori, the support of $g_1(g_2^*)^{-1} = t^{-1}t^{(g_2^*)^{-1}}$ is at most twice the size of the support of t. However, *both of the automorphisms* $(g_2^*)^{-1}$ and $g_1(g_2^*)^{-1}$ must fix $\hat{B}$. Thus,

$$\hat{B}^{g_1(g_2^*)^{-1}} = \hat{B}^{t^{-1}g_2^*t^{-1}(g_2^*)^{-1}} = \hat{B}$$

implies:

$$\hat{B}^{t^{-1}} = C = \hat{B}^{g_2^*t^{-1}(g_2^*)^{-1}} = C^{(g_2^*)^{-1}},$$

so g_2^* fixes C. Therefore,

$$|\mathrm{supp}(g_1(g_2^*)^{-1})| \le 2(|\mathrm{supp}(t)| - 1) = 4n - 2.$$

Thus, $g_1(g_2^*)^{-1}$ fixes at least $4n + 1$ blocks, implying $g_1(g_2^*)^{-1} = 1$, by Fact 5.1 since $\mathcal{D}_1$ is of order $2n$. So, $g_1 = g_2^*$ and g_1 commutes with t. Since g_1

centralizes t and fixes $\hat{B}$, g_1 must fix all elements of supp(t). Thus, g_1 induces an automorphism g_1^* of $\mathcal{D}_{1(\hat{B})}$ fixing *all but at most one block on* p, so g_1^* must fix *all* blocks on p (since $\lambda < k - 1$ for any Hadamard design). Additionally, g_1^* fixes the block $C \cap \hat{B}$ of $\mathcal{D}_{1(\hat{B})}$ which means g_1^* fixes at least $2n$ blocks of $\mathcal{D}_{1(\hat{B})}$. Thus, by Fact 5.1, g_1^* is the identity, as $\mathcal{D}_{1(\hat{B})} = \mathcal{D}$ is of order n. Thus, g_1 induces the identity on $\mathcal{D}_{1(\hat{B})}$ and thus g_1 is the identity on $\mathcal{D}_1$ by Property 1.3 of $\mathcal{D}_1$. Since g_1 is the identity on blocks, $\overline{g_2} = g_1^{\gamma^{-1}}$ is the identity on parallel classes, meaning g_2 is the identity, or a translation. Since g_2 fixes p_∞, it must be the identity (recall that translations are fixed point free). Thus, any $g \in \text{Aut}(\mathcal{D}_2)$ which fixes the point p_∞ must be the identity.

2.6. $|\text{Aut}(\mathcal{D}_2)| \le 8n$ and $|\text{Aut}(\mathcal{D}_2^+)| \le 16n(8n)$. Since p_∞ is fixed by no non-trivial automorphism, $|\text{Aut}(\mathcal{D}_2)|$ is equal to the size of the orbit of p_∞, which is at most $8n$, since p_∞ is off the unique good block of $\mathcal{D}_2$. $\text{Aut}(D_2^+)$ is bounded similarly, as $\mathcal{D}_2^+$ is formed by adding another point, whose orbit size is at most $16n$.

Since $\mathcal{D}_2$ has a point not fixed by any non-trivial automorphism, $\mathcal{D}_2^*$ has a block with that property, call this block $\tilde{B}$. Also, $\mathcal{D}_2$ has a unique good block we'll call C_2 (again, so as not to confuse it with the new B_∞ in the following construction) which induces the unique good parallel class $\overline{C_2}$ of $\mathcal{D}_2^+$. Set $\mathcal{D}_3 = \mathcal{D}_2^* \gamma \mathcal{D}_2^+$ where $\overline{C_2}^\gamma = \tilde{B}$, and γ is otherwise arbitrary. Note that there are $(16n - 2)!$ choices for γ, as $\mathcal{D}_2$ is of order $4n$.

