A Quantum Version of the Désarménien Matrix

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Abstract. We use elements in the quantum hyperalgebra to define a quantum version of the Désarménien matrix. We prove that our matrix is upper triangular with ones on the diagonal and that, as in the classical case, it gives a quantum straightening algorithm for quantum bideterminants. We use our matrix to give a new proof of the standard basis theorem for the q-Weyl module. As well, we show that the standard basis for the q-Weyl module and the basis dual to the standard basis for the q-Schur module are related by the quantum Désarménien matrix.

1. Introduction

Let K be an infinite field, n and r fixed positive integers, and λ a fixed partition of r. In the classical case, the Désarménien matrix corresponding to λ and n is an upper triangular unimodular matrix that plays an interesting role in the representation theory of the general linear group $GL(n, K)$. It was first introduced in [4] where it was defined using Capelli operators which, although it is not pointed out there, are elements in the hyperalgebra for $GL(n, K)$. Its original purpose was to act as tool for writing a given bideterminant as a linear combination of bideterminants which come from semistandard λ-tableaux. In [3], Désarménien showed that the entries in the matrix may be determined in a combinatorial manner by examining pairs of semistandard λ-tableaux with entries from the set $\{1, \ldots, n\}$.

Since its introduction, the Désarménien matrix has appeared in a number of places relating to the polynomial representations of $GL(n, K)$. J. A. Green gives a very nice proof of the Carter-Lusztig basis theorem (see [1]) for the Weyl module $\Delta(\lambda)$ for $GL(n, K)$ in [7]. At the heart of his proof is the Désarménien matrix. The proof directly shows that the Désarménien matrix is the change of basis matrix between the Carter-Lusztig basis for $\Delta(\lambda)$ and the basis dual to the standard basis for the Schur module. In [6], J. A. Green uses a version of the Désarménien matrix to give a basis of codeterminants for the Schur algebra $S(n, r)$.

A symplectic version of the Désarménien matrix is given in [13]. This matrix has many of the properties of the original Désarménien matrix and a version of it has proved useful in providing bases of symplectic codeterminants for the symplectic Schur algebras in [2].

In Section 4 we will define quantum versions of the Capelli operators and use them to define a quantum Désarménien matrix. We prove that our matrix is upper triangular with ones on the diagonal. As well, we show that it can be used to give a quantum straightening

algorithm for quantum bideterminants. Another quantum straightening algorithm is given in [11].

A quantum version of the Carter-Lusztig standard basis theorem for the q-Weyl module corresponding to λ was proved by R. M. Green in [8]. The contravariant dual to the q-Weyl module, called the q-Schur module, has a basis of bideterminants given by semistandard λ-tableaux so the q-Weyl module also has a basis which is dual to this basis. Our Désarménien matrix turns out to be the connection between these two bases. We prove this by giving a new proof of the standard basis theorem from which this fact emerges.

2. Young tableaux

Throughout the article, K shall be an infinite field, n and r shall be fixed positive integers, and λ shall be a fixed partition of r.

A k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers is a *partition* of r if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = r$. The *Young diagram* of shape λ is a collection of r boxes arranged in k left justified rows with the ith row consisting of λ_i boxes. A λ-*tableau* is obtained by filling the Young diagram of shape λ with entries from the set $\{1, 2, \ldots, n\}$. A λ-tableau is called *semistandard* if the entries in its rows are weakly increasing from left to right and strictly increasing from top to bottom.

Let $I(n, r)$ denote the set of r-tuples with entries from the set $\{1, 2, \ldots, n\}$. It is often convenient to associate to a given λ-tableau a sequence $I = (i_1, i_2, \ldots, i_r)$ in $I(n, r)$ and to associate to a sequence I in $I(n, r)$ a λ-tableau T_I. We do this by letting T_I denote the λ-tableau obtained by filling the Young diagram of shape λ canonically down the columns with the entries in I working from left to right and from top to bottom.

Example 2.1 Let $\lambda = (3, 2)$. Then $T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} \end{array}$ is a semistandard λ-tableau. We may write $T = T_I$ where $I = (1, 4, 2, 6, 3)$.

We will often make use of the *basic tableau*, which we denote by T_λ. If s is the number of rows in the Young diagram of shape λ, then T_λ is the λ-tableau in which every entry in the ith row is equal to i, for $1 \leq i \leq s$. We denote by $J(\lambda)$ the sequence in $I(n, r)$ that satisfies $T_\lambda = T_{J(\lambda)}$.

Example 2.2 If $\lambda = (3, 2)$, then $T_\lambda = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$ and $J(\lambda) = (1, 2, 1, 2, 1)$.

The symmetric group S_r acts on $I(n, r)$ on the right by

$$I \cdot \sigma = (i_{1\sigma}, \ldots, i_{r\sigma}), \quad I = (i_1, i_2, \ldots, i_r) \in I(n, r).$$

This action induces an action of S_r on the set of λ-tableaux with entries from $\{1, 2, \ldots, n\}$ by $T_I \cdot \sigma = T_{I \cdot \sigma}$. Let $I = (1, 2, \ldots, r)$. We denote the tableau T_I, for this particular choice of I, by $\hat{T}_\lambda$. The *column stabilizer* of $\hat{T}_\lambda$, denoted $C(\hat{T}_\lambda)$, is the set of $\sigma \in S_r$ which preserve the columns of $\hat{T}_\lambda$ under this action.

3. The quantum hyperalgebra

In this section, we construct the quantum analogue of the hyperalgebra for $GL(n, K)$. We then discuss the quantum analogues of the Schur and Weyl modules.

