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Cyclic Descents and P-Partitions

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Abstract. Louis Solomon showed that the group algebra of the symmetric group $\mathfrak{S}_n$ has a subalgebra called the descent algebra, generated by sums of permutations with a given descent set. In fact, he showed that every Coxeter group has something that can be called a descent algebra. There is also a commutative, semisimple subalgebra of Solomon's descent algebra generated by sums of permutations with the same number of descents: an "Eulerian" descent algebra. For any Coxeter group that is also a Weyl group, Paola Cellini proved the existence of a different Eulerian subalgebra based on a modified definition of descent. We derive the existence of Cellini's subalgebra for the case of the symmetric group and of the hyperoctahedral group using a variation on Richard Stanley's theory of P-partitions.

Keywords: descent algebra, P-partition

1. Introduction

1.1. The symmetric group

The study of permutations by descent sets is a natural generalization of the study of permutations by number of descents: the study of Eulerian numbers. Let $\mathfrak{S}_n$ be the symmetric group on n elements. We think of permutations in $\mathfrak{S}_n$ as bijections

$$\pi : [n] \to [n],$$

where $[n]$ denotes the set $\{1, 2, \ldots, n\}$. For any permutation $\pi \in \mathfrak{S}_n$, we say π has a *descent* in position i if $\pi(i) > \pi(i+1)$. Define the set $\mathrm{Des}(\pi) = \{\, i \mid 1 \leq i \leq n-1, \pi(i) > \pi(i+1)\,\}$ and let $\mathrm{des}(\pi)$ denote the number of elements in $\mathrm{Des}(\pi)$. We call $\mathrm{Des}(\pi)$ the *descent set* of π, and $\mathrm{des}(\pi)$ the *descent number* of π. For example, the permutation $\pi = (\pi(1), \pi(2), \pi(3), \pi(4)) = (1, 4, 3, 2)$ has descent set $\{2, 3\}$ and descent number 2. The number of permutations of $[n]$ with descent number k is denoted by the Eulerian number $A_{n,k+1}$, and we recall that the Eulerian polynomial is defined as

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)+1} = \sum_{i=1}^{n} A_{n,i} t^i.$$

For each subset I of $[n-1]$, let u_I denote the sum, in the group algebra $\mathbb{Q}[\mathfrak{S}_n]$, of all permutations with descent set I. Louis Solomon [19] showed that the linear span of the

u_I forms a subalgebra of the group algebra, called the *descent algebra*. More generally, he showed that one can define a descent algebra for any Coxeter group.

For now consider the descent algebra of the symmetric group. This descent algebra was studied in great detail by Adriano Garsia and Christophe Reutenauer [12]. Jean-Louis Loday [15] proved the existence of a commutative, semisimple subalgebra of Solomon's descent algebra. Sometimes called the "Eulerian subalgebra," it is defined as follows. For $1 \le i \le n$, let E_i be the sum of all permutations in $\mathfrak{S}_n$ with descent number $i-1$. Then the Eulerian descent algebra is the linear span of the E_i. Define

$$\phi(x) = \sum_{\pi\in\mathfrak{S}_n} \binom{x+n-1-\operatorname{des}(\pi)}{n}\pi = \sum_{i=1}^{n} \binom{x+n-i}{n} E_i,$$

which we refer to as the "structure polynomial." Notice that the structure polynomial is a polynomial in x, with coefficients in $\mathbb{Q}[\mathfrak{S}_n]$, of degree n and with no constant term. (In other words, it has exactly as many nonzero terms as there are possible descent numbers.) The Eulerian subalgebra is described by the following multiplication of structure polynomials:

Theorem 1.1 (Gessel) *As polynomials in x and y with coefficients in the group algebra, we have*

$$\phi(x)\phi(y) = \phi(xy). \tag{1}$$

Define elements e_i in the group algebra by $\phi(x) = \sum_{i=1}^{n} e_i x^i$. By examining the coefficients of $x^i y^j$ in (1), it is clear that the e_i are orthogonal idempotents: $e_i^2 = e_i$ and $e_i e_j = 0$ if $i \ne j$. This formulation is essentially an unsigned version of Loday's Théorème 1.7 (see [15]). In particular, for $1 \le k \le n$, we have $\phi(k) = |\lambda^k|$, where Loday defines

$$\lambda^k = \sum_{i=0}^{k-1}(-1)^i \binom{n+i}{i} l^{k-i},$$
$$l^k = (-1)^{k-1} \sum_{\operatorname{des}\pi = k-1} \operatorname{sgn}(\pi)\pi,$$

and $|\lambda^k|$ means that we forget the signs on all the summands.

As an interesting aside, we remark that Theorem 1.1 has connections to the well-known card shuffling analysis of Dave Bayer and Persi Diaconis [2]. If we define

$$\bar{\phi}(x) = \sum_{\pi\in\mathfrak{S}_n} \binom{x+n-1-\operatorname{des}(\pi)}{n}\pi^{-1},$$

then $\bar{\phi}(2)$ is the generating function for the probability distribution for one shuffle of a deck of n cards (as defined in [2]). We use the multiplication rule given by Theorem 1.1 to compute the probability distribution after m (independent) shuffles of the deck (it is not hard to show that $\bar{\phi}(x)$ obeys the same multiplication rule as $\phi(x)$):

$$\underbrace{\bar{\phi}(2)\bar{\phi}(2)\ldots\bar{\phi}(2)}_{m\text{ times}} = \bar{\phi}(2^m),$$

which gives Theorem 1 from [2].

Theorem 1.1 can be proved in several ways, but we will focus on one that employs Richard Stanley's theory of P-partitions. More specifically, the approach taken in this paper follows from work of Ira Gessel—Theorem 1.1 is in fact an easy corollary of Theorem 11 from [14]. Later we will give a proof of Theorem 1.1 that derives from Gessel's work and then extend this method to prove the main results of this paper. While equivalent results can be found elsewhere in the literature, the P-partition approach is self-contained and rather elementary. Further, it often yields q-analogs for formulas like (1) quite naturally.

Now we introduce another descent algebra based on a modified definition of descent. For a permutation $\pi \in \mathfrak{S}_n$ we define a *cyclic descent* at position i if $\pi(i) > \pi(i+1)$, or if $i = n$ and $\pi(n) > \pi(1)$. Define $\mathrm{cDes}(\pi)$ to be the set of cyclic descent positions of π, called the *cyclic descent set*. Let the *cyclic descent number*, $\mathrm{cdes}(\pi)$, be the number of cyclic descents. The number of cyclic descents is between 1 and $n-1$. One can quickly observe that a permutation π has the same number of cyclic descents as both $\pi\omega^i$ and $\omega^i\pi$ for $i = 0, 1, \ldots, n-1$, where ω is the n-cycle $(1\ 2\ \cdots\ n)$. Define the *cyclic Eulerian polynomial* to be

$$A_n^{(c)}(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{cdes}(\pi)} = \sum_{i=1}^{n-1} A_{n,i}^{(c)} t^i,$$

where $A_{n,k}^{(c)}$ is the number of permutations with cyclic descent number k. We can make the following proposition (observed by Jason Fulman in [10], Corollary 1).

