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Blocking sets in $PG(2, q^n)$ from cones of $PG(2n, q)$

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Abstract Let Ω and $\bar{B}$ be a subset of $\Sigma = PG(2n-1, q)$ and a subset of $PG(2n, q)$ respectively, with $\Sigma \subset PG(2n, q)$ and $\bar{B} \not\subset \Sigma$. Denote by K the cone of vertex Ω and base $\bar{B}$ and consider the point set B defined by

$$B = \big(K \setminus \Sigma\big) \cup \{X \in \mathcal{S} \,:\, X \cap K \neq \emptyset\},$$

in the André, Bruck-Bose representation of $PG(2, q^n)$ in $PG(2n, q)$ associated to a regular spread $\mathcal{S}$ of $PG(2n-1, q)$. We are interested in finding conditions on $\bar{B}$ and Ω in order to force the set B to be a minimal blocking set in $PG(2, q^n)$. Our interest is motivated by the following observation. Assume a Property α of the pair $(\Omega, \bar{B})$ forces B to turn out a minimal blocking set. Then one can try to find new classes of minimal blocking sets working with the list of all known pairs $(\Omega, \bar{B})$ with Property α. With this in mind, we deal with the problem in the case Ω is a subspace of $PG(2n-1, q)$ and $\bar{B}$ a blocking set in a subspace of $PG(2n, q)$; both in a mutually suitable position. We achieve, in this way, new classes and new sizes of minimal blocking sets in $PG(2, q^n)$, generalizing the main constructions of [14]. For example, for $q = 3^h$, we get large blocking sets of size $q^{n+2} + 1$ $(n \geq 5)$ and of size greater than $q^{n+2} + q^{n-6}$ $(n \geq 6)$. As an application, a characterization of Buekenhout-Metz unitals in $PG(2, q^{2k})$ is also given.

Keywords Blocking set · André/Bruck-Bose representation · Ovoid.

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1. Introduction

Let Π_n be a finite projective plane of order n. A *blocking set* in Π_n is a point set B intersecting every line and containing none. A point P of B is said to be *essential* if $B \setminus \{P\}$ is not a blocking set, that is if a line ℓ exists meeting B exactly in the point P. When all points of B are essential no proper subset of B is a blocking set and B is called *minimal*.

Let $PG(n, q)$ denote the n–dimensional projective space associated with the $(n + 1)-$dimensional vector space $GF(q)^{n+1}$ over the finite field $GF(q)$ with q elements, q a prime power. Following [12], a *blocking set* in $PG(n, q)$, $n \geq 2$, is defined as a point set B intersecting every hyperplane and containing no line. A blocking set B is called *linear* [21] if its points are defined by the non-zero vectors of a $GF(q')-$vector subspace of $GF(q)^{n+1}$, $GF(q')$ a subfield of $GF(q)$; in this case B is also called $GF(q')-$*linear*. We say that B is *planar* if it is contained in a plane of $PG(n, q)$. The definitions of *essential point* and *minimal blocking set* extend to blocking sets in $PG(n, q)$ in an obvious way. When a subspace S of $PG(n, q)$ meets a blocking set B just in one point P we say that S is *tangent* to B in P. It is straightforward to see that if $PG(h, q)$ is an $h-$dimensional subspace of $PG(n, q)$, $h > 1$, then every blocking set in $PG(h, q)$ is also a blocking set in $PG(n, q)$ and the minimality is preserved.

The above two definitions of blocking set clearly coincide for the Desarguesian plane $PG(2, q)$.

Unfortunately, in the literature the terminology on blocking sets is not yet standard, so sometimes it is possible to find slight variations of the previous definitions. For example, in [6] a blocking set in $PG(n, q)$ is defined as a $1-$*blocking set*. For information on main results and recent developments of blocking set theory we refer the reader to [5, 8, 17, 22, 23, 27, 29, 30]. Here we will survey just some results useful in what follows.

Baer subplanes and unitals in $PG(2, q^2)$ and ovoids in $PG(3, q)$ are examples of extremal minimal blocking sets, in the sense of the following two classical results.

Result 1.1. (A. A. Bruen [8] for $n = 2$; A. Beutelspacher [6] for $n > 2$) *The minimum possible size of a blocking set B in a finite projective space $PG(n, q)$, $n \geq 2$, is $q + \sqrt{q} + 1$ and the bound is attained if, and only if, q is a square and B is a Baer subplane.*

Actually, the result of A. A. Bruen [8] was proved also for non Desarguesian finite projective planes. Moreover, in the case $n > 2$, improved results have been obtained by L.Storme and Sz.Weiner [26].

Result 1.2. (A. A. Bruen and J. A. Thas, [12]) *Let B be a minimal blocking set in $PG(n, q)$. Then we have the following:*

- *if $n = 2$, $|B| \leq q\sqrt{q} + 1$ and equality holds if, and only if, q is a square and B is a unital;*
- *if $n = 3$, $|B| \leq q^2 + 1$ and equality holds if, and only if, B is an ovoid;*
- *if $n \geq 4$, $|B| < \sqrt{q^{n+1}} + 1$.*

For $n > 2$, the notion of ovoid can be generalized to a non singular quadric Q of $PG(n, q)$: it is a point set of Q meeting every generator of Q exactly once. Ovoids of a non singular parabolic quadric $Q(2n, q)$ in $PG(2n, q)$ contain exactly $q^n + 1$ points and it is known that they exist if and only if, $n = 2, 3$ (A. Gunawardena and E. Moorhouse [15] for q odd, J. A. Thas [32] for q even). Very deep results about ovoids of $Q(2n, q)$ have been recently obtained by S. Ball in [2] and by S. Ball, P. Govaerts and L. Storme in [3]. In particular, these authors prove that an ovoid O of $Q(2n, q)$, $n = 2, 3$, meets every elliptic quadric $Q^-(2n - 1, q)$ on $Q(2n, q)$ in 1 mod p points, p the characteristic of $GF(q)$ (*see* [2] for $n = 2$, [3] for $n = 3$). So, since every hyperplane of $PG(2n, q)$ intersecting $Q(2n, q)$ not in a $Q^-(2n - 1, q)$ has some points on O, the following useful result on blocking set can be stated.

Result 1.3. *Every ovoid of a non singular parabolic quadric $Q(2n, q)$ of $PG(2n, q)$, $n = 2, 3$, is a minimal blocking set in $PG(2n, q)$.*

Minimal blocking sets in $PG(2, q)$ of size less than $3(q + 1)/2$ are called *small* and have been intensively studied by several authors; an updated survey on them with a quite complete bibliography can be found in [30, Sect. 3.1]. Here we only recall a result by A. Blokhuis [7] stating the non existence of small blocking sets in $PG(2, p)$, p a prime. Conversely, very few results are known about minimal blocking sets of $PG(n, q)$ whose order is "close" to the bounds of Result 1.2, especially when $n > 2$. These blocking sets are called *large* and we refer to [30, Sect. 3.4] for details. In this direction, in the case of $PG(2, q)$, new interesting results were sketched and announced by A. Gács, T.Szőnyi and Zs.Weiner in [30], but completed and appeared explicitly in [14] afterwards; among them we recall the following.

Result 1.4. (A. Cossidente, A. Gács, C. Mengyán, A. Siciliano, T. Szőnyi and Zs. Weiner, [14]) (*i*) *In $PG(2, q^n)$ there are minimal blocking sets of size $q^{n+1} + 1$, if $n \geq 2$, and minimal blocking sets of size $q^{n+1} + q^{n-3} + 1$, if $n \geq 3$.*

(*ii*) *In $PG(2, q^2)$ there is a minimal blocking set for any size in the interval* $[4q \log q, q\sqrt{q} - q + 2\sqrt{q}]$.

The first part of this result is achieved by generalizing the well known construction for the Buekenhout-Metz unitals [13]. Actually, the authors prove that to some cones in $PG(2n, q)$ with base an ovoid of $PG(3, q)$ there correspond minimal blocking sets in the André, Bruck-Bose representation of $PG(2, q^n)$ in $PG(2n, q)$. The second part of the result is based on a construction that relies on a statistical argument.

