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# Decreasing Subsequences in Permutations and Wilf Equivalence for Involutions

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**Abstract.** In a recent paper, Backelin, West and Xin describe a map $\phi^*$ that recursively replaces all occurrences of the pattern $k \dots 21$ in a permutation $\sigma$ by occurrences of the pattern $(k-1)\dots 21k$. The resulting permutation $\phi^*(\sigma)$ contains no decreasing subsequence of length $k$. We prove that, rather unexpectedly, the map $\phi^*$ commutes with taking the inverse of a permutation.

In the BWX paper, the definition of $\phi^*$ is actually extended to full rook placements on a Ferrers board (the permutations correspond to square boards), and the construction of the map $\phi^*$ is the key step in proving the following result. Let $T$ be a set of patterns starting with the prefix $12\dots k$. Let $T'$ be the set of patterns obtained by replacing this prefix by $k\dots 21$ in every pattern of $T$. Then for all $n$, the number of permutations of the symmetric group $\mathcal{S}_n$ that avoid $T$ equals the number of permutations of $\mathcal{S}_n$ that avoid $T'$.

Our commutation result, generalized to Ferrers boards, implies that the number of *involutions* of $\mathcal{S}_n$ that avoid $T$ is equal to the number of involutions of $\mathcal{S}_n$ avoiding $T'$, as recently conjectured by Jaggard.

**Keywords:** pattern avoiding permutations, Wilf equivalence, involutions, decreasing subsequences, prefix exchange

## 1. Introduction

Let $\pi = \pi_1\pi_2\dots\pi_n$ be a permutation of length $n$. Let $\tau = \tau_1\dots\tau_k$ be another permutation. An *occurrence* of $\tau$ in $\pi$ is a subsequence $\pi_{i_1}\dots\pi_{i_k}$ of $\pi$ that is order-isomorphic to $\tau$. For instance, 246 is an occurrence of $\tau = 123$ in $\pi = 251436$. We say that $\pi$ *avoids* $\tau$ if $\pi$ contains no occurrence of $\tau$. For instance, the above permutation $\pi$ avoids 1234. The set of permutations of length $n$ is denoted by $\mathcal{S}_n$, and $\mathcal{S}_n(\tau)$ denotes the set of $\tau$-avoiding permutations of length $n$.

The idea of systematically studying pattern avoidance in permutations appeared in the mid-eighties [19]. The main problem in this field is to determine $S_n(\tau)$, the cardinality of $\mathcal{S}_n(\tau)$, for any given pattern $\tau$. This question has subsequently been generalized and refined in various ways (see for instance [1, 4, 7, 16], and [15] for a recent survey). However, relatively little is known about the original question. The case of patterns of length 4 is not

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yet completed, since the pattern 1324 still remains unsolved. See [5, 8, 20, 21, 24] for other patterns of length 4.

For length 5 and beyond, all the solved cases follow from three important generic results. The first one, due to Gessel [8, 9], gives the generating function of the numbers $S_n(12\dots k)$. The second one, due to Stankova and West [22], states that $S_n(231\tau) = S_n(312\tau)$ for any pattern $\tau$ on $\{4, 5, \dots, k\}$. The third one, due to Backelin, West and Xin [3], shows that $S_n(12\dots k\tau) = S_n(k\dots 21\tau)$ for any pattern $\tau$ on the set $\{k+1, k+2, \dots, \ell\}$. In the present paper an analogous result is established for pattern-avoiding *involutions*. We denote by $\mathcal{I}_n(\tau)$ the set of involutions avoiding $\tau$, and by $I_n(\tau)$ its cardinality.

The systematic study of pattern avoiding involutions was also initiated in [19], continued in [8, 10] for increasing patterns, and then by Guibert in his thesis [11]. Guibert discovered experimentally that, for a surprisingly large number of patterns $\tau$ of length 4, $I_n(\tau)$ is the $n$th Motzkin number:

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(k+1)!(n-2k)!}.$$

This was already known for $\tau = 1234$ (see [17]), and consequently for $\tau = 4321$, thanks to the properties of the Schensted correspondence [18]. Guibert explained all the other instances of the Motzkin numbers, except for two of them: 2143 and 3214. However, he was able to describe a two-label generating tree for the class $\mathcal{I}_n(2143)$. Several years later, the Motzkin result for the pattern 2143 was at last derived from this tree: first in a bijective way [12], then using generating functions [6]. No simple generating tree could be described for involutions avoiding 3214, and it was only in 2003 that Jaggard [14] gave a proof of this final conjecture, inspired by [2]. More generally, he proved that for $k = 2$ or 3, $I_n(12\dots k\tau) = I_n(k\dots 21\tau)$ for all $\tau$. He conjectured that this holds for all $k$, which we prove here.

We derive this from another result, which may be more interesting than its implication in terms of forbidden patterns. This result deals with a transformation $\phi^*$ that was defined in [3] to prove that $S_n(12\dots k\tau) = S_n(k\dots 21\tau)$. This transformation acts not only on permutations, but on more general objects called *full rook placements on a Ferrers shape* (see Section 2 for precise definitions). The map $\phi^*$ may, at first sight, appear as an *ad hoc* construction, but we prove that it has a remarkable, and far from obvious, property: it commutes with taking the inverse of a permutation, and more generally with the corresponding diagonal reflection of a full rook placement. (By the inverse of a permutation $\pi$ we mean the map that sends $\pi$, seen as a bijection, to its inverse.)

The map $\phi^*$ is defined by iterating a transformation $\phi$, which chooses a certain occurrence of the pattern $k\dots 21$ and replaces it by an occurrence of $(k-1)\dots 21k$. The map $\phi$ itself does *not* commute with taking the inverse of a permutation, and our proof of the commutation theorem is actually quite complicated.

This strongly suggests that we need a better description of the map $\phi^*$, on which the commutation theorem would become obvious. By analogy, let us recall what happened for the Schensted correspondence: the fact that taking the inverse of permutations exchanges the two tableaux only became completely clear with Viennot's description of the correspondence [23].

Actually, since the Schensted correspondence has nice properties regarding the monotone subsequences of permutations, and provides one of the best proofs of the identity $I_n(12\dots k) = I_n(k\dots 21)$, we suspect that the map $\phi^*$ might be related to this correspondence, or to an extension of it to rook placements.

## 2. Wilf equivalence for involutions

One of the main implications of this paper is the following.

**Theorem 1** *Let $k \geq 1$. Let $T$ be a set of patterns, each starting with the prefix $12\dots k$. Let $T'$ be the set of patterns obtained by replacing this prefix by $k\dots 21$ in every pattern of $T$. Then, for all $n \geq 0$, the number of involutions of $\mathcal{S}_n$ that avoid $T$ equals the number of involutions of $\mathcal{S}_n$ that avoid $T'$.*

*In particular, the involutions avoiding $12\dots k\tau$ and the involutions avoiding $k\dots 21\tau$ are equinumerous, for any permutation $\tau$ of $\{k+1, k+2, \dots, \ell\}$.*

This theorem was proved by Jaggard for $k = 2$ and $k = 3$ [14]. It is the analogue, for involutions, of a result recently proved by Backelin, West and Xin for permutations [3]. Thus it is not very surprising that we follow their approach. This approach requires looking at pattern avoidance for slightly more general objects than permutations, namely, *full rook placements on a Ferrers board*.

In what follows, $\lambda$ will be an integer partition, which we represent as a Ferrers board (Figure 1). A full rook placement, or a *placement* for short, is a board $\lambda$, together with a distribution of dots on $\lambda$, such that every row and every column of $\lambda$ contains exactly one dot. This implies that the board has as many rows as columns.

Each cell of the board will be denoted by its coordinates: in the first placement of Figure 1, there is a dot in the cell $(1, 4)$. If the placement has $n$ dots, we associate with it a permutation $\pi$ of $\mathcal{S}_n$, defined by $\pi(i) = j$ if there is a dot in the cell $(i, j)$. The permutation corresponding to the first placement of Figure 1 is $\pi = 4312$. This induces a bijection between placements on the $n \times n$ square and permutations of $\mathcal{S}_n$.

The *inverse* of a placement $p$ on the board $\lambda$ is the placement $p'$ obtained by reflecting $p$ and $\lambda$ with respect to the main diagonal; it is thus a placement on the *conjugate* of $\lambda$, usually denoted by $\lambda'$. This terminology is of course an extension to placements of the classical

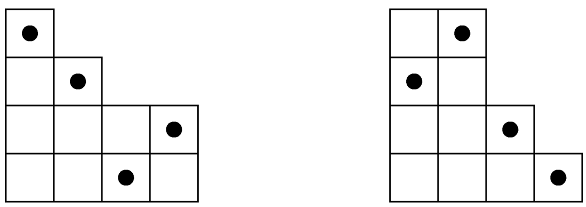


*Figure 1.* A full rook placement on a Ferrers board, and its inverse.

terminology for permutations.

**Definition 2** Let $p$ be a placement on the board $\lambda$, and let $\pi$ be the corresponding permutation. Let $\tau$ be a permutation of $\mathcal{S}_k$. We say that $p$ *contains* $\tau$ if there exists in $\pi$ an occurrence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ of $\tau$ such that the corresponding dots are contained in a *rectangular* sub-board of $\lambda$. In other words, the cell with coordinates $(i_k, \max_j \pi_{i_j})$ must belong to $\lambda$.

The placement of Figure 1 contains the pattern 12, but avoids the pattern 21, even though the associated permutation $\pi = 4312$ contains several occurrences of 21. We denote by $\mathcal{S}_\lambda(\tau)$ the set of placements on $\lambda$ that avoid $\tau$. If $\lambda$ is self-conjugate, we denote by $\mathcal{I}_\lambda(\tau)$ the set of *symmetric* (that is, self-inverse) placements on $\lambda$ that avoid $\tau$. We denote by $S_\lambda(\tau)$ and $I_\lambda(\tau)$ the cardinalities of these sets.

In [2, 3, 22], it was shown that the notion of pattern avoidance in placements is well suited to deal with prefix exchanges in patterns. This was adapted by Jaggard [14] to involutions:

**Proposition 3** *Let $\alpha$ and $\beta$ be two involutions of $\mathcal{S}_k$. Let $T_\alpha$ be a set of patterns, each beginning with $\alpha$. Let $T_\beta$ be obtained by replacing, in each pattern of $T_\alpha$, the prefix $\alpha$ by $\beta$. If, for every self-conjugate shape $\lambda$, $I_\lambda(\alpha) = I_\lambda(\beta)$, then $I_\lambda(T_\alpha) = I_\lambda(T_\beta)$ for every self-conjugate shape $\lambda$.*

Hence Theorem 1 will be proved if we can prove that $I_\lambda(12\dots k) = I_\lambda(k\dots 21)$ for any self-conjugate shape $\lambda$. A simple induction on $k$, combined with Proposition 3, shows that it is actually enough to prove the following:

**Theorem 4** *Let $\lambda$ be a self-conjugate shape. Then $I_\lambda(k\dots 21) = I_\lambda((k-1)\dots 21k)$.*

A similar result was proved in [3] for general (asymmetric) placements: for every shape $\lambda$, one has $S_\lambda(k\dots 21) = S_\lambda((k-1)\dots 21k)$. The proof relies on the description of a recursive bijection between the sets $\mathcal{S}_\lambda(k\dots 21)$ and $\mathcal{S}_\lambda((k-1)\dots 21k)$. What we prove here is that this complicated bijection actually *commutes with taking the inverse of a placement* and this implies Theorem 4.

But let us first describe (and slightly generalize) the transformation defined by Backelin, West and Xin [3]. This transformation depends on $k$, which from now on is supposed to be fixed. Since Theorem 4 is trivial for $k = 1$, we assume $k \geq 2$.