Proposition 1.1 *The cyclic Eulerian polynomial is expressible in terms of the ordinary Eulerian polynomial*:

$$A_n^{(c)}(t) = nA_{n-1}(t).$$

Proof: We will compare the coefficient of t^d on each side of the equation to show $A_{n,d}^{(c)} = nA_{n-1,d}$. Let $\pi \in \mathfrak{S}_{n-1}$ be any permutation of $[n-1]$ such that $\mathrm{des}(\pi)+1 = d$. Let $\tilde{\pi} \in \mathfrak{S}_n$ be the permutation defined by $\tilde{\pi}(i) = \pi(i)$ for $i = 1, 2, \ldots, n-1$ and $\tilde{\pi}(n) = n$. Then we have $\mathrm{des}(\tilde{\pi}) = \mathrm{des}(\pi)$ and $\mathrm{cdes}(\tilde{\pi}) = d$. Let $\langle\tilde{\pi}\rangle = \{\tilde{\pi}\omega^i \mid i = 0, 1, \ldots, n-1\}$, the set consisting of all n cyclic permutations of $\tilde{\pi}$. Every permutation in the set has exactly d cyclic descents. There is a bijection between permutations of $\mathfrak{S}_{n-1}$ and such subsets of $\mathfrak{S}_n$ given by the map

$$\pi \mapsto \langle\tilde{\pi}\rangle,$$

and so the proposition follows. □

Let $E_i^{(c)}$ be the sum in the group algebra of all those permutations with i cyclic descents. Then we define the cyclic structure polynomial

$$\varphi(x) = \frac{1}{n}\sum_{\pi\in\mathfrak{S}_n}\binom{x+n-1-\operatorname{cdes}(\pi)}{n-1}\pi = \frac{1}{n}\sum_{i=1}^{n-1}\binom{x+n-1-i}{n-1}E_i^{(c)}.$$

Notice that $\varphi(x)$ is a polynomial of degree $n-1$ with no constant term, giving as many nonzero terms as possible cyclic descent numbers. Paola Cellini studied cyclic descents more generally in the papers [6–8]. Though her approach is quite different from the one taken in this paper, from her work and Loday's we can derive the following theorem.

Theorem 1.2 *As polynomials in x and y with coefficients in the group algebra of the symmetric group, we have*

$$\varphi(x)\varphi(y) = \varphi(xy).$$

Now if we define elements $e_i^{(c)}$ by $\varphi(x) = \sum_{i=1}^{n-1} e_i^{(c)}x^i$, we see that $(e_i^{(c)})^2 = e_i^{(c)}$ and $e_i^{(c)}e_j^{(c)} = 0$ if $i \neq j$. Therefore the elements $e_i^{(c)}$ are orthogonal idempotents. Fulman relates cyclic descents to card shuffling (now we consider cuts as well as shuffles of the deck); Theorem 2 of [10] is closely related to Theorem 1.2. We will prove Theorem 1.2 in Section 2.2 using formula (1).

That the multiplication for the cyclic structure polynomials is so similar to that of the ordinary structure polynomials is no accident. Indeed, it will be clear from the proof of Theorem 1.2 that the map

$$\pi \mapsto \sum_{\sigma\in\langle\tilde{\pi}\rangle}\sigma,$$

where $\langle\tilde{\pi}\rangle$ is as in the proof of Proposition 1.1, gives an isomorphism of algebras between the ordinary Eulerian descent algebra of $\mathbb{Q}[\mathfrak{S}_{n-1}]$ and the cyclic Eulerian descent algebra of $\mathbb{Q}[\mathfrak{S}_n]$.

1.2. The hyperoctahedral group

Let $\pm[n]$ denote the set $\{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$. Let $\mathfrak{B}_n$ denote the hyperoctahedral group, the group of all bijections $\pi : \pm[n] \to \pm[n]$ with the property that $\pi(-s) = -\pi(s)$, for $s = 0, 1, \ldots, n$. Since the elements of the hyperoctahedral group are uniquely determined by where they map $1, 2, \ldots, n$, we can think of them as signed permutations. For a signed permutation $\pi \in \mathfrak{B}_n$ we will write $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$.

In moving from the symmetric group to the hyperoctahedral group, we define the *descent set* $\operatorname{Des}(\pi)$ of a signed permutation $\pi \in \mathfrak{B}_n$ to be the set of all $i \in \{0, 1, 2, \ldots, n-1\}$ such that $\pi(i) > \pi(i+1)$, where we always take $\pi(0) = 0$. The *descent number* of π is again denoted $\operatorname{des}(\pi)$ and is equal to the cardinality of $\operatorname{Des}(\pi)$.[1] As a simple example, the signed permutation $(-2, 1)$ has descent set $\{0\}$ and descent number 1.

There is an Eulerian subalgebra for the hyperoctahedral group, previously studied by Cellini [7, 8], Francois and Nantel Bergeron [4, 5], N. Bergeron [3], and Fulman [11]. For $1 \leq i \leq n+1$ let E_i be the sum of all permutations in $\mathfrak{B}_n$ with $i-1$ descents. Define the type B structure polynomial

$$\phi(x) = \sum_{\pi \in \mathfrak{B}_n} \binom{x+n-\operatorname{des}(\pi)}{n} \pi = \sum_{i=1}^{n+1} \binom{x+n+1-i}{n} E_i.$$

Chak-On Chow [9] was able to use the theory of P-partitions to prove the following.

Theorem 1.3 (Chow) *As polynomials in x and y with coefficients in the group algebra of $\mathfrak{B}_n$, we have*

$$\phi(x)\phi(y) = \phi(2xy + x + y),$$

or upon substituting

$$\begin{aligned} x &\leftarrow (x-1)/2, \\ y &\leftarrow (y-1)/2, \end{aligned}$$

then

$$\phi((x-1)/2)\phi((y-1)/2) = \phi((xy-1)/2).$$

We therefore have orthogonal idempotents e_i defined by $\phi((x-1)/2) = \sum_{i=0}^{n} e_i x^i$ (notice that shifting the polynomial by 1/2 gives a nontrivial constant term). Just as Theorem 1.1 was equivalent to Théorème 1.7 of [15], Theorem 1.3 is equivalent to Theorem 4.1 from [5]. Further, both Fulman [11] and Bergeron and Bergeron [5] made connections to "signed" card shuffling (flip a card face up or down to change sign).

For a permutation $\pi \in \mathfrak{B}_n$, we define a *type B cyclic descent*, or *augmented descent* at position i if $\pi(i) > \pi(i+1)$ or if $i = n$ and $\pi(n) > 0 = \pi(0)$. If we consider that signed permutations always begin with 0, then augmented descents are the natural generalization of type A cyclic descents.[2] The set of all augmented descent positions is denoted $\operatorname{aDes}(\pi)$, the *augmented descent set*. It is the ordinary descent set of π along with n if $\pi(n) > 0$. The *augmented descent number*, $\operatorname{ades}(\pi)$, is the number of augmented descents. With these definitions, $(-2, 1)$ has augmented descent set $\{0, 2\}$ and augmented descent number 2. Note that while $\operatorname{aDes}(\pi) \subset \{0, 1, \ldots, n\}$, $\operatorname{aDes}(\pi) \neq \emptyset$, and $\operatorname{aDes}(\pi) \neq \{0, 1, \ldots, n\}$. Denote the number of signed permutations in $\mathfrak{B}_n$ with k augmented descents by $A_{n,k}^{(a)}$ and define the *augmented Eulerian polynomial* as

$$A_n^{(a)}(t) = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{ades}(\pi)} = \sum_{i=1}^{n} A_{n,i}^{(a)} t^i.$$

In Section 3.1 we prove the following relation using the theory of P-partitions.