Let q be an indeterminate and let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ denote the ring of Laurent polynomials in q. Let $\mathbb{Q}(q)$ be the field of quotients of $\mathcal{A}$. The *quantum enveloping algebra*, which we denote by $U_{\mathbb{Q}(q)}$, is the associative $\mathbb{Q}(q)$-algebra with generators E_i, F_i, K_j, K_j^{-1} with $1 \le i < n$, $1 \le j \le n$, subject to the relations that follow. We let $K_{i,i+1}$ denote $K_i K_{i+1}^{-1}$.

$$K_i K_i^{-1} = K_i^{-1} K_i = 1 \qquad\qquad K_i K_j = K_j K_i$$
$$K_i E_j = q^{\delta_{i,j} - \delta_{i,j+1}} E_j K_i \qquad\qquad K_i F_j = q^{\delta_{i,j+1} - \delta_{ij}} F_j K_i$$
$$E_i E_j = E_j E_i \quad \text{if } |i - j| > 1 \qquad\qquad F_i F_j = F_j F_i \quad \text{if } |i - j| > 1$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_{i,i+1} - K_{i,i+1}^{-1}}{q - q^{-1}}$$
$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } |i - j| = 1$$
$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad \text{if } |i - j| = 1.$$

The algebra $U_{\mathbb{Q}(q)}$ is a Hopf algebra over $\mathbb{Q}(q)$ with comultiplication

$$\Delta : U_{\mathbb{Q}(q)} \to U_{\mathbb{Q}(q)} \otimes U_{\mathbb{Q}(q)}$$

defined by

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_{i,i+1}, \quad \Delta(F_i) = K_{i,i+1}^{-1} \otimes F_i + F_i \otimes 1, \quad \Delta(K_i) = K_i \otimes K_i.$$
$$(3.1)$$

Note that $U_{\mathbb{Q}(q)}$ possesses other comultiplications which are preferred by some authors.

Let a be a nonnegative integer. Define

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}},$$

and *quantum factorial* as

$$[a]! = [a][a - 1] \dots [1].$$

For $X \in U_{\mathbb{Q}(q)}$ and $a \in \mathbb{N}$, let

$$X^{(a)} = \frac{X^a}{[a]!} \quad \text{and} \quad \binom{K_j}{a} = \prod_{s=1}^{a} \frac{q^{-s+1} K_j - q^{s-1} K_j^{-1}}{q^s - q^{-s}}.$$

The *integral form* $U_{\mathcal{A}}$ of $U_{\mathbb{Q}(q)}$, given in [12], is a quantum version of Kostant's $\mathbb{Z}$-form. It is the Hopf $\mathcal{A}$-subalgebra of $U_{\mathbb{Q}(q)}$ generated by the elements

$$E_i^{(a)}, F_i^{(a)}, K_j, K_j^{-1}, \begin{pmatrix} K_j \\ a \end{pmatrix}, \quad a \in \mathbb{N}, \quad 1 \le i < n, \ 1 \le j \le n.$$

Let t be a unit in K such that $t^2 \ne 1$. We may regard our field K as an $\mathcal{A}$-algebra by letting $q \in \mathcal{A}$ act on K by multiplication by $t \in K$. The quantum hyperalgebra is defined as

$$U_K = U_{\mathcal{A}} \otimes_{\mathcal{A}} K.$$

We will often say that M is a U_K-module, meaning that M is a $U_{\mathcal{A}}$-module so we have a U_K-module via base change. We write $u \in U_K$ to mean the image of u in U_K via the map $\phi : U_{\mathcal{A}} \to U_K$ defined by $\phi(u) = u \otimes 1$.

Let $V_{\mathbb{Q}(q)}$ be the n-dimensional vector space over $\mathbb{Q}(q)$ with basis $v_1, \dots, v_n$. Then $V_{\mathbb{Q}(q)}$ is a $U_{\mathbb{Q}(q)}$-module, called the *natural module*, via

$$E_i v_j = \delta_{i+1,j} v_i, \quad F_i v_j = \delta_{ij} v_{i+1}, \quad K_i v_j = q^{\delta_{ij}} v_j, \quad K_i^{-1} v_j = q^{-\delta_{ij}} v_j.$$

Denote the $U_{\mathcal{A}}$-submodule of $V_{\mathbb{Q}(q)}$ generated by $v_1, \dots, v_n$ by $V_{\mathcal{A}}$. The coassociative comultiplication Δ defined in (3.1) defines a $U_{\mathcal{A}}$-module action on $V_{\mathcal{A}}^{\otimes r}$. We will let $V^{\otimes r}$ denote the U_K-module obtained by base change. Given $I = (i_1, i_2, \dots, i_r) \in I(n, r)$, define $v_I \in V_{\mathcal{A}}$ by $v_I = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$.

Define $z_\lambda \in V_{\mathcal{A}}^{\otimes r}$ by

$$z_\lambda = \sum_{\sigma \in C(\hat{T}_\lambda)} (-q)^{-l(\sigma)} v_{J(\lambda) \cdot \sigma} \tag{3.2}$$

where $l(\sigma)$ denotes the length of the permutation $\sigma \in S_r$ and $J(\lambda)$ and $C(\hat{T}_\lambda)$ are defined in Section 2. Let $\Delta(\lambda)_{\mathcal{A}}$ denote the left $U_{\mathcal{A}}$-submodule of $V_{\mathcal{A}}^{\otimes r}$ generated by z_λ. The *q-Weyl module* is the left U_K-module defined by

$$\Delta_q(\lambda) = \Delta(\lambda)_{\mathcal{A}} \otimes_{\mathcal{A}} K.$$

We now define a quantum version of the algebra of polynomial functions on the set of $n \times n$ matrices over K. As in [15], an $n \times n$ matrix $A = (a_{ij})_{1 \le i, j \le n}$ with entries in an $\mathcal{A}$-algebra is a *q-matrix* if its entries satisfy the following relations:

$$\begin{aligned}
a_{ik} a_{il} &= q^{-1} a_{il} a_{ik} & 1 \le k < l \le n \\
a_{ik} a_{jk} &= q^{-1} a_{jk} a_{ik} & 1 \le i < j \le n \\
a_{il} a_{jk} &= a_{jk} a_{il} & 1 \le i < j \le n, \quad 1 \le k < l \le n \\
a_{ik} a_{jl} - a_{jl} a_{ik} &= (q^{-1} - q) a_{il} a_{jk} & 1 \le i < j \le n, \quad 1 \le k < l \le n.
\end{aligned} \tag{3.3}$$

Define $A_q(n)$ to be the $\mathcal{A}$-algebra with generators x_{ij} subject to the relations which require $X = (x_{ij})_{1 \le i, j \le n}$ to be a q-matrix.