$\mathcal{D}_3$ has the following properties:

3.1. $\mathcal{D}_3$ *has only one good block.* For, since $\mathcal{D}_2$ has no good points by 2.2 and 2.3, $\mathcal{D}_2^*$ has no good blocks, so again, Eq. (3.3) ensures that B_∞ is the only good block of $\mathcal{D}_3$.

3.2. $\mathcal{D}_3$ *has no good points on* B_∞. For, just as with the construction of $\mathcal{D}_2$, such a good point would induce an elation by Theorem 3.7, which would induce a translation of $\mathcal{D}_3^{B_\infty} = \mathcal{D}_2^+$. This is impossible by Lemma 2.1, since $\mathcal{D}_2$ has no good points, again by 2.2 and 2.3.

3.3. $\mathcal{D}_3$ *has no good points off* B_∞. Otherwise by Theorem 3.7, we would have $\mathcal{D}_3 \cong \mathcal{D}_2^* \sigma_0 \mathcal{D}_2^{*+}$, which would imply $(\mathcal{D}_3)_{B_\infty} = \mathcal{D}_2^+ \cong \mathcal{D}_2^{*+}$. But, $\mathcal{D}_2^+$ has a good parallel class, $\overline{C_2}$. However, $\mathcal{D}_2$ has no good points, by Properties 2.2 and 2.3, so $\mathcal{D}_2^*$ has no good blocks, so $\mathcal{D}_2^{*+}$ has no good parallel classes. Thus, $\mathcal{D}_2^{*+} \not\cong \mathcal{D}_2^+$.

3.4. $\mathcal{D}_3$ *is tensor-indecomposable* by Corollary 4.17, as $\mathcal{D}_2$ is, by 2.4.

3.5. $\text{Aut}(\mathcal{D}_3) = 1$. For, any $g \in \text{Aut}(\mathcal{D}_3)$ induces a $g_2 \in \text{Aut}(\mathcal{D}_2^+)$, acting as $\overline{g_2}$ on parallel classes, which must fix $\overline{C_2}$ (as $\overline{C_2}$ is the unique good parallel class of $\mathcal{D}_2^+$), and a $g_1 \in \text{Aut}(\mathcal{D}_2^*)$. By Lemma 3.9: $g_1 = \overline{g_2}^\gamma$, and since $\overline{C}^\gamma = \tilde{B}$, g_1 must fix $\tilde{B}$, implying that g_1 is the identity (because $\tilde{B}$ was chosen to be the block moved by any non-trivial automorphism). Thus, $\overline{g_2}$ is the identity on parallel classes of blocks. Since $\mathcal{D}_2^+$ does not admit translations (again by Lemma 2.1 and the fact $\mathcal{D}_2$ has no good points, by 2.2 and 2.3), g_2 must be the identity. Thus, g fixes all blocks of $\mathcal{D}_3$, so must be the identity.

3.6. *There are at least* $\frac{(16n-2)!}{2^{10}n^3}$ *non-isomorphic choices for* $\mathcal{D}_3$. Clearly there are $(16n - 2)!$ choices for γ. By Fact 3.4, a given $\mathcal{D}_2^* \gamma \mathcal{D}_2^+$ can only be isomorphic to $|\text{Aut}(\mathcal{D}_2^*) \gamma \text{Aut}(\mathcal{D}_2^+)| \le 2^{10}n^3$ other such designs, by the bounds in Property 2.6.

6. GMW designs

GMW designs are symmetric designs arising from certain difference sets, and have the same parameters as projective spaces. For a definition and basic properties, see [2, Section VI, 17]. Additionally, [8, Theorem 1] gives the full automorphism group of such designs, as well as determining when they are isomorphic. This is also proven in [3, Theorem 1.2], without the use of the classification of finite simple groups. We will not need the full generality of the theorems in those papers, but use the following:

Fact 6.1. *For any integer $N > 1$, there exists a Hadamard GMW design $\mathcal{D}$ with parameters $2 - (2^{3N} - 1, 2^{3N-1} - 1, 2^{3N-2} - 1)$, such that $Aut(\mathcal{D}) \cong \Gamma L(N, 8)$ in its natural action on non-zero vectors of $\mathbb{F}_8^N$.*

The field size 8 here is entirely arbitrary, any power of 2 greater than 4 will do. Note that while the designs in question have the same parameters as a projective space of dimension $3N - 1$ over $\mathbb{F}_2$, they are not isomorphic to projective spaces, as their automorphism groups are too small. Other differences are also apparent.