Proposition 1.2 *The number of signed permutations with $i+1$ augmented descents is 2^n times the number of unsigned permutations with i descents, $0 \leq i \leq n-1$. In other words, $A_n^{(a)}(t) = 2^n A_n(t)$.*

Define the type B cyclic structure polynomial as

$$\psi(x) = \sum_{\pi \in \mathfrak{B}_n} \binom{x+n-\mathrm{ades}(\pi)}{n} \pi = \sum_{i=1}^{n} \binom{x+n-i}{n} E_i^{(a)},$$

where $E_i^{(a)}$ is the sum of all signed permutations with i augmented descents.

Theorem 1.4 *As polynomials in x and y with coefficients in the group algebra of the hyperoctahedral group we have*

$$\psi(x)\psi(y) = \psi(2xy),$$

or upon substituting

$$x \leftarrow x/2,$$
$$y \leftarrow y/2,$$

then

$$\psi(x/2)\psi(y/2) = \psi(xy/2).$$

We get orthogonal idempotents $e_i^{(a)}$ defined by $\psi(x/2) = \sum_{i=1}^{n} e_i^{(a)} x^i$. We will prove Theorem 1.4 in Section 3.2 using modified types of P-partitions called *augmented P-partitions*. Also in Section 3.2 we prove the following theorem for which we have no type A equivalent.

Theorem 1.5 *As polynomials in x and y with coefficients in the group algebra of the hyperoctahedral group we have*

$$\psi(x)\phi(y) = \psi(2xy + x),$$

or upon substituting

$$x \leftarrow x/2,$$
$$y \leftarrow (y-1)/2,$$

then

$$\psi(x/2)\phi((y-1)/2) = \psi(xy/2).$$

This formula tells us that $e_i^{(a)} e_i = e_i^{(a)}$ and that $e_i^{(a)} e_j = 0$ if $i \neq j$. In other words, the augmented Eulerian descent algebra is an ideal in the subalgebra of the group algebra formed by the span of the e_i and the $e_i^{(a)}$. This relationship shows up again in the case of peak algebras of type A, and forms the basis of future work as discussed in Section 5. See the paper of Marcelo Aguiar, N. Bergeron, and Kathryn Nyman [1] for more. We also point out here that when taken together, Theorems 1.3, 1.4, and 1.5, imply Cellini's Theorem A from [7]. Indeed, for positive integers k, Cellini's elements x_{2k+1} are equivalent to $\bar{\phi}(k)$, and the x_{2k+2} are equivalent to $\bar{\psi}(k)$. Here, as in Section 1.1, $\bar{\phi}$ and $\bar{\psi}$ are the same as ϕ and ψ except that we replace the coefficient of π with the coefficient of π^{-1}. See also Lemma 4 of [11].

2. The cyclic descent algebra

2.1. Definitions

Throughout this paper we will use P to denote a partially ordered set, or poset. The partial order on P is denoted $<_P$, or simply $<$ when the meaning is clear. Our posets will always be finite and for a poset of n elements, the elements will be distinctly labeled by the numbers $1, 2, \ldots, n$.

Definition 2.1 Let $X = \{x_1, x_2, \ldots\}$ be a countable, totally ordered set. For a given poset P, a *P-partition*[3] is a function $f : [n] \to X$ such that:

- $f(i) \leq f(j)$ if $i <_P j$
- $f(i) < f(j)$ if $i <_P j$ and $i > j$ in $\mathbb{Z}$

For our purposes we usually think of X as a subset of the positive integers. Let $\mathcal{A}(P)$ denote the set of all P-partitions. When X has finite cardinality k, then the number of P-partitions is finite. In this case, define the *order polynomial*, denoted $\Omega_P(k)$, to be the number of P-partitions $f : [n] \to X$.

We will consider any permutation $\pi \in \mathfrak{S}_n$ to be a poset with the total order $\pi(s) <_\pi \pi(s+1)$. For example, the permutation $\pi = (3, 2, 1, 4)$ has $3 <_\pi 2 <_\pi 1 <_\pi 4$ as a poset. With this convention, the set of all π-partitions is easily characterized. The set $\mathcal{A}(\pi)$ is the

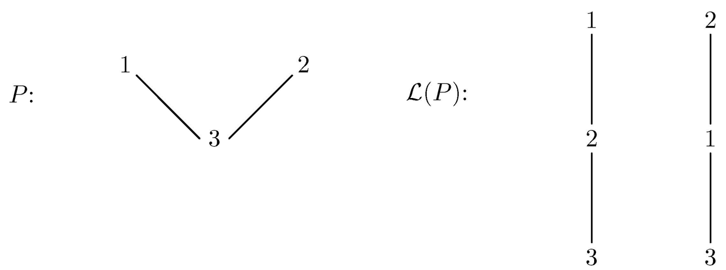

Figure 1. Linear extensions of a poset P.

set of all functions $f : [n] \to X$ such that (if we take X to be the positive integers)

$$1 \leq f(\pi_1) \leq f(\pi_2) \leq \cdots \leq f(\pi_n),$$

and whenever $\pi(s) > \pi(s+1)$, then $f(\pi(s)) < f(\pi(s+1))$, $s = 1, 2, \ldots, n-1$. The set of all π-partitions where $\pi = (3, 2, 1, 4)$ is all maps f such that $1 \leq f(3) < f(2) < f(1) \leq f(4)$.

For a poset P of order n, let $\mathcal{L}(P)$ denote the set of all permutations of $[n]$ which extend P to a total order. This set is sometimes called the set of "linear extensions" of P. For example let P be the poset defined by $1 >_P 3 <_P 2$. In "linearizing" P we form a total order by retaining all the relations of P but introducing new relations so that any element is comparable to any other. See Figure 1. In this case, 1 and 2 are not comparable, so we have exactly two ways of linearizing P: $3 < 2 < 1$ and $3 < 1 < 2$. These correspond to the permutations (3, 2, 1) and (3, 1, 2). Let us make the following observation.

Observation 2.1 *A permutation π is in $\mathcal{L}(P)$ if and only if $i <_P j$ implies $\pi^{-1}(i) < \pi^{-1}(j)$.*

In other words, if i is "below" j in the Hasse diagram of the poset P, it had better be below j in any linear extension of the poset. We also now prove what is sometimes called the fundamental theorem (or lemma) of P-partitions.

