Pointed and multi-pointed partitions of type *A* and *B*

F. Chapoton · B. Vallette

Abstract The aim of this paper is to define and study pointed and multi-pointed partition posets of type *A* and *B* (in the classification of Coxeter groups). We compute their characteristic polynomials, incidence Hopf algebras and homology groups. As a corollary, we show that some operads are Koszul over $\mathbb{Z}$.

Introduction

For every finite Weyl group W, there exists a generalized partition poset (*cf.* [2]) defined through the hyperplane arrangement of type W. In the case A_{n-1}, this poset is the usual poset of partitions of $\{1, \dots, n\}$. B. Fresse proved in [6] that this poset also arises from the theory of operads. A pointed and a multi-pointed variation of this poset were defined by the second author in [14], once again in the context of Koszul duality of operads. In this article, we study the main properties of these two types of posets. One motivation for this article was the idea that there should also exist a pointed partition poset and a multi-pointed partition poset for other Weyl groups. Here we propose a definition for the pointed partition posets of type B and check that it satisfies most of the properties which are expected in general and hold in type A. This definition was guessed by similarity, but we hope that there is a general definition of geometric nature, to be found.

    Let us summarize briefly what properties the generalized pointed partition poset associated to a Weyl group should have. Let h be the Coxeter number and n be the rank of the Weyl

F. Chapoton
Institut Camille Jordan, Universitè Claude Bernard Lyon 1,
Bâtiment Braconnier, 21 avenue Claude Bernard, 69622 Villeurbanne Cedex France
e-mail: chapoton@math.univ-lyon1.fr

B. Vallette (✉)
Laboratoire J.-A. Dieudonnè, Université de Nice Sophia-Antipolis,
Parc Valrose, 06108 Nice Cedex, France
e-mail: brunov@math.unice.fr

group W. Then its characteristic polynomial should be $(x - h)^n$; the number of maximal elements should be h, with a transitive action of the Weyl group. Also the characteristic polynomial of any maximal interval should be $(x - 1)(x - h)^{n-1}$ and the homology must be concentrated in maximal dimension. We prove that all these properties hold in type A and B.

One can remark that the expected characteristic polynomial is the same as the characteristic polynomial of the hyperplane arrangement called the Shi arrangement [1, 7]. One difference is that there is no action of the Weyl group on the Shi arrangement. Going to the limit where parallel hyperplanes come together gives the so-called double Coxeter arrangement [12], which is no longer a hyperplane arrangement in the usual sense. Still the double Coxeter arrangement is free, and all its degrees are the Coxeter number. Maybe the pointed partition poset should be thought of as the missing intersection poset for the double Coxeter arrangement.

It may be worth noting that there is a family of posets which are probably related to the pointed partition posets of type A, in a rather non-evident way. It is made of some posets on forests of labeled rooted trees, introduced by J. Pitman in [8], which seem to share the same characteristic polynomials. Maybe there is a homotopy equivalence between these posets of forests and the posets of pointed partitions.

Let us now state what are the expected properties of the multi-pointed partition posets associated to a Weyl group. Let $e_1, \ldots, e_n$ be the exponents of W. Then the characteristic polynomial should be $\prod_{i=1}^{n}(x - (h + e_i))$ and the homology of the poset must be concentrated in maximal dimension. This is true in type A. We were unable to guess what should be the poset of type B, even if we have some evidence that it should exist.

The multi-pointed partition poset should be related to the so-called Catalan arrangement [1]. The characteristic polynomials coincide, but the posets are different in general. As the Weyl group acts on the Catalan arrangement, one may wonder if the action is the same on the top homology of both posets, which should have the same dimension. If this is true, this may come from a homotopy equivalence between the (realizations of the) two posets.

Just as in the Shi case, one can take the limit of the Catalan arrangement where parallel hyperplanes come together. This gives the triple Coxeter arrangement [13]. This triple Coxeter arrangement is free, and its degrees are the roots of the characteristic polynomial of the Catalan arrangement.

In type A, just as for the usual partition lattices, the pointed and multi-pointed partition posets give rise to interesting actions of the symmetric groups $\mathbb{S}_n$ on their homology groups. We use the relations with the theory of Koszul operads (based on representations of $\mathbb{S}_n$), described by the second author in [14], to compute this action on the homology groups of the pointed and multi-pointed posets. On the other way, the fact that the various posets are Cohen-Macaulay over $\mathbb{Z}$ implies that some operads are Koszul over $\mathbb{Z}$, which is an important result in the study of the deformations of algebraic structures.

Since the posets studied here have nice properties for their intervals and products, we can associate them an incidence Hopf algebra.

Conventions

All posets are implicitly finite. A poset Π is said to be *bounded* if it admits one minimal and one maximal element (denoted by $\hat{0}$ and $\hat{1}$). It is *pure* if for any $x \leqslant y$, all maximal chains between x and y have the same length. If a poset is both bounded and pure, it is called a *graded* poset. A pure poset with a minimal element is called *ranked*.

For a graded or ranked poset Π with rank function rk, the *characteristic polynomial* is defined by the following formula:

$$\chi(x) := \sum_{a \in \Pi} \mu(a) x^{n - \mathrm{rk}(a)}, \tag{1}$$

where μ denotes the Möbius function of the poset Π and n is the rank of the maximal elements.

Let $\mathbb{K}$ be the ring $\mathbb{Z}$ or any field. Denote by $[n]$ the set $\{1, \ldots, n\}$.

1. Pointed partition posets

In this section, we give the definitions of the pointed partition posets and some of their basic properties.

1.1. Definitions

1.1.1. Type A

First, we recall the definition, introduced in [14], of the pointed partition poset of type A_{n-1}.

Definition 1 (Pointed partition). A *pointed partition* of $[n]$ is a partition of $[n]$ together with the choice of one element inside each block, called the *pointed element* of this block.

The order relation on the set of pointed partitions of $[n]$ is defined as follows. The underlying partitions must be related by the refinement order of partitions and the set of pointed elements of the finer partition π must contain the set of pointed elements of the other one ν. In this case, one gets $\pi \leqslant \nu$. For instance, one has $\{\overline{1}\}\{\overline{3}\}\{2\overline{4}\} \leqslant \{\overline{1}3\}\{2\overline{4}\}$. We denote this poset by Π_n^A. The example of Π_3^A is given in Figure 1.

From the very definition of pointed partitions as non-empty sets of pointed sets, one can see that the exponential generating function for the graded cardinalities of the posets of pointed partitions of type A is given by $\frac{e^{xue^u} - 1}{x}$. Indeed the exponential generating function
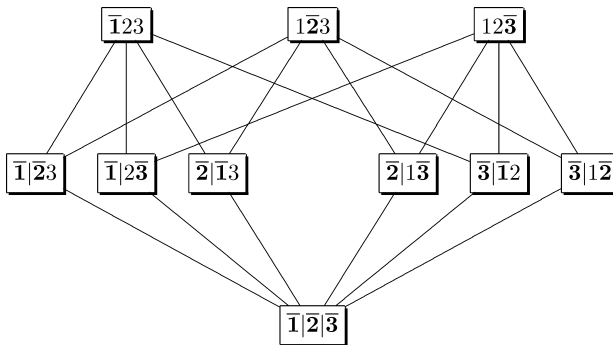

Fig. 1 The poset Π_3^A

for the pointed sets is ue^u and the exponential generating function for the non-empty sets is $e^u - 1$. The additional variable x takes the number of parts into account.

The symmetric group on $[n]$ acts by automorphisms on the poset Π_n^A. The set of maximal elements has cardinality n and the action of the symmetric group is the natural transitive permutation action. Hence all maximal intervals are isomorphic as posets. We denote this poset by $\Pi_{n,1}^A$.

The main property of this family of posets is the following one.

Proposition 1.1. *Each interval of Π_n^A is isomorphic to a product of posets $\Pi_{\lambda_1,1}^A \times \cdots \times \Pi_{\lambda_k,1}^A$, where $\lambda_1 + \cdots + \lambda_k \leqslant n$.*

Proof: First, any interval can be decomposed into a product according to the parts of the coarser partition. One can therefore assume that the maximal element of the interval is a single block. One can then replace, in each element of the interval, each block of the minimal element by a single element. This provides a isomorphism with some interval $\Pi_{\lambda,1}^A$. $\qquad\square$

Hence the Möbius number of a pointed partition is the product of the Möbius numbers of its parts. This property will allow us to work by induction.