It can be checked (see [15]) that $A_q(n)$ is a $U_{\mathcal{A}}$-module with action

$$E_i x_{kl} = \delta_{i+1,l} x_{ki}, \quad F_i x_{kl} = \delta_{il} x_{k,i+1}, \quad K_i x_{kl} = q^{\delta_{il}} x_{kl}, \quad K_i^{-1} x_{kl} = q^{-\delta_{il}} x_{kl} \qquad (3.4)$$

and $E_i(PQ) = P(E_i Q) + (E_i P)(K_{i,i+1} Q)$, $F_i(PQ) = (K_{i,i+1}^{-1} P)(F_i Q) + (F_i P)Q$, $K_i(PQ) = (K_i P)(K_i Q)$ where $P, Q \in A_q(n)$. Of course we get a U_K-module by base change and we also denote this module by $A_q(n)$.

If $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_r)$ belong to $I(n, r)$ with $i_1 < i_2 < \cdots < i_r$, define $\det_q X_J^I$ in $A_q(n)$ by

$$\det_q X_J^I = \begin{cases} \displaystyle\sum_{\sigma \in S_r} (-q)^{-\ell(\sigma)} x_{i_1 j_{\sigma(1)}} x_{i_2 j_{\sigma(2)}} \ldots x_{i_r j_{\sigma(r)}} & \text{if } j_1 < j_2 < \cdots < j_r \\[2ex] \displaystyle\sum_{\sigma \in S_r} (-q)^{-\ell(\sigma)} x_{i_{\sigma^{-1}(1)} j_1} x_{\sigma^{-1}(2) j_2} \ldots x_{i_{\sigma^{-1}(r)} j_r} & \text{otherwise.} \end{cases}$$

Given λ-tableaux S and T, where S is column increasing, the *quantum bideterminant* $(S : T) \in A_q(n)$ is defined as

$$(S : T) = \left(\det_q X_{T(1)}^{S(1)}\right)\left(\det_q X_{T(2)}^{S(2)}\right) \cdots \left(\det_q X_{T(s)}^{S(s)}\right)$$

where s is the number of columns in the Young diagram of shape λ and $T(i)$ (respectively $S(i)$) denotes the subsequence corresponding to the entries in the ith column of T (respectively S). In this article, we are only interested in bideterminants of the form $(T_\lambda : T)$, where T_λ is the basic λ-tableau defined in Section 2. We take $[T]$ to represent the bideterminant $(T_\lambda : T)$. Note that λ is fixed throughout, so T is always a λ-tableau.

Let $\nabla(\lambda)_{\mathcal{A}}$ be the $\mathcal{A}$-span of the quantum bideterminants $[T]$, where the set runs over all λ-tableaux T with entries from the set $\{1, \ldots, n\}$. The fact that $\nabla(\lambda)_{\mathcal{A}}$ is a $U_{\mathcal{A}}$-invariant submodule of $A_q(n)$ follows from [14, Proposition 2.1]. The *q-Schur module*, denoted $\nabla_q(\lambda)$ is the U_K-module given by base change.

There is an antiautormorphism $\tau : U_{\mathcal{A}} \to U_{\mathcal{A}}$ defined by

$$\tau(E_i) = F_i, \quad \tau(F_i) = E_i, \quad \tau(K_i) = K_i. \qquad (3.5)$$

We denote the induced map on U_K by τ as well.

If M is a U_K-module, the contravariant dual of M, denoted M°, is just the dual space M^* with action defined as

$$u f(m) = f(\tau(u)m), \quad u \in U_K, \quad m \in M, \quad f \in M^*.$$

In Section 5, we define a contravariant form on $V^{\otimes r}$ and use it to explicitly construct the contravariant dual to $\nabla_q(\lambda)$. The resulting module will be isomorphic to $\Delta_q(\lambda)$. For a proof of the following, see [5, Proposition 4.1.6].

Theorem 3.1 *The q-Weyl module is the contravariant dual to* $\nabla_q(\lambda)$; *that is*

$$(\nabla_q(\lambda))^\circ \cong \Delta_q(\lambda).$$

We shall frequently use the following standard basis theorem for $\nabla_q(\lambda)$. There are several different proofs of this result; see for instance [9] or [10].

Theorem 3.2 *The set* $\{[T] : T \text{ is a semistandard } \lambda\text{-tableau}\}$ *forms a K-basis for* $\nabla_q(\lambda)$.

4. A quantum version of the Désarménien matrix

In this section we define a quantum version of the Désarménien matrix. We introduce a number of technical lemmas which will ease our task. Using (3.4), we may determine the action of E_i, F_i or K_i on a bideterminant $[T]$. For a discussion of this action on bideterminants, see [11, p. 7 and Example 2.2].

To define quantum analogues of the Capelli operators, we first define E_{ij} and F_{ij} recursively in U_K as follows:

$$E_i = E_{i,i+1}, \ E_{ij} = E_i E_{i+1,j} - q^{-1} E_{i+1,j} E_i \quad \text{and} \quad F_{ij} = F_{i+1,j} F_i - q^{-1} F_i F_{i+1,j}.$$

It can be quite difficult to compute $E_{ij}[T]$ for an arbitrary λ-tableau T. We will frequently be encountering tableaux for which the task becomes simplified, as illustrated by the following example and Lemma 4.2.