Theorem 2.1 (FTPP) *The set of all P-partitions of a poset P is the disjoint union of the set of π-partitions of all linear extensions π of P :*

$$\mathcal{A}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{A}(\pi).$$

Proof: The proof follows from induction on the number of incomparable pairs of elements of P. If there are no incomparable pairs, then P has a total order and already represents a permutation. Suppose i and j are incomparable in P. Let P_{ij} be the poset formed from P by introducing the relation $i < j$. Then it is clear that $\mathcal{A}(P) = \mathcal{A}(P_{ij}) \coprod \mathcal{A}(P_{ji})$. We continue to split these posets (each with strictly fewer incomparable pairs) until we have a collection of totally ordered chains corresponding to distinct linear extensions of P. $\square$

Corollary 2.1

$$\Omega_P(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega_\pi(k).$$

The fundamental theorem tells us that in order to study P-partitions for a given poset, we can focus on the π-partitions for its linear extensions—a more straightforward task. In particular, counting π-partitions is not too difficult. Notice that for any permutation π and any positive integer k,

$$\binom{k+n-1-\mathrm{des}(\pi)}{n} = \left(\!\!\binom{k-\mathrm{des}(\pi)}{n}\!\!\right),$$

where $\left(\!\binom{a}{b}\!\right)$ denotes the "multi-choose" function—the number of ways to choose b objects from a set of a objects with repetitions. Another interpretation of $\left(\!\binom{a}{b}\!\right)$ is the number of integer solutions to the set of inequalities

$$1 \le i_1 \le i_2 \le \cdots \le i_b \le a.$$

With this in mind, $\binom{k+n-1-\mathrm{des}(\pi)}{n}$ is the same as the number of solutions to

$$1 \le i_1 \le i_2 \le \cdots \le i_n \le k - \mathrm{des}(\pi).$$

Better still, we can say it is the number of solutions (though not in general the same *set* of solutions) to

$$1 \le i_1 \le i_2 \le \cdots \le i_n \le k \quad \text{and } i_s < i_{s+1} \text{ if } s \in \mathrm{Des}(\pi). \tag{2}$$

For example, the number of solutions to $1 \le i < j \le 4$ is the same as the number of solutions to $1 \le i \le j-1 \le 4-1$ or the solutions to $1 \le i \le j' \le 3$. In general, if we take $f(\pi(s)) = i_s$ it is clear that f is a π-partition if and only if $(i_1, i_2, \ldots, i_n)$ is a solution to (2), which gives

$$\Omega_\pi(k) = \binom{k+n-1-\mathrm{des}(\pi)}{n}.$$

Before moving on, let us point out that in order to prove that the formulas in this paper hold as polynomials in x and y, it will suffice to prove that they hold for all pairs of positive integers. It is not hard to prove this fact, and we use it in each of the proofs presented in this paper.

2.2. Proofs for the case of the symmetric group

We will now prove Theorem 1.1 using the theory of P-partitions.

Proof of Theorem 1.1: If we write out $\phi(xy) = \phi(x)\phi(y)$ using the definition, we have

$$\begin{aligned}
&\sum_{\pi \in \mathfrak{S}_n} \binom{xy + n - 1 - \operatorname{des}(\pi)}{n} \pi \\
&\quad = \sum_{\sigma \in \mathfrak{S}_n} \binom{x + n - 1 - \operatorname{des}(\sigma)}{n} \sigma \sum_{\tau \in \mathfrak{S}_n} \binom{y + n - 1 - \operatorname{des}(\tau)}{n} \tau \\
&\quad = \sum_{\sigma, \tau \in \mathfrak{S}_n} \binom{x + n - 1 - \operatorname{des}(\sigma)}{n} \binom{y + n - 1 - \operatorname{des}(\tau)}{n} \sigma\tau
\end{aligned}$$

If we equate the coefficients of π we have

$$\binom{xy + n - 1 - \operatorname{des}(\pi)}{n} = \sum_{\sigma\tau = \pi} \binom{x + n - 1 - \operatorname{des}(\sigma)}{n} \binom{y + n - 1 - \operatorname{des}(\tau)}{n}. \tag{3}$$

Clearly, if formula (3) holds for all π, then formula (1) is true. Let us interpret the left hand side of this equation.

Let $x = k$, and $y = l$ be positive integers. Then the left hand side of equation (3) is just the order polynomial $\Omega_\pi(kl)$. To compute this order polynomial we need to count the number of π-partitions $f : [n] \to X$, where X is some totally ordered set with kl elements. But instead of using $[kl]$ as our image set, we will use a different totally ordered set of the same cardinality. Let us count the π-partitions $f : [n] \to [l] \times [k]$. This is equal to the number of solutions to

$$\begin{aligned}
&(1, 1) \le (i_1, j_1) \le (i_2, j_2) \le \cdots \le (i_n, j_n) \le (l, k) \quad \text{and} \\
&(i_s, j_s) < (i_{s+1}, j_{s+1}) \quad \text{if } s \in \operatorname{Des}(\pi).
\end{aligned} \tag{4}$$

Here we take the *lexicographic ordering* on pairs of integers. Specifically, $(i, j) < (i', j')$ if $i < i'$ or else if $i = i'$ and $j < j'$.

To get the result we desire, we will sort the set of all solutions to (4) into distinct cases indexed by subsets $I \subset [n - 1]$. The sorting depends on π and proceeds as follows. Let $F = ((i_1, j_1), \ldots, (i_n, j_n))$ be any solution to (4). For any $s = 1, 2, \ldots, n - 1$, if $\pi(s) < \pi(s + 1)$, then $(i_s, j_s) \le (i_{s+1}, j_{s+1})$, which falls into one of two mutually exclusive cases:

$$i_s \le i_{s+1} \quad \text{and } j_s \le j_{s+1}, \quad \text{or} \tag{5}$$

$$i_s < i_{s+1} \quad \text{and } j_s > j_{s+1}. \tag{6}$$

If $\pi(s) > \pi(s+1)$, then $(i_s, j_s) < (i_{s+1}, j_{s+1})$, which means either:

$$i_s \leq i_{s+1} \quad \text{and } j_s < j_{s+1}, \quad \text{or} \tag{7}$$
$$i_s < i_{s+1} \quad \text{and } j_s \geq j_{s+1}, \tag{8}$$

also mutually exclusive. Define $I_F = \{s \in [n-1] \setminus \mathrm{Des}(\pi) \mid j_s > j_{s+1}\} \cup \{s \in \mathrm{Des}(\pi) \mid j_s \geq j_{s+1}\}$. Then I_F is the set of all s such that either (6) or (8) holds for F. Notice that in both cases, $i_s < i_{s+1}$. Now for any $I \subset [n-1]$, let S_I be the set of all solutions F to (5) satisfying $I_F = I$. We have split the solutions of (4) into 2^{n-1} distinct cases indexed by all the different subsets I of $[n-1]$.

Say $\pi = (2, 1, 3)$. Then we want to count the number of solutions to

$$(1, 1) \leq (i_1, j_1) < (i_2, j_2) \leq (i_3, j_3) \leq (l, k),$$

which splits into four distinct cases:

$$\begin{aligned}
\emptyset &: i_1 \leq i_2 \leq i_3 \quad && j_1 < j_2 \leq j_3\\
\{1\} &: i_1 < i_2 \leq i_3 \quad && j_1 \geq j_2 \leq j_3\\
\{2\} &: i_1 \leq i_2 < i_3 \quad && j_1 < j_2 > j_3\\
\{1, 2\} &: i_1 < i_2 < i_3 \quad && j_1 \geq j_2 > j_3.
\end{aligned}$$

We now want to count all the solutions contained in each of these cases and add them up. For a fixed subset I we will use the theory of P-partitions to count the number of solutions for the set of inequalities first for the j_s's and then for the i_s's. Multiplying will give us the number of solutions in S_I; we do the same for the remaining subsets and sum to obtain the final result. For $I = \{1\}$ in the example above, we would count first the number of integer solutions to $j_1 \geq j_2 \leq j_3$, with $1 \leq j_s \leq k$, and then we multiply this number by the number of solutions to $1 \leq i_1 < i_2 \leq i_3 \leq l$ to obtain the cardinality of $S_{\{1\}}$. We will now carry out the computation in general.