In general, the pointed partition posets are not lattices. They are bounded below, pure posets and the rank of a pointed partition π of $[n]$ is equal to n minus the number of blocks of π.

1.1.2. Type B

Let us define the pointed partition poset of type B_n.

Let $[-n]$ be the set $\{-1, \ldots, -n\}$. Recall first the description of the usual partitions of type B_n. They are the partitions of the set $[n] \sqcup [-n]$ such that there is at most one block containing some opposite indices and the other blocks come in opposite pairs. The block with opposite elements is called the *zero block*.
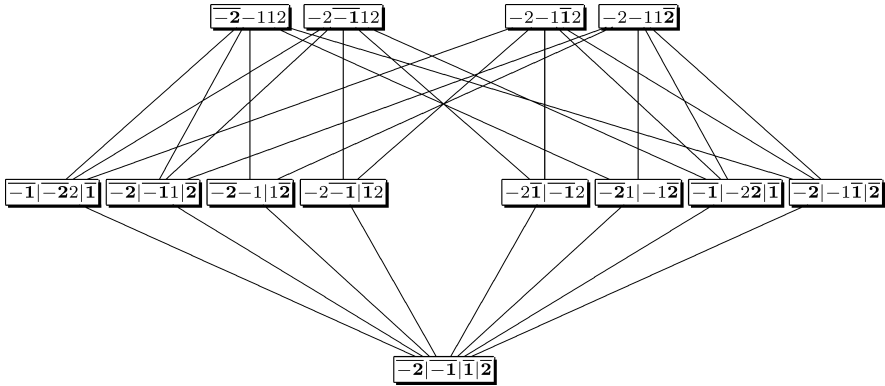
Definition 2 (Pointed partition of type B). A *pointed partition of type B_n* is a partition of type B_n together with the choice of an element of the zero block and the choice for each pair of opposite blocks of a pair of opposite elements. The chosen elements are called *pointed*.

For example, $\{\overline{3}, -2\}\{\overline{-1}, 1\}\{2, \overline{-3}\}$ is a pointed partition of type B_3.

The order relation is as follows. The underlying partitions must be related by the refinement order of partitions and the set of pointed elements of the finer partition must contain the set of pointed elements of the other one. For instance, one has $\{\overline{3}\}\{\overline{-2}\}\{\overline{-1}, 1\}\{\overline{2}\}\{\overline{-3}\} \leqslant \{\overline{3}, -2\}\{\overline{-1}, 1\}\{2, \overline{-3}\}$. We denote these posets by Π_n^B. The example of Π_2^B is given in Figure 2.

The hyperoctahedral group of signed permutations of $[n]$ acts by automorphisms on the poset Π_n^B. The set of maximal elements has cardinality $2n$. Once again, the transitive action of the hyperoctahedral groups on maximal intervals shows that they are all isomorphic as posets. We denote this poset by $\Pi_n^{B'}$.

In general, the pointed partition posets of type B are not lattices. They are bounded below, pure posets and the rank of a pointed partition π of type B is equal to n minus the number of pairs of opposite blocks of π.

Fig. 2 The poset Π_2^B

Remark 1.2. We consider also a variation of pointed partitions of type B such that no element is pointed in the zero block. Such partitions are called *pointed partitions of type β*. The partial order between pointed partitions of type β is defined like the order in Π_n^B. We denote these graded posets by Π_n^β.

1.2. Characteristic polynomials

1.2.1. *Characteristic polynomials in type* A

The aim of this section is to compute the characteristic polynomials of the posets of pointed partitions of type A. The proof uses the subposets of pointed partitions for which a fixed subset of $[n]$ is contained in the set of pointed elements. Up to isomorphism, these subposets only depend on the cardinality of the fixed subset of pointed indices. For $1 \leq i \leq n$, let $\Pi_{n,i}^A$ be the poset where the indices in $[i]$ are pointed. This poset has rank $n - i$. Recall that the pointed partition poset of type A_{n-1} is denoted by Π_n^A and has rank $n - 1$.

Theorem 1.3. *For $1 \leq i \leq n$, the characteristic polynomial of $\Pi_{n,i}^A$ is*

$$\chi_{n,i}^A(x) = (x - i)(x - n)^{n-1-i}, \tag{2}$$

and its constant term $C_{n,i}^A$ is $(-1)^{n-i}i\, n^{n-1-i}$. The characteristic polynomial of Π_n^A is

$$\chi_n^A(x) = (x - n)^{n-1}. \tag{3}$$

Proof: The proof proceeds by induction on n. The statement of the Theorem is clearly true for $n = 1$.

From now on fix $n \geq 2$ and assume that the Theorem is proved for smaller n. Then the proof is by decreasing induction on i from n to 1. In the case $i = n$, the poset $\Pi_{n,n}^A$ has just one element, so its characteristic polynomial is 1, which is the expected value.

So assume now that i is smaller than n. Suppose first that i is at least 2. Using the decomposition of a partition into its parts, which gives a product for the Möbius number, the

constant term $C_{n,i}^A$ is given by

$$\sum_{n_1,\dots,n_i} \prod_{j=1}^{i} \frac{C_{n_j,1}^A}{(n_j-1)!}(n-i)!, \tag{4}$$

where the sum runs over integers $n_j \geq 1$ with sum n.

Since $n_j < n$ for all j, we know by induction that $C_{n_j,1}^A = (-1)^{n_j-1} n_j^{n_j-2}$ and Lemma 1.5 then allows to compute the resulting sum:

$$C_{n,i}^A = (-1)^{n-i} i\, n^{n-1-i}, \tag{5}$$

as expected, when i is at least 2.

Let us now compute $\chi_{n,i}^A$. By Möbius inversion on subsets of $[n]$ strictly containing $[i]$, it is given by

$$i\, n^{n-1-i}(-1)^{n-i} + \sum_{[i] \subsetneq S \subseteq [n]} (-1)^{|S|+i+1} \chi_{n,|S|}^A x^{|S|-i}. \tag{6}$$

By induction on i, one gets

$$in^{n-1-i}(-1)^{n-i} + \sum_{j=i+1}^{n} (-1)^{j+i+1} \binom{n-i}{j-i}(x-j)(x-n)^{n-1-j} x^{j-i}. \tag{7}$$

Then using Lemma 1.4, one finds the expected formula for $\chi_{n,i}^A$.

Let us consider now the case $i = 1$.

Here we can not use induction to compute $C_{n,1}^A$, as Formula (4) becomes trivial when $i = 1$. Instead we use the fact that the poset $\Pi_{n,1}^A$ is bounded, hence $\chi_{n,1}^A$ must vanish at $x = 1$ and this property characterizes the constant term if the others coefficients are known. So let us guess what the constant term is and check later that the result vanishes at $x = 1$.

Let us therefore compute $\chi_{n,1}^A$ as before, assuming that $C_{n,1}^A = (-1)^{n-1} n^{n-2}$. By Möbius inversion, it is given by

$$n^{n-2}(-1)^{n-1} + \sum_{[1] \subsetneq S \subseteq [n]} (-1)^{|S|} \chi_{n,|S|}^A x^{|S|-1}. \tag{8}$$

By induction on i, one gets

$$n^{n-2}(-1)^{n-1} + \sum_{j=2}^{n} (-1)^j \binom{n-1}{j-1}(x-j)(x-n)^{n-1-j} x^{j-1}. \tag{9}$$

Then using Lemma 1.4 again, one finds the expected formula for $\chi_{n,1}^A$. This formula vanishes at $x = 1$, so the guess for the constant term was correct.

Let us consider now the case of Π_n^A. In this case, we just have to use our knowledge of the other characteristic polynomials. Let us now compute χ_n^A as before. By Möbius inversion, it

is given by

$$\sum_{\emptyset \subsetneq S \subseteq [n]} (-1)^{|S|+1} \chi^A_{n,|S|} x^{|S|-1}. \tag{10}$$

By the previous results, one gets

$$\sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} (x-j)(x-n)^{n-1-j} x^{j-1}. \tag{11}$$

Then using Lemma 1.4 one last time, one finds the expected formula for χ^A_n.