Example 4.1   Let $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 & 3 & 6 & 8 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} \end{array}$. Then

$$
\begin{aligned}
E_{36}[T] &= (E_3 E_{46} - q^{-1} E_{46} E_3)[T] \\
&= E_3 E_{46}[T], \text{ since } E_3[T] = 0 \text{ (see [11, Example 2.2])} \\
&= E_3(E_4 E_5 - q^{-1} E_5 E_4)[T] \\
&= E_3 E_4 E_5[T] \text{ since } E_3 E_5 E_4[T] = 0.
\end{aligned}
$$

Lemma 4.2 *Suppose that T is a column increasing λ-tableau that coincides with T_λ in the first $i - 1$ rows. Suppose also that if m is an integer with $i < m < j$, then m does not appear in the ith row of T. Then $E_{ij}[T] = E_i E_{i+1} \ldots E_{j-1}[T]$.*

Proof: We fix j and induct on i in the operator E_{ij}. If $i = j - 1$, the result is trivial. Suppose that $E_{kj}[T] = E_k E_{k+1} \ldots E_{j-1}[T]$ for $i < k \le j - 1$. If an $i + 1$ appears in T, our hypotheses force an i to appear in a column above it so $E_i[T] = 0$. Thus $E_{ij}[T] = E_i E_{i+1,j}[T] - q^{-1} E_{i+1,j} E_i[T] = E_i E_{i+1} \ldots E_{j-1}[T]$.   □

Given $I \in I(n, r)$, define $d(I)$ to be the number of pairs (a, b) which satisfy $a < b$ and $i_a < i_b$. For instance, if $I = (1, 2, 1, 4, 3)$, then $d(I) = 7$.

Lemma 4.3 *Suppose that $j > i$. Then $E_{ij}^r(x_{1j}^r) = [r]! x_{1i}^r$.*

Proof: We will show that $E_{ij}^s[T] = [s]! \sum_I q^{-d(I)}[T_I]$ for $1 \le s \le r$ where the sum runs over all r-tuples I which contain s entries equal to i and $r - s$ entries equal to j and T is the one-row tableau with r entries equal to j. Then, if we let $s = r$, this is precisely the statement of the Lemma. The statement holds for $s = 1$ since

$$E_{ij}[T] = E_i E_{i+1} \ldots E_{j-1}[T] = E_i E_{i+1} \ldots E_{j-2} \sum_M q^{-d(M)}[T_M] = \sum_J q^{-d(J)}[T_J],$$

where the sum runs over the r-tuples J which contain one i and $r - 1$ entries equal to j.

Now suppose that for all $1 \le m \le s$ we have $E_{ij}^m[T] = [m]! \sum_I q^{-d(I)}[T_I]$ where the sum runs over all r-tuples I which contain m entries equal to i and $r - m$ entries equal to j. Then $E_{ij}^s[T] = [s-1]! \sum_I q^{-d(I)} E_{ij}[T_I]$ where the sum is over the r-tuples I which contain $s - 1$ entries equal to i and $r - (s - 1)$ entries equal to j. Now $\sum_I E_{ij}[T_I] = \sum_J \gamma_{IJ}[T_J]$ where $\gamma_{IJ} \in \mathbb{Z}[q, q^{-1}]$ and the sum runs over all J which contain s entries equal to i and $r - s$ entries equal to j. Furthermore, $\gamma_{IJ} = 0$ in case J cannot be formed by changing a j in I to an i. Thus $E_{ij}^s[T] = [s - 1]! \sum_J \sum_I q^{-d(I)} \gamma_{IJ}[T_J]$, where the sum runs over all J which contain s entries equal to i and $r - s$ entries equal to j and all I which contain $s - 1$ entries equal to i and $r - (s - 1)$ entries equal to j. We now prove that for each J with s entries equal to i and $r - s$ entries equal to j , we have $\sum_I q^{-d(I)} \gamma_{IJ} = [s] q^{-d(J)}$, where the sum is over all I which yield J when a j in I is changed to an i. Fix such a J. Let I_1 be the r-tuple that comes from changing the first i from the right in J to a j, I_2 the r-tuple that comes from changing the second i from the right in J to a j, etc. Then $\sum_I q^{-d(I)} \gamma_{IJ} = \sum_{k=1}^s q^{-d(I_k)} \gamma_{I_k J}$.

For each k with $1 \le k \le s$, let α_k be the number of j's to the right of the kth i from the right in J. Then $d(I_k) = \sum_{i=1}^s \alpha_i - \alpha_k + (s - k) = d(J) - \alpha_k + (s - k)$. To examine $\gamma_{I_k J}$ we apply $E_i E_{i+1} \ldots E_{j-1}$ to $[T_{I_k}]$ and look at the coefficient of the portion of this sum that gives J. Since $q^{-\alpha_j}$ is introduced by the E_{j-1} operator and q^{k-1} is introduced by the E_i operator, $\gamma_{I_k J} = q^{-\alpha_k} q^{k-1}$.
It follows that

$$\sum_{k=1}^s q^{-d(I_k)} \gamma_{I_k J} = \sum_{k=1}^s q^{-d(J)} q^{-s+2k-1} = \sum_{k=1}^s q^{-d(J)} [s]$$

and this completes the proof.

$\square$

Lemma 4.4 *Let T be a semistandard λ-tableau that coincides with T_λ in the first $i - 1$ rows. Suppose that there are r entries equal to j in the ith row of T. Suppose also that if m is an integer with $i < m < j$ then m does not occur in the ith row of T. Then $E_{ij}^r[T] = [r]![S]$*

where S is identical to T except that the r entries equal to j in the ith row of T have been replaced by i's.

Proof: Suppose that the r entries equal to j appear in the ith row of T in columns a through $b = a + r$. We argue that applying E_{ij}^r affects only the portion of the tableau in which these j's occur. The lemma then follows from Lemma 4.3.

From our assumptions on T, it follows that if $k > b$ there are no entries between $i - 1$ and $j + 1$ in the kth column $T(k)$ of T. Thus, the operators $E_i, \ldots, E_{j-1}$ do not affect these columns and $K_{\ell,\ell+1}^{-1}[T(k)] = [T(k)]$ for $k > b$ and $i \leq \ell \leq j - 1$.