In this paper, in the same spirit of Result 1.4(*i*), we introduce some more general constructions consisting of cones in $PG(2n, q)$ of base a blocking set in a suitable subspace of $PG(2n, q)$ such that minimal blocking sets in André, Bruck-Bose representations of $PG(2, q^n)$ are achieved. In this way we can exhibit new classes and new sizes of minimal blocking sets in $PG(2, q^n)$. Some of these blocking sets are large and sometimes their sizes lie in the interval of Result 1.4(*ii*). We note that, in this last case, our constructions are purely geometrical and do not rely on statistical arguments as the corresponding ones of Result 1.4(*ii*). As an application of our results we give a characterization of Buekenhout-Metz unitals in

$PG(2, q^{2k})$, also showing how the existence of non Buekenhout-Metz unitals depends on that of a special kind of blocking sets in projective spaces. We point out that, to get new results by our constructions, we especially need examples and properties of minimal blocking sets which are not planar. As we will see in the next sections, some useful results can be found in [16] and [31]; here we only recall the following.

Result 1.5. (U. Heim, [16]) *Let $q = p^h$ be a power of a prime p. For every integer $d > 2$, there exist blocking sets in $PG(d, q)$, not contained in a hyperplane, of size $(d-1)q - (d-3)q/p + 1$, if $h > 2$, and of size $(d-1)q - (d-3)(q+1)/2 + d - 1$, if $q = p$ is an odd prime.*

2. Preliminaries

Let us briefly recall the well known *André, Bruck-Bose representation* of the plane $PG(2, q^n)$. Let $\mathcal{S}$ be a *regular* $(n-1)$-spread of a hyperplane $\Sigma = PG(2n-1, q)$ in $\Sigma' = PG(2n, q)$. A point-line geometry $\Pi = \Pi(\mathcal{S})$, isomorphic to $PG(2, q^n)$, can be defined in the following way [1, 9, 10]: (i) the *points* are the points of $\Sigma' \setminus \Sigma$ (*affine* points) and the elements of $\mathcal{S}$, (ii) the lines are the n-dimensional subspaces of Σ' which intersect Σ in an element of $\mathcal{S}$ (*affine* lines) and the $(n-1)$-spread $\mathcal{S}$, (iii) the *point-line incidences* are inherited from Σ'.

The incidence structure $\Pi = \Pi(\mathcal{S})$ can also be defined without the assumption that the spread $\mathcal{S}$ is regular and, in this case, Π is a translation plane [9]. As we are interested in Desarguesian planes, for the rest of the paper we do not care about this more general context. However, we point out that most of our results extend to finite translation planes in a very natural way.

Now let Ω and $\bar{B}$ be a subset of Σ and a subset of Σ' not contained in Σ, respectively. Denote by $K = K(\Omega, \bar{B})$ the cone of vertex Ω and base $\bar{B}$, i.e.

$$K = K(\Omega, \bar{B}) = \bigcup_{\bar{P} \in \bar{B}} \langle \bar{P}, \Omega \rangle, \qquad (1)$$

and consider the subset $B = B(\Omega, \bar{B})$ of Π defined by

$$B = B(\Omega, \bar{B}) = \big(K \setminus \Sigma\big) \cup \{X \in \mathcal{S} \, : \, X \cap K \neq \emptyset\}. \qquad (2)$$

We are interested in finding conditions on $\bar{B}$ and Ω in order to force the set B to be a minimal blocking set in Π. Our interest is motivated by the following observation. Assume a Property α of the pair $(\Omega, \bar{B})$ forces B to turn out a minimal blocking set. Then one can try to find new classes of minimal blocking sets working with the list of all known pairs $(\Omega, \bar{B})$ with Property α. With this in mind, we deal with the problem in the case Ω is a subspace of Σ and $\bar{B}$ a minimal blocking set in a subspace of Σ'; both in a mutually suitable position.

We point out that the notation introduced in this section will be used for the rest of the paper, even without explicitly recalling it.

3. Construction 1

Let Y be a fixed element of the spread $\mathcal{S}$ of Σ and let Ω be a hyperplane of Y. Let Γ' be an $(n+1)$-dimensional subspace of Σ' such that $\Gamma' \cap \Omega = \emptyset$ and assume that $\bar{B}$ is a subset of Γ' not contained in Σ. Note that, since $\Gamma' \cap \Omega = \emptyset$, the intersection of Γ' and Y is a point T and

$$\langle \bar{P}, \Omega \rangle \cap \langle \bar{P}', \Omega \rangle = \Omega, \qquad (3)$$

for any distinct points $\bar{P}, \bar{P}' \in \bar{B}$. Moreover, define $K = K(\Omega, \bar{B})$ and $B = B(\Omega, \bar{B})$ by (1) and (2), respectively. If $\bar{B} \cap \Sigma \subseteq Y$, the number γ of elements of $\mathcal{S}$ distinct from Y and meeting K is 0, otherwise $\gamma \geq q^{n-1}$. The bound $\gamma = q^{n-1}$ is attained if, and only if, $\bar{B} \cap \Sigma$ is a point not in Y. The size of B is given by

$$|B| = |K \setminus \Sigma| + \gamma + 1 = q^{n-1}|\bar{B} \setminus \Sigma| + \gamma + 1. \qquad (4)$$

Proposition 3.1. *B is a blocking set of the plane Π if, and only if, the following properties are fulfilled:*

(i) *$\bar{B}$ meets every hyperplane of Γ' not through T,*
(ii) *$\bar{B}$ contains no hyperplane of Γ',*
(iii) *$\bar{B}$ contains no line through T,*
(iv) *a spread element $X \in \mathcal{S}$ exists such that $X \cap K = \emptyset$.*

Proof: Let $\bar{B}$ satisfy the four properties above. Let S_n be an n-dimensional subspace of Σ' not contained in Σ and assume $S_n \cap \Omega = \emptyset$. Then $dim\langle S_n, \Omega \rangle = 2n - 1$ and, as a consequence, $dim(\langle S_n, \Omega \rangle \cap \Gamma') = n$. This implies that there exists a point $\bar{P} \in \bar{B} \cap \langle S_n, \Omega \rangle \cap \Gamma'$ and hence $S_n \cap \langle \bar{P}, \Omega \rangle \neq \emptyset$. Moreover S_n is not contained in K. Actually, under this assumption, (3) implies that the n−dimensional subspace $\langle S_n, \Omega \rangle \cap \Gamma'$ is contained in $\bar{B}$; a contradiction by (ii). In conclusion, the cone K blocks any n-dimensional subspace S_n of Σ' defining a line in Π not through Y and no such S_n is contained in it. To conclude that B is a blocking set of Π, it is enough to note that $B \cap \mathcal{S} \neq \emptyset$, as $Y \in B$, and no line of Π through Y is contained in B by (iii) and (iv).

Conversely, assume that B is a blocking set in Π and suppose that there exists an n−dimensional subspace S'_n of Γ' not through T such that $\bar{B} \cap S'_n = \emptyset$. The subspace spanned by S'_n and Ω is a hyperplane H of Σ'; so $H \cap \Sigma$ is a hyperplane of Σ and must contain a unique spread element Z, which turns out to be distinct from Y. Now, if S_n is an n-dimensional subspace of H through Z not contained in Σ, there exists a common point P of S_n and K, as B is a blocking set in Π. Then P is on a line joining a point of Ω and a point $\bar{P}$ of $\bar{B}$ which must belong to $S'_n = H \cap \Gamma'$, a contradiction. So, $\bar{B}$ blocks every hyperplane of Γ' not on T. Moreover, it is straightforward to prove (iii) and (iv) and, as a consequence, $\Gamma = \Gamma' \cap \Sigma$ is not contained in $\bar{B}$. Finally, assume the existence of an $n-$dimensional subspace S'_n other than Γ and contained in $\bar{B}$. As already noted, the $(2n-1)-$dimensional subspace $S_{2n-1} = \langle S'_n, \Omega \rangle$ contains an element Z of the spread $\mathcal{S}$, so we can consider an $n-$dimensional subspace S_n through Z and contained in S_{2n-1}. Since every point of S_n is on a line meeting S'_n and Ω, S_n should be contained in K; a contradiction, as B does not contain lines of Π. It follows the validity of (ii), finishing the proof. □

The minimum size of a subset $\bar{B}$ verifying the Properties (i)-(iv) is $q+1$; in this case $\bar{B}$ is a line of Γ' meeting Σ in a point distinct from T and the corresponding B is a minimal blocking set of $PG(2, q^n)$ of size $q^n + q^{n-1} + 1$. Actually, $q^n + q^{n-1} + 1$ is the smallest size of a blocking set of $PG(2, q^n)$ of type $B(\Omega, \bar{B})$, as easily follows from Result 1.1. Moreover, it follows from (4) that, if a blocking set $B = B(\Omega, \bar{B})$ in $PG(2, q^n)$ has size $q^n + q^{n-1} + 1$, then $\bar{B}$ is a line of Γ' meeting Σ in a point distinct from T.