For any particular $I \subset [n-1]$, form the poset P_I of the elements $1, 2, \ldots, n$ by $\pi(s) <_{P_I} \pi(s+1)$ if $s \notin I$, $\pi(s) >_{P_I} \pi(s+1)$ if $s \in I$. We form a "zig-zag" poset of n elements labeled consecutively by $\pi(1), \pi(2), \ldots, \pi(n)$, with downward zigs corresponding to the elements of I. For example, if $I = \{2, 3\}$ for $n = 5$, then P_I has $\pi(1) < \pi(2) > \pi(3) > \pi(4) < \pi(5)$. See Figure 2.

For any solution in S_I, let $f : [n] \to [k]$ be defined by $f(\pi(s)) = j_s$ for $1 \leq s \leq n$. We will show that f is a P_I-partition. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (5) tells us that $f(\pi(s)) = j_s \leq j_{s+1} = f(\pi(s+1))$. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (7) tells us that $f(\pi(s)) = j_s < j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (6) gives us that $f(\pi(s)) = j_s > j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (8) gives us that $f(\pi(s)) = j_s \geq j_{s+1} = f(\pi(s+1))$. In other words, we have verified that f is a P_I-partition. So for any particular solution in S_I, the j_s's can be thought of as a P_I-partition. Conversely, any P_I-partition f gives a solution in S_I since if $j_s = f(\pi(s))$, then $((i_1, j_1), \ldots, (i_n, j_n)) \in S_I$ if and only if $1 \leq i_1 \leq \cdots \leq i_n \leq l$ and $i_s < i_{s+1}$ for all $i \in I$. We can therefore turn our attention to counting P_I-partitions.

P_I:

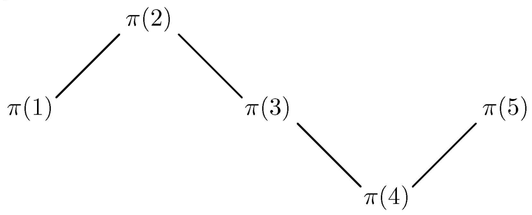

Figure 2. The "zig-zag" poset P_I for $I = \{2, 3\} \subset [5]$.

Let $\sigma \in \mathcal{L}(P_I)$. Then for any σ-partition f, we get a chain

$$1 \le f(\sigma(1)) \le f(\sigma(2)) \le \cdots \le f(\sigma(n)) \le k$$

with $f(\sigma(s)) < f(\sigma(s+1))$ if $s \in \mathrm{Des}(\sigma)$. The number of solutions to this set of inequalities is

$$\Omega_\sigma(k) = \binom{k+n-1-\mathrm{des}(\sigma)}{n}.$$

Recall by Observation 2.1 that $\sigma^{-1}\pi(s) < \sigma^{-1}\pi(s+1)$ if $\pi(s) <_{P_I} \pi(s+1)$, i.e., if $s \notin I$. If $\pi(s) >_{P_I} \pi(s+1)$ then $\sigma^{-1}\pi(s) > \sigma^{-1}\pi(s+1)$ and $s \in I$. We get that $\mathrm{Des}(\sigma^{-1}\pi) = I$ if and only if $\sigma \in \mathcal{L}(P_I)$. Set $\tau = \sigma^{-1}\pi$. The number of solutions to

$$1 \le i_1 \le \cdots \le i_n \le l \quad \text{and} \quad i_s < i_{s+1} \quad \text{if } s \in \mathrm{Des}(\tau)$$

is given by

$$\Omega_\tau(l) = \binom{l+n-1-\mathrm{des}(\tau)}{n}.$$

Now for a given I, the number of solutions in S_I is

$$\sum_{\substack{\sigma \in \mathcal{L}(P_I) \\ \sigma\tau = \pi}} \binom{k+n-1-\mathrm{des}(\sigma)}{n} \binom{l+n-1-\mathrm{des}(\tau)}{n}.$$

Summing over all subsets $I \subset [n-1]$, we can write the number of all solutions to (4) as

$$\sum_{\sigma\tau = \pi} \binom{k+n-1-\mathrm{des}(\sigma)}{n} \binom{l+n-1-\mathrm{des}(\tau)}{n},$$

and so we have derived formula (3). $\square$

We now have a taste of how P-partitions can be used. We are ready to go on and prove Theorem 1.2.

Proof of Theorem 1.2: If we write out the definition for $\varphi(x)$ in the statement of Theorem 1.2, multiply both sides by n^2, and equate coefficients, we have for any $\pi \in \mathfrak{S}_n$,

$$n\binom{xy+n-1-\operatorname{cdes}(\pi)}{n-1} = \sum_{\sigma\tau=\pi}\binom{x+n-1-\operatorname{cdes}(\sigma)}{n-1}\binom{y+n-1-\operatorname{cdes}(\tau)}{n-1}.$$

For some i, we can write $\pi = \nu\omega^i$ where ω is the n-cycle $(\,1\ 2\ \ldots\ n\,)$ and $\nu = (n, \nu(2), \ldots, \nu(n))$. Observe that $\operatorname{cdes}(\pi) = \operatorname{cdes}(\nu) = \operatorname{des}(\nu)$. Form the permutation $\hat{\nu} \in \mathfrak{S}_{n-1}$ by $\hat{\nu}(s) = \nu(s+1)$, $s = 1, 2, \ldots, n-1$. Then we can see that $\operatorname{cdes}(\pi) = \operatorname{des}(\hat{\nu}) + 1$. We have

$$\binom{xy+n-1-\operatorname{cdes}(\pi)}{n-1} = \binom{xy+(n-1)-1-\operatorname{des}(\hat{\nu})}{n-1}.$$

Now we can apply Eq. (3) to give us

$$\begin{aligned}&\binom{xy+(n-1)-1-\operatorname{des}(\hat{\nu})}{n-1} \\ &\quad= \sum_{\sigma\tau=\hat{\nu}}\binom{x+(n-1)-1-\operatorname{des}(\sigma)}{n-1}\binom{y+(n-1)-1-\operatorname{des}(\tau)}{n-1}.\end{aligned} \tag{9}$$