If $i < m < j$ and m appears in T, it must appear below the ith row in some column $T(k)$ with $k < a$. Suppose that $[S]$ is a bideterminant in the sum $E_{j-1}[T]$ that comes from changing a j below the ith row of T to a $j - 1$. Any bideterminant in the sum $E_i(E_{i+1} \ldots E_{j-2}[S])$ comes from changing an $i + 1$ below the ith row to an i in some bideterminant $[S']$ in the sum $E_{i+1} \ldots E_{j-1}[S]$. There is an i in the ith row above this i, so $E_i[S'] = 0$ and $E_i(E_{i+1} \ldots E_{j-2}[S]) = 0$. Thus all bideterminants in the sum $E_{j-1}[T]$ come from changing a j in the ith row of T to a $j - 1$. Continuing this argument, we see that every bideterminant in the sum $E_i E_{i+1} \ldots E_{j-1}[T]$ comes from changing a j in the ith row to an i. Inductively, it follows that E_{ij}^r affects only those j's in the ith row of T.   □

We now define operators E_T and F_T in U_K. The operators F_T also appear in [8]. Given a semistandard λ-tableau T with $k \leq n$ rows, define

$$E_T = \prod_{\substack{1 \leq i \leq k \\ i < j \leq n}} E_{ij}^{(\gamma_{ij})} \quad \text{and } F_T = \prod_{\substack{1 \leq i \leq k \\ i < j \leq n}} F_{ij}^{(\gamma_{ij})} \tag{4.1}$$

where γ_{ij} is the number of entries equal to j in row i of T. The product E_T is ordered

$$E_T = E_{kn}^{(\gamma_{kn})} \ldots E_{k,k+1}^{(\gamma_{k,k+1})} \ldots E_{2n}^{(\gamma_{2n})} \ldots E_{23}^{(\gamma_{23})} E_{1n}^{(\gamma_{1n})} \ldots E_{13}^{(\gamma_{13})} E_{12}^{(\gamma_{12})} \text{ and}$$

$$F_T = F_{12}^{(\gamma_{12})} F_{13}^{(\gamma_{13})} \ldots F_{1n}^{(\gamma_{1n})} F_{23}^{(\gamma_{23})} \ldots F_{2n}^{(\gamma_{2n})} \ldots F_{k,k+1}^{(\gamma_{k,k+1})} \ldots F_{kn}^{(\gamma_{kn})}.$$

Definition 4.5 Given a semistandard λ-tableau T and a column increasing λ-tableau S, define $\Omega_q(S, T) = c$ where c is the coefficient of $[T_\lambda]$ in the expansion of $E_T[S]$ into a linear combination of basis elements.

Order the set of λ-tableaux by declaring $S < T$ if the row sequence of S is less than the row sequence of T with respect to the lexicographic order on $I(n, r)$. In Example 4, for instance, $T_1 < T_2$.

Definition 4.6 The quantum Désarménien matrix is the matrix

$$\Omega_q = [\Omega_q(T_I, T_J)]_{I,J \in \mathcal{I}_\lambda}$$

where $\mathcal{I}_\lambda = \{I \in I(n, r) : T_I \text{ is a semistandard } \lambda\text{-tableau}\}$.

We begin with an example to motivate the next theorem.

Example 4.7 Let $T = \begin{array}{|c|c|c|} \hline 1 & 1 & 6 \\ \hline 5 & 5 \\ \hline \end{array}$. Then $E_T[T] = E_{25}^{(2)}E_{16}[T] = [\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array}]$ so $\Omega_q(T, T)$ $= 1$.

Theorem 4.8 *Suppose that T is a semistandard λ-tableau. Then $\Omega_q(T, T) = 1$.*

Proof: By Lemma 4.4, we have $E_{12}^{\gamma_{12}}[T] = [\gamma_{12}]![U]$ where U is the single tableau that results from changing the γ_{12} entries equal to 2 in the first row of T to ones. Thus, $E_{12}^{(\gamma_{12})}[T] = [U]$. Suppose that $E_{1,j-1}^{(\gamma_{1,j-1})} \ldots E_{12}^{(\gamma_{12})}[T] = [U]$ where the first $\gamma_{11}+\gamma_{12}+\cdots+\gamma_{1,j-1}$ columns of the first row of U contain ones while the remainder of U is identical to T. By Lemma 4.4, we have $E_{1j}^{(\gamma_{1j})}[U] = [U']$, where the entries in the first $\gamma_{11} + \gamma_{12} + \cdots + \gamma_{1j}$ columns of the first row of U' are ones and the remainder of U coincides with T. This proves that $E_{1n}^{(\gamma_{1n})} \cdots E_{12}^{(\gamma_{12})}[T] = [U]$ where the first row of U consists entirely of ones and the remainder of U' is identical to T. The general result follows by repeating the argument for each of the rows in T. □

If V is a U_K-module and $\chi = (\chi_1, \ldots, \chi_n)$ an n-tuple of non-negative integers, the *weight space* associated to χ is the subspace

$$V^\chi = \{v \in V : K_i v = q^{\chi_i} v, \ 1 \le i \le n\}.$$

A vector $v \in V^\chi$ is a *weight vector* of weight χ. The weight of a bideterminant $[T]$ in $\nabla_q(\lambda)$ is $\chi = (\chi_1, \ldots, \chi_n)$ where χ_i is equal to the number of i's that occur in the tableau T. In particular, $[T_\lambda]$ is the unique bideterminant in $\nabla_q(\lambda)$ with weight λ. We will require the following fact, which is easily shown by induction.

Lemma 4.9 *Suppose that $[T]$ has weight $\chi = (\chi_1, \chi_2, \ldots, \chi_n)$. Then $E_{ij}[T]$ belongs to the weight space $\Delta_q(\lambda)^\alpha$ where $\alpha = (\chi_1, \ldots, \chi_i + 1, \ldots, \chi_j - 1, \ldots, \chi_n)$.*

Theorem 4.10 *Suppose that S and T are column-increasing λ-tableaux and suppose that T is semistandard.*

1. *If $\Omega_q(S, T) \ne 0$ then $[S]$ and $[T]$ have the same weight.*
2. *If S is semistandard and $\Omega_q(S, T) \ne 0$ then $S \le T$.*

Proof: By Lemma 4.9, if $[S]$ and $[T]$ do not have the same weight, E_T takes $[S]$ and $[T]$ to different weight spaces. By Theorem 4, $E_T[T]$ lies in the weight space $\nabla_q(\lambda)^\lambda$. Thus $\Omega_q(S, T) = 0$, which completes the proof of 1.