Observation 6.2. The designs of Fact 6.1 have no good points or blocks, and are tensor-indecomposable.

Proof: Since the automorphism group is point/block transitive, one good point/block would imply that all points/blocks are good, contradicting the Dembowski-Wagner Theorem [2, Theorem XII.2.10], as these designs are not projective spaces. Also, the transitive automorphism group ensures that these designs are tensor-indecomposable by Lemma 4.10.   □

Lemma 6.3. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a GMW design as given in Fact 6.1. Then $Aut(\mathcal{D}^+) = Aut(\mathcal{D})$.*

Proof: Set $G = Aut(\mathcal{D}^+)$. We need to show that G fixes the point p_∞. Assume that p_∞ is moved. Then G is 2-transitive on points, as the subgroup fixing p_∞ is already known to be transitive on the remaining points.
Claim: $G \cong A\Gamma L(N, 8)$. By 6.1, we have $G_{p_\infty} = Aut(\mathcal{D}) \cong \Gamma L(N, 8)$. Consider the subgroup $T < G_{p_\infty}$ consisting of transvections fixing a common hyperplane (see [17, Ch. 4] for relevant definitions). $|T| = 8^{N-1}$, T fixes 8^{N-1} points (including p_∞), and any other such subgroup of $G_{p_\infty} \cong \Gamma L(N, 8)$ is conjugate to T in G_{p_∞}. Set $H = \text{Fix}(T) \subset \mathcal{P} \cup \{p_\infty\}$ and consider the incidence structure $\mathcal{D}' = (\mathcal{P} \cup \{p_\infty\}, H^G)$. Since G is 2-transitive, this is a design. Some properties of $\mathcal{D}'$:

1. If $g \in G$ and $p_\infty \in H^g$, then $H^g \in H^{G_{p_\infty}}$ as $H^g = \text{Fix}(T^g)$, so T and T^g are G_{p_∞} conjugate.
2. Thus, the number of blocks of $\mathcal{D}'$ on p_∞ is the number of conjugates of T in G_{p_∞}, which in turn is the number of hyperplanes of a vector space of dimension N over $\mathbb{F}_8$, which is $\frac{8^N - 1}{7}$. Thus, $\mathcal{D}'$ has parameters $v = 8^N$, $k = |H| = 8^{N-1}$, and

$r = \frac{8^N - 1}{7}$. Therefore, the parameters of $\mathcal{D}'$ are the same as $\mathcal{AG}(N, 8)$. However, we do not yet know that the $\mathcal{D}'$ is, in fact, classical.

3. Let $H_1 \neq H_2 \in H^G$ with $H_1 \cap H_2 \neq \phi$. By transitivity of $\mathrm{Aut}(\mathcal{D}')$, we may assume $p_\infty \in H_1 \cap H_2$. The remaining points of $H_1 \cap H_2$ are those points fixed by 2 distinct conjugates of T in G_{p_∞}. But these are just the non-zero points contained in the intersection of 2 hyperplanes of $\mathbb{F}_8^N$. Thus, $|H_1 \cap H_2| = 8^{N-2}$. Furthermore, there are exactly 9 conjugates of T pointwise fixing $H_1 \cap H_2$, so there are exactly 9 blocks containing $H_1 \cap H_2$. Since these 9 blocks cannot pairwise intersect outside of $H_1 \cap H_2$, they partition $\mathcal{P} \cup \{p_\infty\} - (H_1 \cap H_2)$.

By [5, Theorem 1], the above 3 properties imply $\mathcal{D}' \cong \mathcal{AG}(N, 8)$, proving the claim that $G \cong A\Gamma\mathrm{L}(N, 8)$. Therefore, $G \cong G_{p_\infty} \ltimes V$, where $V = \mathbb{F}_8^N$.