Corollary 3.2. *If $\bar{B}$ is a blocking set of Γ', then B is a blocking set of Π.*

Proof: It is enough to remark that every blocking set in Γ' fulfills Properties (i)-(iv) of Proposition 3.1. □

Proposition 3.1 has a kind of converse, in the following sense.

Proposition 3.3. *Let $\bar{B}'$ be a subset of Σ' not contained in Σ and disjoint from Ω. Define $K(\Omega, \bar{B}')$ and $B(\Omega, \bar{B}')$ by* (1) *and* (2)*, respectively, and assume that $B(\Omega, \bar{B}')$ is a blocking set of Π. Then there exist an* $(n+1)$*-dimensional subspace Γ' of Σ' skew to Ω and a subset $\bar{B}$ of Γ' verifying Properties* (i)-(iv) *of Proposition* 3.1 *such that* $B(\Omega, \bar{B}) = B(\Omega, \bar{B}')$.

Proof: Let Γ' be an $(n+1)$-dimensional subspace of Σ' intersecting Y in a point $T \notin \Omega$ and define $\bar{B} = K(\Omega, \bar{B}') \cap \Gamma'$. It is straightforward to see that $K(\Omega, \bar{B}) = K(\Omega, \bar{B}')$, so $B(\Omega, \bar{B}) = B(\Omega, \bar{B}')$, concluding the proof. □

Note that if $\bar{B}$ fulfills Properties (i)-(iv) of Proposition 3.1, then $\bar{B} \cup \{T\}$ meets every hyperplane of Γ' and, if $\bar{B}$ contains a subset $\bar{B}'$ which is either a blocking set or a line, then $B(\Omega, \bar{B}')$ is a blocking set in $PG(2, q^n)$ and $B(\Omega, \bar{B}') \subseteq B(\Omega, \bar{B})$. So, as we are interested in minimal blocking sets and we know the structure of $B(\Omega, \bar{B})$ when $\bar{B}$ is a line, w.l.o.g. we suppose for the rest of the section that $\bar{B}$ is a minimal blocking set of Γ'. Moreover, we assume that $\Gamma = \Gamma' \cap \Sigma$ is a tangent hyperplane of $\bar{B}$ at a point Q, i.e.

$$\Gamma' \cap \Sigma \cap \bar{B} = \{Q\}.$$

By Corollary 3.2, B is a blocking set of Π and, in order to check its minimality, we distinguish the following two cases in the next subsections: $Q \in Y$ and $Q \notin Y$.

3.1. Construction $1a$

Under the assumption $Q \in Y$, i.e. $Q = T$, we have

$$B = \big(K \setminus \Sigma\big) \cup \{Y\}$$

and the size of B is given by

$$|B| = |K \setminus \Sigma| + 1 = q^{n-1}(|\bar{B}| - 1) + 1. \qquad (5)$$

By next proposition the line intersection numbers of B can be determined.

Proposition 3.4. *Let S_n be a line of Π other than $\mathcal{S}$. If S_n contains Y and ℓ is the line $S_n \cap \Gamma'$, then*

$$|B \cap S_n| = q^{n-1}|(\ell \cap \bar{B}) \setminus \{Q\}| + 1. \qquad (6)$$

If S_n does not contain Y and $S'_n = \langle S_n, \Omega\rangle \cap \Gamma'$, then

$$|B \cap S_n| = |\bar{B} \cap S'_n|. \qquad (7)$$

Moreover, Equality (7) *holds for any hyperplane S'_n of Γ' not through Q and for each of the q^{n-1} lines S_n of Π contained in $\langle \Omega, S'_n\rangle$.*

Proof: Equality (6) is straightforward; so assume S_n is not on Y. Then

$$B \cap S_n = (K \setminus \Sigma) \cap S_n = \bigcup_{\bar{P} \in \bar{B}} (\langle \bar{P}, \Omega\rangle \cap S_n) = \bigcup_{\bar{P} \in \bar{B} \cap S'_n} (\langle \bar{P}, \Omega\rangle \cap S_n),$$

where $S'_n = \langle S_n, \Omega\rangle \cap \Gamma'$. Since $\langle \bar{P}, \Omega\rangle \cap \langle \bar{P}', \Omega\rangle = \Omega$ for any distinct points $\bar{P}, \bar{P}' \in \bar{B}$ and $dim(\langle \bar{P}, \Omega\rangle \cap S_n) = 0$ if $\bar{P} \in \bar{B} \cap S'_n$, we obtain (7). Now assume S'_n is a hyperplane of Γ' not through Q and consider the $(2n-2)-$subspace $H = \langle S'_n \cap \Sigma, \Omega\rangle$. Since H is a hyperplane of Σ, there exists a unique element $X \in \mathcal{S}$, $X \neq Y$, contained in H. If S_n is one of the q^{n-1} lines of Π on X contained in $\langle S'_n, \Omega\rangle$, then $\langle S_n, \Omega\rangle = \langle S'_n, \Omega\rangle$, $\langle S_n, \Omega\rangle \cap \Gamma' = S'_n$ and Equality (7) follows. □

The above proposition allows us to prove the minimality of B.

Proposition 3.5. *B is a minimal blocking set of* Π.

Proof: The line $\mathcal{S}$ is a tangent to B at the point Y, so Y is an essential point of B. Let P be an affine point of B, i.e. $P \in K \setminus \Sigma$ and let $\bar{P}$ be the unique point of $\bar{B}$ such that $P \in \langle \bar{P}, \Omega\rangle$. Since $\bar{B}$ is a minimal blocking set of Γ', an n-dimensional subspace S'_n of Γ' exists such that $S'_n \cap \bar{B} = \{\bar{P}\}$. Then, by Proposition

3.4, $|B \cap S_n| = 1$ for each line of Π contained in $\langle \Omega, S'_n \rangle$. Note that $\langle \Omega, S'_n \rangle$ contains some lines of Π, since $\langle \Omega, S'_n \rangle \cap \Sigma$ is a hyperplane of Σ and consequently it contains an element of the spread $\mathcal{S}$. On the other hand, as $P \in \langle \Omega, S'_n \rangle$, there exists one line of Π through P contained in $\langle \Omega, S'_n \rangle$, hence P is an essential point of B. $\square$

Construction $1a$ generalizes the following already known constructions:

- If the base $\bar{B}$ of the cone K is an ovoid in a 3-dimensional space contained in Γ', then we get the *ovoidal cone construction* [14], also described in [30, Section 3.4]. Note that when $n = 2$, this is exactly the well known construction for the Buekenhout-Metz unitals [13].
- If the base $\bar{B}$ of the cone K is a planar blocking set of Γ', then we get a construction equivalent to the Construction 2.12 as described in the comment after Proposition 3.24 of [30]. Indeed, following the notation of [30], consider the blocking set $B^* = B$" obtained by Construction 2.12 in the above mentioned comment. This blocking set can be seen as a sort of "cone" in $\pi = PG(2, q^h)$ with vertex $\bar{V}'$ (a $(h-2)-$dimensional projective subspace over $GF(q)$, projection of V' from P onto π) contained in a point $\bar{R}$ of π (projection of the subspace R from P onto π) and with base a minimal blocking set $\bar{B}$ of a subplane $PG(2, q)$ of π such that $\bar{R} \cap PG(2, q) = \bar{R} \cap \bar{B}$ is a point over $GF(q)$ not belonging to $\bar{V}'$. This is exactly the representation in $PG(2, q^h)$ of a minimal blocking set obtained by Construction $1a$ with $n = h$, choosing as base of the cone K a minimal planar blocking set.

Moreover, the linearity is preserved, in the sense of the next proposition.