This concludes the inductive proof of the Theorem.   □

Lemma 1.4. *For $0 \leq i \leq n$, one has the following equality:*

$$i\, n^{n-1-i} (-1)^{n-i} + \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} (x-(j+i))(x-n)^{n-1-i-j} x^j$$

$$= (x-i)(x-n)^{n-1-i}. \tag{12}$$

Proof: Introduce a new variable y to get a homogeneous identity of degree $n-i$. Then replace x by 1 and y by $(1-y)/n$. The resulting identity is easy to check.   □

Lemma 1.5. *For $1 \leq i \leq n$, one has the following equation:*

$$\sum_{n_1,\ldots,n_i} \prod_{j=1}^{i} \frac{n_j^{n_j-1}}{n_j!} = i \frac{n^{n-1-i}}{(n-i)!}, \tag{13}$$

where the sum runs over integers $n_j \geq 1$ with sum n.

Proof: Classical, see for example Proposition 2.5 in [15].   □

1.2.2. Characteristic polynomials in type B

Let us compute the characteristic polynomials of the posets of pointed partitions in type B. The proof uses the subposets where i and $-i$ are pointed for i in a fixed subset of $[n]$. Up to isomorphism, these subposets only depend on the cardinality of the fixed subset of pointed pairs of indices. Let $\Pi^B_{n,i}$ be the poset where the indices in $[i]$ and $[-i]$ are pointed. By convention, let $\Pi^B_{n,0}$ denote the pointed partition poset Π^B_n. Recall that $\Pi^{B'}_n$ denotes a maximal interval in Π^B_n.

Theorem 1.6. *For $0 \leq i \leq n$, the characteristic polynomial of $\Pi^B_{n,i}$ is*

$$\chi^B_{n,i}(x) = (x-2n)^{n-i}, \tag{14}$$

with constant term $C_{n,i}^{B} = (-2n)^{n-i}$. The characteristic polynomial of $\Pi_{n}^{B'}$ is

$$\chi_{n}^{B'}(x) = (x - 1)(x - 2n)^{n-1}, \tag{15}$$

and its constant term is $C_{n}^{B'} = (-1)^{n}(2n)^{n-1}$.

Proof: The proof proceeds by induction on n. The statement of the Theorem is clearly true for $n = 1$.

From now on fix $n \geq 2$ and assume that the Theorem is proved for smaller n. Then the proof is by decreasing induction on i from n to 0. In the case $i = n$, the poset $\Pi_{n,n}^{B}$ has just one element, so its characteristic polynomial is 1, which is the expected value.

So assume now that i is smaller than n. Suppose first that i is at least 1. Using the decomposition of a partition of type B into its parts, which gives a product formula for the Möbius number, the constant term $C_{n,i}^{B}$ is given by

$$\sum_{n_1,\dots,n_i,m} \prod_{j=1}^{i} \frac{C_{n_j,1}^{A}}{(n_j - 1)!} \frac{C_{m}^{B'}}{m!} 2^{n-m-i} \, 2m \, (n - i)!, \tag{16}$$

where the sum runs over integers $n_j \geq 1$ and an integer $m \geq 0$ with sum n.

Using the results for type A, induction on n to know $C_{m}^{B'}$ and Lemma 1.8 to compute the resulting sum, one gets that

$$C_{n,i}^{B} = (-2n)^{n-i}, \tag{17}$$

as expected.

Let us now compute $\chi_{n,i}^{B}$. By Möbius inversion, one has the following equation:

$$\chi_{n,i}^{B} = (-2n)^{n-i} + \sum_{[i] \subsetneq S \subseteq [n]} (-1)^{|S|+i+1} \chi_{n,|S|}^{B} x^{|S|-i}. \tag{18}$$

One gets by induction on i that

$$\chi_{n,i}^{B} = (-2n)^{n-i} + \sum_{j=i+1}^{n} (-1)^{j+i+1} \binom{n-i}{j-i} (x - 2n)^{n-j} x^{j-i}, \tag{19}$$

from which the expected formula follows through the binomial formula.

There remains to compute $\chi_{n,0}^{B}$. For this, we need first to compute the characteristic polynomial $\chi_{n}^{B'}$ of a maximal interval $\Pi_{n}^{B'}$. Let us choose the maximal interval of elements where n is pointed. Elements of this interval are of two distinct shapes: either n is in the zero block or both n and $-n$ are pointed. Let us split the computation of $\chi_{n}^{B'}$ accordingly, as the sum of an unknown constant term, of $x \chi_{n,1}^{B}$ (already known) for the terms when n and $-n$ are pointed and of the remaining terms when n is in the zero block and the complement to the zero block is not empty.

Let us compute this third part. There is a bijection between the set of such partitions and the set of triples (S, π, ϵ) where S is a non-empty subset of $[n] \setminus \{n\}$, π is a pointed partition of type A on the set S and ϵ is the choice of a sign for each element of S up to complete change of sign of each block of π. The set S is the positive half of the complement of the

zero block and π is the rest of the partition without its signs. Hence, one gets

$$\sum_{\emptyset \subsetneq S \subseteq [n-1]} \sum_{\pi \in \Pi_{|S|}^A} 2^{\mathrm{rk}(\pi)} \mu_{|S|}^A(\pi) C_{n-|S|}^{B'} x^{|S|-\mathrm{rk}(\pi)}, \tag{20}$$

where $\mathrm{rk}(\pi)$ is the rank in the poset $\Pi_{|S|}^A$. This can be rewritten using induction on n as

$$\sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^{n-j} (2n-2j)^{n-j-1} 2^{j-1} x \chi_j^A(x/2). \tag{21}$$

Then one can use the known results for type A to obtain

$$\sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^{n-j} (2n-2j)^{n-j-1} x(x-2j)^{j-1}. \tag{22}$$

Using Lemma 1.7, this is seen to be

$$-(x-2n)^{n-1} + (-2n)^{n-1}. \tag{23}$$

As $\chi_n^{B'}$ has to vanish at $x = 1$ because $\Pi_n^{B'}$ is bounded, one finds that its constant term is $(-1)^n (2n)^{n-1}$. Hence $\chi_n^{B'}$ is $(x-1)(x-2n)^{n-1}$ as expected.

Now one can complete the proof by computing $\chi_{n,0}^B$ by Möbius inversion, just as for $\chi_{n,i}^B$ for $i > 0$, because one now knows that $C_n^{B'} = (-1)^n (2n)^{n-1}$. $\qquad\square$

Lemma 1.7. *For all $n \geq 1$, one has*

$$\sum_{j=0}^{n-1} \binom{n-1}{j} (y+uj)^{n-j-1} (x-uj)^{j-1} = x^{-1}(x+y)^{n-1}. \tag{24}$$

Proof: First set $m = n - 1$ and rewrite it as

$$\sum_{j=0}^{m} \binom{m}{j} (y+ju)^{m-j} (x-ju)^{j-1} = x^{-1}(x+y)^m. \tag{25}$$

Replace y by $y + m$ and u by -1. The identity becomes

$$\sum_{j=0}^{m} \binom{m}{j} (y+m-j)^{m-j} (x+j)^{j-1} = x^{-1}(x+y+m)^m, \tag{26}$$

which is one of many forms of the classical Abel binomial identity, see [9] for example. $\square$

Lemma 1.8. *For $1 \leq i \leq n$, one has the following equation:*

$$\sum_{n_1,\ldots,n_i,m} \prod_{j=1}^{i} \frac{n_j^{n_j-1}}{n_j!} \frac{m^m}{m!} = \frac{n^{n-i}}{(n-i)!}, \tag{27}$$

where the sum runs over integers $n_j \geq 1$ and an integer $m \geq 0$ with sum n.

Proof: Using the notations of Zvonkine [15], let

$$Y = \sum_{n \geq 1} \frac{n^{n-1}}{n!} u^n \text{ and } Z = \sum_{n \geq 1} \frac{n^n}{n!} u^n. \tag{28}$$

Then it is known ([15, Proposition 2.5]) that

$$Y^i = i \sum_{n \geq i} \frac{n^{n-i-1}}{(n-i)!} u^n. \tag{29}$$

Applying the Euler operator $D = u \partial_u$, one gets

$$i\, Y^{i-1} Z = i \sum_{n \geq i} \frac{n^{n-i}}{(n-i)!} u^n. \tag{30}$$

and the result follows because $Z = Y(1 + Z)$.   □

1.2.3. Characteristic polynomials in type β

Recall that Π_n^β is the poset of partitions of type B_n where all blocks but the zero block are pointed, which was defined in Remark 1.2. These posets appear as intervals in the posets Π^B.