Finally, we return to the original design $\mathcal{D}^+$ by considering the (point-transitive, hence point-regular) action of $V = \mathbb{F}_8^N$ on the blocks of $\mathcal{D}^+$. Since there are $2^{3N-1} - 1$ parallel classes in $\mathcal{D}^+$, V must fix at least one: $\{B, B^c\}$. Then, V has a subgroup W of index 2 fixing B, hence acting regularly on the points of B. Thus, B can be identified with W, a hyperplane of $\mathcal{AG}(V)$, so by transitivity, all blocks can be identified with hyperplanes. Therefore, $\mathcal{D}^+ \cong \mathcal{AG}(3n, 2)$, which in turn implies that $\mathcal{D} \cong \mathcal{PG}(3N - 1, 2)$, and $\mathrm{Aut}(\mathcal{D}) \cong GL(3N, 2)$ contradicting Fact 6.1.   □

Note that this proof can also be done by simply citing [16, Theorem 1.1], which classifies *all* affine designs admitting a 2-transitive action on points. However, the above proof is included for completeness, and does not require the classification of finite simple groups.   □

7. Isolating given finite groups

In light of Theorem 3.10, we now show that any group is isomorphic to the intersection of two conjugate copies of $\Gamma\mathrm{L}(N, 8)$ within the symmetric group $\mathbf{S}_{8^N - 1}$ for suitably chosen N. The lemma below is adapted from [8]. A similar argument can be found in [7, Lemma 10.3].

Lemma 7.1. *Let G be a finite group, with $N > 4|G| + 2$. There exists $\sigma \in \mathbf{S}_{8^N - 1}$ such that $G \cong \Gamma\mathrm{L}(N, 8) \cap \Gamma\mathrm{L}(N, 8)^\sigma$.*

Proof: Throughout, we will be considering the action of the groups $\mathbf{S}_{8^N - 1}$, G and $\Gamma\mathrm{L}(N, 8)$ on non-zero vectors of $\mathbb{F}_8^N$, so notation involving span and direct sum will always be assumed to have the zero vector removed. Let $K = \mathbb{F}_{8^2}$ and set

$$\mathbb{F}_8^N = (\oplus_g K x_g) \oplus K u \oplus \langle Y \rangle,$$

where Y is some set of more than $2|G|$ linearly independent vectors. G acts on $\mathbb{F}_8^N$ via $(K x_g)^h = K x_{gh}$ while pointwise fixing Ku and $\langle Y \rangle$. Fix $y_0 \in Y_0 \subseteq Y$ where $|Y_0| = 2|G|$.

Now, set π to be the product of 2 disjoint cycles π_1 and π_2, both pointwise fixing $K x_1$ and $K u$, with π_1 a 6-cycle permuting all but 1 non-zero vector of a 1-space of

$Kx_1 \oplus Ku$, and π_2 a $(8^4 - 2(8^2) - 5)$-cycle acting on the remaining non-zero vectors of $Kx_1 \oplus Ku$. Set π' to be the product of 2 disjoint cycles π'_1 and π'_2, both pointwise fixing Kx_1 and Ky_0, with π'_1 a 6-cycle permuting all but 1 non-zero vector of a 1-space of $Kx_1 \oplus Ky_0$ and π'_2 a $(8^4 - 2(8^2) - 6)$-cycle on $Kx_1 \oplus Ky_0$, whose support contains the remaining non-zero vector of the 1-space spanned by the support of π'_1. Set π_* to be the product of 2 disjoint cycles π_{1*} and π_{2*}, both pointwise fixing Ku and $\langle Y \rangle$, where π_{1*} is a 6-cycle on all but 1 non-zero vector of a 1-space, and π_{2*} a $(8^{2+|Y|} - 8^{|Y|} - 8^2 - 5)$-cycle on the remaining non-zero vectors of $Ku \oplus \langle Y \rangle$. Whenever $1 \neq g \in G$ let π_g denote the product of 2 disjoint cycles, both pointwise fixing $Kx_1 \oplus \langle x_g \rangle$ and $Kx_1 \oplus \langle Y_0 \rangle$, π_{1g} is a 6-cycle permuting all but 1 non-zero vector of a 1-space, and π_{2g} a cycle of length greater than $8^{2+|Y_0|}$ whose support spans exactly $Kx_1 \oplus \langle x_g \rangle \oplus \langle Y_0 \rangle$. Furthermore, choose π_{2g} in such a way that the length of the cycle differs for distinct g. This is possible, as $|G| < 8^{3+|Y_0|} - (8^3 + 8^{|Y_0|} + 8^{2+|Y_0|} + 6)$. For all $h \in G$ define σ to be