**Definition 5** (*The transformation $\phi$*) Let $p$ be a placement containing $k\dots 21$, and let $\pi$ be the associated permutation. To each occurrence of $k\dots 21$ in $p$, there corresponds a decreasing subsequence of length $k$ in $\pi$. The *$\mathcal{A}$-sequence* of $p$, denoted by $\mathcal{A}(p)$, is the smallest of these subsequences for the lexicographic order.

The corresponding dots in $p$ form an occurrence of $k\dots 21$. Rearrange these dots cyclically so as to form an occurrence of $(k-1)\dots 21k$. The resulting placement is defined to be $\phi(p)$.

If $p$ avoids $k\dots 21$, we simply define $\phi(p) := p$. The transformation $\phi$ is also called the *$\mathcal{A}$-shift.*

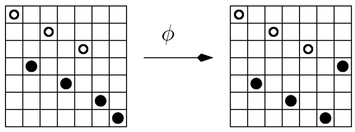



*Figure 2.* The $\mathcal{A}$-shift on the permutation 7 4 6 3 5 2 1, when $k = 4$.

An example is provided by Figure 2 (the letters of the $\mathcal{A}$-sequence are underlined, and the corresponding dots are black). It is easy to see that the $\mathcal{A}$-shift decreases the inversion number of the permutation associated with the placement (details will be given in the proof of Corollary 11). This implies that after finitely many iterations of $\phi$, there will be no more decreasing subsequences of length $k$ in the placement. We denote by $\phi^*$ the iterated transformation, that recursively transforms *every* pattern $k \dots 21$ into $(k-1)\dots 21k$. For instance, with the permutation $\pi = 7\ 4\ 6\ 3\ 5\ 2\ 1$ of Figure 2 and $k = 4$, we find

$$\pi = 7\ \underline{4}\ 6\ \underline{3}\ 5\ \underline{2}\ \underline{1} \to \underline{7}\ \underline{3}\ 6\ \underline{2}\ 5\ \underline{1}\ 4 \to 3\ 2\ 6\ 1\ 5\ 7\ 4 = \phi^*(\pi).$$

The main property of $\phi^*$ that was proved and used in [3] is the following:

**Theorem 6** (*The BWX bijection*) *For every shape $\lambda$, the transformation $\phi^*$ induces a bijection from $\mathcal{S}_\lambda((k-1)\dots 21k)$ to $\mathcal{S}_\lambda(k\dots 21)$.*

The key to our paper is the following rather unexpected theorem.

**Theorem 7** (*Global commutation*) *The transformation $\phi^*$ commutes with taking the inverse of a placement.*

For instance, with $\pi$ as above, we have

$$\pi^{-1} = 7\ \underline{6}\ \underline{4}\ \underline{2}\ 5\ 3\ \underline{1} \to \underline{7}\ \underline{4}\ \underline{2}\ \underline{1}\ 5\ 3\ 6 \to 4\ 2\ 1\ 7\ 5\ 3\ 6 = \phi^*(\pi^{-1})$$

and we observe that

$$\phi^*(\pi^{-1}) = (\phi^*(\pi))^{-1}.$$

Note, however, that $\phi(\pi^{-1}) \neq (\phi(\pi))^{-1}$. Indeed, $\phi(\pi^{-1}) = 7\ 4\ 2\ 1\ 5\ 3\ 6$ while $(\phi(\pi))^{-1} = 6\ 4\ 2\ 7\ 5\ 3\ 1$, so that the elementary transformation $\phi$, that is, the $\mathcal{A}$-shift, does not commute with taking the inverse.

Theorems 6 and 7 together imply that $\phi^*$ induces a bijection from $\mathcal{I}_\lambda((k-1)\dots 21k)$ to $\mathcal{I}_\lambda(k\dots 21)$, for every self-conjugate shape $\lambda$. This proves Theorem 4, and hence Theorem 1. The rest of the paper is devoted to proving Theorem 7, which we call the theorem of *global commutation*. By this, we mean that taking the inverse of a placement commutes with the global transformation $\phi^*$ (but not with the elementary transformation $\phi$).

**Remark** At first sight, our definition of the $\mathcal{A}$-sequence (Definition 5), does not seem to coincide with the definition given in [3]. Let $a_k \dots a_2 a_1$ denote the $\mathcal{A}$-sequence of the placement $p$, with $a_k > \dots > a_1$. We identify this sequence with the corresponding set of dots in $p$. The dot $a_k$ is the lowest dot that is the leftmost point in an occurrence of $k \dots 21$ in $p$. Then $a_{k-1}$ is the lowest dot such that $a_k a_{k-1}$ is the beginning of an occurrence of $k \dots 21$ in $p$, and so on.

However, in [3], the dot $a_k$ is chosen as above, but then each of the next dots $a'_{k-1}, \dots, a'_1$ is chosen to be as far *left*, as possible, and not as *low* as possible. Let us prove that the two procedures give the same sequence of dots. Assume not, and let $a_j \neq a'_j$ be the first (leftmost) point where the two sequences differ. By definition, $a_j$ is lower than $a'_j$, and to the right of it. But then the sequence $a_{k-1} \dots a_{j+1} a'_j a_j \dots a_2 a_1$ is an occurrence of the pattern $k \dots 21$ in $p$, which is smaller than $a_k \dots a_2 a_1$ for the lexicographic order, a contradiction.

The fact that the $\mathcal{A}$-sequence can be defined in two different ways will be used very often in the paper.

2. At this stage, we have reduced the proof of Theorem 1 to the proof of the global commutation theorem, Theorem 7.

## 3. From local commutation to global commutation

In order to prove that $\phi^*$ commutes with the taking the inverse of placements, it would naturally be tempting to prove that $\phi$ itself commutes with this operation. However, this is not the case, as shown above. Given a placement $p$ and its inverse $p'$, we thus want to know how the placements $\phi(p)$ and $\phi(p')'$ differ.

**Definition 8** For any shape $\lambda$ and any placement $p$ on $\lambda$, we define $\psi(p)$ by

$$\psi(p) := \phi(p')'.$$

Thus $\psi(p)$ is also a placement on $\lambda$.

In other words, $\psi(p)$ is obtained by flipping $p$ around the diagonal, then applying $\phi$, and flipping the resulting placement back. In what follows, when we invoke a *symmetry argument* we are referring to the symmetry underlying the above definition. Note that $\psi^m(p) = (\phi^m(p'))'$, so that the theorem of global commutation, Theorem 7, can be restated as $\psi^* = \phi^*$.

Combining the above definition of $\psi$ with Definition 5 gives an alternative description of $\psi$.

**Lemma 9** (*The transformation* $\psi$) *Let $p$ be a placement containing $k \dots 21$. Let $b_1$, $b_2, \dots, b_k$ be defined recursively as follows: For all $j$, $b_j$ is the leftmost dot such that*

*$b_j \dots b_2 b_1$ ends an occurrence of $k \dots 21$ in $p$. We call $b_k \dots b_2 b_1$ the $\mathcal{B}$-sequence of $p$, and denote it by $\mathcal{B}(p)$.*

*Rearrange the $k$ dots of the $\mathcal{B}$-sequence cyclically so as to form an occurrence of $(k-1)\dots 21k$: the resulting placement is $\psi(p)$.*

*If $p$ avoids $k \dots 21$, then $\psi(p) = p$. The transformation $\psi$ is also called the $\mathcal{B}$-shift.*

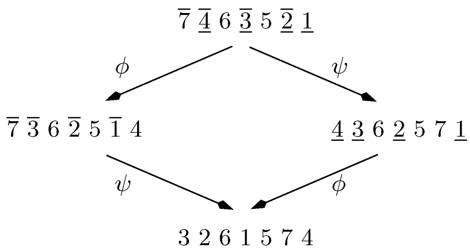
According to the first remark that concludes Section 2, we can alternatively define $b_j$, for $j \geq 2$, as the *lowest* dot such that $b_j \dots b_2 b_1$ ends an occurrence of $k \dots 21$ in $p$.

We have seen that, in general, $\phi$ does not commute with taking the inverse. That is, $\phi(p) \neq \psi(p)$ in general. The above lemma tells us that $\phi(p) = \psi(p)$ if and only if the $\mathcal{A}$-sequence and the $\mathcal{B}$-sequence of $p$ coincide. If they do *not* coincide, then we still have the following remarkable property, whose proof is deferred to the very end of the paper.

**Theorem 10** (*Local commutation*) *Let $p$ be a placement for which the $\mathcal{A}$- and $\mathcal{B}$-sequences do not coincide. Then $\phi(p)$ and $\psi(p)$ still contain the pattern $k \dots 21$, and*

$$\phi(\psi(p)) = \psi(\phi(p)).$$

For instance, for the permutation of Figure 2 and $k = 4$, we have the following commutative diagram, in which the underlined (resp. overlined) letters correspond to the $\mathcal{A}$-sequence (resp. $\mathcal{B}$-sequence):




A classical argument, which is sometimes stated in terms of *locally confluent* and *globally confluent rewriting systems* (see [13] and references therein), will show that Theorem 10 implies $\psi^* = \phi^*$, and actually the following more general corollary.

**Corollary 11** *Let $p$ be a placement. Any iterated application of the transformations $\phi$ and $\psi$ yields ultimately the same placement, namely $\phi^*(p)$. Moreover, all the minimal sequences of transformations that yield $\phi^*(p)$ have the same length.*

Before we prove this corollary, let us illustrate it. We think of the set of permutations of length $n$ as the set of vertices of an oriented graph, the edges of which are given by the maps $\phi$ and $\psi$. Figure 3 shows a connected component of this graph. The dotted edges represent $\phi$ while the plain edges represent $\psi$. The dashed edges correspond to the cases where $\phi$ and $\psi$ coincide. We see that all the paths that start at a given point converge to the same point.

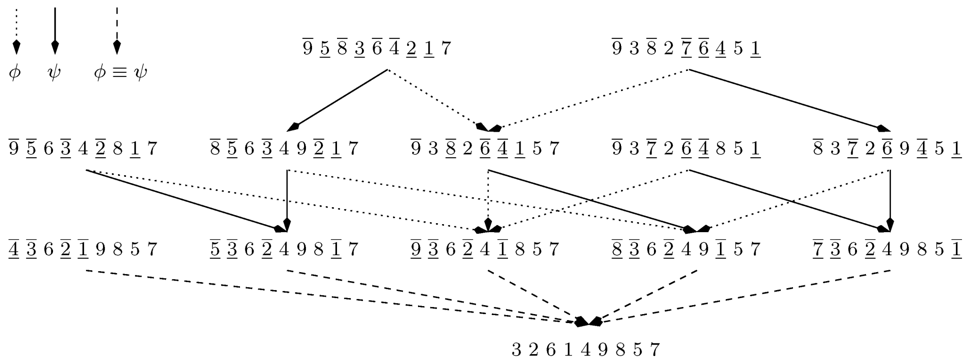



*Figure 3.* The action of $\phi$ and $\psi$ on a part of $\mathcal{S}_9$, for $k = 4$.

**Proof:** For any placement $p$, define the inversion number of $p$ as the inversion number of the associated permutation $\pi$ (that is, the number of pairs $(i, j)$ such that $i < j$ and $\pi_i > \pi_j$). Assume $p$ contains at least one occurrence of $k \dots 21$, and let $i_1 < \cdots < i_k$ be the positions (abscissae) of the elements of the $\mathcal{A}$-sequence of $p$. It is straightforward to verify that $\text{inv}(p) - \text{inv}(\phi(p)) \geq k - 1$. In fact, we have

$$\text{inv}(p) - \text{inv}(\phi(p)) = k - 1 + 2 \sum_{m=1}^{k-1} \text{Card}\,\{i : i_m < i < i_{m+1} \text{ and } \pi_{i_1} > \pi_i > \pi_{i_{m+1}}\}.$$

In particular, $\text{inv}(\phi(p)) < \text{inv}(p)$. By symmetry, together with the fact that $\text{inv}(\pi^{-1}) = \text{inv}(\pi)$, it follows that $\text{inv}(\psi(p)) < \text{inv}(p)$ too.