Proposition 3.6. *The blocking set B is linear in $PG(2, q^n)$ if, and only if, $\bar{B}$ is a linear blocking set in Γ'.*

Proof: Throughout the proof we represent Σ', Γ' and Ω as the projective spaces associated with the $GF(q)$–vector spaces V, U and L, respectively. Assume $\bar{B}$ is a $GF(q')$-linear blocking set of Γ', where $GF(q')$ is a subfield of $GF(q)$ and $q = q'^h$. This means that the points of $\bar{B}$ are defined by the non zero vectors of an h-dimensional vector subspace W of U over $GF(q')$; i.e. $\bar{B} = \{\bar{P} = \langle \mathbf{w} \rangle \ : \ \mathbf{w} \in W \setminus \{\mathbf{0}\}\}$. Now, if $P = \langle \mathbf{v} \rangle$ is a point in $K \setminus \Omega$ and $\bar{P} = \langle \mathbf{w} \rangle$ is the unique point of $\bar{B}$ such that $P \in \langle \bar{P}, \Omega \rangle$, then we can write $\mathbf{v} = \mathbf{u} + \alpha \mathbf{w}$, with $\mathbf{u} \in L$, $\mathbf{w} \in W$ and $\alpha \in GF(q) \setminus \{0\}$. This implies that $P = \langle \alpha^{-1} \mathbf{v} \rangle$, where $\alpha^{-1}\mathbf{v} \in \langle L, W \rangle_{GF(q')}$, i.e. the points of K, and hence the points of B, are defined by the non zero vectors of the $GF(q')$-vector subspace $\langle L, W \rangle_{GF(q')}$ of V, which has dimension nh. Then B is a $GF(q')$-linear blocking set of Π. Conversely, it is easy to see that the linearity of B implies that of $\bar{B}$. $\square$

Our aim is to find new families of minimal blocking sets in $PG(2, q^n)$ choosing as base of the cone K some suitable minimal blocking sets of Γ'. To this end, among some classes of non planar blocking sets of $PG(3, q)$ constructed by *G. Tallini* in [31], we selected the following five examples $\bar{B}_i$, that are minimal:

$$\bar{B}_1 = (r \setminus \pi) \cup T \quad \text{with} \quad | \bar{B}_1 | = 2q + 1 \, ,$$

where r is a line, π is a plane not containing r and T is a set of $(q+1)$ non collinear points of π having the point $\pi \cap r$ as a nucleus;

$$\bar{B}_2 = (r \setminus \{N_1, N_2\}) \cup (K_1 \cup K_2) \text{ with } \mid \bar{B}_2 \mid = 3q + 1\,,$$

where r and r' are skew lines, $q > 2$, N_1 and N_2 are distinct points on r and K_i $(i = 1, 2)$ is a $(q+1)$-set in the plane $\pi_i = \langle N_i, r' \rangle$, disjoint from r', having N_i as a nucleus and satisfying the following property: $(\star)$ *every point of* r' *is on at least one line of* π_i *different from* r' *and disjoint from* K_i;

$$\bar{B}_3 = (r \setminus \{N_1, N_2, N_3\}) \cup K_1 \cup K_2 \cup K_3 \text{ with } \mid \bar{B} \mid = 4q + 1\,,$$

where r and r' are skew lines, $q > 2$ is even, N_1, N_2 and N_3 are three distinct points on r, K_1 is a $(q+1)$-set in the plane $\pi_1 = \langle N_1, r' \rangle$ disjoint from r', having N_1 as a nucleus and satisfying property $(\star)$, K_i $(i = 2, 3)$ is the projection of K_1 on the plane $\pi_i = \langle N_i, r' \rangle$ from the point N_j, where $\{i, j\} = \{2, 3\}$;

$$\bar{B}_4 = (\ell_1 \cup \ell_2 \cup \ell_3) \setminus (r_1 \cup r_2) \cup \{P_1, P_2\} \text{ with } \mid \bar{B} \mid = 3q - 1\,,$$

where ℓ_1, ℓ_2, ℓ_3 are distinct lines of a regulus of a hyperbolic quadric, $q > 2$, r_1, r_2 are two distinct lines of the opposite regulus, P_i is a point on r_i $(i = 1, 2)$ such that $P_i \notin \cup_{j=1}^{3} \ell_j$;

$$\bar{B}_5 = \left(\mathcal{O} \setminus \{\cup_{i=1}^{h} C_i\}\right) \cup \{\cup_{i=1}^{h} N_i\} \text{ with } \mid \bar{B} \mid = q(q-h) + 1\,,$$

where $\mathcal{O}$ is an ovoid of $PG(3, q)$, q is even, $\pi_1, \ldots, \pi_h$ $(1 \le h \le q-2)$ are distinct planes through an external line r to $\mathcal{O}$ intersecting $\mathcal{O}$ in the $(q+1)$-arcs $C_1, \ldots, C_h$ with nuclei $N_1, \ldots, N_h$ respectively.

Now, let S_3 be a 3-dimensional subspace of Γ' and let $\bar{B}_i$ be one of the previous examples of blocking sets of S_3 with $\Gamma \cap Y \cap \bar{B}_i = \{Q\}$ and having $S_3 \cap \Gamma$ as a tangent plane. Then, via the cone K having $\bar{B}_i$ as base, we get minimal blocking sets B_i of $PG(2, q^n)$ $(n \ge 2)$ of the following sizes:

$$\mid B_1 \mid = 2q^n + 1 \ (n \ge 2), \qquad \mid B_2 \mid = 3q^n + 1 \ (q > 2,\ n \ge 2),$$
$$\mid B_3 \mid = 4q^n + 1 \ (q > 2 \text{ even}, n \ge 2), \quad \mid B_4 \mid = 3q^n - 2q^{n-1} + 1 \ (q > 2),$$
$$\mid B_5 \mid = kq^n + 1 \ (q \text{ even}, 2 \le k \le q-1).$$

The sizes of B_2 (if q is even) and B_3 (if $q > 4$) seem to be new in the spectrum of known cardinalities of minimal blocking sets of $PG(2, q^n)$ (see [30]), even compared with the interval of Result 1.4 (*ii*). Also the sizes of the blocking sets B_5 should be new when either n is odd and $k \ge 3$ or n is even and $3 \le k < 4n \log q$. Computing the intersection numbers with respect to lines one can verify that B_1 and B_4 are not contained in the union of four lines, hence they are not isomorphic to the examples of the same size obtained by the so-called *IMI construction* (see [18, 19, 30]). Similarly, it is possible to prove that B_2 (q odd) is not contained in the union of three conics through a point, then it cannot be obtained by the *parabola construction* described in [28]. Some of the above remarks can be summarized in the following result.

Proposition 3.7. *In* $PG(2, q^n)$, $n \geq 2$, *there exist minimal blocking sets of sizes* $kq^n + 1$, q *even and* $3 \leq k \leq q - 1$.

The next proposition gives some further sizes for minimal blocking sets in $PG(2, q^n)$.

Proposition 3.8. *In* $PG(2, q^n)$, *for every* $d = 3, 4, \ldots, n + 1$, *there exist minimal blocking sets* B *of size:* $|B| = (d - 1)q^n - (d - 3)q^n/p + 1$, *if* $q = p^h$ *and* $h > 2$, *and* $|B| = (d - 1)q^n - (d - 3)\frac{(q+1)}{2}q^{n-1} + d - 1$, *if* q *is an odd prime.*

Proof: By (5), it is enough to use as base of the cone K a blocking set $\bar{B}$ of type described in Result 1.5. □

Now, let O be an ovoid of the parabolic quadric $Q(2r, q)$ $(r = 2, 3)$ of $PG(2r, q)$ and, if $r = 2$, assume that O is non classical, i.e. O is not an elliptic quadric of $PG(3, q)$. By Result 1.3, the ovoid O is a minimal blocking set of $PG(2r, q)$ of size $q^r + 1$. If $n \geq 2r - 1$, $PG(2r, q)$ can be embedded as a subspace in Γ' in such a way that $\Gamma \cap Y \cap O = \Gamma \cap O = \{Q\}$ and we can consider the minimal blocking set B_O of $PG(2, q^n)$ obtained via the cone K with O as a base. Then

$$|B_O| = q^{n+r-1} + 1, \qquad (8)$$

where $n \geq 3$ if $r = 2$ and $n \geq 5$ if $r = 3$.

Proposition 3.9. (i) *In* $PG(2, q^n)$, $n \geq 3$, *with either* $q = p^h$ *with* p *an odd prime and* $h > 1$ *or* $q = 2^{2e+1}$ *with* $e \geq 1$, *there exist minimal blocking sets of size* $q^{n+1} + 1$, *not obtained via the ovoidal cone construction* [14].
(ii) *In* $PG(2, q^n)$, $n \geq 5$ *and* $q = 3^h$ *with* $h \geq 1$, *there exist minimal blocking sets of size* $q^{n+2} + 1$.