For each pair of permutations $\sigma, \tau \in \mathfrak{S}_{n-1}$ such that $\sigma\tau = \hat{\nu}$, define the permutations $\tilde{\sigma}, \tilde{\tau} \in \mathfrak{S}_n$ as follows. For $s = 1, 2, \ldots, n-1$, let $\tilde{\sigma}(s) = \sigma(s)$ and $\tilde{\tau}(s+1) = \tau(s)$. Let $\tilde{\sigma}(n) = n$ and $\tilde{\tau}(1) = n$. Then by construction we have $\tilde{\sigma}\tilde{\tau} = \nu$ and a quick observation tells us that $\operatorname{cdes}(\tilde{\sigma}) = \operatorname{des}(\sigma) + 1$ and $\operatorname{cdes}(\tilde{\tau}) = \operatorname{des}(\tau) + 1$. On the other hand, from any pair of permutations $\tilde{\sigma}, \tilde{\tau} \in \mathfrak{S}_n$ such that $\tilde{\sigma}\tilde{\tau} = \nu$, $\tilde{\sigma}(n) = n$, we can construct a pair of permutations $\sigma, \tau \in \mathfrak{S}_{n-1}$ such that $\sigma\tau = \hat{\nu}$ by reversing the process. Observe now that if $\tilde{\sigma}(n) = n$ and $\tilde{\sigma}\tilde{\tau} = \nu$, then $\tilde{\tau}(1) = n$. Therefore we have that (9) is equal to

$$\begin{aligned}&\sum_{\substack{\tilde{\sigma}\tilde{\tau}=\nu\\ \tilde{\sigma}(n)=n}}\binom{x+n-1-\operatorname{cdes}(\tilde{\sigma})}{n-1}\binom{y+n-1-\operatorname{cdes}(\tilde{\tau})}{n-1} \\ &= \sum_{\substack{\tilde{\sigma}(\tilde{\tau}\omega^i)=\pi\\ \tilde{\sigma}(n)=n}}\binom{x+n-1-\operatorname{cdes}(\tilde{\sigma})}{n-1}\binom{y+n-1-\operatorname{cdes}(\tilde{\tau}\omega^i)}{n-1} \\ &= \sum_{\substack{(\tilde{\sigma}\omega^{n-j})(\omega^j\tilde{\tau}\omega^i)=\pi\\ \tilde{\sigma}(n)=n}}\binom{x+n-1-\operatorname{cdes}(\tilde{\sigma}\omega^{n-j})}{n-1}\binom{y+n-1-\operatorname{cdes}(\omega^j\tilde{\tau}\omega^i)}{n-1} \\ &= \sum_{\substack{\sigma\tau=\pi\\ \sigma(j)=n}}\binom{x+n-1-\operatorname{cdes}(\sigma)}{n-1}\binom{y+n-1-\operatorname{cdes}(\tau)}{n-1},\end{aligned}$$

where the last two formulas hold for any $j \in [n]$. Notice that the number of cyclic descents of $\tau = \omega^j \tilde{\tau} \omega^i$ is still the same as the number of cyclic descents of $\tilde{\tau}$. We take the sum over all $j = 1, \ldots, n$, yielding

$$n\binom{xy + n - 1 - \operatorname{cdes}(\pi)}{n-1} = \sum_{\sigma\tau=\pi} \binom{x + n - 1 - \operatorname{cdes}(\sigma)}{n-1}\binom{y + n - 1 - \operatorname{cdes}(\tau)}{n-1}$$

as desired. □

3. The augmented descent algebra

3.1. Definitions and observations

Here we present the definitions and basic results we will need to prove the remaining theorems. Proofs of some of these basic facts are identical to the proofs of analogous statements for ordinary P-partitions and may be omitted. It bears mentioning that the following definitions for type B posets and type B P-partitions, though taken from Chow [9], derive from earlier work by Victor Reiner [18]. In [17], Reiner extends this concept to any Coxeter group.

Definition 3.1 A $\mathfrak{B}_n$ *poset* is a poset P whose elements are $0, \pm 1, \pm 2, \ldots, \pm n$ such that if $i <_P j$ then $-j <_P -i$.

Note that if we are given a poset with $n + 1$ elements labeled by $0, a_1, \ldots, a_n$ where $a_i = i$ or $-i$, then we can extend it to a $\mathfrak{B}_n$ poset of $2n + 1$ elements. See Figures 3 and 4.

Let $X = \{x_0, x_1, x_2, \ldots\}$ be a countable, totally ordered set with total order

$$x_0 < x_1 < x_2 < \cdots.$$

Then define $\pm X$ to be the set $\{\ldots, -x_1, x_0, x_1, \ldots\}$ with total order

$$\cdots < -x_2 < -x_1 < x_0 < x_1 < x_2 < \cdots.$$

Definition 3.2 For any $\mathfrak{B}_n$ poset P, a *P-partition of type B* is a function $f : \pm[n] \to \pm X$ such that:

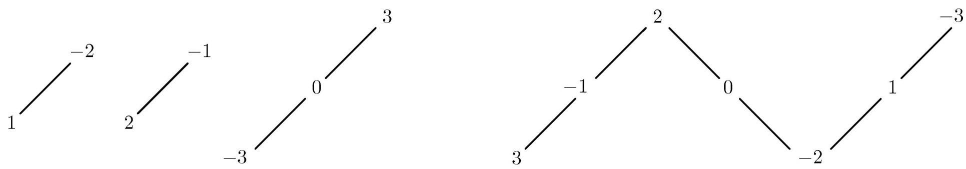

Figure 3. Two $\mathfrak{B}_3$ posets.

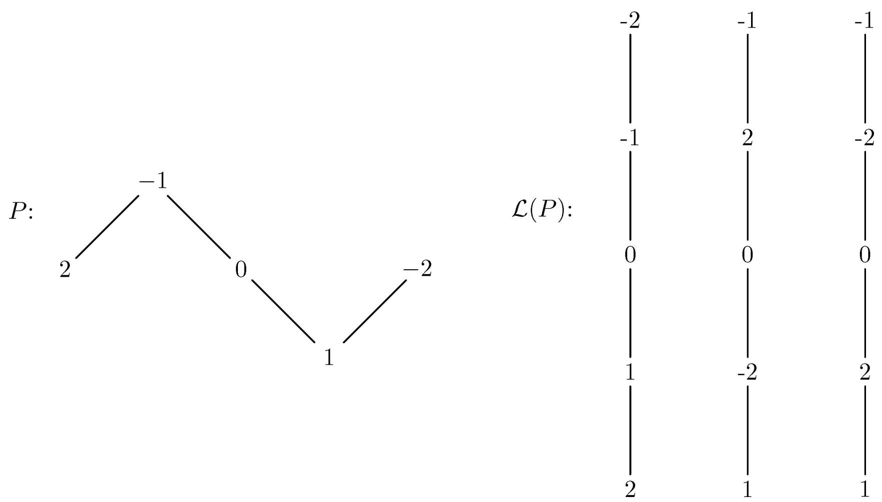

Figure 4. Linear extensions of a $\mathfrak{B}_2$ poset P.

- $f(i) \leq f(j)$ if $i <_P j$
- $f(i) < f(j)$ if $i <_P j$ and $i > j$ in $\mathbb{Z}$
- $f(-i) = -f(i)$

Note that type B P-partitions differ from ordinary P-partitions only in the addition of the property $f(-i) = -f(i)$. Let $\mathcal{A}(P)$ denote the set of all type B P-partitions. We usually think of X as a subset of the nonnegative integers, and when X has finite cardinality $k + 1$, then the *type B order polynomial*, denoted $\Omega_P(k)$, is the number of P-partitions $f : \pm[n] \to \pm X$.