We encode the compositions of the maps $\phi$ and $\psi$ by words on the alphabet $\{\phi, \psi\}$. For instance, if $u$ is the word $\phi\psi^2$, then $u(p) = \phi\psi^2(p)$. Let us prove, by induction on $\text{inv}(p)$, the following two statements:

1. If $u$ and $v$ are two words such that $u(p)$ and $v(p)$ avoid $k \dots 21$, then $u(p) = v(p)$.

2. Moreover, if $u$ and $v$ are minimal for this property (that is, for any non-trivial factorization $u = u_0 u_1$, the placement $u_1(p)$ still contains an occurrence of $k \dots 21$—and similarly for $v$), then $u$ and $v$ have the same length.

If the first property holds for $p$, then $u(p) = v(p) = \phi^*(p)$. If the second property holds, we denote by $L(p)$ the length of any minimal word $u$ such that $u(p)$ avoids $k \dots 21$.

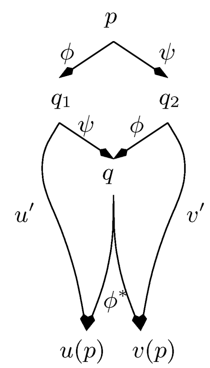
If $\pi$ is the identity, then the two results are obvious. They remain obvious, with $L(p) = 0$, if $p$ does not contain any occurrence of $k \dots 21$.

Now assume $p$ contains such an occurrence, and $u(p)$ and $v(p)$ avoid $k \dots 21$. By assumption, neither $u$ nor $v$ is the empty word. Let $f$ (resp. $g$) be the rightmost letter of $u$ (resp. $v$), that is, the *first* transformation that is applied to $p$ in the evaluation of $u(p)$ (resp. $v(p)$). Write $u = u'f$ and $v = v'g$.

If $f(p) = g(p)$, let $q$ be the placement $f(p)$. Given that $\mathrm{inv}(q) < \mathrm{inv}(p)$, and that the placements $u(p) = u'(q)$ and $v(p) = v'(q)$ avoid $k \dots 21$, both statements follow by induction.

If $f(p) \neq g(p)$, we may assume, without loss of generality, that $f = \phi$ and $g = \psi$. Let $q_1 = \phi(p)$, $q_2 = \psi(p)$ and $q = \phi(\psi(p)) = \psi(\phi(p))$ (Theorem 10). The induction hypothesis, applied to $q_1$, gives $u'(q_1) = \phi^*(\psi(q_1)) = \phi^*(q)$, that is, $u(p) = \phi^*(q)$ (see the figure below). Similarly, $v'(q_2) = \phi^*(q_2) = \phi^*(q)$, that is, $v(p) = \phi^*(q)$. This proves the first statement. If $u$ and $v$ are minimal for $p$, then so are $u'$ and $v'$ for $q_1$ and $q_2$ respectively. By the first statement of Theorem 10, $q_1$ and $q_2$ still contain the pattern $k \dots 21$, so $L(q) = L(q_1) - 1 = L(q_2) - 1$, and the words $u'$ and $v'$ have the same length. Consequently, $u$ and $v$ have the same length too. □

**Note.** We have reduced the proof of Theorem 1 to the proof of the local commutation theorem, Theorem 10. The last two sections of the paper are devoted to this proof, which turns out to be unexpectedly complicated. There is no question that one needs to find a more illuminating description of $\phi^*$, or of $\phi \circ \psi$, which makes Theorems 7 and 10 clear.




## 4. The local commutation for permutations

In this section, we prove that the local commutation theorem holds for permutations (it will be extended to placements in the next section). This will be done by a long sequence of lemmas and propositions that combine to prove local commutation, in Theorem 32. First, in Section 4.1, we introduce some definitions and prove some technical lemmas that we then use in Section 4.2 to show what happens to various parts of a permutation under the $\mathcal{A}$-shift and the $\mathcal{B}$-shift. The composition of the two shifts is described in Section 4.3.

To begin with, let us study a big example, and use it to describe the contents and the structure of this section. This example is illustrated in Figure 4.

**Example** Let $\pi$ be the following permutation of length 21:

$$\pi = 17\ 21\ 20\ 16\ 19\ 18\ 13\ 15\ 11\ 14\ 12\ 8\ 10\ 9\ 7\ 4\ 2\ 6\ 5\ 3\ 1.$$

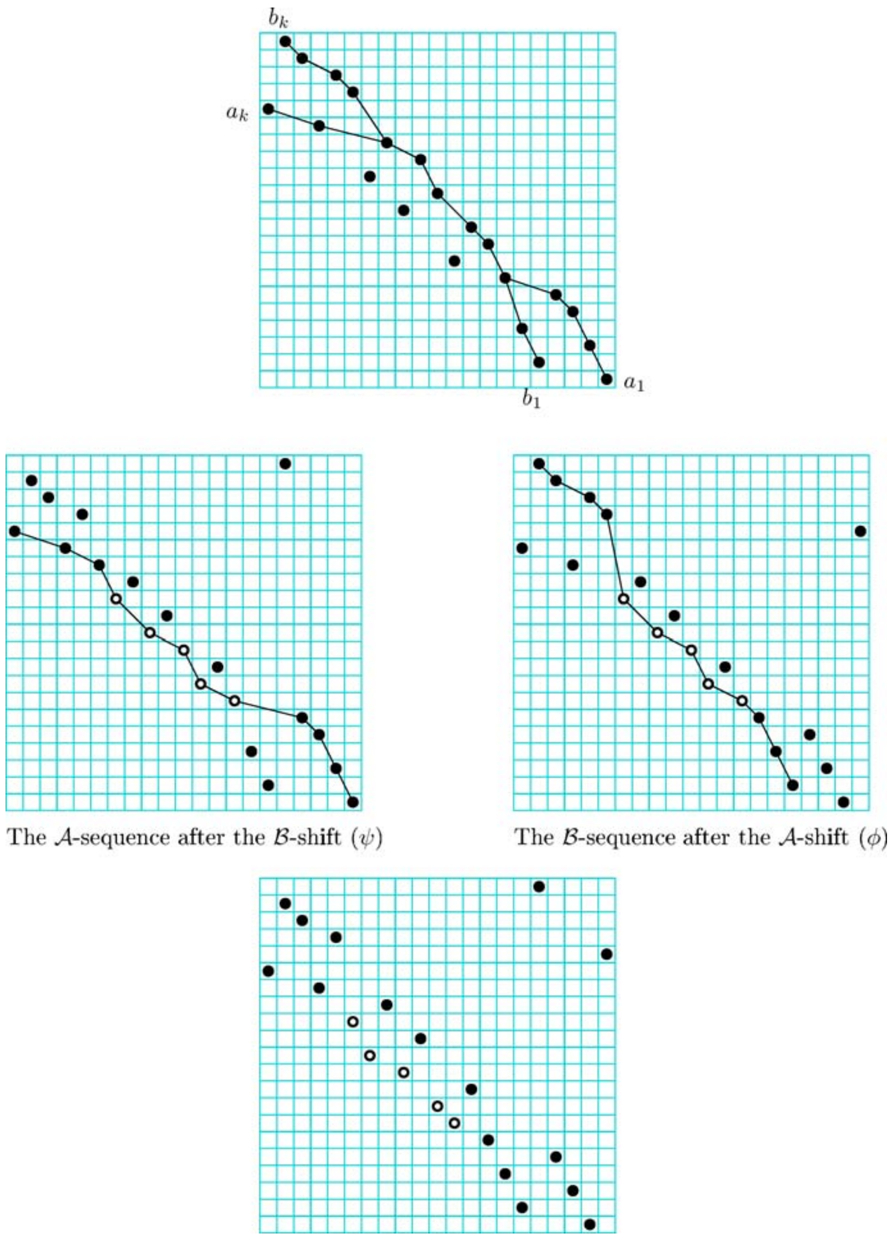



*Figure 4*. Top: A permutation $\pi$, with its $\mathcal{A}$- and $\mathcal{B}$-sequences shown. Left: After the $\mathcal{B}$-shift. Right: After the $\mathcal{A}$-shift. Bottom: After the composition of $\phi$ and $\psi$.

1. Let $k = 12$. The $\mathcal{A}$-sequence of $\pi$ is

$$\mathcal{A}(\pi) = 17\ 16/15\ 14\ 12\ 10\ 9\ 7/6\ 5\ 3\ 1,$$

while its $\mathcal{B}$-sequence is

$$\mathcal{B}(\pi) = 21\ 20\ 19\ 18/15\ 14\ 12\ 10\ 9\ 7/4\ 2.$$

Observe that the intersection of $\mathcal{A}(\pi)$ and $\mathcal{B}(\pi)$ (delimited by '/') consists of the letters 15 14 12 10 9 7, and that they are consecutive both in $\mathcal{A}(\pi)$ and $\mathcal{B}(\pi)$. Also, $\mathcal{B}$ contains more letters than $\mathcal{A}$ before this intersection, while $\mathcal{A}$ contains more letters than $\mathcal{B}$ after the intersection. We prove that this is always true in Section 4.1 below (Propositions 21 and 22).
2. Let us now apply the $\mathcal{B}$-shift to $\pi$. One finds:

$$\psi(\pi) = 17\ 20\ 19\ 16\ 18\ 15\ 13\ 14\ 11\ 12\ 10\ 8\ 9\ 7\ 4\ 2\ 21\ 6\ 5\ 3\ 1.$$

The new $\mathcal{A}$-sequence is now $\mathcal{A}(\psi(\pi)) = 17\ 16/15\ 13\ 11\ 10\ 8\ 7/6\ 5\ 3\ 1$. Observe that all the letters of $\mathcal{A}(\pi)$ that were before or after the intersection with $\mathcal{B}(\pi)$ are still in the new $\mathcal{A}$-sequence, as well as the first letter of the intersection. We prove that this is always true in Section 4.2 (Propositions 28 and 29). In this example, the last letter of the intersection is still in the new $\mathcal{A}$-sequence, but this is not true in general.
By symmetry with respect to the main diagonal, after the $\mathcal{A}$-shift, the letters of $\mathcal{B}$ that were before or after the intersection are in the new $\mathcal{B}$-sequence, as well as the first letter of $\mathcal{A}$ following the intersection (Corollary 30). This can be checked on our example:

$$\phi(\pi) = 16\ 21\ 20\ 15\ 19\ 18\ 13\ 14\ 11\ 12\ 10\ 8\ 9\ 7\ 6\ 4\ 2\ 5\ 3\ 1\ 17,$$

and the new $\mathcal{B}$-sequence is $\mathcal{B}(\phi(\pi)) = 21\ 20\ 19\ 18/13\ 11\ 10\ 8\ 7\ 6\ /4\ 2$.
3. Let $a_i = b_j$ denote the first (leftmost) point in $\mathcal{A}(\pi) \cap \mathcal{B}(\pi)$, and let $a_d = b_e$ be the last point in this intersection. We have seen that after the $\mathcal{B}$-shift, the new $\mathcal{A}$-sequence begins with $a_k \dots a_i = 17\ 16\ 15$, and ends with $a_{d-1} \dots a_1 = 6\ 5\ 3\ 1$. The letters in the center of the new $\mathcal{A}$-sequence, that is, the letters replacing $a_{i-1} \dots a_d$, are $x_{i-1} \dots x_d = 13\ 11\ 10\ 8\ 7$. Similarly, after the $\mathcal{A}$-shift, the new $\mathcal{B}$-sequence begins with $b_k \dots b_{j+1} = 21\ 20\ 19\ 18$, and ends with $a_{d-1} b_{e-1} \dots b_1 = 6\ 4\ 2$. The central letters are again $x_{i-1} \dots x_d = 13\ 11\ 10\ 8\ 7$! (See Figure 4). This is not a coincidence; we prove in Section 4.3 that this always holds (Proposition 31).
4. We finally combine all these properties to describe explicitly how the maps $\phi \circ \psi$ and $\psi \circ \phi$ act on a permutation $\pi$, and conclude that they yield the same permutation if the $\mathcal{A}$- and $\mathcal{B}$-sequences of $\pi$ do not coincide (Theorem 32).