Proof: Examples of non classical ovoids of $Q(4, q)$ are known only for $q = p^h$, $h > 1$, p an odd prime (Kantor ovoids), for $q = 3^h$, $h > 2$, (Thas-Payne ovoids), for $q = 3^{2h+1}$, $h > 0$ (Ree-Tits slice ovoids), for $q = 3^5$ (Penttila-Williams ovoid) and for $q = 2^{2e+1}$, $e > 1$ (Tits ovoids) (see for instance [20, 32]). The known ovoids of $Q(6, q)$ are the Thas-Kantor ovoids of $Q(6, q)$ with $q = 3^h$ and $h \geq 1$ and the Ree-Tits ovoids of $Q(6, q)$ with $q = 3^{2h+1}$, $h > 0$ (see for instance [20, 32]). Then by (8), (i) and (ii) follow from these two remarks, respectively. □

3.2. Construction 1b

Suppose that $Q \notin Y$ and let Z be the unique element of $\mathcal{S}$ such that $Q \in Z$. Moreover, recall the notation $\Gamma \cap Y = \{T\}$. By (4) the size of B is given by

$$|B| = q^{n-1}(|\bar{B}| - 1) + q^{n-1} + 1 = q^{n-1}|\bar{B}| + 1$$

and the line intersection numbers of B can be determined as in Construction $1a$.

Proposition 3.10. *Let S_n be a line of Π other than $\mathcal{S}$. If S_n contains Y and ℓ is the line $S_n \cap \Gamma'$, then*

$$|B \cap S_n| = q^{n-1}|\ell \cap \bar{B}| + 1. \qquad (9)$$

If S_n does not contain Y and $S_n' = \langle S_n, \Omega\rangle \cap \Gamma'$, then

$$|B \cap S_n| = |\bar{B} \cap S_n'|. \qquad (10)$$

Moreover, Equality (10) *holds for any hyperplane S_n' of Γ' not through T and for each of the q^{n-1} lines S_n of Π contained in $\langle \Omega, S_n'\rangle$.*

Remark 3.11. Note that, if S_n is a line of Π passing through a point of $B \cap \mathcal{S}$ different from Y, then $Q \in S_n' = \langle S_n, \Omega\rangle \cap \Gamma'$. $\square$

Proposition 3.12. *Let P be an affine point of B and let $\bar{P} = \langle P, \Omega\rangle \cap \Gamma'$. Then P is an essential point of B if and only if there exists in Γ' a tangent hyperplane to $\bar{B}$ at the point $\bar{P}$ not through T.*

Proof: If S_n is a line of Π tangent to B at the point P, then $S_n' = \langle S_n, \Omega\rangle \cap \Gamma'$ is a hyperplane of Γ' not through T tangent to $\bar{B}$ at the point $\bar{P}$, and conversely. $\square$

The above proposition shows that the minimality of B does not automatically follow from the minimality of $\bar{B}$, as in Construction $1a$; to this end we need some extra conditions on $\bar{B}$. We say that $\bar{B}$ satisfies Condition $(*)$ with respect to the point T if:

$(*)$ for each point $\bar{P} \in \bar{B} \setminus \{Q\}$ there exists a tangent hyperplane to $\bar{B}$ passing through $\bar{P}$ not containing T.

Corollary 3.13. *The affine points of B are essential points of B if, and only, if $\bar{B}$ satisfies Condition $(*)$ w.r.t. the point T.*

Remark 3.14. If $\bar{B}$ is a minimal blocking set of Γ' contained in an h-dimensional subspace S_h of Γ' with $h \le n$ and $T \notin S_h$, then $\bar{B}$ satisfies Condition $(*)$ w.r.t. the point T. $\square$

Now, let X be a point of $B \cap \mathcal{S}$ different from Y and let $S_{n-1} = \langle X, \Omega\rangle \cap \Gamma$. Then the intersection numbers of B with respect to the lines of Π through X, different from $\mathcal{S}$, are determined by the intersection numbers of $\bar{B}$ with respect to the hyperplanes of Γ', different from Γ, containing S_{n-1}. Conversely, if S_{n-1} is a hyperplane of Γ passing through Q and not containing T, then the intersection numbers of $\bar{B}$ with respect to the hyperplanes of Γ', different from Γ, containing S_{n-1}, determine the intersection numbers of B with respect to the lines of Π through the unique element X of $\mathcal{S}$ contained in $\langle S_{n-1}, \Omega\rangle \cap \Gamma$ (see Prop.3.10). Hence, we have:

Proposition 3.15. *Let $X \in B \cap \mathcal{S}$, with $X \neq Y$. Then X is an essential point of B if and only if there exists a hyperplane S'_n of Γ', different from Γ, tangent to $\bar{B}$ and containing the subspace $\langle X, \Omega \rangle \cap \Gamma$. Also, the number of essential points of B on $\mathcal{S} \setminus \{Y\}$ is equal to the number of hyperplanes of Γ passing through Q, not containing T and contained in a tangent hyperplane to $\bar{B}$ different from Γ.*

If B' is a minimal blocking set of Π contained in B, by the previous results, we get

Corollary 3.16. *If $\bar{B}$ satisfies Condition $(*)$ w.r.t. the point T, then*

$$|B'| = q^{n-1}(|\bar{B}| - 1) + t_Q + 1,$$

where t_Q is the number of hyperplanes of Γ passing through Q not containing T and contained in a tangent hyperplane to $\bar{B}$ different from Γ.

Proof: By Corollary 3.13 and Proposition 3.15, we only have to check that Y is an essential point of B. If Y is not an essential point of B, then each line of Γ' passing through T and not contained in Γ contains a point of $\bar{B}$ different from Q, i.e. $|\bar{B}| \geq q^n + 1$. Since $|\bar{B}| \leq q^{\frac{n+2}{2}} + 1$ (see Result 1.2), we have that $n = 2$, $|\bar{B}| = q^2 + 1$ and hence $\bar{B}$ is an ovoid of Γ'. But, in this case, $\bar{B}$ does not satisfy Condition $(*)$ with respect to the point T, contradicting our assumption. □

Denote by S'_h the h-dimensional space spanned by $\bar{B}$ and let $S_{h-1} = S'_h \cap \Gamma$. Since $\bar{B}$ is contained in S'_h, it is a blocking set of S'_h with respect to the hyperplanes.

Proposition 3.17. (1) *If $h \leq n$ and $T \notin S_h$, then*

$$|B'| = q^{n-1}(|\bar{B}| - 1) + q^{n-h} + l_Q(q^{n-h+1} - q^{n-h}) + 1,$$

where l_Q is the number of hyperplanes of S_{h-1} passing through Q contained in a hyperplane of S'_h tangent to $\bar{B}$ different from S_{h-1}; in particular $0 \leq l_Q \leq q^{h-2} + \cdots + q + 1$.

(2) *If $h \leq n$, $T \in S_h$ and $\bar{B}$ satisfies Condition $(*)$ w.r.t. the point T, then*

$$|B'| = q^{n-1}(|\bar{B}| - 1) + s_Q q^{n-h+1} + 1,$$

where s_Q is the number of hyperplanes of S_{h-1} passing through Q, not containing T and contained in a hyperplane of S'_h tangent to $\bar{B}$ different from S_{h-1}; in particular $0 \leq s_Q \leq q^{h-2}$.

Proof: Suppose that $h \leq n$ and that $T \notin S_h$. By Remark 3.14 $\bar{B}$ satisfies Condition $(*)$ w.r.t. the point T and hence all the affine points of B are essential points. Then, to determine the size of B', by Corollary 3.16, we have to determine the number t_Q of hyperplanes of Γ through Q, not containing T and contained in a tangent hyperplane to $\bar{B}$ different from Γ. It is easy to see that each hyperplane of Γ containing S_{h-1} and not containing T is contained in a tangent hyperplane to $\bar{B}$ different from Γ, hence such hyperplanes determine q^{n-h} essential points of B on the line $\mathcal{S}$. Now, suppose that

S_{n-1} is a hyperplane of Γ passing through Q, not passing through T and not containing S_{h-1}, then $dim(S_{h-1} \cap S_{n-1}) = h - 2$. In this case S_{n-1} determines an essential point of $B \cap \mathcal{S}$ if and only if there exists a hyperplane of S'_h tangent to $\bar{B}$, different from S_{h-1}, containing $S_{h-1} \cap S_{n-1}$. Since through each $(h-2)$-dimensional subspace of S_{h-1} through Q there pass $q^{n-h+1} - q^{n-h}$ hyperplanes of Γ not containing T, we get $t_Q = q^{n-h} + l_Q(q^{n-h+1} - q^{n-h})$, where l_Q is the number of $(h-2)$-dimensional subspaces of S_{h-1} passing through Q contained in a hyperplane of S'_h tangent to $\bar{B}$ different from S_{h-1}. In a similar way it is possible to prove (2). $\square$

If $\bar{B}$ is a blocking set of Γ' contained in a plane π and $T \in \pi$, then it is possible to verify that our construction is equivalent to Construction 2.12 described in [30, Proposition 3.24 Case 1]. Also, if $\bar{B}$ is an ovoid of a 3-dimensional space S_3 and $T \notin S_3$, then we get the examples constructed in [14, Theorem 2.8].