Theorem 1.9. *The characteristic polynomial of Π_n^β is*

$$\chi_n^\beta = (x - 1)(x - (2n + 1))^{n-1}, \tag{31}$$

and its constant term is $C_n^\beta = (-1)^n (2n + 1)^{n-1}$.

Proof: Let us prove the Theorem by induction on n. It is clearly true if $n = 1$. Let us decompose the poset Π_n^β according to the size j of the complement of the zero block. Then the characteristic polynomial is

$$\sum_{j=0}^n \binom{n}{j} \sum_{\pi \in \Pi_j^A} \mu_j^A(\pi) 2^{\text{rk}(\pi)} C_{n-j}^\beta x^{j-\text{rk}(\pi)}, \tag{32}$$

which can be rewritten as

$$\sum_{j=0}^n \binom{n}{j} 2^{j-1} \chi_j^A(x/2) C_{n-j}^\beta x. \tag{33}$$

By known results on type A and induction, the only unknown term is the constant term C_n^β, which is therefore fixed by the fact that the characteristic polynomial must vanish at $x = 1$.

Let us assume that this constant term has the expected value. One has to compute

$$\sum_{j=0}^{n} \binom{n}{j} x(x-2j)^{j-1}(-1)^{n-j}(2(n-j)+1)^{n-j-1}. \tag{34}$$

Decomposing the binomial coefficient into $\binom{n-1}{j-1} + \binom{n-1}{j}$ and using twice the Abel binomial formula, one gets

$$(x-1)(x-(2n+1))^{n-1}, \tag{35}$$

which vanishes at $x = 1$. This concludes the induction and the proof. □

1.3. Homology

The aim of this section is to compute the homology of the pointed partition posets of type A and B. In the two cases, we show that the homology is concentrated in top dimension. For the pointed partition poset of type A, we use the relation between Koszul duality for operads and partition posets proved by the second author in [14] in order to compute the top homology groups in terms of $\mathbb{S}_n$-modules. As a corollary, we get that the operad $\mathcal{P}erm$ is Koszul over $\mathbb{K}$.
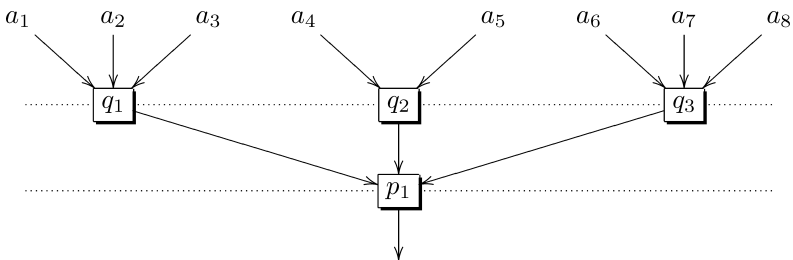
For the different notions encountered in this section (*totally semi-modular, Cohen-Macaulay*), we refer to the article of A. Björner and M. Wachs [3].

1.3.1. *Koszul duality for operads and partition posets*

In this section, we recall the results of [14] between Koszul operads and partition posets.

Consider the symmetric monoidal category (*Set*, $\times$) of sets, equipped with the cartesian product $\times$. We call an $\mathbb{S}$-*Set* a collection $(\mathcal{P}_n)_{n \in \mathbb{N}^*}$ of sets $\mathcal{P}_n$ equipped with an action of the group $\mathbb{S}_n$. The elements of $\mathcal{P}_n$ are to be thought of as operations with n inputs acting on algebraic structures. In order to represent the compositions of such operations (see Figure 3), we define a monoidal product $\circ$ in the category of $\mathbb{S}$-Sets by the formula

$$(\mathcal{P} \circ \mathcal{Q})_n := \coprod_{\substack{1 \leqslant k \leqslant n \\ i_1 + \cdots + i_k = n}} \mathcal{P}_k \times_{\mathbb{S}_k} (\mathcal{Q}_{i_1} \times \cdots \times \mathcal{Q}_{i_k}).$$

Fig. 3 Example of composition of operations in $(\mathcal{P} \circ \mathcal{Q})_8$

Definition 3 (Set-operad). A monoid $(\mathcal{P}, \mu, \eta)$ in the monoidal category $(\mathbb{S}\text{-sets}, \circ)$ is called a *set-operad*.

Let $(\mathcal{P}, \mu, \eta)$ be a set-operad. We use the elements of $\mathcal{P}$ to enrich classical partitions of $[n]$ and get a family of posets of partition type.

Let I be a set $\{x_1, \dots, x_n\}$ of n elements. We consider

$$\mathcal{P}(I) := \left(\coprod_{\substack{f : \text{bijection} \\ [n] \to I}} \mathcal{P}_n \right)_\sim ,$$

where the equivalence relation $\sim$ is given by $(\nu_n, f) \sim (\nu_n.\sigma, f \circ \sigma^{-1})$ with $\sigma \in \mathbb{S}_n$.

Definition 4 ($\mathcal{P}$-partition). A $\mathcal{P}$-partition of $[n]$ is a set of components $\{B_1, \dots, B_t\}$ such that each B_j belongs to $\mathcal{P}_{i_j}(I_j)$ where $i_1 + \cdots + i_t = n$ and $\{I_j\}_{1 \leqslant j \leqslant t}$ is a partition of $[n]$.

For every n, we define an order $\leqslant$ between $\mathcal{P}$-partitions of $[n]$ with the refinement of partitions and the composition product $\mu : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ of the set-operad $\mathcal{P}$. This poset is denoted $\Pi_{\mathcal{P}}(n)$.

The family of posets associated to the operad $\mathcal{C}om$ of commutative algebras is the classical partition poset (see [6]). We proved in [14] that the poset $\Pi_{\mathcal{P}erm}(n)$ associated to the operad $\mathcal{P}erm$ is the poset of pointed partitions Π_n^A (Proposition 12).

An algebraic operad is an operad in the category of $\mathbb{K}$-modules with the tensor product $\otimes$. To a set-operad $\mathcal{P}$, one can consider the algebraic operad generated by the free $\mathbb{K}$-modules on $\mathcal{P} : \widetilde{\mathcal{P}}(n) := \mathbb{K}[\mathcal{P}_n]$. We proved in [14], that the order complex $\Delta_*(\Pi_{\mathcal{P}}(n))$ is isomorphic to the part of degree n of the normalized bar construction $\mathcal{N}_*(\widetilde{\mathcal{P}})(n)$ of the operad $\widetilde{\mathcal{P}}$. This chain complex is quasi-isomorphic to the bar construction $\mathcal{B}_*(\widetilde{\mathcal{P}})(n)$ of $\widetilde{\mathcal{P}}$ by a Theorem of B. Fresse [6]. An algebraic operad is said to be *Koszul* if the homology of its bar construction is concentrated in top dimension. Therefore, an operad is Koszul if and only if the homology of the posets $\Pi_{\mathcal{P}}(n)$ is concentrated in top dimension. The hereditary property of the family of posets $\{\Pi_{\mathcal{P}}(n)\}_n$ gives that the operad is Koszul if and only if the posets are Cohen-Macaulay.

Moreover, the top homology groups are isomorphic to the Koszul dual (co)operad, which allows us to describe them in terms of $\mathbb{S}_n$-modules.

1.3.2. Homology of Π^A

Unlike the classical partition poset Π_n, which is a semi-modular lattice, the pointed partition poset $\Pi_{n,1}^A$ is not a lattice. Nevertheless, one has

Lemma 1.10. *For every $n \in \mathbb{N}^*$, the poset $\Pi_{n,1}^A$ is totally semi-modular.*

Proof: First, we prove that the poset $\Pi_{n,1}^A$ is semi-modular for every $n \in \mathbb{N}^*$. Let X and Y be two different pointed partitions of $[n]$ covering a third pointed partition T. Denote the blocks of T by $T = \{T_1, T_2, \dots, T_{k+1}\}$ and the pointed element of T_i by t_i. Therefore, the pointed partitions X and Y are obtained from T by the union of two blocks T_i and T_j and a choice of a pointed element between t_i and t_j. (We will often choose to denote these blocks by T_1 and T_2 for convenience). There are three possible cases.