To prove 2 we assume that $[S]$ and $[T]$ have the same weight. Suppose that $S > T$ and let t be the first entry in the row sequence of T which differs from the corresponding entry s in the row sequence of S so that $s > t$. Let i be the row of the Young diagram in which t occurs in T. Suppose that the number of t's in the ith row of S is equal to k so $\gamma_{it} > k$. Thus

$E_T[S] = \ldots E_{it}^{(\gamma_{it}-k)} E_{it}^{(k)} \ldots E_{12}^{(\gamma_{12})}[S]$ where $\gamma_{it} - k > 0$. Repeatedly applying Theorem 4.4 we have $E_{it}^k \ldots E_{13}^{\gamma_{13}} E_{12}^{\gamma_{12}}[S] = [k]! \ldots [\gamma_{13}]![\gamma_{12}]![U]$ where the first $i-1$ rows of U and the first $\gamma_{ii} + \gamma_{i,i+1} + \cdots + k$ columns of the ith row of U coincide with T_λ and the remainder of U coincides with S. But there are no t's in the ith row of U and any which appear below the ith are in the first $\gamma_{ii} + \gamma_{i,i+1} + \cdots + k$ columns. Since each of these columns contains an i in a row above such a t, we have $E_{it}^{\gamma_{it}-k}[U] = 0$ so $E_T[S] = 0$. $\square$

Corollary 4.11 *The quantum Désarménien matrix is an invertible upper triangular matrix.*

The following lemma allows us to deduce a straightening algorithm for quantum bide-terminants. For an alternative straightening algorithm, see [11].

Lemma 4.12 *Suppose that S and T are column-increasing λ-tableaux and that T is semistandard. Suppose also that $[S]$ and $[T]$ have the same weight. Then*

$$E_T[S] = \Omega_q(S, T)[T_\lambda].$$

Proof: Since $[S]$ and $[T]$ have the same weight, $E_T[S]$ and $E_T[T]$ both lie in the weight space $\nabla_q(\lambda)^\lambda$. Thus $E_T[S] = c[T_\lambda] = \Omega_q(S, T)[T_\lambda]$. $\square$

To get a q-straightening algorithm, we use the fact that any bideterminant with weight χ can be expressed as a linear combination of semistandard bideterminants of weight χ and then call on the Désarménien matrix to give us the coefficients in the decomposition. Let $\Omega_{q,\chi}$ be the submatrix of Ω_q that runs over the $\Omega_q(S, T)$ where $[S]$ and $[T]$ have weight χ. Let $\mathcal{I}_\lambda^\chi = \{I \in \mathcal{I}_\lambda : [T_I] \text{ has weight } \chi\}$. If U is a column increasing λ-tableau such that $[U]$ has weight χ, then $[U] = \sum_{I \in \mathcal{I}_\lambda^\chi} a_I[T_I]$. If $J \in \mathcal{I}_\lambda^\chi$ then

$$\begin{aligned}
\Omega_q(U, T_J)[T_\lambda] &= E_{T_J}[U] \text{ by Lemma 4} \\
&= \sum_{I \in \mathcal{I}_\lambda^\chi} a_I E_{T_J}[T_I] \\
&= \sum_{I \in \mathcal{I}_\lambda^\chi} a_I \Omega_q(T_I, T_J)[T_\lambda].
\end{aligned}$$

Thus $[\Omega_q(U, T_I)]_{I \in \mathcal{I}_\lambda^\chi} = [a_I]_{I \in \mathcal{I}_\lambda^\chi} \Omega_{q,\chi}$ and since $\Omega_{q,\chi}$ is invertible, we have

$$[a_I]_{I \in \mathcal{I}_\lambda^\chi} = [\Omega_q(U, T_I)]_{I \in \mathcal{I}_\lambda^\chi}(\Omega_{q,\chi})^{-1}. \tag{4.2}$$

Example 4.13 Let $\lambda = (2, 1)$, $n = 3$, and $\chi = (1, 1, 1)$. There are two semistandard λ-tableaux which give bideterminants of weight χ;

$$T_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}.$$

To find $\Omega_q(T_1, T_2)$, we compute $E_{T_2}[T_1] = E_{13}[T_1] = (q - q^{-1})[T_\lambda]$. Thus,

$$\Omega_q^\chi = \begin{pmatrix} 1 & q - q^{-1} \\ 0 & 1 \end{pmatrix}.$$

Let $T = \boxed{\begin{smallmatrix} 2 & 1 \\ 3 & \end{smallmatrix}}$. Then $E_{T_1}[T] = E_{23}E_{12}[T] = q[T_\lambda]$, so $\Omega_q(T, T_1) = q$. Furthermore, $\Omega_q(T, T_2) = -1$. Let a_{T_1} and a_{T_2} be the coefficients of $[T_1]$ and $[T_2]$ respectively in the straightening decomposition of $[T]$. Then

$$(a_{T_1}, a_{T_2}) = (q, -1)\begin{pmatrix} 1 & q^{-1} - q \\ 0 & 1 \end{pmatrix} = (q, -q^2), \text{ and } [T] = q[T_1] - q^2[T_2].$$

5. The connection between two bases of $\Delta_q(\lambda)$

A quantum version of the Carter-Lusztig basis theorem was proved by R. M. Green, [8]. The q-Weyl module also has basis dual to the basis of $\nabla_q(\lambda)$ in Theorem 3. We would like to discover the relationship between these two bases. We shall prove the standard basis theorem using the quantum Désarménien matrix at the end of this section. Our proof also shows that the standard basis for $\Delta_q(\lambda)$ and the basis dual to the basis of semistandard bideterminants for $\nabla_q(\lambda)$ are connected by the quantum Désarménien matrix.

We first construct the contravariant dual to $\nabla_q(\lambda)$. We begin with a map $\phi : V^{\otimes r} \to \nabla_q(\lambda)$.