Corollary 3.18. *If $\bar{B}$ is a planar blocking set and $T \notin \langle \bar{B} \rangle = \pi$, then*

$$|B'| = q^{n-1}(|\bar{B}| - 1) + q^{n-2} + 1,$$

if $\Gamma \cap \pi$ is the unique tangent line to $\bar{B}$ passing through Q in π, and

$$|B| = |B'| = q^{n-1}|\bar{B}| + 1,$$

if there exist at least two tangent lines to $\bar{B}$ in π passing through Q. In particular, if q is a square and $\bar{B}$ is a unital of π, we get minimal blocking sets in $PG(2, q^n)$ of size $q^n\sqrt{q} + q^{n-2} + 1$.

Corollary 3.19. *In $PG(2, q^n)$, q even, there exist minimal blocking sets of size $kq^n + q^{n-1} + 1$, with $2 \leq k \leq q - 1$.*

Proof: If $\bar{B}$ is one of examples $\bar{B}_5$ of Section 3.1 contained in a 3-dimensional subspace S_3 of Γ', with $Q = N_i$ for some i, and $T \notin S_3$ $(n \geq 3)$, then $l_Q = q + 1$ and hence $|B| = |B'| = kq^n + q^{n-1} + 1$ $(2 \leq k \leq q - 1)$. $\square$

Finally, let $\bar{B} = O$ be a non classical ovoid of the parabolic quadric $Q(2r, q)$, $r = 2, 3$ (see Result 1.3). Since $\dim\langle O \rangle = 2r$, if $n \geq 2r$, we can choose $T \notin \langle O \rangle$ and, by Proposition 3.17 (1), we get from O a minimal blocking set B_O of $PG(2, q^n)$. If $r = 2$ and $n \geq 4$, we have

$$|B_O| = q^{n+1} + q^{n-4} + l_Q(q^{n-3} - q^{n-4}) + 1,$$

where $0 \leq l_Q \leq q^2 + q + 1$. If $r = 3$ and $n \geq 6$, we have

$$|B_O| = q^{n+2} + q^{n-6} + l_Q(q^{n-5} - q^{n-6}) + 1,$$

where $0 \leq l_Q \leq q^4 + q^3 + q^2 + q + 1$. Applying this construction to the known examples of ovoids of $Q(4, q)$ and $Q(6, q)$ (see for instance [20, 32]), we obtain the following existence results.

Proposition 3.20. *(i) In $PG(2, q^n)$, $n \geq 4$ with either $q = p^h$, p odd prime, and $h > 1$ or $q = 2^{2e+1}$ with $e \geq 1$, there exist minimal blocking sets of size $q^{n+1} + q^{n-4} + l_Q(q^{n-3} - q^{n-4}) + 1$, for some positive integer $l_Q \leq q^2 + q + 1$.*
(ii) In $PG(2, q^n)$, $n \geq 6$ and $q = 3^h$ with $h \geq 1$, there exist minimal blocking sets of size $q^{n+2} + q^{n-6} + l_Q(q^{n-5} - q^{n-6}) + 1$ for some positive integer $l_Q \leq q^4 + q^3 + q^2 + q + 1$.

4. Construction 2

In this section we give a generalization of Construction $1a$. More precisely, under the assumption $dim\Omega \leq n - 2$, we investigate when a slight variation of a construction of type $1a$ still produces a blocking set of Π. To do this, we need some more notation.

Let Y and Ω be a fixed element of $\mathcal{S}$ and an s-dimensional subspace of Y, respectively, with $0 \leq s \leq n - 2$. Let Γ be a $(2n - s - 2)$-dimensional subspace of Σ disjoint from Ω and put $\Theta = Y \cap \Gamma$. For every spread element X other than Y, let $I_{n-1}(X)$ be the $(n - 1)$-dimensional subspace $\langle \Omega, X \rangle \cap \Gamma$. Note that $I_{n-1}(X)$ is disjoint from Θ, for any $X \in \mathcal{S} \setminus \{Y\}$.

Now let Γ' be a $(2n - s - 1)$-dimensional subspace of Σ' disjoint from Ω such that $\Gamma = \Gamma' \cap \Sigma$ and denote by $\mathcal{F} = \mathcal{F}(\mathcal{S}, \Omega)$ the family of n-dimensional subspaces of Γ' containing an $(n - 1)$-dimensional subspace of type $I_{n-1}(X)$. Let $\bar{B}$ be an *$\mathcal{F}$-blocking set* of Γ', i.e. a blocking set of Γ' with respect to the n-dimensional subspaces belonging to $\mathcal{F}$, such that $\bar{B} \cap \Gamma = \Theta$. Finally, define K and B by (1) and (2), respectively, and note that under our assumption:

$$B = (K \setminus \Sigma) \cup \{Y\}.$$

Proposition 4.1. *B is a blocking set of the plane Π of size*

$$|B| = q^{s+1} \left[|\bar{B}| - (q^{n-s-2} + \cdots + q + 1) \right] + 1.$$

Proof: Let S_n be an n-dimensional subspace of Σ' defining a line of Π not passing through Y. Then $dim\langle S_n, \Omega \rangle = n + s + 1$, hence $dim(\langle S_n, \Omega \rangle \cap \Gamma') = n$, and $I_n = \langle S_n, \Omega \rangle \cap \Gamma'$ is an element of $\mathcal{F}$. This implies that there exists a point $\bar{P} \in \bar{B} \cap I_n$ and $S_n \cap \langle \bar{P}, \Omega \rangle \neq \emptyset$. It follows that the cone K blocks all the n-dimensional subspaces of Σ' defining a line of Π not passing through Y and hence B is a blocking set of Π. □

As in the case of Construction $1a$, the line intersection numbers of B can be easily determined.

Proposition 4.2. *If S_n is a line of Π passing through Y, then*

$$|B \cap S_n| = q^{s+1}|(\bar{B} \cap \Gamma'_{n-s-1}) \setminus \Theta| + 1, \qquad (11)$$

where $\Gamma'_{n-s-1} = S_n \cap \Gamma'$.
If S_n is a line of Π not passing through Y, then

$$|B \cap S_n| = |\bar{B} \cap I_n|,$$

where $I_n = (\langle \Omega, S_n \rangle \cap \Gamma') \in \mathcal{F}$. Also, if $I_n \in \mathcal{F}$ and $I_n \not\subset \Gamma$, then

$$|B \cap S_n| = |\bar{B} \cap I_n|,$$

for each of the q^{s+1} lines S_n of Π contained in $\langle \Omega, I_n \rangle$.

Proof: If S_n is a line of Π passing through Y, then

$$S_n \cap B = (S_n \cap (K \setminus \Omega)) \cup \{Y\} = \left(\bigcup_{\bar{P} \in (\bar{B} \setminus \Theta) \cap \Gamma'_{n-s-1}} \left(\langle \Omega, \bar{P} \rangle \cap S_n \right) \right) \cup \{Y\},$$

where $\Gamma'_{n-s-1} = S_n \cap \Gamma'$, hence (11) follows.

Let S_n be a line of Π not passing through Y and let $X \in S_n \cap \mathcal{S}$. As in the proof of Proposition 3.4, we have that $|B \cap S_n| = |\bar{B} \cap I_n|$, where $I_n = \langle \Omega, S_n \rangle \cap \Gamma'$ and $I_n \in \mathcal{F}$. Now, let $I_n \in \mathcal{F}$, I_n not contained in Γ, then there exists $X \in \mathcal{S}$, $X \neq Y$, such that $\langle \Omega, I_n \rangle \cap \Gamma = \langle \Omega, X \rangle \cap \Gamma$. Hence $\langle \Omega, I_n \rangle$ contains q^{s+1} lines S_n of Π such that $|B \cap S_n| = |\bar{B} \cap I_n|$. □

By Proposition 4.2, the minimality of B as a blocking set easily follows from that of $\bar{B}$.