- π^h on $(Kx_1 \oplus Ku)^h = Kx_h \oplus Ku$,
- π'^h on $(Kx_1 \oplus Ky_0)^h = Kx_h \oplus Ky_0$,
- π_g^h on $(Kx_1 \oplus \langle x_g \rangle \oplus \langle Y_0 \rangle)^h = Kx_h \oplus \langle x_{gh} \rangle \oplus \langle Y_0 \rangle$ whenever $g \neq 1$, and
- π_* on $Ku \oplus \langle Y \rangle$.

Then: $G \leq \Gamma L(N, 8) \cap \Gamma L(N, 8)^{\sigma}$, as all elements of G commute with σ; we are going to prove equality here.

Now, suppose we have $\alpha, \beta \in \Gamma L(N, 8)$ such that $\alpha = \beta^{\sigma}$. Then we have $\alpha^{-1}\beta = \sigma^{-1}\sigma^{\beta} \in \Gamma L(N, 8)$. But the support of $\sigma^{-1}\sigma^{\beta}$ has size at most:

$$2|\text{supp}(\sigma)| \leq 2(8^4 2|G| + 8^{3+|Y_0|}|G|(|G| - 1) + 8^{2+|Y|}) < 8^N - 8^{N-1}$$

and therefore $\alpha^{-1}\beta$ is the identity, as no non-trivial element of $\Gamma L(N, 8)$ can fix more than 8^{N-1} non-zero vectors of $\mathbb{F}_8^N$. Thus, σ centralizes α. Then α permutes the cycles of σ. Since $8^{2+|Y|} - 8^{|Y|} > 8^{3+|Y_0|} - 8^{2+|Y_0|}$, π_{2*} is a longer cycle than any of the π_{2g}, and hence is the longest cycle of σ. Thus, α must stabilize the support of π_{2*}. Also, α permutes the 6-cycles of σ, hence must permute the 1-spaces they determine. Thus, α must fix the single vector in the intersection of those 1-spaces and the support of π_{2*}. Since α fixes this vector and commutes with π_{2*}, it must fix all vectors in the support of π_{2*}. This set contains a basis for $Ku \oplus \langle Y \rangle$, so α must be the identity on this subspace. In particular, α is linear.

If $(Kx_1 \oplus Ku)^{\alpha} = Kx_h \oplus Ku$ then we may replace α with αh^{-1} and assume $Kx_1 \oplus Ku$ is left invariant by α. This means that α commutes with π, hence must stabilize the support of π_2. Again, this support intersects the 1-spaces determined by all the 6-cycles of σ in a single vector, implying that α pointwise fixes the support of π_2. Again, this set contains a basis of $Kx_1 \oplus Ku$, so α induces the identity on $Kx_1 \oplus Ku$. Note that if $|G| = 1$ we are finished, as in that case $\mathbb{F}_8^N = Kx_1 \oplus Ku \oplus \langle Y \rangle$, so we have shown that α is the identity. Therefore, we may assume $|G| > 1$.