### 4.1. *The $\mathcal{A}$-sequence and the $\mathcal{B}$-sequence*

**Definition 12** (Labels) Let $\pi = \pi_1\pi_2 \ldots \pi_n$ be a permutation. For $1 \leq i \leq n$, let $\ell_i$ be the maximal length of a decreasing subsequence in $\pi$ that starts at $\pi_i$. The *length sequence*, or *$\ell$-sequence*, of $\pi$ is $\ell(\pi) = \ell_1\ell_2 \ldots \ell_n$. Alternatively, it can be defined recursively as follows: $\ell_n = 1$ and, for $i < n$,

$$\ell_i = \max\{\ell_m\} + 1, \tag{1}$$

where the maximum is taken over all $m > i$ such that $\pi_m < \pi_i$.

We refer to the entries of the $\ell$-sequence as *labels* and say that the label $\ell_i$ is *associated to* the letter $\pi_i$ in $\pi$. Also, if $x = \pi_i$ then, abusing notation, we let $\ell(x) = \ell_i$.

Given a subsequence $s = \pi_{i_1}\pi_{i_2} \ldots \pi_{i_k}$ of $\pi$, we say that $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_k}$ is the subsequence of $\ell(\pi)$ *associated to s*.

Here is an example, where we have written the label of $\pi_i$ below $\pi_i$ for each $i$:

$$\begin{array}{l} \pi = 3\ 7\ 4\ 9\ 1\ 8\ 5\ 6\ 2 \\ \phantom{\pi = }\ 2\ 3\ 2\ 4\ 1\ 3\ 2\ 2\ 1 \end{array}$$

The subsequence of $\ell(\pi)$ associated to 741 is 3, 2, 1.

**Lemma 13** *The subsequence of $\ell(\pi)$ associated to a decreasing subsequence $x_m \ldots x_2x_1$ in $\pi$ is strictly decreasing. In particular, $\ell(x_i) \geq i$ for all i.*

**Proof:** Obvious, by definition of the labels. □

**Lemma 14** *Let $x_1, \ldots, x_i$ be, from left to right, the list of letters in $\pi$ that have label m. Then $x_1 < x_2 < \cdots < x_i$.*

**Proof:** If $x_j > x_{j+1}$ then, since $x_j$ precedes $x_{j+1}$, we would have $\ell(x_j) > \ell(x_{j+1})$, contrary to assumption. □

**Definition 15** (Successor sequence) Let $x$ be a letter in $\pi$, with $\ell(x) = m$. The *successor sequence $s_ms_{m-1} \ldots s_1$ of x* is the sequence of letters of $\pi$ such that $s_m = x$ and, for $i \leq m$, $s_{i-1}$ is the first (leftmost) letter after $s_i$ such that $\ell(s_{i-1}) = \ell(s_i) - 1$. In this case, we say that $s_{m-1}$ is the *label successor of x*.

**Lemma 16** *The successor sequence of x is a decreasing sequence.*

**Proof:** By definition of the labels, one of the letters labelled $\ell(x) - 1$ that are to the right of $x$ is smaller than $x$. By Lemma 14, the leftmost of them, that is, the label successor of $x$, is smaller than $x$. □

Let us now rephrase, in terms of permutations, the definitions of the $\mathcal{A}$-sequence and $\mathcal{B}$-sequence (Definition 5 and Lemma 9).

Given a permutation $\pi$ that contains a decreasing subsequence of length $k$, the $\mathcal{A}$-sequence of $\pi$ is the sequence $\mathcal{A}(\pi) = a_k a_{k-1} \dots a_1$, where for all $i$, $a_i$ is the *smallest* letter in $\pi$ such that $a_k \dots a_{i+1} a_i$ is the *prefix* of a decreasing sequence of $\pi$ of length $k$. The $\mathcal{B}$-sequence of $\pi$ is $\mathcal{B}(\pi) = b_k b_{k-1} \dots b_1$, where for all $i$, $b_i$ is the *leftmost* letter such that $b_i b_{i-1} \dots b_1$ is the *suffix* of a decreasing sequence of length $k$. According to the remark at the end of Section 2, the letter $a_i$ can alternatively be chosen as *left* as possible (for $i < k$), and the letter $b_i$ as *small* as possible (for $i > 1$).

The three simple lemmas above, as well as Lemma 17 below, will be used frequently, but without specific mention, in the remainder of this section. From now on we denote by $\mathcal{A} = a_k \dots a_1$ and $\mathcal{B} = b_k \dots b_1$ the $\mathcal{A}$- and $\mathcal{B}$-sequences of $\pi$. Strictly speaking, $\mathcal{A}$ should be denoted $\mathcal{A}(\pi)$ (and likewise for $\mathcal{B}$), and occasionally we will use this notation, when ambiguity could otherwise arise. The next lemma characterizes the $\mathcal{A}$-sequence in terms of labels.

**Lemma 17** *The letter $a_k$ is the leftmost letter in $\pi$ with label $k$ and, for $i < k$, $a_i$ is the first letter after $a_{i+1}$ that has label $i$. In particular, the $\mathcal{A}$-sequence of $\pi$ is the successor sequence of $a_k$, and the subsequence of $\ell(\pi)$ associated to $\mathcal{A}(\pi)$ is $k, k-1, \dots, 1$.*

**Proof:** Clearly, the label of $a_k$ must be at least $k$. If it is larger than $k$, then the label successor of $a_k$ is smaller than $a_k$ and is the first letter of a decreasing sequence of length $k$, a contradiction. Hence the label of $a_k$ must be exactly $k$. Now given that $a_k$ has to be as small as possible, Lemma 14 implies that $a_k$ is the leftmost letter having label $k$.

We then proceed by decreasing induction on $i$. Since $a_i$ is smaller than, and to the right of, $a_{i+1}$, its label must be at most $i$. Since $a_k \dots a_i$ is the prefix of a decreasing sequence of length $k$, the label of $a_i$ must be at least $i$, and hence, exactly $i$. Since we want $a_i$ to be as small as possible, it has to be the first letter after $a_{i+1}$ with label $i$ (Lemma 14). $\square$

The following lemma, and some variations of its argument, will be used several times in what follows.

**Lemma 18** *Let $i \le j$. Suppose $\pi$ contains a decreasing sequence of the form $b_{j+1} x_j \dots x_i$ such that $x_i$ precedes $b_i$. Then $\ell(x_i) < \ell(b_i)$.*

**Proof:** First, observe that, by definition of $\mathcal{B}$, one actually has $x_i < b_i$ (otherwise, the $\mathcal{B}$-sequence could be extended). Suppose that $\ell(x_i) \ge \ell(b_i)$. In particular, then, $\ell(x_i) \ge i$. Let us write $\ell(x_i) = i + r$, with $r \ge 0$. Let $x_i x_{i-1} \dots x_{i-r+1}$ be the successor sequence of $x_i$, so that $\ell(x_p) = p + r$ for all $p \le i$. Now, $\ell(x_{i-m}) \ge \ell(b_{i-m})$ for all $m \in [0, i-1]$, because the labels of the $\mathcal{B}$-sequence are strictly decreasing. Thus, if $x_{i-m} < b_{i-m}$ then $x_{i-m}$ must precede $b_{i-m}$, for otherwise $\ell(b_{i-m}) > \ell(x_{i-m})$.

Recall that $x_i < b_i$. Let $m$ be the largest integer with $m < i$ such that $x_{i-m} < b_{i-m}$, which implies that $x_{i-m}$ precedes $b_{i-m}$ (clearly, $m \ge 0$). If $m = i-1$ then $x_1 < b_1$, so $x_1$ precedes $b_1$, and thus the sequence

$$b_k \dots b_{j+1} x_j x_{j-1} \dots x_1$$

is decreasing, has length $k$ and ends to the left of $b_1$, which contradicts the definition of the $\mathcal{B}$-sequence. Thus $m < i-1$. Now, $x_{i-m} < b_{i-m}$ and $x_{i-m}$ precedes $b_{i-m}$, but $x_{i-m-1} > b_{i-m-1}$. Note that $x_{i-m}$ precedes $b_{i-m-1}$ since it precedes $b_{i-m}$. Thus, the sequence

$$b_k \dots b_{j+1} x_j x_{j-1} \dots x_{i-m} b_{i-m-1} \dots b_1$$

is decreasing and has length $k$. Since $x_{i-m}$ precedes $b_{i-m}$, the definition of $\mathcal{B}$ implies that $x_{i-m}$ would have been chosen instead of $b_{i-m}$ in $\mathcal{B}$. This is a contradiction, so $\ell(x_i) < \ell(b_i)$. □

**Lemma 19** *Assume the label successor of $b_m$ does not belong to $\mathcal{B}$. Then no letter of the successor sequence of $b_m$ belongs to $\mathcal{B}$, apart from $b_m$ itself.*

**Proof:** Let $x$ be the label successor of $b_m$, and let $x_r \dots x_1$ be the successor sequence of $x$, with $x_r = x$. Assume one of the $x_i$ belongs to $\mathcal{B}$, and let $x_{s-1} = b_j$ be the leftmost of these. The successor sequence of $b_m$ thus reads $b_m x_r \dots x_s b_j x_{s-2} \dots x_1$. By assumption, $s \leq r$.

We want to prove that the sequence $x_r \dots x_s$ is longer than $b_{m-1} \dots b_{j+1}$, which will contradict the definition of the $\mathcal{B}$-sequence of $\pi$. We have $\ell(x_r) + 1 = \ell(b_m) > \ell(b_{m-1})$, so that $\ell(x_r) \geq \ell(b_{m-1})$. Hence by Lemma 18, $b_{m-1}$ precedes $x_r$. This implies that

$$\ell(x_r) > \ell(b_{m-1}),$$

for otherwise the label successor of $b_m$ would be $b_{m-1}$ instead of $x_r$. At the other end of the sequence $x_r \dots x_s$, we naturally have

$$\ell(x_s) = 1 + \ell(b_j) \leq \ell(b_{j+1}).$$

Given that the $x_i$ form a successor sequence, while the labels of $\mathcal{B}$ are strictly decreasing, the above two inequalities imply that $x_r \dots x_s$ is longer than $b_{m-1} \dots b_{j+1}$, as desired. □

**Lemma 20** *Assume that $a_d = b_e$ with $e > 1$, and that $b_{e-1}$ does not belong to $\mathcal{A}$. Then $d > 1$ and $b_{e-1}$ precedes $a_{d-1}$. Moreover, $b_{e-1} < a_{d-1}$ and $\ell(b_{e-1}) < \ell(a_{d-1}) = d - 1$.*

*By symmetry, if $a_i = b_j$ with $i < k$ and $a_{i+1}$ does not belong to $\mathcal{B}$, then $j < k$ and $a_{i+1}$ precedes $b_{j+1}$. Moreover, $a_{i+1} < b_{j+1}$.*

**Proof:** If $d = 1$, then $a_k \dots a_1$ is a decreasing sequence of length $k$ that ends to the left of $b_1$ and this contradicts the definition of $\mathcal{B}$. Hence $d > 1$.