We can think of any signed permutation $\pi \in \mathfrak{B}_n$ as a $\mathfrak{B}_n$ poset with the total order $\pi(s) <_\pi \pi(s+1)$, $0 \leq s \leq n-1$. For example, the signed permutation $(-2, 1)$ has $-1 <_\pi 2 <_\pi 0 <_\pi -2 <_\pi 1$ as a poset. Note that $\mathcal{A}(\pi)$ is the set of all functions $f : \pm[n] \to \pm X$ such that for $0 \leq s \leq n$, $f(-s) = -f(s)$ and

$$x_0 = f(\pi(0)) \leq f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)),$$

where $f(\pi(s)) < f(\pi(s+1))$ whenever $\pi(s) > \pi(s+1)$, $s = 0, 1, \ldots, n-1$. For example, the type B π-partitions where $\pi = (-2, 1)$ are all maps f such that $x_0 < f(-2) \leq f(1)$. As before,

Observation 3.1 *For a type B poset P, a signed permutation π is in $\mathcal{L}(P)$ if and only if $i <_P j$ implies $\pi^{-1}(i) < \pi^{-1}(j)$.*

We have a fundamental theorem for P-partitions of type B.

Theorem 3.1 (FTPPB) *The set of all type B P-partitions of a $\mathfrak{B}_n$ poset P is the disjoint union of the set of π-partitions of all linear extensions π of P:*

$$\mathcal{A}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{A}(\pi).$$

Corollary 3.1

$$\Omega_P(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega_\pi(k).$$

Similarly to the type A case, it is easy to compute the order polynomial $\Omega_\pi(k)$ for any permutation $\pi \in \mathfrak{B}_n$. Any π-partition $f : \pm[n] \to \pm[k]$ is determined by where we map $\pi(1), \pi(2), \ldots, \pi(n)$ since $f(-i) = -f(i)$ (and so $f(0) = 0$). To count them we can take $f(\pi(s)) = i_s$, and look at the number of integer solutions to the set of inequalities

$$0 \le i_1 \le i_2 \le \cdots \le i_n \le k, \quad \text{and } i_s < i_{s+1} \text{ if } s \in \mathrm{Des}(\pi).$$

This number is the same as the number of solutions to

$$1 \le i_1 \le i_2 \le \cdots \le i_n \le k + 1 - \mathrm{des}(\pi),$$

which we know to be $\left(\!\binom{k+1-\mathrm{des}(\pi)}{n}\!\right)$. We have

$$\Omega_\pi(k) = \binom{k + n - \mathrm{des}(\pi)}{n}.$$

Now we have the tools to prove Theorem 1.3.

Proof of Theorem 1.3: We will omit the details, since they are essentially the same as for Theorem 1.1. The main difference is that we want to count π-partitions $f : \pm[n] \to \pm[l] \times \pm[k]$. We notice that because of the property $f(-s) = -f(s)$, this is just like counting all $f : [n] \to \{0, 1, \ldots, l\} \times \{-k, \ldots, -1, 0, 1, \ldots, k\}$ where for $s = 1, 2, \ldots, n$, $f(\pi(s)) = (i_s, j_s)$ with $(0, 0) \le (i_s, j_s) \le (l, k)$ in the lexicographic order. See Figure 5.

This choice of image set X has $2kl + k + l + 1$ elements, and so for each π we can count all these maps with $\Omega_\pi(2kl + k + l) = \binom{2kl+k+l-\mathrm{des}(\pi)}{n}$. We use similar arguments to those of Theorem 1.1 for splitting the lexicographic solutions to

$$(0, 0) \le (i_1, j_1) \le \cdots \le (i_n, j_n) \le (l, k) \quad \text{and } (i_s, j_s) < (i_{s+1}, j_{s+1}) \text{ if } s \in \mathrm{Des}(\pi).$$

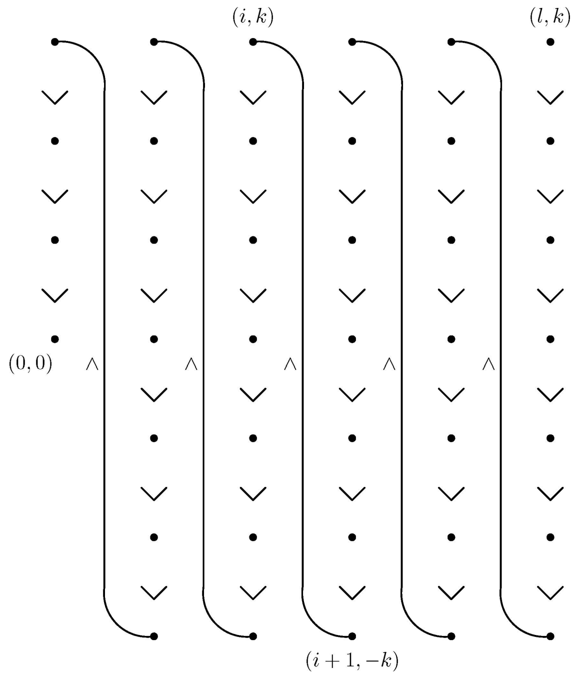

Figure 5. The lexicographic order on $\{0, 1, \ldots, l\} \times \{-k, \ldots, -1, 0, 1, \ldots, k\}$ with $(0, 0) \leq (i_s, j_s) \leq (l, k)$.

Once we have properly grouped the set of solutions it is not much more work to obtain the crucial formula:

$$\Omega_\pi(2kl + k + l) = \sum_{\sigma\tau=\pi} \Omega_\sigma(k)\Omega_\tau(l).$$

□

We now give the definition of an augmented P-partition and basic tools related to their study. Let $X = \{x_0, x_1, \ldots, x_\infty\}$ be a countable, totally ordered set with a maximal element x_∞. The total ordering on X is given by

$$x_0 < x_1 < x_2 < \cdots < x_\infty.$$

Define $\pm X$ to be $\{-x_\infty, \ldots, -x_1, x_0, x_1, \ldots, x_\infty\}$ with the total order

$$-x_\infty < \cdots < -x_1 < x_0 < x_1 < \cdots < x_\infty.$$

Definition 3.3 For any $\mathfrak{B}_n$ poset P, an *augmented P-partition* is a function $f : \pm[n] \to \pm X$ such that:

- $f(i) \leq f(j)$ if $i <_P j$
- $f(i) < f(j)$ if $i <_P j$ and $i > j$ in $\mathbb{Z}$
- $f(-i) = -f(i)$
- if $0 < i$ in $\mathbb{Z}$, then $f(i) < x_\infty$.

Note that augmented P-partitions differ from P-partitions of type B only in the addition of maximal and minimal elements of the image set $\pm X$ and in the last criterion. Let $\mathcal{A}^{(a)}(P)$ denote the set of all augmented P-partitions. When X has finite cardinality $k+1$ (and so $\pm X$ has cardinality $2k+1$), then the *augmented order polynomial*, denoted $\Omega_P^{(a)}(k)$, is the number of augmented P-partitions.

For any signed permutation $\pi \in \mathfrak{B}_n$, note that $\mathcal{A}^{(a)}(\pi)$ is the set of all functions $f : \pm[n] \to \pm X$ such that for $0 \leq s \leq n$, $f(-s) = -f(s)$ and

$$x_0 = f(\pi(0)) \leq f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)) \leq x_\infty,$$

and $f(\pi(s)) < f(\pi(s+1))$ whenever $\pi(s) > \pi(s+1)$. In addition, we have $f(\pi(n)) < x_\infty$ whenever $\pi(n) > 0$. The set of all augmented π-partitions where $\pi = (-2, 1)$ is all maps f such that $x_0 < f(-2) \leq f(1) < x_\infty$.