α must permute the cycles of length $8^4 - 2(8^2) - 5$, as well as the cycles of length $8^4 - 2(8^2) - 6$. Thus, for all $h \in G$, there exists $h^*, \overline{h} \in G$ satisfying $(Kx_h \oplus Ku)^{\alpha} = Kx_{\overline{h}} \oplus Ku$ and $(Kx_h \oplus Ky_0)^{\alpha} = Kx_{h^*} \oplus Ky_0$. Then $(Kx_h)^{\alpha} \subseteq$

$(Kx_{\overline{h}} \oplus Ku) \cap (Kx_{h^*} \oplus Ky_0)$ implies that $(Kx_h)^\alpha = Kx_{h^*} = Kx_{\overline{h}}$, since Ku and Ky_0 are fixed by α. Thus, the "basic" subspaces Kx_h are permuted by α.

Let $g \neq 1$. Then the cycles π_{2_g} are of distinct length for different g. α must permute these cycles, so for each h there is some h' such that

$$(Kx_h \oplus \langle x_{gh} \rangle \oplus \langle Y_0 \rangle)^\alpha = Kx_{h'} \oplus \langle x_{gh'} \rangle \oplus \langle Y_0 \rangle.$$

Then $Kx_{h^*} \oplus \langle x_{gh} \rangle^\alpha \subset Kx_{h'} \oplus \langle x_{gh'} \rangle \oplus \langle Y_0 \rangle$ so that $h' = h^*$, since $Kx_{h'}$ is the only "basic" subspace of $Kx_{h'} \oplus \langle x_{gh'} \rangle \oplus \langle Y_0 \rangle$. This in turn implies that $\langle x_{gh} \rangle^\alpha = \langle x_{gh'} \rangle$. Since $\langle x_{gh} \rangle^\alpha \subset (Kx_{gh})^\alpha = Kx_{(gh)^*}$, it follows that $(gh)^* = gh' = gh^*$ for all $g \neq 1$ and h. Since 1^* is already known to be 1, setting $h = 1$, we see $g^* = g$, so that α stabilizes each Kx_g. Thus, $(Kx_g \oplus Ku)^\alpha = Kx_g \oplus Ku$, since $\alpha = 1$ on Ku. As before, this means that α must centralize π_2^h, and fix the vector it has in common with the 1-spaces determined by the 6-cycles, so α must pointwise fix the support of π_2^h. Again, this support contains a basis for $Kx_h \oplus Ku$, and therefore $\alpha = 1$ on this subspace. Thus, α is the identity, and we have shown $G = \Gamma L(N, 8) \cap \Gamma L(N, 8)^\sigma$. $\square$

Corollary 7.2. *For any finite group G and $N > 4|G| + 2$, there exists a tensor-indecomposable Hadamard design on $v = 2^{3N+1} - 1$ points with automorphism group isomorphic to G, a unique good block and no good points.*

Proof: Let $\mathcal{D}$ be a GMW design on $2^{3N} - 1$ points with $\mathrm{Aut}(\mathcal{D}) \cong \Gamma L(N, 8)$, as in Fact 6.1. Since $\mathcal{D}$ is an abelian difference set design, it is self-dual. Thus, the permutation σ defined in Lemma 7.1, which acts on the points of $\mathcal{D}$, can be dualized to a permutation σ^* which acts on blocks of $\mathcal{D}$. Form $\mathcal{D}_G = \mathcal{D}\hat{\sigma}\mathcal{D}^+$, where $\hat{\sigma} = \sigma_0 \sigma^*$. $\mathcal{D}_G$ has $2^{3N+1} - 1$ points. Since $\mathcal{D}$ has no good blocks by Observation 6.2, B_∞ is the unique good block of $\mathcal{D}\sigma\mathcal{D}^+$ by Eq. (3.3). Thus, Corollary 3.10 gives $\mathrm{Aut}(\mathcal{D}\sigma\mathcal{D}^+) \cong G$ as $\mathrm{Aut}(\mathcal{D}) = \mathrm{Aut}(\mathcal{D}^+)$ by Lemma 6.3. A good point off B_∞ would yield $\mathcal{D}_G \cong \mathcal{D}\sigma_0\mathcal{D}^+$ by Fact 3.6. However, $\Gamma L(N, 8) \leq \mathrm{Aut}(\mathcal{D}\sigma_0\mathcal{D}^+)$, stabilizing B_∞, by Corollary 3.10 and $\Gamma L(N, 8)$ is much larger than G. A good point on B_∞ would induce an elation by Theorem 3.7 which in turn would induce a translation of $\mathcal{D}^+$, contradicting Fact 2.1 and Observation 6.2. Since $\mathcal{D}$ is tensor-indecomposable by Observation 6.2, so is $\mathcal{D}_G$ by Corollary 4.17. $\square$