Corollary 4.3. *If $\bar{B}$ is a minimal $\mathcal{F}$-blocking set of Γ', then B is a minimal blocking set of Π.*

Remark 4.4. If $dim\Omega = n - 2$, then $\mathcal{F}$ is the family of all n-dimensional subspaces of Γ'; in this case Construction 2 exactly comes from Construction 1a. □

It seems natural at this point to investigate when a blocking set obtained by Construction 1a can be also achieved by Construction 2, with $dim\Omega < n - 2$. To do this, let us give some more preliminaries.

Under the assumption $n = mt$, $1 < t < n$, a unique normal $(m-1)$-spread $\mathcal{S}^*$ of $\Sigma = PG(2mt - 1, q)$ can be associated with the regular $(n-1)$-spread $\mathcal{S}$ of Σ so that $\mathcal{S}^*$ induces on each element $X \in \mathcal{S}$ a normal $(m-1)$-spread $\mathcal{S}^*(X)$ [24, 25]. We define an $\mathcal{S}^*-$subspace of Σ as a subspace T of Σ which is union of elements of $\mathcal{S}^*$. As a consequence, an $\mathcal{S}^*-$subspace T, other than a spread element of $\mathcal{S}^*$, has dimension of type $dm - 1$, with $2 \leq d \leq 2t$. The spread $\mathcal{S}^*$, together with the $\mathcal{S}^*-$subspaces, is an incidence structure $PG(\mathcal{S}^*)$ isomorphic to $PG(2t-1, q^m)$. Here, the $(d-1)-$dimensional subspaces, $0 < d-1 < 2t-1$, are the $(m-1)-$spreads $\mathcal{S}^*(T)$ induced by $\mathcal{S}^*$ on the $\mathcal{S}^*-$subspaces T of dimension $dm-1$ of Σ. Finally, an incidence structure $PG^+(\mathcal{S}^*)$ isomorphic to $PG(2t, q^m)$ can be defined in the following way: (i) the points are the points of $\Sigma' \setminus \Sigma$ and the elements of $\mathcal{S}^*$, (ii) the d-dimensional subspaces, $0 < d < 2t$, are the dm-dimensional subspaces of Σ' which intersect Σ in an $\mathcal{S}^*-$subspace of dimension $dm-1$ and the $d-$dimensional subspaces of $PG(\mathcal{S}^*)$, (iii) the incidences are inherited from the inclusion relation.

Note that $PG(\mathcal{S}^*)$ is a hyperplane of $PG^+(\mathcal{S}^*)$ and $\mathcal{S}' = \{\mathcal{S}^*(X) : X \in \mathcal{S}\}$ turns out to be a regular $(m-1)$-spread of $PG(\mathcal{S}^*)$. It follows that we can work at the same time with two André, Bruck-Bose representations of $PG(2, q^n)$: the usual one in $PG(2n, q)$ and a second one in $PG^+(\mathcal{S}^*) \cong PG(2t, q^m)$.

Now let B be a blocking set of $PG(2, q^n)$ obtained by Construction $1a$ in $PG^+(\mathcal{S}^*)$ $\cong PG(2t, q^m)$, i.e. B is associated with a cone K in $PG^+(\mathcal{S}^*)$ of vertex a $(t-2)$-dimensional subspace $\mathcal{S}^*(\Omega)$ of $PG(\mathcal{S}^*)$ contained in an element $\mathcal{S}^*(Y)$ of $\mathcal{S}'$ and having as base a minimal blocking set $\bar{B}$ of a $(t+1)$-dimensional subspace Γ' of $PG^+(\mathcal{S}^*)$ disjoint from $\mathcal{S}^*(\Omega)$. Note that, under our assumptions, the blocking set B of $PG(2, q^n)$ is defined in $PG(2n, q)$ by a cone K^* of vertex Ω, with $\Omega \subset Y \in \mathcal{S}$. Moreover, since K is union of $(t-1)$-dimensional subspaces of $PG^+(\mathcal{S}^*)$, K^* is union of $(t-1)m$-dimensional subspaces of $PG(2n, q)$ and, if $\bar{B}^* = K^* \cap \Gamma^*$, we get

$$K^* = \bigcup_{\bar{P} \in \bar{B}^*} \langle \Omega, \bar{P} \rangle.$$

The last equality proves that $\bar{B}^*$ is an $\mathcal{F}$-blocking set in Γ^*, where $\mathcal{F} = \mathcal{F}(\Omega, \mathcal{S})$ and B is the blocking set of $PG(2n, q)$ associated with Ω and $\bar{B}^*$ by Construction 2.

The above remark shows how and why Construction 2 always works when n is not a prime and the vertex Ω of the cone is an $\mathcal{S}^*-$subspace of $PG(2n-1, q)$: *the central point is the existence of the $\mathcal{F}$-blocking set $\bar{B}^*$ in* Γ^*. In this case, of course, we do not obtain new examples, since we simply construct the same blocking set in different André, Bruck-Bose representations of the plane $PG(2, q^n)$. So, to try to get some new examples of blocking sets by Construction 2, Ω should not be chosen as an $\mathcal{S}^*-$subspace of Σ and, under such assumption, the problem reduces to finding a minimal $\mathcal{F}(\Omega, \mathcal{S})$-blocking set of Γ'.

For example, we can apply our considerations to state the following characterization of Buekenhout-Metz unitals in $PG(2, q^4)$.

Proposition 4.5. *Let $\mathcal{U}$ be a unital in the representation of $\Pi = PG(2, q^4)$ in $\Sigma' = PG(8, q)$ obtained by Construction 2 using:*

- *a line Ω of an element Y of the regular 3-spread $\mathcal{S}$ of $\Sigma = PG(7, q)$;*
- *a 6-dimensional subspace Γ' of Σ' not contained in Σ and meeting Y in a line $\Theta \in \mathcal{S}^*(Y)$ disjoint from Ω, where $\mathcal{S}^*$ is the $1-$spread induced on Σ by $\mathcal{S}$;*
- *the family $\mathcal{F} = \mathcal{F}(\Omega, \mathcal{S})$ of $4-$dimensional subspaces of Γ' associated with Ω and $\mathcal{S}$;*
- *a minimal $\mathcal{F}-$blocking set $\bar{B}$ of Γ' of size q^4+q+1 intersecting Y in the line Θ.*

Then $\mathcal{U}$ is a Buekenhout-Metz unital if, and only if, Ω is a spread element of $\mathcal{S}^*$.

Proof: If Ω is a spread element of $\mathcal{S}^*$, we can get $\mathcal{U}$ also by Construction 1, using the representation of Π in $PG^+(\mathcal{S}^*) = PG(4, q^2)$. In this way $\mathcal{U}$ is associated with a cone K in $PG^+(\mathcal{S}^*)$ of vertex $\Omega \in \mathcal{S}^*(Y) \subset PG(\mathcal{S}^*)$ and having as base a minimal blocking set $\bar{B}^*$ of size $q^4 + 1$ of Γ'; here Γ' is considered as a $3-$dimensional subspace $PG(3, q^2)$ of $PG^+(\mathcal{S}^*)$. Then, by Result 1.2, $\bar{B}^*$ is an ovoid of Γ' and $\mathcal{U}$ turns out to be a Buekenhout-Metz unital.

Now note that to every plane π of Σ', with $\pi \cap \Sigma$ a line ℓ contained in a spread element X of $\mathcal{S}$, there corresponds a set X_π of $q^2 + 1$ collinear points in the representation of Π in Σ'. Actually, X_π is a Baer subline if, and only if, ℓ is an element of the induced spread $\mathcal{S}^*$ [4, 24]. It turns out that X_π is not a Baer subline of Π when ℓ is not an element of the induced spread $\mathcal{S}^*$. So, assuming that Ω is not a spread element of $\mathcal{S}^*$, there exists no line of Π through the point corresponding to Ω and meeting $\mathcal{U}$ in a Baer subline. This proves that $\mathcal{U}$ is not a Buekenhout-Metz unital, since it is well known that through every point of such a unital there exists at least one line meeting the unital in a Baer subline. □

Remark 4.6. Last proposition shows how the existence of a non Buekenhout-Metz unital in $PG(2, q^4)$ depends on that of a minimal $\mathcal{F}(\Omega, \mathcal{S})-$ blocking set $\bar{B}$ of Γ' of size $q^4 + q + 1$ and intersecting Y in the line Θ, when the line Ω is not a spread element of $\mathcal{S}^*$. □

We note that the proof of Proposition 4.5 can be suitably modified to obtain the following more general result.