1. The pointed partitions X and Y are obtained by the union of the same blocks T_1 and T_2. While t_1 is emphasized in X, t_2 is emphasized in Y. Since X is different from Y in the bounded poset $\Pi^A_{n,1}$, k must be greater than 2. Consider the pointed partition Z obtained from T by the union of T_1, T_2 and T_3 where t_3 is pointed. Therefore, Z covers X and Y.
2. The pointed partition X is obtained from T by the union of T_1 and T_2 with t_1 emphasized and Y is obtained by the union of T_3 and T_4 with t_3 emphasized. Consider the pointed partition Z obtained from T by the union of T_1 with T_2 and the union of T_3 with T_4 where t_1 and t_3 are emphasized. This pointed partition Z covers both X and Y.
3. The pointed partition X is obtained from T by the union of T_1 and T_2 with t_i emphasized ($i = 1, 2$) and Y is obtained by the union of T_2 and T_3 with t_j emphasized ($j = 2, 3$). We consider the pointed partition Z obtained by the union of T_1, T_2 and T_3. If i is equal to 1, we point out the element t_1 in Z. Otherwise, if i is equal to 2, we point out the element t_j in Z. The resulting pointed partition Z covers X and Y.

We can now prove that the poset $\Pi^A_{n,1}$ is totally semi-modular for every $n \in \mathbb{N}^*$. Let $[U, V]$ be an interval of $\Pi^A_{n,1}$. The poset $[U, V]$ is isomorphic to a product $\Pi^A_{\lambda_1,1} \times \cdots \times \Pi^A_{\lambda_k,1}$ of semi-modular posets. Therefore, $[U, V]$ is semi-modular. $\qquad\square$

As a corollary, we get

Theorem 1.11. *The posets $\Pi^A_{n,1}$ are CL-shellable and Cohen-Macaulay.*

Proof: CL-shellability follows from total semi-modularity by [3, Corollary 5.2]. Then the Cohen-Macaulay property follows from shellability. $\qquad\square$

Remark 1.12. We do not know whether the posets $\Pi^A_{n,1}$ admit an EL-labelling. The relation with operads allows us to compute the homology.

Theorem 1.13. *The operad $\mathcal{P}erm$ is a Koszul operad over $\mathbb{K}$ (the ring $\mathbb{Z}$ or any field). The homology of the posets $\Pi^A_{n,1}$ is concentrated in top dimension. Moreover, the homology of the posets Π^A_n with coefficients in $\mathbb{K}$ is given by the following isomorphism of $\mathbb{S}_n$-modules*

$$H_i(\Pi^A_n) \cong \begin{cases} \mathcal{RT}(n)^* \otimes sgn_{\mathbb{S}_n} & \text{if } i = n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{RT}(n)$ is the $\mathbb{S}_n$-module induced by the free $\mathbb{K}$-module on the set of rooted trees *(cf. [4]).*

Proof: The set operad $\mathcal{P}erm$ gives rise to a family of posets $(\Pi_{\mathcal{P}erm}(n))_{n \in \mathbb{N}^*}$ which are isomorphic to $(\Pi^A_n)_{n \in \mathbb{N}^*}$. For every maximal pointed partition α of the shape $\{1, \ldots, \bar{\mathbf{i}}, \ldots, n\}$, the interval $[\hat{0}, \alpha]$ is isomorphic to $\Pi^A_{n,1}$. Therefore, Theorem 10 of [14] gives that the operad $\mathcal{P}erm$ is Koszul if and only if the posets $\Pi^A_{n,1}$ are Cohen-Macaulay. Since these posets are CL-shellable, they are Cohen-Macaulay over the ring $\mathbb{Z}$ and over every field.

Once again, the results of [14] show that the homology groups of Π^A_n of top dimension are isomorphic, as $\mathbb{S}_n$-modules, to the Koszul dual operad of $\mathcal{P}erm$, which is the operad $\mathcal{P}re\mathcal{L}ie(n)$. And the operad $\mathcal{P}re\mathcal{L}ie$ is known to be described by the representations of the symmetric groups on rooted trees. $\qquad\square$

1.3.3. Homology of Π^B

Once again, we show that the posets $\Pi_n^{B'}$ are totally semi-modular, CL-shellable and Cohen-Macaulay, for every $n \in \mathbb{N}^*$.

To do that, we need to understand the intervals of Π_n^B. Let us introduce a variation of the posets Π_n^B and Π_n^β such that there is at most one pointed element in the zero block. The partial order is defined like the order of Π_n^B and Π_n^β. The only difference is that if $\pi \leqslant \nu$ then the number of pointed elements in the zero block of ν is greater than the number of pointed elements in the zero block of π. We denote these posets by $\Pi_n^{\beta B}$. The maximal interval of $\Pi_n^{\beta B}$ such that the maximal element is a zero block with one pointed element is denoted $\Pi_n^{\beta B'}$.

Proposition 1.14. *Each interval of Π_n^B is isomorphic to a product of posets of the shape* $\Pi_\lambda^\beta \times \Pi_{\lambda_1, 1}^A \times \cdots \times \Pi_{\lambda_k, 1}^A$ *or of the shape* $\Pi_\lambda^{\beta B'} \times \Pi_{\lambda_1, 1}^A \times \cdots \times \Pi_{\lambda_k, 1}^A$, *where* $\lambda + \lambda_1 + \cdots + \lambda_k \leq n$.

Proof: Let $[U, V]$ be an interval of Π_n^B. There are two possible cases.

If the pointed elements of the zero blocks of U and V are the same, then the refinement of the zero block of V corresponds to a poset of type Π_λ^β. The refinement of the other blocks of V with blocks of U corresponds to posets of type $\Pi_{\lambda_i, 1}^A$.

If the pointed elements of the zero blocks of U and V are different, then the refinement of the zero block of V corresponds to a poset of type $\Pi_\lambda^{\beta B'}$. $\qquad\square$

Lemma 1.15. *For every $n \in \mathbb{N}^*$, the poset $\Pi_n^{B'}$ is totally semi-modular.*

Proof: With Proposition 1.14, it is enough to show that the posets Π_n^β and $\Pi_n^{\beta B'}$ are semi-modular.

Let X and Y cover $T = \{-T_{k+1}, \dots, -T_1, T_0, T_1, \dots, T_{k+1}\}$ in Π_n^β. If the zero block of X and Y is T_0, then the proof is the same as in the case $\Pi_{n,1}^A$. Otherwise, there are three cases.

1. If X is given by $-T_1 \cup T_0 \cup T_1$ and Y by $T_2 \cup T_3$ (and $-T_2 \cup -T_3$), then we consider Z defined by $-T_1 \cup T_0 \cup T_1$ and $T_2 \cup T_3$, with the same choice of pointed elements.
2. If X is given by $-T_1 \cup T_0 \cup T_1$ and Y by $T_1 \cup T_2$ (and $-T_1 \cup -T_2$), then we consider Z defined by $-T_2 \cup -T_1 \cup T_0 \cup T_1 \cup T_2$, with no pointed elements.
3. If X is given by $-T_1 \cup T_0 \cup T_1$ and Y by $-T_2 \cup T_0 \cup T_2$, then we consider Z defined by $-T_2 \cup -T_1 \cup T_0 \cup T_1 \cup T_2$, with no pointed elements.

In each case, the partition Z covers both X and Y.

Let X and Y cover $T = \{-T_{k+1}, \dots, -T_1, T_0, T_1, \dots, T_{k+1}\}$ in $\Pi_n^{\beta B'}$. The proof is mainly the same as in the case Π_n^β, except some choices of pointed elements in the zero blocks. The only new case is the following one.

When X and Y are obtained from T by $-T_1 \cup T_0 \cup T_1$ but a different choice of pointed element. Therefore, we consider Z defined by $-T_2 \cup -T_1 \cup T_0 \cup T_1 \cup T_2$ with a pointed element coming from T_2. $\qquad\square$

Therefore, using the results of [3], we have the following theorem.

Theorem 1.16. *The posets $\Pi_n^{B'}$ are CL-shellable and Cohen-Macaulay, for $n \in \mathbb{N}^*$.*

Remark 1.17. Since the theory of operads is based on representations of the symmetric groups $\mathbb{S}_n$, we can not use it here to compute the homology groups of Π_n^B.

1.4. Extended pointed partition posets

Let us define $\widehat{\Pi}_n^A$ as the bounded poset obtained from Π_n^A by adding of a maximal element $\hat{1}$.

Theorem 1.18. *The poset $\widehat{\Pi}_n^A$ is totally semi-modular, CL-shellable and Cohen-Macaulay. Its homology is concentrated in top dimension and has dimension $(n-1)^{n-1}$.*

Proof: Let us prove that this poset is semi-modular first. The proof is essentially the same as for the poset Π_n^A. Only the first case can be different, when two blocks are gathered in two different ways and there is no other block. Then $\hat{1}$ covers both.