We have $a_{d-1} < b_e$ and $\ell(a_{d-1}) \geq \ell(b_{e-1})$. By Lemma 18, this implies that $b_{e-1}$ precedes $a_{d-1}$.

By the definition of $\mathcal{A}$, we have $b_{e-1} < a_{d-1}$, for otherwise $b_{e-1}$ could be inserted in $\mathcal{A}$.

Finally, if $\ell(b_{e-1}) = \ell(a_{d-1})$ then $b_{e-1}$ would be the next letter after $a_d$ in the $\mathcal{A}$-sequence, since $b_{e-1}$ precedes $a_{d-1}$. □

**Proposition 21** *If $b_e \in \mathcal{A}$ and $b_{e-1} \notin \mathcal{A}$ then $b_m \notin \mathcal{A}$ for all $m < e$. Consequently, the intersection of $\mathcal{A}$ and $\mathcal{B}$ is a (contiguous) segment of each sequence.*

**Proof:** Suppose not, so there is an $m < e - 1$ with $b_m \in \mathcal{A}$. Let $d$ and $p$ be such that $a_d = b_e$ and $a_p = b_m$. By Lemma 20, $\ell(b_{e-1}) < \ell(a_{d-1})$, so there are more letters in the $\mathcal{A}$-sequence than in the $\mathcal{B}$-sequence between $b_e$ and $b_m$. But then the sequence

$$b_k \dots b_e a_{d-1} \dots a_p b_{m-1} \dots b_2$$

has length at least $k$, which contradicts the definition of the $\mathcal{B}$-sequence. Hence the intersection of $\mathcal{A}$ and $\mathcal{B}$ is formed of consecutive letters of $\mathcal{B}$. By symmetry, it also consists of consecutive letters of $\mathcal{A}$. □

The preceding proposition will be used implicitly in the remainder of this section.

**Proposition 22** *If $\mathcal{A}$ and $\mathcal{B}$ intersect but do not coincide, then the $\mathcal{A}$-sequence contains more letters than the $\mathcal{B}$-sequence after the intersection and the $\mathcal{B}$-sequence contains more letters than the $\mathcal{A}$-sequence before the intersection. In particular, if $a_1$ belongs to $\mathcal{B}$ or $b_k$ belongs to $\mathcal{A}$, then the $\mathcal{A}$-sequence and the $\mathcal{B}$-sequence coincide.*

**Proof:** Let $b_e = a_d$ be the last letter of the intersection. The $\mathcal{A}$-sequence has exactly $d - 1$ letters after the intersection. Let us first prove that $d \geq 2$. Assume $d = 1$. Then by Lemma 20, $e = 1$. Let $a_i = b_i$ be the largest element of $\mathcal{A} \cap \mathcal{B}$. By assumption, $i < k$. By Lemma 20, $a_{i+1}$ precedes $b_{i+1}$ and is smaller. This contradicts the definition of $\mathcal{B}$. Hence $d > 1$.

If the $\mathcal{B}$-sequence contains any letters after the intersection, then $\ell(b_{e-1}) < d - 1$, according to Lemma 20, so the $\mathcal{B}$-sequence can contain at most $d - 2$ letters after the intersection. This proves the first statement. The second one follows by symmetry (or by subtraction). □

**Lemma 23** *Assume $\ell(b_k) = k$. Then the $\mathcal{A}$-sequence and $\mathcal{B}$-sequence coincide.*

**Proof:** Assume the two sequences do not coincide. If they intersect, their last common point being $b_e = a_d$, then Proposition 22 shows that the sequence

$$b_k \dots b_e a_{d-1} \dots a_1$$

is decreasing and has length $> k$. This implies that $\ell(b_k) > k$, a contradiction.

Let us now assume that the two sequences do not intersect. By definition of the $\mathcal{A}$-sequence, $a_k < b_k$. If $b_k$ precedes $a_k$, then $\ell(b_k) > \ell(a_k) = k$, another contradiction. Thus $a_k$ precedes $b_k$. Let us prove by decreasing induction on $h$ that $a_h$ precedes $b_h$ for all $h$. If this is true for some $h \in [2, k]$, then $a_h$ is in the $\mathcal{A}$-sequence, $a_{h-1}$ and $b_{h-1}$ lie to its right and have the same label. Since $a_{h-1}$ is chosen in the $\mathcal{A}$-sequence, it must be left of $b_{h-1}$. By induction, we conclude that $a_1$ precedes $b_1$, which contradicts the definition of the $\mathcal{B}$-sequence. □

### 4.2. *The $\mathcal{A}$-sequence after the $\mathcal{B}$-shift*

We still denote by $\mathcal{A} = a_k \dots a_1$ and $\mathcal{B} = b_k \dots b_1$ the $\mathcal{A}$- and $\mathcal{B}$-sequences of a permutation $\pi$. Recall that the $\mathcal{B}$-shift performs a cyclic shift of the elements of the $\mathcal{B}$-sequence, and is denoted $\psi$. We begin with a sequence of lemmas that tell us how the labels evolve during the $\mathcal{B}$-shift.

**Lemma 24** (*The order of $\mathcal{A}$*) *Assume $\mathcal{A}$ and $\mathcal{B}$ do not coincide. In $\psi(\pi)$, the letters $a_k, \dots, a_2, a_1$ appear in this order. In particular, $\ell(a_i) \geq i$ in $\psi(\pi)$, and $\psi(\pi)$ contains the pattern $k \dots 21$.*

**Proof:** The statement is obvious if $\mathcal{A}$ and $\mathcal{B}$ do not intersect. Otherwise, let $a_i = b_j$ be the first (leftmost) letter of $\mathcal{A} \cap \mathcal{B}$ and let $a_d = b_e$ be the last letter of $\mathcal{A} \cap \mathcal{B}$. By Proposition 22, $j < k$. Hence when we do the $\mathcal{B}$-shift, the letters $a_i, \dots, a_d$ move to the left, while the other letters of $\mathcal{A}$ do not move. Moreover, the letter $a_i$ will replace $b_{j+1}$, which, by Lemma 20, is to the right of $a_{i+1}$. Hence the letters $a_k, \dots, a_2, a_1$ appear in this order after the $\mathcal{B}$-shift. □

**Lemma 25** *Let $x \leq b_k$. Then the label of $x$ cannot be larger in $\psi(\pi)$ than in $\pi$.*

**Proof:** We proceed by induction on $x \in \{1, \dots, b_k\}$, and use the definition (1) (in Definition 12) of the labels. The result is obvious for $x = 1$. Take now $x \geq 2$, and assume the labels of $1, \dots, x-1$ have not increased. If $x \notin \mathcal{B}$, all the letters that are smaller than $x$ and to the right of $x$ in $\psi(\pi)$ were already to the right of $x$ in $\pi$, and have not had a label increase by the induction hypothesis. Thus the label of $x$ cannot have increased. The same argument applies if $x = b_k$.

Assume now that $x = b_m$, with $m < k$. Then $b_m$ has moved to the place of $b_{m+1}$ during the $\mathcal{B}$-shift. The letters that are smaller than $b_m$ and were already to the right of $x$ in $\pi$ have not had a label increase. Thus they cannot entail a label increase for $b_m$. The letters that are smaller than $b_m$ and lie between $b_{m+1}$ and $b_m$ in $\pi$ have label at most $\ell(b_m) - 1$ in $\pi$ (Lemma 18), and hence in $\psi(\pi)$, by the induction hypothesis. Thus they cannot entail a label increase for $b_m$ either. □

Note that the label of letters larger than $b_k$ may increase, as shown by the following example, where $k = 3$:

$$\begin{array}{ll} \pi = 3\ \overline{7}\ \overline{4}\ \mathbf{8}\ \overline{1}\ 5\ 6\ 2 \rightarrow & \psi(\pi) = 3\ 4\ 1\ \mathbf{8}\ 7\ 5\ 6\ 2 \\ \phantom{\pi =}\ 2\ 3\ 2\ \mathbf{3}\ 1\ 2\ 2\ 1 & \phantom{\psi(\pi) =}\ 2\ 2\ 1\ \mathbf{4}\ 3\ 2\ 2\ 1. \end{array}$$

**Lemma 26** (*The labels of $\mathcal{A}$*) *Assume $\mathcal{A}$ and $\mathcal{B}$ do not coincide. The labels associated to the letters $a_k, \dots, a_1$ do not change during the $\mathcal{B}$-shift.*

**Proof:** By Lemma 24, the label of $a_i$ cannot decrease and by Lemma 25, it cannot increase either. □

**Lemma 27** (*The labels of* $\mathcal{B}$) *Let* $m < k$. *The label associated to* $b_m$ *does not change during the* $\mathcal{B}$*-shift* (*although* $b_m$ *moves left*).

**Proof:** By Lemma 25, the label of $b_m$ cannot increase. Assume that it decreases, and that $m$ is minimal for this property. Let $x$ be the label successor of $b_m$ in $\pi$. Then $x$ is still to the right of $b_m$ in $\psi(\pi)$, and this implies that its label has decreased too. By the choice of $m$, the letter $x$ does not belong to $\mathcal{B}$. Let $x_r \dots x_1$ be the successor sequence of $x$ in $\pi$, with $x_r = x$. By Lemma 19, none of the $x_i$ are in $\mathcal{B}$. Consequently, the order of the $x_i$ is not changed during the shift, so the label of $x$ cannot have decreased, a contradiction. Thus the label of $b_m$ cannot decrease. □

**Proposition 28** (*The prefix of* $\mathcal{A}$) *Assume* $\mathcal{A}$ *and* $\mathcal{B}$ *do not coincide. Assume* $a_k, \dots, a_{i+1}$ *do not belong to* $\mathcal{B}$, *with* $0 \le i \le k$. *The* $\mathcal{A}$*-sequence of* $\psi(\pi)$ *begins with* $a_k \dots a_{i+1}$ *and even with* $a_k \dots a_i$ *if* $i > 0$.

**Proof:** We first show that $a_k$ is the first letter of $\mathcal{A}(\psi(\pi))$. Suppose not. Let $x$ be the first letter of the new $\mathcal{A}$-sequence. Then $x$ has label $k$ in $\psi(\pi)$ and is smaller than $a_k$, since $a_k$ still has label $k$ in $\psi(\pi)$, by Lemma 26. Since $x$ was already smaller than $a_k$ in $\pi$, it means that the label of $x$ has changed during the $\mathcal{B}$-shift (otherwise it would have been the starting point of the original $\mathcal{A}$-sequence). By Lemma 25, the label of $x$ has actually decreased. In other words, the label of $x$ is larger than $k$ in $\pi$.

But then the successor sequence of $x$ in $\pi$ must contain a letter with label $k$, and this letter is smaller than $x$ and hence smaller than $a_k$, which contradicts the choice of $a_k$. Thus the first letter of the new $\mathcal{A}$-sequence is $a_k$.

We now prove that no letter can be the first (leftmost) letter that replaces one of the letters $a_{k-1}, \dots, a_i$ in the new $\mathcal{A}$-sequence. Assume that the $\mathcal{A}$-sequence of $\psi(\pi)$ starts with $a_k \dots a_{p+1}x$, with $i \le p < k$ and $x \ne a_p$. Then $x$ has label $p$ in $\psi(\pi)$, and $a_p$ has label $p$ as well (Lemma 26). Since $x$ is chosen in $\mathcal{A}(\psi(\pi))$ instead of $a_p$, this means that $a_k, \dots, a_{p+1}, x, a_p$ come in this order in $\psi(\pi)$, and that $x < a_p$. Let us prove that the letters $a_k, \dots, a_{p+1}, x, a_p$ also come in this order in $\pi$. Since $a_k, \dots, a_{p+1}$ do not belong to $\mathcal{B}$, they cannot have moved during the shift, so it is clear that $x$ follows $a_{p+1}$ in $\pi$. Moreover, $x$ must precede $a_p$ in $\pi$, otherwise we would have $p = \ell(a_p) > \ell(x)$ in $\pi$, contradicting Lemma 25.