Proposition 4.7. *Let k be a positive integer. Let $\mathcal{U}$ be a unital in the representation of $\Pi = PG(2, q^{2k})$ in $\Sigma' = PG(4k, q)$ obtained by Construction 2 using:*

- *a $(k-1)-$dimensional subspace Ω of an element Y of the regular $(2k-1)$-spread $\mathcal{S}$ of $\Sigma = PG(4k-1, q)$;*
- *a $3k$-dimensional subspace Γ' of Σ' not contained in Σ and meeting Y in a $(k-1)-$dimensional subspace $\Theta \in \mathcal{S}^*(Y)$ disjoint from Ω, where $\mathcal{S}^*$ is the $(k-1)-$spread induced on Σ by $\mathcal{S}$;*
- *the family $\mathcal{F} = \mathcal{F}(\Omega, \mathcal{S})$ of $(2k+1)-$dimensional subspaces of Γ' associated with Ω and $\mathcal{S}$;*
- *a minimal $\mathcal{F}-$blocking set $\bar{B}$ of Γ' of size $q^{2k} + q^{k-1} + q^{k-2} + \cdots + q + 1$ intersecting Y in the subspace Θ.*

Then $\mathcal{U}$ is a Buekenhout-Metz unital if, and only if, Ω is a spread element of $\mathcal{S}^*$.

Finally, we explicitly remark that Proposition 4.5 is a slight variation of the result contained in Sect.3.4 of [4] and Proposition 4.7 is a generalization of Theorem 3.4 of [4].

5. Construction 3

The present section deals with a second variation of Construction 1a, which essentially is a way of looking at Construction 2.7 of [30] in the André, Bruck-Bose representation of $PG(2, q^n)$ in $PG(2n, q)$. As usual, we start from a fixed spread element Y of $\mathcal{S}$ and a subspace Ω of Y; let $n - s$ be the dimension of Ω, $2 \leq s \leq n + 1$. Now let Γ'_s be an s-dimensional subspace of Σ' not contained in Σ, such that $\Gamma'_s \cap Y$ is an $(s - 2)$-dimensional subspace disjoint from Ω, and denote by Γ_{s-2} and Γ_{s-1} the subspaces $\Gamma'_s \cap Y$ and $\Gamma'_s \cap \Sigma$, respectively. Moreover, let $\bar{B}$ be a blocking set with respect to the set of all lines of Γ'_s and define K and B by (1) and (2), respectively. Since $dim(\langle Y, \Gamma_{s-1} \cap X\rangle) = -1, 0$, for every element X of $\mathcal{S}$, the size of B is given by

$$|B| = |K \setminus \Omega| + 1 = q^{n-s+1}|\bar{B} \setminus \Gamma_{s-2}| + 1$$

and, using the same arguments as in Propositions 3.2 and 3.4, the following can be proved.

Proposition 5.1. *B is a blocking set of Π. Moreover, denoted by S_n a line of Π other than $\mathcal{S}$, we have :*

- *if $Y \notin S_n$ and $S_n \cap \Sigma \cap K = \emptyset$, then*

$$|B \cap S_n| = |\bar{B} \cap \ell|,$$

  where ℓ is the line $\langle S_n, \Omega\rangle \cap \Gamma'_s$;
- *if $Y \notin S_n$ and $S_n \cap \Sigma \cap K \neq \emptyset$, then*

$$|B \cap S_n| = |(\bar{B} \cap \ell) \setminus \Sigma| + 1,$$

  where ℓ is the line $\langle S_n, \Omega\rangle \cap \Gamma'_s$;
- *if $Y \in S_n$, then*

$$|B \cap S_n| = q^{n-s+1}|(S_n \cap \Gamma'_s \cap \bar{B}) \setminus \Sigma| + 1.$$

As in the case of Construction 1b, the minimality of B does not automatically follow from that of $\bar{B}$. Next proposition and its two corollaries correspond to Proposition 2.8 and Theorem 2.9 of [30]; for sake of completeness we give here a proof of them in our context.

Proposition 5.2. *Let P be an affine point of B, i.e. $P \in B \setminus \Sigma$, and let $\bar{P}$ be the unique point of $\bar{B} \cap \langle \bar{P}, \Omega\rangle$. Then P is an essential point of B if, and only if, there exists a line ℓ of Γ'_s through $\bar{P}$ which is tangent to $\bar{B}$ and disjoint from Γ_{s-2}.*

Proof: From the line intersection numbers we get that there exists in Π a tangent line S_n to B through P if, and only if, $\ell = \langle S_n, \Omega\rangle \cap \Gamma'_s$ is a tangent line of $\bar{B}$ through $\bar{P}$ and $\ell \cap \Gamma_{s-2} = \emptyset$; this concludes the proof. □

The next corollaries point out two special cases in which the minimality of $\bar{B}$ gives some strong information about B.

Corollary 5.3. *If $\bar{B} \cap \Sigma = \Gamma_{s-2}$, then*

$$|B| = q^{n-s+1}(|\bar{B}| - (q^{s-2} + q^{s-3} + \cdots + q + 1)) + 1$$

and B is a minimal blocking set of Π.

Proof: It is enough to remark that, under our assumption, the condition of Proposition 5.2 is satisfied for all affine points of B and $B \cap \mathcal{S} = \{Y\}$. □

Corollary 5.4. *If $|\bar{B}| < 2q^{s-1}$, then all the affine points of B are essential.*

Proof: Firstly remark that, if for each $\bar{P} \in \bar{B}$ the number $\tau_{\bar{P}}$ of tangent lines of $\bar{B}$ in Γ'_s through $\bar{P}$ is greater than $q^{s-2} + q^{s-3} + \cdots + q + 1$, then the condition of Proposition 5.2 is satisfied by the affine points of B that all turn out essential. On the other hand, by Lemma 2.11 of [30], if $|\bar{B}| < 2q^{s-1}$, then $\tau_{\bar{P}} > q^{s-2} + q^{s-3} + \cdots + q + 1$ for each point of $\bar{B}$; so our assertion follows. □

Proposition 5.5. *Let X be a point of $B \cap \mathcal{S}$. If $X \neq Y$, then a unique point $\bar{P} \in \bar{B} \cap \Sigma$ exists such that $X \cap \langle \bar{P}, \Omega \rangle \neq \emptyset$. Moreover, X is an essential point of B if, and only if, either $X = Y$ or $X \neq Y$ and there exists in Γ'_s a tangent line of $\bar{B}$ through $\bar{P}$ and not contained in Σ.*

Proof: If $X \in B \cap \mathcal{S}$, with $X \neq Y$, then $X \cap K \neq \emptyset$ and a point $\bar{P} \in \bar{B} \cap \Sigma$ exists such that $X \cap \langle \bar{P}, \Omega \rangle \neq \emptyset$. Moreover, as X and Y are disjoint, $\bar{P}$ is unique and $|X \cap \langle \bar{P}, \Omega \rangle| = 1$.

If S_n is a line through X other than $\mathcal{S}$, then $\ell = \langle S_n, \Omega \rangle \cap \Gamma'_s$ is a line through $\bar{P}$ not contained in Σ and $|S_n \cap B| = |(\ell \setminus \{\bar{P}\}) \cap \bar{B}| + 1$.

Finally, if S_n is a line through Y, then $|B \cap S_n| = q^{n-s+1}|(S_n \cap \Gamma'_s \cap \bar{B}) \setminus \Sigma| + 1$ and, by an easy counting argument, one can see that there exists a line S_n through Y such that $S_n \cap \Gamma'_s = \Gamma_{s-2}$. It follows that $|B \cap S_n| = 1$, i.e. Y is an essential point of B. This finishes the proof. □

Corollary 5.6. *The blocking set B is minimal if, and only if, all points of $\bar{B}$ verify the conditions of Propositions* 5.2 *and* 5.5. *If B verifies the conditions of Proposition* 5.2, *then B contains a minimal blocking set B' of Π such that*

$$q^{n-s+1}|\bar{B} \setminus \Sigma| + 1 \leq |B'| \leq q^{n-s+1}|\bar{B} \setminus \Gamma_{s-2}| + 1.$$

Final remark. We plan to show in a forthcoming paper how some of our constructions can be generalized in order to achieve new minimal blocking sets in $PG(m, q^n)$, $m > 2$. □

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