Now any interval in $\widehat{\Pi}_n^A$ is either an interval in Π_n^A, hence semi-modular, or an interval $[\pi, \hat{1}]$. Such an interval is isomorphic to a poset $\widehat{\Pi}_\lambda^A$, hence semi-modular either.

From this, one deduces the shellability and Cohen-Macaulay property. This implies the concentration of the homology in top dimension.

The Möbius number of $\widehat{\Pi}_n^A$ is given by the opposite of the value at $x = 1$ of the characteristic polynomial of Π_n^A. This gives the Euler characteristic, hence here the dimension of the homology. $\qquad\square$

Remark 1.19. The action of the symmetric groups on the top homology of the posets $\widehat{\Pi}_n^A$ certainly deserves further study. It should be related to the vertebrates (twice-pointed trees) and to the generators of the free pre-Lie algebras as Lie algebras.

1.5. Incidence Hopf algebra in type A

Let us consider the set $\mathcal{F}$ of isomorphism classes of all intervals in all posets Π_n^A for $n \geq 1$. Then it follows from Section 1.1.1 that a set of representatives of isomorphism classes is provided by arbitrary (possibly empty) products of the intervals $\Pi_{n,1}^A$ for $n \geq 2$. This family of intervals is therefore closed under products and taking subintervals. Such a family is called *hereditary* in [10].

Hence one can introduce the incidence Hopf algebra of this family of intervals. For short, let a_n be the isomorphism class of $\Pi_{n,1}^A$ for $n \geq 2$. Then a Hopf algebra structure is defined on the polynomial algebra $H(\mathcal{F})$ in the a_n by the following coproduct:

$$\Delta\, a_n := \sum_{\pi \in \Pi_{n,1}^A} [\hat{0}, \pi] \otimes [\pi, \hat{1}], \qquad (36)$$

where $[\,]$ denotes the isomorphism class of the underlying interval.

By convention, let a_1 be the unit of the polynomial algebra in the variables a_n for $n \geq 2$. It corresponds to the class of the trivial interval.

Let us decompose the coproduct according to the rank of π. One gets

$$\Delta\, a_n = \sum_{k=1}^{n} \left(\sum_{\substack{\pi \in \Pi_{n,1}^A \\ \mathrm{rk}(\pi)=n-k}} \prod_{i=1}^{k} a_{\pi_i} \right) \otimes a_k, \qquad (37)$$

where the π_i denotes the size of the blocks of the pointed partition π ($\sum_{i=1}^{k} \pi_i = n$).

Then decomposing the pointed partition π into its parts, with a distinguished part containing 1, one gets

$$\Delta \, a_n = \sum_{k=1}^{n} \left(\sum_{\pi_1=1}^{n+1-k} \sum_{\substack{\pi_2,\ldots,\pi_k \geq 1 \\ \pi_1+\cdots+\pi_k=n}} \frac{(n-1)!}{(\pi_1-1)!\pi_2!\ldots\pi_k!} \frac{\prod_{i=2}^{k} \pi_i}{(k-1)!} \prod_{i=1}^{k} a_{\pi_i} \right) \otimes a_k. \qquad (38)$$

This formula can be rewritten as

$$\Delta \frac{a_n}{(n-1)!} = \sum_{k=1}^{n} \left(\sum_{\substack{\pi_1,\pi_2,\ldots,\pi_k \geq 1 \\ \pi_1+\cdots+\pi_k=n}} \frac{a_{\pi_1}\ldots a_{\pi_k}}{(\pi_1-1)!\ldots(\pi_k-1)!} \right) \otimes \frac{a_k}{(k-1)!}. \qquad (39)$$

This has the following interpretation.

Theorem 1.20. *The incidence Hopf algebra of the family of pointed partition posets of type A is isomorphic to the Hopf algebra structure on the polynomial algebra in the variables $(a_n)_{n\geq 2}$ given by the composition of formal power series of the following shape :*

$$x + \sum_{n\geq 2} a_n \frac{x^n}{(n-1)!}. \qquad (40)$$

Proof: This follows from the explicit formula (39) for the coproduct on the generators. $\quad\square$

As a corollary, the Möbius numbers of the intervals $\Pi_{n,1}^A$ can be deduced from the fact that the inverse for composition of $x \exp(x)$ is the Lambert W function whose Taylor expansion is known to be

$$W(x) = \sum_{n\geq 1} (-1)^{n-1} n^{n-2} \frac{x^n}{(n-1)!}. \qquad (41)$$

2. Multi-pointed partition posets

In this section, we give the definition of the multi-pointed partition poset of type A and its basic properties.

2.1. Type A

Let us define the multi-pointed partition poset of type A_{n-1}.

Definition 5 (Multi-pointed partition). A *multi-pointed partition* of $[n]$ is a partition of $[n]$ together with the choice of a non-empty subset of each block, called the *pointed subset* of this block.

The order relation is as follows. First the underlying partitions must be related by the refinement order of partitions. Then if two partitions are related by the gathering of two blocks, the set of pointed elements of the big block is either one of the sets of pointed elements
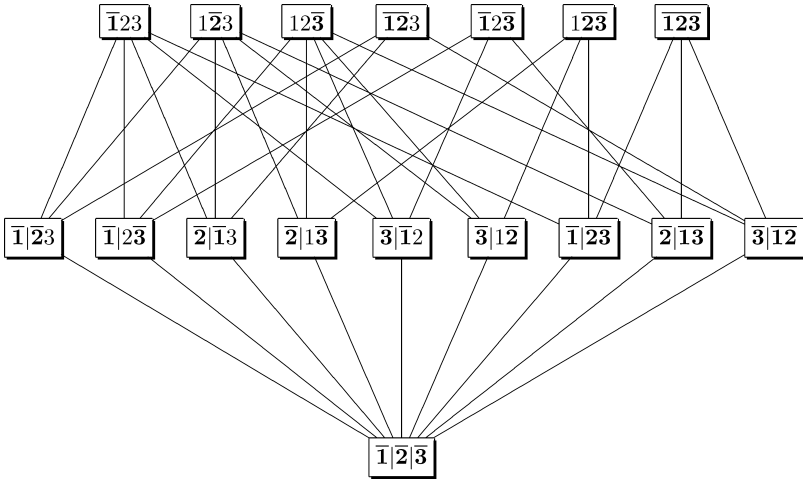
Fig. 4　The poset $\mathrm{M\Pi}_3^A$

of the two small blocks or their union. For instance, one has $\{\overline{\mathbf{1}}2\}\{3\overline{\mathbf{56}}\}\{\overline{\mathbf{47}}8\} \leqslant \{123\overline{\mathbf{56}}\}\{\overline{\mathbf{47}}8\}$. The poset of multi-pointed partitions of type A_{n-1} is denoted by $\mathrm{M\Pi}_n^A$. The example of the poset $\mathrm{M\Pi}_3^A$ is displayed in Figure 4.

As the multi-pointed partitions are just non-empty sets of pairs made of a non-empty set and a set, the generating series for the graded cardinality is given by $\frac{e^{xe^u(e^u-1)}-1}{x}$.

Of course, the symmetric group $\mathbb{S}_n$ acts on the poset $\mathrm{M\Pi}_n^A$.

Let us denote by $\mathrm{M\Pi}_{n,i}^{A'}$ the maximal interval under a partition with a single block and i pointed elements. Up to isomorphism, this does not depend on the choice of the pointed elements. By convention, let $\mathrm{M\Pi}_{n,0}^{A'}$ denote the multi-pointed partition poset $\mathrm{M\Pi}_n^A$ of type A_{n-1}.

The following proposition will play a crucial role in the sequel.

Proposition 2.1. *Each interval of $\mathrm{M\Pi}_n^A$ is isomorphic to a product of posets $\mathrm{M\Pi}_{\lambda_1, \nu_1}^{A'} \times \cdots \times \mathrm{M\Pi}_{\lambda_k, \nu_k}^{A'}$, where $\lambda_1 + \cdots + \lambda_k \leqslant n$ and $1 \leqslant \nu_i \leqslant \lambda_i$.*

Proof: As for the pointed partition posets, any interval can be decomposed into a product according to the parts of the coarser partition. One can therefore assume that the maximal element of the interval is a single block. One can then replace, in each element of the interval, each block of the minimal element by a single element. This provides an isomorphism with some interval $\mathrm{M\Pi}_{\lambda, \nu}^{A'}$.　　　　　　　　　　　　　　　　　$\square$

2.2. Characteristic polynomials in type A

Let us compute the characteristic polynomials of the posets of multi-pointed partitions of type A. The proof uses the subposets of elements where a fixed subset of $[n]$ is pointed. Up to isomorphism, these subposets only depend on the cardinality of the fixed subset of pointed indices. For $1 \leq i \leq n$, let $\mathrm{M\Pi}_{n,i}^A$ be the poset where the indices in $[i]$ are pointed.