Thus $a_k, \dots, a_{p+1}, x, a_p$ come in this order in $\pi$, and Lemma 25 implies that $\ell(x) \ge p$ in $\pi$. By definition of the $\mathcal{A}$-sequence, $\ell(x)$ cannot be equal to $p$. Hence $\ell(x) > p$, which forces $\ell(a_{p+1}) > p + 1$, a contradiction.

Since no letter can be the *first* letter replacing one of $a_{k-1}, \dots, a_{i+1}, a_i$ in the new $\mathcal{A}$-sequence, these letters form the prefix of the new $\mathcal{A}$-sequence. □

The example presented at the beginning of this section shows that the next letter of the $\mathcal{A}$-sequence, namely $a_{i-1}$, may not belong to the $\mathcal{A}$-sequence after the $\mathcal{B}$-shift.

**Proposition 29** (*The suffix of* $\mathcal{A}$) *Assume* $\mathcal{A}$ *and* $\mathcal{B}$ *intersect but do not coincide. Let* $a_{d-1}$ *be the first letter of* $\mathcal{A}$ *after* $\mathcal{A} \cap \mathcal{B}$. *After the* $\mathcal{B}$*-shift, the* $\mathcal{A}$*-sequence ends with* $a_{d-1} \dots a_1$.

**Proof:** Observe that the existence of $a_{d-1}$ follows from Proposition 22.

Most of the proof will be devoted to proving that $a_{d-1}$ still belongs to the $\mathcal{A}$-sequence after the $\mathcal{B}$-shift. Suppose not. Let $a_m = b_p$ be the rightmost letter of $\mathcal{A} \cap \mathcal{B}$ that still belongs to the $\mathcal{A}$-sequence after the $\mathcal{B}$-shift (such a letter does exist, by Proposition 28). Let $a_d = b_e$ be the rightmost letter of $\mathcal{A} \cap \mathcal{B}$. (Note that $d - e = m - p$.) The $\mathcal{A}$-sequence of $\psi(\pi)$ ends with $a_m x_{m-1} \dots x_d x_{d-1} y_{d-2} \dots y_1$, with $\ell(x_j) = j$, and $x_j \neq a_j$ for $m-1 \geq j \geq d-1$. Let us prove that none of the $x_j$ were in the original $\mathcal{B}$-sequence. If $x_j$ were in the original $\mathcal{B}$-sequence, its label in $\pi$ would have been $j$ (Lemma 27). But for $m-1 \geq j \geq d$, the only letter of $\mathcal{B}(\pi)$ having label $j$ is $a_j$, and by Lemma 20, no letter in $\mathcal{B}(\pi)$ has label $d-1$. Thus the $x_j$ cannot have been in $\mathcal{B}(\pi)$. This guarantees that they have not moved during the $\mathcal{B}$-shift. Moreover, since they are smaller than $b_k$, their labels cannot have increased during the shift (Lemma 25).

Let us prove that for $m-1 \geq h \geq d-1$, the letter $x_h$ precedes $a_h$ in $\psi(\pi)$. We proceed by decreasing induction on $h$. First, $a_m$ belongs to $\mathcal{A}(\psi(\pi))$ by assumption, the letters $x_{m-1}$ and $a_{m-1}$ are to its right and have the same label, and $x_{m-1}$ is chosen in the $\mathcal{A}$-sequence of $\psi(\pi)$, which implies that it precedes $a_{m-1}$. Now assume that $x_h$ precedes $a_h$ in $\psi(\pi)$, with $m-1 \geq h \geq d$. The letter $x_h$ belongs to $\mathcal{A}(\psi(\pi))$, the letters $x_{h-1}$ and $a_{h-1}$ are on its right and have the same label, and $x_{h-1}$ is chosen in the new $\mathcal{A}$-sequence, which implies that it precedes $a_{h-1}$. Finally, $x_{d-1}$ precedes $a_{d-1}$ in $\psi(\pi)$, and is smaller than it.

Let us focus on $x_{d-1}$. Assume first that it is to the right of $a_d$ in $\pi$. Since $\ell(x_{d-1}) \geq d-1$ in $\pi$, there is a letter $y$ in the successor sequence of $x_{d-1}$ that has label $d-1$ and is smaller than $a_{d-1}$, which contradicts the choice of $a_{d-1}$ in the original $\mathcal{A}$-sequence.

Thus $x_{d-1}$ is to the left of $a_d$ in $\pi$, and hence to the left of $b_{e-1}$. The sequence $b_{p+1} x_{m-1} \dots x_{d-1}$ is a decreasing sequence of $\pi$ of the same length as $b_{p+1} b_p \dots b_e$, and $x_{d-1}$ precedes $b_{e-1}$. By Lemma 18, this implies that $\ell(x_{d-1}) < \ell(b_{e-1})$. But $\ell(x_{d-1}) \geq d-1$, so that $\ell(b_{e-1}) \geq d = \ell(b_e)$, which is impossible.

We have established that $a_{d-1}$ belongs to the $\mathcal{A}$-sequence after the $\mathcal{B}$-shift. Assume now that $a_{d-1}, a_{d-2}, \dots, a_h$ all belong to the new $\mathcal{A}$-sequence, but not $a_{h-1}$, which is replaced by a letter $x_{h-1}$. This implies that $x_{h-1} < a_{h-1}$. By Lemma 25, the label of $x_{h-1}$ was at least $h-1$ in $\pi$. Also, $x_{h-1}$ was to the right of $a_h$ in $\pi$. Thus in the successor sequence of $x_{h-1}$ in $\pi$, there was a letter $y$, at most equal to $x_{h-1}$, that had label $h-1$ and was smaller than $a_{h-1}$, which contradicts the choice of $a_{h-1}$ in the original $\mathcal{A}$-sequence. ☐

### *4.3. The composition of $\phi$ and $\psi$*

We have seen that the beginning and the end of the $\mathcal{A}$-sequence are preserved after the $\mathcal{B}$-shift. By symmetry, we obtain a similar result for the $\mathcal{B}$-sequence after the $\mathcal{A}$-shift.

**Corollary 30** *Assume $\mathcal{A}$ and $\mathcal{B}$ intersect but do not coincide. Let $a_i = b_j$ be the leftmost element of $\mathcal{A} \cap \mathcal{B}$ and let $a_d = b_e$ be the rightmost element of $\mathcal{A} \cap \mathcal{B}$. After the $\mathcal{A}$-shift, the $\mathcal{B}$-sequence begins with $b_k \dots b_{j+1}$ and ends with $a_{d-1} b_{e-1} \dots b_1$.*

**Proof:** This follows from Propositions 28 and 29, together with symmetry. Namely, since by Proposition 28 the first (largest) letter of the intersection still belongs to the $\mathcal{A}$-sequence after the $\mathcal{B}$-shift, the *place* of the last (smallest) letter of the intersection still belongs to the $\mathcal{B}$-sequence after the $\mathcal{A}$-shift. After the $\mathcal{A}$-shift, the letter in this place is $a_{d-1}$. The rest of the claim follows directly from symmetry, together with the propositions mentioned. ☐

It remains to describe how the intersection of the $\mathcal{A}$- and $\mathcal{B}$-sequences is affected by the two respective shifts. In short, we will show that the intersection is affected in the same way, regardless of the order in which we perform the two shifts.

**Proposition 31** (*The intersection of $\mathcal{A}$ and $\mathcal{B}$*) *Assume $\mathcal{A}$ and $\mathcal{B}$ intersect but do not coincide. Let $a_i = b_j$ be the leftmost element of $\mathcal{A} \cap \mathcal{B}$ and let $a_d = b_e$ be the rightmost element of $\mathcal{A} \cap \mathcal{B}$. Let $a_k \dots a_i x_{i-1} \dots x_d a_{d-1} \dots a_1$ be the $\mathcal{A}$-sequence of $\psi(\pi)$. Let $b_k \dots b_{j+1} y_{i-1} \dots y_d a_{d-1} b_{e-1} \dots b_1$ be the $\mathcal{B}$-sequence of $\phi(\pi)$. Then $x_m = y_m$ for all $m$. Moreover, $x_m$ lies at the same position in $\psi(\pi)$ and $\phi(\pi)$.*

**Proof:** First, note that the above form of the two sequences follows from Propositions 28, 29 and Corollary 30. Note also that if $i = d$, that is, the intersection is reduced to a single point, then there is nothing to prove.

Our first objective is to prove that the sequences $\mathcal{X} = x_{i-1} \dots x_d$ and $\mathcal{Y} = y_{i-1} \dots y_d$ are the $\mathcal{A}$- and $\mathcal{B}$-sequences of length $i - d$ of the same word (the generalization of the notion of $\mathcal{A}$- and $\mathcal{B}$-sequences to words with distinct letters is straightforward).

By definition of the $\mathcal{A}$-sequence of $\psi(\pi)$, $\mathcal{X}$ is the smallest sequence of length $i - d$ (for the lexicographic order) that lies between $a_i$ and $a_{d-1}$ in $\psi(\pi)$. By this, we mean that it lies between $a_i$ and $a_{d-1}$ both in position and in value.

Let $p_m$ denote the position of $b_m$ in $\pi$. Let us show that $\mathcal{X}$ actually lies between the positions $p_{j+1}$ and $p_e$ (Figure 5). The first statement is clear, since $p_{j+1}$ is the position of $a_i$ in $\psi(\pi)$. In order to prove that $x_d$ is to the left of $p_e$ in $\psi(\pi)$, we proceed as at the beginning of the proof of Proposition 29. We may assume $x_d \neq a_d$ (otherwise, $x_d$ is definitely to the left of $p_e$). Let $a_m$ be the rightmost letter of $\mathcal{A} \cap \mathcal{B}$ that belongs to the $\mathcal{A}$-sequence after the $\mathcal{B}$-shift (such a letter does exist, and $d < m \leq i$). The $\mathcal{A}$-sequence of $\psi(\pi)$ ends with $a_m x_{m-1} \dots x_d a_{d-1} \dots a_1$, with $\ell(x_j) = j$ for all $j$, and $x_j \neq a_j$ for $m - 1 \geq j \geq d$.

Let us prove, by a decreasing induction on $j \in [d, m - 1]$, that the letter $x_j$ precedes $a_j$ for all $j$. First, $a_m$ belongs to $\mathcal{A}(\psi(\pi))$ by assumption, the letters $x_{m-1}$ and $a_{m-1}$ are to its right and have the same label, and $x_{m-1}$ is chosen in the new $\mathcal{A}$-sequence, which implies that it precedes $a_{m-1}$. Now assume that $x_h$ precedes $a_h$ in $\psi(\pi)$, with $m - 1 \geq h > d$. The letter $x_h$ belongs to $\mathcal{A}(\psi(\pi))$, the letters $x_{h-1}$ and $a_{h-1}$ are on its right and have the same label, and $x_{h-1}$ is chosen in the new $\mathcal{A}$-sequence, which implies that $x_{h-1}$ precedes $a_{h-1}$, and concludes our proof that $x_j$ precedes $a_j$. In particular, $x_d$ is to the left of $a_d$, and hence to the left of the position $p_e$.

We can summarize the first part of this proof by saying that $\mathcal{X}$ is the smallest sequence of length $i - d$ in $\psi(\pi)$ that lies in position between $p_{j+1}$ and $p_e$ and in value between $a_i$ and $a_{d-1}$. In other words, let $u$ be the word obtained by retaining in $\psi(\pi)$ only the letters that lie between $p_{j+1}$ and $p_e$ in position and between $a_i$ and $a_{d-1}$ in value. Then $\mathcal{X}$ is the $\mathcal{A}$-sequence of length $i - d$ of $u$.