Proposition 5.1 *The K-linear map $\phi : V^{\otimes r} \to \nabla_q(\lambda)$ given by $\phi(v_I) = [T_I]$ is a U_K-epimorphism.*

Proof: It is clear that ϕ is an epimorphism, so we need only prove that $\phi(uv_J) = u[T_J]$ for each r-tuple J and each $u \in U_K$. Suppose first that $\lambda = (1^r)$ so that $\phi(v_J) = [T_J]$ where T_J is a one-column tableau. Let a denote the number of entries equal to i in T_J. If $a \geq 2$ then $\phi(K_i v_J) = q^a[T_J] = 0 = K_i\phi(v_J)$. If $a < 2$, we also have $\phi(K_i v_J) = q^a[T_J] = K_i\phi(v_J)$.

To prove $\phi(E_i v_J) = E_i\phi(v_J)$, we may assume that J contains at least one $i + 1$, for otherwise equality follows trivially. Suppose first that the number of entries equal to $i + 1$ in J is larger than two. Then $E_i\phi(v_J) = E_i[T_J] = 0 = \phi(E_i[T_J])$. Next suppose that the number of entries equal to $i + 1$ is less than or equal to two. Then if J also contains an i, $E_i v_J = \sum a_M v_M$ where each M in the sum has at least two entries equal to i, from which it follows that $\phi(E_i v_J) = 0 = E_i\phi(v_J)$.

Now suppose that J contains no entries equal to i and one or two entries equal to $i + 1$. If J contains exactly one $i + 1$, then $E_i v_J = v_M$ where M is identical to J except that the $i + 1$ has been replaced with an i. Thus, $\phi(E_i v_J) = [T_M] = E_i[T_J]$. Now suppose that J contains exactly two entries equal to $i + 1$ and no entries equal to i. Then $E_i\phi(v_J) = E_i[T_J] = 0$. As well, $\phi(E_i v_J) = [T_M] + q^{-1}[T_{M'}]$ where the tableaux T_M and $T_{M'}$ are the same except that an i and an $i + 1$ have been interchanged. But $[T_M] + q^{-1}[T_{M'}] = 0$ (see [14, Proposition 2.1]) so $\phi(E_i v_J) = 0$. The proof that $\phi(F_i v_J) = F_i\phi(v_J)$ is similar.

For the general case, let $\mu = (\mu_1, \ldots, \mu_s)$ be the conjugate of the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$. Given $I \in I(n, r)$, let $I = (i_{11}, i_{21}, \ldots, i_{\mu_1 1}, \ldots, i_{1s}, \ldots, i_{\mu_s s})$, $I_1 = (i_{11}, i_{21}, \ldots, i_{\mu_1 1}), \ldots, I_s = (i_{1s}, i_{2s}, \ldots, i_{\mu_s s})$ so that $v_I = v_{I_1} \otimes v_{I_2} \otimes \cdots \otimes v_{I_s}$. The map $\theta : V^{\otimes r} \to \nabla_q(\mu_1) \otimes \nabla_q(\mu_2) \otimes \cdots \otimes \nabla_q(\mu_s)$ defined by

$$\theta(v_{I_1} \otimes v_{I_2} \otimes \cdots \otimes v_{I_s}) = \phi(v_{I_1}) \otimes \phi(v_{I_2}) \otimes \cdots \otimes \phi(v_{I_s})$$

is a U_K-homomorphism. As well, the map $\psi : \nabla_q(\mu_1) \otimes \nabla_q(\mu_2) \otimes \ldots \otimes \nabla_q(\mu_s) \to \nabla_q(\lambda)$ given by $\psi([T_{I_1}] \otimes [T_{I_2}] \otimes \cdots \otimes [T_{I_s}]) = [T_{I_1}][T_{I_2}] \ldots [T_{I_s}]$ is a U_K-homomorphism. Thus, $\phi = \psi \circ \theta$ is a U_K-homomorphism as well. $\qquad\square$

If V and W are two U_K-modules, a bilinear form $\langle , \rangle : V \times W \to K$ is said to be U_K-contravariant if for all $u \in U_K$, $v \in V$, and $w \in W$, we have $\langle uv, w \rangle = \langle v, \tau(u)w \rangle$, where τ is the antiautomorphism defined in (3.5). We want to define a U_K-contravariant form on $V^{\otimes r}$ and use it to construct the q-Weyl module as is done in the classical case by J. A. Green in [7]. To do so, we introduce the following notation. Given $I \in I(n, r)$, define $\beta(I)$ to be the number of pairs (a, b) for which $a < b$ and $i_a \neq i_b$. For example, if $I = (1, 3, 2, 1)$ then $\beta(I) = 5$. Define a bilinear form $\langle , \rangle_q : V^{\otimes r} \times V^{\otimes r} \to K$ by

$$\langle v_I, v_J \rangle_q = q^{\beta(I)} \delta_{IJ}.$$

For example, $\langle v_1 \otimes v_3 \otimes v_2 \otimes v_1, v_1 \otimes v_3 \otimes v_2 \otimes v_1 \rangle = q^5$.

Theorem 5.2 *The form $\langle , \rangle_q : V^{\otimes r} \times V^{\otimes r} \to K$ is a U_K-contravariant form.*

Proof: Let $I, J \in I(n, r)$ and let a be the number of entries equal to i which occur in I and let b denote the number of entries equal to i which occur in J. Then $\langle K_i v_I, v_J \rangle_q = q^a \langle v_I, v_J \rangle_q$ and $\langle v_I, K_i v_J \rangle_q = q^b \langle v_I, v_J \rangle_q$. Both are equal to zero if $I \neq J$ and if $I = J$ then $a = b$ so $\langle K_i v_I, v_J \rangle_q = \langle v_I, K_i v_J \rangle_q$.