Let us introduce the following convenient (if not traditional) notation:

$$\langle n \rangle = \prod_{j=1}^{n} (x - j). \tag{42}$$

Let us remark that $\mathrm{M}\Pi_{n,n}^{A}$ and $\mathrm{M}\Pi_{n,n}^{A'}$ are both isomorphic to the classical partition poset of type A_{n-1} whose characteristic polynomial is known to be $\langle n - 1 \rangle$.

Theorem 2.2. *For* $1 \leq i \leq n$, *the characteristic polynomial of* $\mathrm{M}\Pi_{n,i}^{A}$ *is*

$$\mathrm{M}\chi_{n,i}^{A}(x) = \frac{x - 2i}{x - i} \frac{\langle i \rangle \langle 2n - 1 \rangle}{\langle n + i \rangle}, \tag{43}$$

and its constant term $MC_{n,i}^{A}$ *is* $(-1)^{n-1} 2 \frac{i!(2n-1)!}{(n+i)!}$. *For* $0 \leq i \leq n$, *the characteristic polynomial of* $\mathrm{M}\Pi_{n,i}^{A'}$ *is*

$$\mathrm{M}\chi_{n,i}^{A'}(x) = \frac{\langle i \rangle \langle 2n - i - 1 \rangle}{\langle n \rangle}, \tag{44}$$

and its constant term $MC_{n,i}^{A'}$ *is* $(-1)^{n-1} \frac{i!(2n-i-1)!}{n!}$.

Proof: Let us prove the Theorem by induction on n. It is clearly true for $n = 1$. Let us now assume that it is proved for smaller n. The proof goes in three steps. The first step is to compute $\mathrm{M}\chi_{n,i}^{A'}$ by decreasing induction on i for $i > 0$. The statement is clear if $i = n$. Let us assume that the chosen pointed elements are $[i]$. The poset $\mathrm{M}\Pi_{n,i}^{A'}$ can be decomposed according to the size and number of pointed elements of the block p_1 containing 1. Let J be the intersection of p_1 with $[i]$. This is exactly the set of pointed elements of p_1. Let S be the complement of J in p_1, contained in $[n] \setminus [i]$. Then the result is

$$\sum_{[1] \subseteq J \subseteq [i]} \sum_{\emptyset \subseteq S \subseteq [n] \setminus [i]} \sum_{\pi \in \mathrm{M}\Pi_{n-|S|-|J|,i-|J|}^{A'}} \mathrm{M}\chi_{|J|+|S|,|J|}^{A'} \mu(\pi) x^{n-\mathrm{rk}(\pi)-|S|-|J|+1}, \tag{45}$$

where $\mathrm{rk}(\pi)$ is the rank in the poset $\mathrm{M}\Pi_{n-|S|-|J|,i-|J|}^{A'}$. Hence one gets the following equation for $\mathrm{M}\chi_{n,i}^{A'}$:

$$\sum_{j=1}^{i} \binom{i-1}{j-1} \sum_{s=0}^{n-i} \binom{n-i}{s} MC_{j+s,j}^{A'} \mathrm{M}\chi_{n-j-s,i-j}^{A'} x. \tag{46}$$

The only unknown term is the constant term when $s = n - i$ and $j = i$. This coefficient is determined by the fact that $\mathrm{M}\chi_{n,i}^{A'}$ must vanish at $x = 1$. So let us assume that it has the expected value and check later that the result vanishes at $x = 1$. One therefore has to

compute

$$\sum_{j=1}^{i} \binom{i-1}{j-1} \sum_{s=0}^{n-i} \binom{n-i}{s} (-1)^{j+s-1} \frac{j!(2s+j-1)!}{(s+j)!}$$
$$\times \frac{\langle i-j\rangle\langle 2n-2s-j-i-1\rangle}{\langle n-j-s\rangle} x^1. \tag{47}$$

Using Lemma 2.3 to compute the inner summation on s and then Lemma 2.4 to compute the remaining summation on j, one gets the expected formula for $M\chi_{n,i}^{A'}$. As this formula vanishes at $x=1$, the guess for the constant term was correct.

The second step is to compute $M\chi_{n,i}^{A}$ by decreasing induction on i. By a decomposition as above according to the block containing 1, one gets the following equation for $M\chi_{n,i}^{A}$:

$$\sum_{j=1}^{i} \binom{i-1}{j-1} \sum_{s=0}^{n-i} \binom{n-i}{s} MC_{j+s,j}^{A} M\chi_{n-j-s,i-j}^{A} x. \tag{48}$$

The only unknown term is the constant term when $s=n-i$ and $j=i$. This coefficient is given by

$$\sum_{[i]\subseteq S\subseteq[n]} MC_{n,|S|}^{A}. \tag{49}$$

This quantity is computed in Lemma 2.5 and found to be as expected. To determine $M\chi_{n,i}^{A}$, one therefore has to compute

$$\sum_{j=1}^{i} \binom{i-1}{j-1} \sum_{s=0}^{n-i} \binom{n-i}{s} (-1)^{j+s-1} 2 \frac{j!(2s+2j-1)!}{(s+2j)!}$$
$$\times \frac{(x-2(i-j))}{x-(i-j)} \frac{\langle i-j\rangle\langle 2n-2s-2j-1\rangle}{\langle n-2j-s+i\rangle} x. \tag{50}$$

Using Lemma 2.3 to compute the inner summation on s and then Lemma 2.4 to compute the remaining summation on j, one gets the expected formula for $M\chi_{n,i}^{A}$.

The last step is to compute $M\chi_{n,0}^{A'}$ by Möbius inversion on non-empty subsets of $[n]$. Indeed it is clear that

$$x M\chi_{n,0}^{A'} = \sum_{\emptyset \subsetneq S\subseteq[n]} (-1)^{|S|+1} M\chi_{n,|S|}^{A}. \tag{51}$$

So we have to show that

$$x \frac{\langle 2n-1\rangle}{\langle n\rangle} = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} (x-2j) \frac{\langle j-1\rangle\langle 2n-1\rangle}{\langle n+j\rangle}. \tag{52}$$

This can be restated as the vanishing of

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (x - 2j) \frac{\langle j - 1 \rangle}{\langle n + j \rangle}. \tag{53}$$

In hypergeometric terms, this is equivalent to

$${}_3F_2 (-n, y, y/2 + 1; y/2, y + n + 1; 1) = 0. \tag{54}$$

This follows from a known identity, see Appendix (III.9) in [11] for example, with $a = y$ and $b = y/2 + 1$. $\qquad\square$

Lemma 2.3. *For all $m \geq 0$ and $j, k \geq 1$, one has*

$$\sum_{s=0}^{m} \binom{m}{s} \frac{(-1)^s j(2s + j - 1)!}{(s + j)!} \frac{(x - k)\langle 2(m - s) + k - 1 \rangle}{\langle m - s + k \rangle}$$

$$= \frac{(x - (j + k))\langle 2m + j + k - 1 \rangle}{\langle m + j + k \rangle} \tag{55}$$

Proof: Once reformulated using the Pochhammer symbol, this is a direct consequence of the product formula associated to the following hypergeometric function:

$$\psi_x(\theta) = {}_2F_1 \left(\frac{x}{2}, \frac{1 + x}{2}; 1 + x; \theta \right) = \left(\frac{2}{1 + \sqrt{1 - \theta}} \right)^x, \tag{56}$$

which can be found for example as Formula (5), page 101 in [5]. More precisely, one takes the constant term with respect to y in the Taylor coefficients with respect to θ of the identity

$$\psi_y(\theta)\psi_x(\theta) = \psi_{x+y}(\theta). \tag{57}$$
$$\qquad\square$$

Lemma 2.4. *For all $k \geq 1$, one has*

$$x \sum_{j=1}^{k} \binom{k - 1}{j - 1} (-1)^{j-1} (j - 1)! \langle k - j - 1 \rangle = \langle k - 1 \rangle. \tag{58}$$

Proof: Once reformulated using Pochhammer symbols, this becomes equivalent to

$${}_2F_1 (-k, 1; -y - k + 1; 1) = \frac{y + k}{k}, \tag{59}$$

which is just one instance of the Gauss identity. $\qquad\square$

Lemma 2.5. *One has the following identity:*

$$\sum_{j=i}^{n} \binom{n-i}{j-i} \frac{j!(2n-j-1)!}{n!} = 2\frac{i!(2n-1)!}{(n+i)!}. \tag{60}$$

Proof: Once reformulated using hypergeometric functions, this becomes equivalent to

$${}_2F_1(i+1, -m; -2m-i+1; 1) = 2\frac{(m+i)!(2m+2i-1)!}{(m+2i)!(2m+i-1)!}, \tag{61}$$

which is just another instance of the Gauss identity. □

2.3. Homology of $M\Pi^A$

In this section, we compute the homology of the posets $M\Pi_{n,0}^A$. Once again, we use the methods of [14] and Koszul duality for operads. As a corollary, we get that the operad $\mathcal{C}om\mathcal{T}rias$ is Koszul over $\mathbb{K}$.