By symmetry, $\mathcal{Y}$ is the $\mathcal{B}$-sequence of length $i - d$ of the word $v$ obtained by retaining in $\phi(\pi)$ the letters that lie between $p_{j+1}$ and $p_e$ in position and between $a_i$ and $a_{d-1}$ in value.

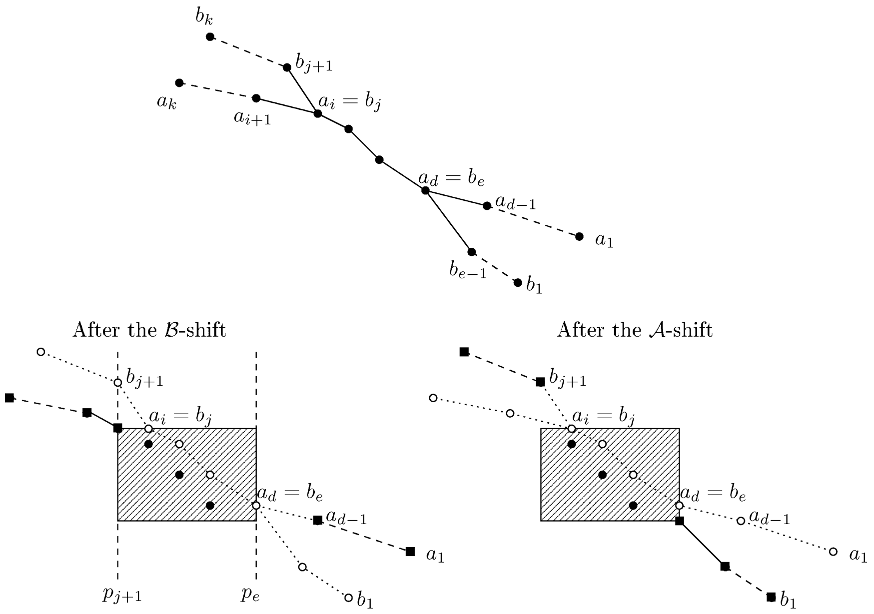



*Figure 5.* The $\mathcal{A}$- and $\mathcal{B}$-sequences in $\pi$ (top), and what happens to them after the $\mathcal{B}$-shift (left) and the $\mathcal{A}$-shift (right). Only the black discs and squares belong to the permutations. The squares show some letters of the new $\mathcal{A}$-sequence (left) or new $\mathcal{B}$-sequence (right). The interior of the shaded rectangle contains the letters of $u$.

But the words $u$ and $v$ actually coincide, for they contain

- the letters of $\pi$ that do not belong to $\mathcal{A}$ or $\mathcal{B}$ and lie between $p_{j+1}$ and $p_e$ in position and between $a_i$ and $a_{d-1}$ in value. These letters keep in $\psi(\pi)$ and $\phi(\pi)$ the position they had in $\pi$,
- the letters $b_{j-1}, \ldots, b_e$, placed at positions $p_j, \ldots, p_{e+1}$ (see Figure 5).

Observe also that $u$ does not contain any decreasing sequence of length larger than $i - d$, because otherwise, we could use this sequence to extend the $\mathcal{A}$-sequence of $\psi(\pi)$. Hence we have a word $u$ with distinct letters, with its $\mathcal{A}$- and $\mathcal{B}$-sequences (of length $i - d$) and we know that there is no longer decreasing sequence in $u$. In particular, the rightmost letter of its $\mathcal{B}$-sequence, $y_{i-1}$, has label $i - d$, and Lemma 23 implies that $\mathcal{X}$ and $\mathcal{Y}$ coincide. □

**Theorem 32** (*Local commutation for permutations*) *Let $\pi$ be a permutation for which the $\mathcal{A}$- and $\mathcal{B}$-sequences do not coincide. Then $\phi(\pi)$ and $\psi(\pi)$ still contain the pattern $k \ldots 21$, and $\phi(\psi(\pi)) = \psi(\phi(\pi))$.*

**Proof:** The first statement follows from Lemma 24, plus symmetry.

Assume first that $\mathcal{A}$ and $\mathcal{B}$ are disjoint. By Proposition 28, the $\mathcal{A}$-sequence is unchanged after the $\mathcal{B}$-shift. Thus the permutation $\phi(\psi(\pi))$ can be obtained by shifting $\mathcal{A}$ and $\mathcal{B}$ in $\pi$ in parallel. By symmetry, this is also the result of applying $\psi \circ \phi$ to $\pi$.

Let us now assume that $\mathcal{A}$ and $\mathcal{B}$ intersect. Following the notation of Proposition 31, let $a_k \dots a_i x_{i-1} \dots x_d a_{d-1} \dots a_1$ be the $\mathcal{A}$-sequence of $\psi(\pi)$, and let $b_k \dots b_{j+1} x_{i-1} \dots x_d a_{d-1} b_{e-1} \dots b_1$ be the $\mathcal{B}$-sequence of $\phi(\pi)$. Clearly, the only letters that can move when we apply $\phi \circ \psi$ (or $\psi \circ \phi$) to $\pi$, are those of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{X}$. We need to describe at which place each of them ends. We denote by $p(x)$ the position of the letter $x$ in $\pi$ (note that $p(x) = \pi^{-1}(x)$).

Let us begin with the transformation $\phi \circ \psi$. That is, the $\mathcal{B}$-shift is applied first. It is easy to see what happens to the letters that lie far away from the intersection of $\mathcal{A}$ and $\mathcal{B}$ (Figure 5). During the $\mathcal{B}$-shift, the letter $b_k$ is sent to $p(b_1)$ and then it does not move during the $\mathcal{A}$-shift (it is too big to belong to the new $\mathcal{A}$-sequence). Similarly, for $j + 1 \le h < k$, and for $1 \le h < e$, the letter $b_h$ is sent to $p(b_{h+1})$, and then does not move. As far as the $\mathcal{A}$-sequence is concerned, we see that $a_h$ does not move during the $\mathcal{B}$-shift, for $1 \le h \le d - 1$ and $i + 1 \le h \le k$. Then, during the $\mathcal{A}$-shift, $a_k$ is sent to $p(a_1)$, and the letter $a_h$ moves to $p(a_{h+1})$ for $1 \le h < d - 1$ and $i + 1 \le h < k$.

It remains to describe what happens to $a_{d-1}, \dots, a_i$, and to the $x_h$. The letter $a_i$ moves to $p(b_{j+1})$ first, and then, being an element of the new $\mathcal{A}$-sequence, it moves to $p(a_{i+1})$. The letter $a_{d-1}$ only moves during the $\mathcal{A}$-shift, and it moves to the position of $x_d$ in $\psi(\pi)$. For $d \le h < i - 1$, the letter $x_h$ moves to the position of $x_{h+1}$ in $\psi(\pi)$. The letter $x_{i-1}$ moves to the position of $a_i$ in $\psi(\pi)$, that is, to $p(b_{j+1})$. Finally, the letters $a_h$, with $d \le h < i$, which are not in $\mathcal{X}$ move only during the $\mathcal{B}$-shift and end up at $p(a_{h+1})$.

Let us put together our results: When we apply $\phi \circ \psi$,

- $x_{i-1}$ moves to $p(b_{j+1})$,
- $x_h$ moves to the position of $x_{h+1}$ in $\psi(\pi)$, for $d \le h < i - 1$,
- $a_k$ is sent to $p(a_1)$ and $b_k$ to $p(b_1)$,
- $a_{d-1}$ moves to the position of $x_d$ in $\psi(\pi)$,
- the remaining $a_h$ and $b_h$ move respectively to $p(a_{h+1})$ and $p(b_{h+1})$.

Now a similar examination, together with the fact that each $x_h$ lies in the same position in $\psi(\pi)$ and $\phi(\pi)$ (Proposition 31), shows that applying $\psi \circ \phi$ results exactly in the same moves. □

## 5. Local commutation: from permutations to rook placements

The aim of this section is to derive the local commutation for placements (Theorem 10) from the commutation theorem for permutations (Theorem 32). We begin with a few simple definitions and lemmas.

A *corner cell* $c$ of a Ferrers shape $\lambda$ is a cell such that $\lambda \backslash \{c\}$ is still a Ferrers shape. If $p$ is a placement on $\lambda$ containing $k \dots 21$, with $\mathcal{A}$-sequence $a_k \dots a_1$, then the *$\mathcal{A}$-rectangle* of $p$, denoted by $R_{\mathcal{A}}$, is the largest rectangle of $\lambda$ whose top row contains $a_k$. Symmetrically, the *$\mathcal{B}$-rectangle* of $p$, denoted by $R_{\mathcal{B}}$, is the largest rectangle of $\lambda$ whose rightmost column contains $b_1$ (where $b_k \dots b_1$ is the $\mathcal{B}$-sequence of $p$). By definition of the $\mathcal{A}$- and $\mathcal{B}$-sequences, $R_{\mathcal{B}}$ is at least as high, and at most as wide, as $R_{\mathcal{A}}$. See the leftmost placement of Figure 6 for an example.

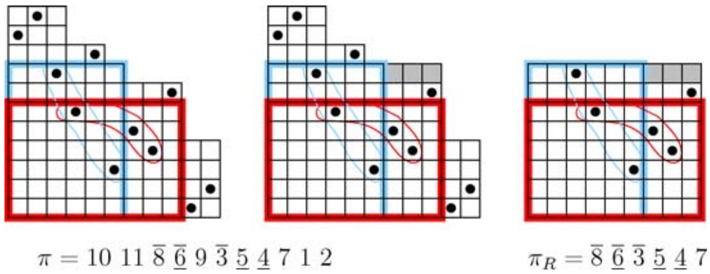



*Figure 6.* Left: A placement $p$, its $\mathcal{A}$- and $\mathcal{B}$-sequences (for $k = 3$), and the rectangles $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$. Center: The placement $p^R$. Right: The placement $p_R$ and the corresponding subsequence of $\pi$.

In the following lemmas, $p$ is supposed to be a placement on the board $\lambda$, containing the pattern $k \ldots 21$.

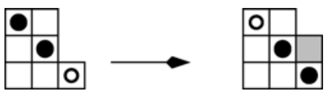
**Lemma 33** *Let $c$ be a corner cell of $\lambda$ that does not contain a dot and is not contained in $R_{\mathcal{A}}$. Let $q$ be the placement obtained by deleting $c$ from $p$. Then the $\mathcal{A}$-sequences of $p$ and $q$ are the same.*

**Proof:** After the deletion of $c$, the sequence $a_k \ldots a_1$ remains an occurrence of $k \ldots 21$ in $q$. Since the deletion of a cell cannot create new occurrences of this pattern, $a_k \ldots a_1$ remains the smallest occurrence for the lexicographic order. □

**Lemma 34** *Adding an empty corner cell $c$ to a row located above $R_{\mathcal{A}}$ does not change the $\mathcal{A}$-sequence. By symmetry, adding an empty corner cell to a column located to the right of $R_{\mathcal{B}}$ does not change the $\mathcal{B}$-sequence.*

**Proof:** Assume the $\mathcal{A}$-sequence changes, and let $\mathcal{A}' = a'_k \ldots a'_1$ be the $\mathcal{A}$-sequence of the new placement $q$. Observe that $a_k \ldots a_1$ is still an occurrence of $k \ldots 21$ in $q$. By the previous lemma, $c$ belongs to $R_{\mathcal{A}'}$, the $\mathcal{A}$-rectangle of $q$. However, by assumption, $c$ is above $R_{\mathcal{A}}$. This implies that the top row of $R_{\mathcal{A}'}$ is higher than the top row of $R_{\mathcal{A}}$, so that $a'_k$ is higher (that is, larger) than $a_k$. This contradicts the definition of the $\mathcal{A}$-sequence of $q$. □

**Remark.** The lemma is not true if the new cell is not added above $R_{\mathcal{A}}$, as shown by the following example, where $k = 2$. The $\mathcal{A}$-sequence is shown with black disks.