Let $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_r)$. Applying (3.1), we have

$$F_i(v_I) = \left(K_{i,i+1}^{-1} v_{i_1}\right) \otimes \cdots \otimes \left(K_{i,i+1}^{-1} v_{i_{r-1}}\right) \otimes \left(F_i v_{i_r}\right) + \cdots + \left(F_i v_{i_1}\right) \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \text{ and}$$

$$E_i(v_J) = v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes \left(E_i v_{j_r}\right) + \cdots + \left(E_i v_{j_1}\right) \otimes \left(K_{i,i+1} v_{j_2}\right) \otimes \ldots \otimes \left(K_{i,i+1} v_{j_r}\right).$$

We will prove that $\langle K_{i,i+1}^{-1} v_{i_1} \otimes \cdots \otimes K_{i,i+1}^{-1} v_{i_{k-1}} \otimes F_i v_{i_k} \otimes \cdots \otimes v_{i_r}, v_J \rangle_q$ is equal to $\langle v_I, v_{j_1} \otimes \cdots \otimes E_i v_{j_k} \otimes K_{i,i+1} v_{j_{k+1}} \otimes \cdots \otimes K_{i,i+1} v_{j_r} \rangle_q$ for a fixed k with $1 \leq k \leq r$. We may assume that the r-tuples I and J coincide before and after the kth place so we replace all such i_l's in the first form by j_l's. We may also assume that $v_{i_k} = v_i$ and $v_{j_k} = v_{i+1}$. Thus $I = (j_1, \ldots, j_{k-1}, i, j_{k+1}, \ldots, j_r)$, $J = (j_1, \ldots, j_{k-1}, i+1, j_{k+1}, \ldots, j_r)$. Suppose that there are a_1 entries equal to i which occur in the r-tuple J prior to the kth place and suppose that there are a_2 entries equal to $i+1$ which occur in J prior to the kth place. Let b_1 denote the number of entries equal to i which appear in J after the kth place and b_2 the number of

entries equal to $i + 1$ that appear after the kth place. Then

$$\left\langle K_{i,i+1}^{-1} v_{j_1} \otimes \cdots \otimes K_{i,i+1}^{-1} v_{j_{k-1}} \otimes F_i v_i \otimes \cdots \otimes v_{j_r}, v_J \right\rangle_q = q^{-a_1} q^{a_2} q^{\beta(J)}, \quad \text{and}$$

$$\left\langle v_I, v_{j_1} \otimes \cdots \otimes E_i v_{i+1} \otimes K_{i,i+1} v_{j_{k+1}} \otimes \cdots \otimes K_{i,i+1} v_{j_r} \right\rangle_q = q^{b_1} q^{-b_2} q^{\beta(I)}.$$

But $q^{-a_1} q^{a_2} q^{\beta(J)} = q^{-a_1} q^{a_2} q^{\beta(I)+a_1+b_1-a_2-b_2} = q^{b_1} q^{-b_2} q^{\beta(I)}$ and this completes the proof. $\square$

We now take $V_{\lambda,q}$ to be the orthogonal complement to $N = \ker\phi$ with respect to the form $\langle, \rangle_q$. It is clear that $V_{\lambda,q}$ is a U_K-submodule of $V^{\otimes r}$ and the form $(,)_q : V_{\lambda,q} \times \nabla_q(\lambda) \to K$ defined by

$$(x, \phi(y))_q = \langle x, y \rangle_q, \quad x \in V_{\lambda,q}, \ y \in V^{\otimes r} \tag{5.1}$$

is a non-degenerate contravariant form since $\langle, \rangle_q$ is non-degenerate and contravariant and ϕ is a U_K-homomorphism. Thus $V_{\lambda,q} \cong (\nabla_q(\lambda))^\circ$ via the map $\psi : V_{\lambda,q} \to (\nabla_q(\lambda))^\circ$ defined by

$$\psi(x)(y) = (x, y)_q, \quad x \in V_{\lambda,q}, \ y \in \nabla_q(\lambda).$$

One can adjust [7, Proposition 2.7e] appropriately to get a quantum version of that result from which it follows that $V_{\lambda,q} \cong \Delta_q(\lambda)$.

The following proof of the standard basis theorem for $\Delta_q(\lambda)$ shows that this basis and the basis dual to the basis of Theorem 3 are related by the Désarménien matrix. Recall the definitions of F_T and z_λ from (4.1) and (3.2).

Standard basis theorem— *The set $\{F_T z_\lambda : T \text{ is a semistandard } \lambda\text{-tableau}\}$ is a K-basis for $\Delta_q(\lambda)$.*

Proof: By definition $z_\lambda \in \Delta_q(\lambda)$. Let T_I and T_J be semistandard λ-tableaux and let $(,)_q$ be the non-degenerate contravariant form on $\Delta_q(\lambda) \times \nabla_q(\lambda)$ defined in (5.1). Then

$$\left(F_{T_I} z_\lambda, [T_J]\right)_q = \left(z_\lambda, E_{T_I}[T_J]\right)_q = (z_\lambda, \Omega_q(T_J, T_I)[T_\lambda] + \sum_M a_M[T_M])_q,$$

where each $[T_M]$ in the sum has weight different from λ. But for each M in the sum, we have $(z_\lambda, [T_M])_q = \langle z_\lambda, v_M \rangle_q = 0$. Thus

$$(F_{T_I} z_\lambda, [T_J])_q = (z_\lambda, \Omega_q(T_J, T_I)[T_\lambda])_q = \Omega_q(T_J, T_I)\langle z_\lambda, v_{J(\lambda)} \rangle_q = q^{\beta(J(\lambda))} \Omega_q(T_J, T_I).$$

So for each $J \in \mathcal{I}_\lambda$ we have $(F_{T_I} z_\lambda, [T_J])_q = q^{\beta(J(\lambda))} \Omega_q(T_J, T_I)$. Since $\Omega_q = [\Omega_q(T_J, T_I)]_{I \in \mathcal{I}_\lambda}$ is an upper triangular invertible matrix and the set of semistandard λ-tableaux forms a K-basis for $\nabla_q(\lambda)$, the set $\{F_T z_\lambda : T \text{ is a semistandard } \lambda\text{-tableau}\}$ is a K-basis for $\Delta_q(\lambda)$. $\square$

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