We show that each maximal interval of $M\Pi_{n,0}^A$ is totally semi-modular. Therefore, the homology of $M\Pi_{n,0}^A$ is concentrated in top dimension. And we use Koszul duality theory for operads to compute this homology in terms of $\mathbb{S}_n$-modules.

Lemma 2.6. *For every $n \in \mathbb{N}^*$ and every $1 \leqslant i \leqslant n$, the poset $M\Pi_{n,i}^{A'}$ is totally semi-modular.*

Proof: Since each interval of $M\Pi_{n,i}^{A'}$ is isomorphic to a product $M\Pi_{\lambda_1,v_1}^{A'} \times \cdots \times M\Pi_{\lambda_k,v_k}^{A'}$, where $\lambda_1 + \cdots + \lambda_k \leqslant n$ and $1 \leqslant v_j \leqslant \lambda_j$, it is enough to show that every $M\Pi_{n,i}^{A'}$ is a semi-modular poset.

Let X and Y be two different multi-pointed partitions of $[n]$ covering a third multi-pointed partition T in $M\Pi_{n,i}^{A'}$. Denote the blocks of T by $T = \{T_1, \ldots, T_{k+1}\}$ and the set of pointed elements of T_i by $\mathcal{T}_i$. Therefore, the multi-pointed partitions X and Y are obtained from T by the union of two blocks T_i and T_j and a choice of a pointed elements between $\mathcal{T}_i, \mathcal{T}_j$ or both. (We will often choose to denote these blocks by T_1 and T_2 for convenience). There are three possible cases.

1. The multi-pointed partitions X and Y are obtained by the union of the same blocks T_1 and T_2. Since X is different from Y in the bounded poset $M\Pi_{n,i}^{A'}$, k must be greater than 2. Consider the multi-pointed partition Z obtained from T by the union of T_1, T_2 and T_3 where the set $\mathcal{T}_3$ is pointed. Therefore, Z covers X and Y.
2. The multi-pointed partition X is obtained from T by the union of T_1 and T_2 with the set $\mathcal{X}_1$ of pointed elements. The multi-pointed partition Y is obtained by the union of T_3 and T_4 with the set $\mathcal{Y}_2$ of pointed elements. Consider the multi-pointed partition Z obtained from T by the union of T_1 with T_2 and the union of T_3 with T_4 where the set $\mathcal{X}_1 \cup \mathcal{Y}_2$ of element is emphasized. This multi-pointed partition Z covers both X and Y.
3. The multi-pointed partition X is obtained from T by the union of T_1 and T_2 and Y is obtained by the union of T_2 and T_3 where $\mathcal{Y}_2$ denotes the set of pointed chosen elements. If only the elements of $\mathcal{T}_1$ or the elements of $\mathcal{T}_2$ are emphasized in X, then we built the same kind of covering partition Z as in the proof of Lemma 1.10. If the elements of $\mathcal{T}_1 \cup \mathcal{T}_2$ are pointed in X, we consider the multi-pointed partition Z given by the union $T_1 \cup T_2 \cup T_3$ where only the elements of $\mathcal{T}_3$ are pointed, if the elements of $\mathcal{T}_2$ are not pointed in Y,

and where the elements of $\mathcal{T}_1 \cup \mathcal{Y}_2$ are pointed otherwise. In any case, the multi-pointed partition Z covers X and Y.   □

As a consequence, using results of [3], we have

Theorem 2.7. *The posets* $\mathrm{M}\Pi_{n,i}^{A'}$ *are CL-shellable and Cohen-Macaulay.*

Then the relation with operads allows us to determine the homology, as follows.

Theorem 2.8. *The operad* $\mathcal{C}om\mathcal{T}rias$ *of commutative trialgebras is a Koszul operad over* $\mathbb{K}$ *(the ring* $\mathbb{Z}$ *or any field). The homology of the posets* $\mathrm{M}\Pi_{n,i}^{A'}$ *is concentrated in top dimension. Moreover, the homology of the posets* $\mathrm{M}\Pi_n^{A}$ *with coefficients in* $\mathbb{K}$ *is given by the following isomorphism of* $\mathbb{S}_n$-*modules*

$$H_i(\mathrm{M}\Pi_n^A) \cong \begin{cases} \mathcal{L}ie \circ \mathcal{M}ag(n)^* \otimes sgn_{\mathbb{S}_n} & \text{if } i = n - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{L}ie \circ \mathcal{M}ag(n)$ *is the* $\mathbb{S}_n$-*module induced by plethysm or equivalently by the operadic composition of the operad* $\mathcal{L}ie$, *of Lie algebras, with the operad* $\mathcal{M}ag$, *of magmatic algebras.*

Proof: The partition posets associated to the operad $\mathcal{C}om\mathcal{T}rias$ are isomorphic to the posets $\mathrm{M}\Pi_n^A$, for $n \in \mathbb{N}^*$. Since the Koszul dual operad of the operad $\mathcal{C}om\mathcal{T}rias$ is the operad $\mathcal{P}ost\mathcal{L}ie$, which is isomorphic as $\mathbb{S}$-module to the composition $\mathcal{L}ie \circ \mathcal{M}ag$ (*cf.* [14]), we conclude by the same arguments as in the proof of Theorem 1.13.   □

References

1. C. A. Athanasiadis, Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes. *Bull. London Math. Soc.*, **36**(3) (2004), 294–302.
2. A. Björner and M. Wachs, Geometrically constructed bases for homology of partition lattices of types A, B and D. *Electron. J. Combin.*, 11, no. 2, Research Paper 3, (2004/05), 26 pp. (electronic).
3. A. Björner and M. Wachs, On lexicographically shellable posets. *Trans. Amer. Math. Soc.*, **277**(1) (1983), 323–341.
4. F. Chapoton and M. Livernet, Pre-Lie algebras and the rooted trees operad. *IMRN*, **8** (2001), 395–408.
5. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions. Vol. I.* Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981.
6. B. Fresse, Koszul duality of operads and homology of partition posets. *in "Homotopy theory and its applications (Evanston, 2002)", Contemp. Math.*, **346** (2004), 115–215.
7. P. Headley, On a family of hyperplane arrangements related to the affine Weyl groups. *J. Algebraic Combin.*, **6**(4) (1997), 331–338.
8. J. Pitman, Coalescent random forests. *J. Combin. Theory Ser. A*, **85**(2) (1999), 165–193.
9. J. Riordan, *Combinatorial identities*. John Wiley & Sons Inc., New York, 1968.
10. W. R. Schmitt, Incidence Hopf Algebra. *J. of Pure and Appl. Algebra*, **96** (1994), 299–330.
11. L. J. Slater, *Generalized hypergeometric functions*. Cambridge University Press, Cambridge, 1966.
12. L. Solomon and H. Terao, The double Coxeter arrangement. *Comment. Math. Helv.*, **73**(2) (1998), 237–258.
13. H. Terao, Multiderivations of Coxeter arrangements. *Invent. Math.*, **148**(3) (2002), 659–674.
14. B. Vallette, Homology of generalized partition posets. preprint arXiv:math.AT/0405312.
15. D. Zvonkine, An algebra of power series arising in the intersection theory of moduli spaces of curves and in the enumeration of ramified coverings of the sphere. preprint arXiv:math.AG/0403092.