Let $R$ be the smallest rectangle containing both $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$. It is possible that $R$ is not contained in $\lambda$. Let $p^R$ be the placement obtained by adding the cells of $R \setminus \lambda$ to $p$. The previous lemma implies the following corollary, illustrated by the central placement of Figure 6.

**Corollary 35** *The placements $p$ and $p^R$ have the same $\mathcal{A}$-sequence and the same $\mathcal{B}$-sequence.*

**Proof:** All the new cells are above $R_{\mathcal{A}}$ and to the right of $R_{\mathcal{B}}$. □

In what follows, the definitions of the $\mathcal{A}$- and $\mathcal{B}$-sequences, and of the maps $\psi$ and $\phi$, are extended in a straightforward manner to *partial* rook placements (some rows and columns may contain no dot). We extend them similarly to words with distinct letters.

**Lemma 36** *Let $p$ be a partial rook placement containing the pattern $k \dots 21$. If we delete a row located above $a_k$, the $\mathcal{A}$-sequence will not change. A symmetric statement holds for the deletion of a column located to the right of $b_1$.*

**Proof:** The sequence $a_k \dots a_1$ is still an occurrence of $k \dots 21$ in the new placement, and deleting a row cannot create a new occurrence of this pattern. □

**Proposition 37** *Let $\pi$ be the permutation associated with a placement $p$ containing $k \dots 21$. There exists a subsequence of $\pi$ that has the same $\mathcal{A}$- and $\mathcal{B}$-sequences as $p$. One such subsequence is $\pi_R$, the subsequence of $\pi$ corresponding to the dots contained in $R$.*

**Proof:** By Corollary 35, we can assume that $R$ is included in $\lambda$. By Lemma 36, we can assume that $\lambda = R$, which concludes the proof. □

We shall denote by $p_R$ the (partial) placement obtained from $p^R$ by deleting all rows above $R$ and all columns to the right of $R$ (third placement in Figure 6).

**Lemma 38** *Let $i \leq j < k$. In $\psi(p)$, the maximum length of a decreasing sequence starting at $b_j$ and ending at $b_i$ is $j - i + 1$. One such sequence is of course $b_j b_{j-1} \dots b_i$.*

**Proof:** Clearly, it suffices to prove the statement under the assumption that $b_j$ and $b_i$ are the only letters in the sequence that are shifted elements of the $\mathcal{B}$-sequence of $p$, which we now assume.

Suppose that there exists in $\psi(p)$ a longer decreasing sequence, of the form $b_j x_j x_{j-1} \dots x_{i+1} b_i$, where the $x$'s do not belong to the $\mathcal{B}$-sequence of $p$. Then $b_k \dots b_{j+1} x_j \dots x_{i+1} b_i \dots b_1$ is an occurrence of the pattern $k \dots 21$ in $p$. The fact that $x_{i+1}$ comes before $b_i$ in $\psi(p)$ means that $x_{i+1}$ precedes $b_{i+1}$ in $p$. This contradicts the construction of the $\mathcal{B}$-sequence of $p$ (Lemma 9). □

The following proposition is the last technical difficulty we meet in the proof of the commutation theorem.

**Proposition 39** *Assume the $\mathcal{A}$- and $\mathcal{B}$-sequences of $p$ do not coincide. Then $\psi(p)$ contains the pattern $k \dots 21$, and its $\mathcal{A}$-sequence begins with $a_k$.*

**Proof:** Let $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$ denote the $\mathcal{A}$- and $\mathcal{B}$-rectangles of $p$. They form sub-boards of $\lambda$. Let $R$ be the smallest rectangle containing $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$.

Let us first prove that there exists in $\psi(p)$ an occurrence of $k \dots 21$ starting with $a_k$. First, since $p$ and $p_R$ have the same $\mathcal{B}$-sequence (Proposition 37), the map $\psi$ acts in the same way on these two placements. This means that $\psi(p_R)$ can be obtained from $\psi(p)$ by deleting the rows above $R_{\mathcal{B}}$ and to the right of $R_{\mathcal{A}}$, and by adding the cells of $R \setminus \lambda$. Then, by Proposition 28, $\psi(p_R)$ contains an occurrence of $k \dots 21$ starting with $a_k$, namely, the $\mathcal{A}$-sequence of $\psi(p_R)$. These dots are all contained in $R_{\mathcal{A}}$, and so they form, in $\psi(p)$ also, an occurrence of $k \dots 21$ starting with $a_k$.

Now let $x_k \dots x_1$ be the $\mathcal{A}$-sequence of $\psi(p)$, and assume that $x_k \neq a_k$ (which implies that $x_k < a_k$). We will derive from this assumption a contradiction, which will complete the proof.

If none of the values $x_j$ were in $\mathcal{B}(p)$, then they would form an occurrence of $k \dots 21$ in $p$, which would be smaller than $a_k \dots a_1$, a contradiction. Hence at least one of the $x_j$ is in $\mathcal{B}(p)$. Let $x_\ell = b_m$ (resp. $x_{i+1} = b_j$) be the leftmost (resp. rightmost) of these. Then $x_k, \dots, x_{\ell+1}$ and $x_i, \dots, x_1$ are in the same places in $p$ as in $\psi(p)$.

We consider two cases:

**Case 1**. Suppose first that one of the $x_r$, for $1 \leq r \leq i$, lies "above" the $\mathcal{B}$-sequence in $p$. By this we mean that there exists an $s$ such that $b_s < x_r$ and $b_s$ precedes $x_r$ in $p$. Let $r \leq i$ be maximal such that $x_r$ satisfies this condition. Let $s$ be maximal such that $b_s$ satisfies this condition for $x_r$. Clearly, $s < k$, because $b_s < x_r < x_k < a_k < b_k$.

The maximality of $s$ implies that $b_{s+1} > x_r$. In fact, $b_{s+1}$ is the smallest element of $\mathcal{B}$ that is larger than $x_r$. Consider, in $p$, the decreasing sequence

$$x_k \dots x_{\ell+1} b_m \dots b_{s+1} x_r \dots x_1.$$

It is an occurrence of a decreasing pattern, which, given that $x_k < a_k$, cannot be as long as $a_k \dots a_1$. That is,

$$k - \ell + m - s + r < k. \tag{2}$$

Assume for the moment that $r < i$. By maximality of $r$, we know that $x_{r+1}$ precedes $b_s$ in $p$. Let us show that it actually precedes $b_{s+1}$ (and thus precedes $b_s$ in $\psi(p)$). If not, $x_{r+1}$ lies between $b_{s+1}$ and $b_s$. But $x_{r+1} > b_s$, since $x_r > b_s$, and $x_{r+1} < b_{s+1}$ by maximality of $r$. Thus $x_{r+1}$ lies between $b_{s+1}$ and $b_s$ in position and in value, which contradicts the definition of the $\mathcal{B}$-sequence of $p$. Hence $x_{r+1}$ precedes $b_{s+1}$, so the sequence $x_\ell \dots x_{r+1} b_s$ in $\psi(p)$ is decreasing and has $\ell - r + 1$ elements. But this sequence has $x_\ell = b_m$ and $b_s$ as its endpoints, so, by Lemma 38, it has at most $m - s + 1$ points. In other words, $\ell - r + 1 \leq m - s + 1$, or $\ell - r \leq m - s$, contradicting (2).

Now if $r = i$, we have $s < j$ (since $b_s < x_i$ and $b_j > x_i$). The sequence $x_\ell \dots x_{i+1}$ in $\psi(p)$ is decreasing and has $\ell - i$ elements. But this sequence has $x_\ell = b_m$ and $b_j$ as its endpoints, so, by Lemma 38, it has at most $m - j + 1$ points. In other words, $\ell - i \leq m - j + 1$. But, since $s < j$, this contradicts (2).

**Case 2**. We now assume that for each $x_r$ among $x_i, \ldots, x_1$ there is no $s$ such that $b_s < x_r$ and $b_s$ precedes $x_r$ in $p$.

Lemma 38, applied to the subsequence $b_m = x_\ell, x_{\ell-1}, \ldots, x_{i+1} = b_j$ of $\psi(p)$, implies that $\ell - i \leq m - j + 1$. That is, $i - j \geq \ell - m - 1$. Now,

$$b_k \ldots b_{j+1} x_i \ldots x_1$$

is a decreasing sequence in $p$ of length $k - j + i \geq k + \ell - m - 1$. At most $k - 1$ of its elements can precede $b_1$, for else $b_1$ could not be the rightmost letter of $\mathcal{B}(p)$. Hence, since $b_1$ itself does not occur in this sequence, at least $\ell - m$ of its elements must be preceded by $b_1$, that is, $x_{\ell-m}, \ldots, x_1$ all lie to the right of $b_1$. Recall that none of the letters $x_i, \ldots, x_1$ are to the right of *and* above any $b_s$, so $x_{\ell-m}, \ldots, x_1$ must be smaller than $b_1$. But then

$$x_k \ldots x_{\ell+1} b_m \ldots b_1 x_{\ell-m} \ldots x_1$$

is an occurrence of the pattern $k \ldots 21$ in $p$, with $x_k < a_k$, which contradicts the definition of the $\mathcal{A}$-sequence. □

We are finally ready for a proof of the local commutation theorem, which we restate.

**Theorem** (*same as Theorem 10*) Let $p$ be a placement for which the $\mathcal{A}$- and $\mathcal{B}$-sequences do not coincide. Then $\phi(p)$ and $\psi(p)$ still contain the pattern $k \ldots 21$, and

$$\phi(\psi(p)) = \psi(\phi(p)).$$

**Proof:** As above, let $R$ be the smallest rectangle containing $R_\mathcal{A}$ and $R_\mathcal{B}$. The first statement follows from Proposition 39 and symmetry.

We want to prove that the map $\phi \circ \psi$ acts in the same way on the placements $p$, $p^R$ and $p_R$. If we prove this, then, by symmetry, the same holds for the map $\psi \circ \phi$. But the commutation theorem for permutations (Theorem 32) states that $\phi(\psi(p_R)) = \psi(\phi(p_R))$. Thus $\phi(\psi(p)) = \psi(\phi(p))$, and we will be done.

By Corollary 35 and Proposition 37, the placements $p$, $p^R$ and $p_R$ have the same $\mathcal{B}$-sequence. Consequently, $\psi$ acts in the same way on these three placements. In other words,

- $\psi(p^R)$ is obtained by adding to $\psi(p)$ the cells of $R \setminus \lambda$; we summarize this by writing $\psi(p^R) = \psi(p)^R$,
- $\psi(p_R)$ is obtained by deleting from $\psi(p^R)$ the rows above $R$ and the columns to the right of $R$.

It only remains to prove that $\psi(p)$, $\psi(p^R)$ and $\psi(p_R)$ have the same $\mathcal{A}$-sequence.

By Proposition 39, the $\mathcal{A}$-sequence of $\psi(p)$ starts with $a_k$. This means that the $\mathcal{A}$-rectangle of $\psi(p)$ coincides with the $\mathcal{A}$-rectangle of $p$. Hence Lemma 34, applied to $\psi(p)$, implies that $\psi(p)$ and $\psi(p)^R$ have the same $\mathcal{A}$-sequence. But $\psi(p)^R = \psi(p^R)$, so that $\psi(p)$ and

$\psi(p^R)$ have the same $\mathcal{A}$-sequence. The $\mathcal{A}$-sequence of $\psi(p^R)$, being contained in the $\mathcal{A}$-rectangle of $p$, is contained in $R$. By Lemma 36, the $\mathcal{A}$-sequences of $\psi(p_R)$ and $\psi(p^R)$ coincide. □

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