Proof of Theorem 1.1: Given a Hadamard design $\mathcal{D}$ of order n, form $\mathcal{D}_3$ of order $8n$ with $\mathrm{Aut}(\mathcal{D}_3) = 1$ as in Theorem 5.2. Given G, form $\mathcal{D}_G$ of order 2^{3N-1} with $\mathrm{Aut}(\mathcal{D}_G) \cong G$ as in Corollary 7.2. Since neither $\mathcal{D}_3$ nor $\mathcal{D}_G$ has good points, we have $\mathrm{Aut}(\mathcal{D}_G \otimes \mathcal{D}_3) \cong G$ by Corollary 4.15. $\mathcal{D}_G \otimes \mathcal{D}_3$ is a Hadamard design of order $2^{3N+4}n$. There are $\frac{(16n-2)!}{2^{10}n^3}$ non-isomorphic choices for $\mathcal{D}_3$, inducing non-isomorphic designs $\mathcal{D}_3 \otimes \mathcal{D}_G$ by the unique factorization of Corollary 4.14. $\square$

References

1. L. Babai, "On the abstract group of automorphisms," in *Combinatorics*, *Proc. Eighth British Comb. Conf.* H.N.V. Temperley (Ed.), Cambridge U. Press, Cambridge, 1981 pp. 1–40.
2. T. Beth, D. Jungnickel, and H. Lenz, *Design Theory*, Cambridge University Press, Cambridge, 1999.
3. U. Dempwolff and W.M. Kantor, Distorting symmetric designs. in preparation
4. D. Jungnickel, "The number of designs with classical parameters grows exponentially," *Geometriae Dedicata* **16** (1984), 167–178.
5. W.M. Kantor, "Characterizations of finite projective and affine spaces," *Can. J. Math.* **21** (1969), 64–75.
6. W.M. Kantor, "Automorphism Groups of Hadamard Matrices," *J. Comb. Theory* **6** (1969), 279–281.
7. W.M. Kantor, "Automorphisms and isomorphisms of symmetric and affine designs," *J. Alg. Comb.* **3** (1994), 301–338.
8. W.M. Kantor, "Note on GMW designs," *Europ. J. Comb.* (22) (2001), 63–69.
9. W.M. Kantor, unpublished manuscript.
10. M.E. Kimberley, "On the construction of certain Hadamard designs," *Math Z.* **119** (1971), 41–59.
11. C. Lam, S. Lam, and V.D. Tonchev, "Bounds on the number of Hadamard designs of even order," *J. Comb. Designs* **9** (2001), 363–378.
12. E. Merchant, "Exponentially many Hadamard designs," *Des. Codes. and Cryptogr.* **38**(2) (2006), 297–308.
13. C.W. Norman, "Hadamard designs with no non-trivial automorphisms," *Geom. Ded.* **2** (1976), 201–204.
14. C.W. Norman, "Non-isomorphic Hadamard designs," *J. Combin. Theory (A)* **21** (1976), 336–344.
15. R.E.A.C. Paley, "On Orthogonal Matrices," *J. Math. Phys.* **12** (1933), 311–320.
16. O. Pfaff, "The classification of doubly transitive affine designs," *Des., Codes, Cryptogr.* **1** (1991), 207–217.
17. D.E. Taylor, *The Geometry of the Classical Groups*, Heldermann Verlag, Berlin, 1992.
18. J.A. Todd, "A combinatorial problem," *J. Math. Phys.* **12** (1933), 321–333.