We have a fundamental theorem for augmented P-partitions.

Theorem 3.2 (FTAPP) *The set of all augmented P-partitions of a $\mathfrak{B}_n$ poset P is the disjoint union of the set of π-partitions of all linear extensions π of P :*

$$\mathcal{A}^{(a)}(P) = \coprod_{\pi \in \mathcal{L}(P)} \mathcal{A}^{(a)}(\pi).$$

Corollary 3.2

$$\Omega_P^{(a)}(k) = \sum_{\pi \in \mathcal{L}(P)} \Omega_\pi^{(a)}(k).$$

As in our previous cases, it is fairly easy to compute the augmented order polynomial for totally ordered chains. The number of augmented π-permutations $f : \pm[n] \to \pm[k]$ is equal to the number of integer solutions to the set of inequalities

$$0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq k = i_{n+1}, \quad \text{and } i_s < i_{s+1} \quad \text{if } s \in \mathrm{aDes}(\pi).$$

Therefore, just as with other types of order polynomials, we can express the augmented order polynomial as a binomial coefficient,

$$\Omega_\pi^{(a)}(k) = \binom{k+n-\mathrm{ades}(\pi)}{n}.$$

To give one example of the usefulness of augmented P-partitions, we conclude this subsection with proof of Proposition 1.2, claiming $A_n^{(a)}(t) = 2^n A_n(t)$.

Proof of Proposition 1.2: From the general theory of P-partitions in Stanley's book [20], we have

$$\sum_{k\geq 0} \Omega_P(k)t^k = \frac{\sum_{\pi\in\mathcal{L}(P)} t^{\mathrm{des}(\pi)+1}}{(1-t)^{|P|+1}}.$$

Let P be an antichain—that is, a poset with no relations—of n elements. Then $\Omega_P(k) = k^n$ since each of the n elements of P is free to be mapped to any of k places. Furthermore, $\mathcal{L}(P) = \mathfrak{S}_n$, so we get the following equation,

$$\sum_{k\geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Now let P be the poset given by an antichain of $2n+1$ elements labeled $0, \pm 1, \pm 2, \ldots, \pm n$. The number of augmented P-partitions $f : \pm[n] \to \pm[k]$ is determined by the choices for $f(1), f(2), \ldots, f(n)$, which can take any of the values in $\{-k, -k+1, \ldots, k-2, k-1\}$. Therefore $\Omega_P^{(a)}(k) = (2k)^n$. For $\mathfrak{B}_n$ posets P, it is not difficult to show that we have the identity

$$\sum_{k\geq 0} \Omega_P^{(a)}(k)t^k = \frac{\sum_{\pi\in\mathcal{L}(P)} t^{\mathrm{ades}(\pi)}}{(1-t)^{n+1}}.$$

For our antichain we have $\mathcal{L}(P) = \mathfrak{B}_n$, and therefore

$$\frac{A_n^{(a)}(t)}{(1-t)^{n+1}} = \sum_{k\geq 0}(2k)^n t^k = 2^n \sum_{k\geq 0} k^n t^k = \frac{2^n A_n(t)}{(1-t)^{n+1}},$$

so the proposition is proved. □

3.2. *Proofs for the case of the hyperoctahedral group*

The following proofs will follow the same basic structure as the proof of Theorem 1.1, but with some important changes in detail. In both cases we will rely on a slightly different total ordering on the integer points (i, j), where i and j are bounded both above and below. Let us now define the *augmented lexicographic order*. See Figure 6.

Consider all points (i, j) with $0 \leq i \leq l$, $-k \leq j \leq k$. We have $(i, j) < (i', j')$ if $i < i'$ or else if $i = i'$ and $j < j'$ as before, except in the important special case that follows. We now say $(i, j) = (i', j')$ in one of two situations. Either

$$i = i' \text{ and } j = j'$$

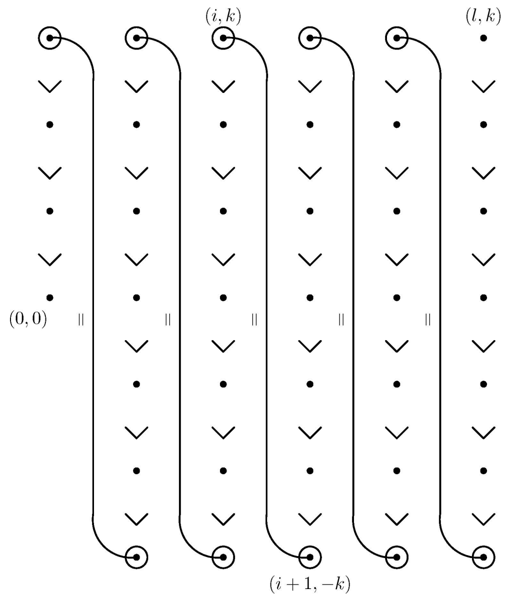

Figure 6. The augmented lexicographic order.

or

$$i+1=i' \text{ and } j=k=-j'.$$

If we have $0 \leq i \leq l$, $-2 \leq j \leq 2$, then in augmented lexicographic order, the first few points $(0,0) \leq (i,j) \leq (l,2)$ are:

$$\begin{aligned}(0,0) < (0,1) < (0,2)& \\ &= (1,-2) < (1,-1) < (1,0) \\ &< (1,1) < (1,2) \\ &= (2,-2) < (2,-1) < (2,0) < \cdots\end{aligned}$$

To be more precise, what we have done is to form equivalence classes of points and to introduce a total order on these equivalence classes. If $j \neq \pm k$, then the class represented by (i,j) is just the point itself. Otherwise, the classes consist of the two points (i,k) and $(i+1,-k)$. When we write $(i,j)=(i',j')$, what we mean is that the two points are in the same equivalence class. In the proofs that follow, it will be important to remember the original points as well as the equivalence classes to which they belong. This special ordering will be very apparent in deriving the q-analogs of Theorem 1.4 and Theorem 1.5.

We will now prove Theorem 1.4.

Proof of Theorem 1.4: As before, we equate coefficients and prove that a simpler formula,

$$\binom{2kl+n-\mathrm{ades}(\pi)}{n} = \sum_{\sigma\tau=\pi} \binom{k+n-\mathrm{ades}(\sigma)}{n}\binom{l+n-\mathrm{ades}(\tau)}{n}, \tag{10}$$

holds for any $\pi \in \mathfrak{B}_n$.

We recognize the left-hand side of Eq. (10) as $\Omega_\pi^{(a)}(2kl)$, so we want to count augmented P-partitions $f : \pm[n] \to \pm X$, where X is a totally ordered set of order $2kl+1$. We interpret this as the number of solutions, in the augmented lexicographic ordering, to

$$(0,0) \le (i_1, j_1) \le (i_2, j_2) \le \cdots \le (i_n, j_n) \le (l, 0), \tag{11}$$

where we have

- $0 \le i_s \le l$,
- $-k < j_s \le k$ if $\pi(s) < 0$,
- $-k \le j_s < k$ if $\pi(s) > 0$, and
- $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if $s \in \mathrm{aDes}(\pi)$.

Let us clarify. There are $2kl + l + 1$ points (i, j) with $0 \le i \le l$ and $-k \le j \le k$, not including the points $(0, j)$ with $j < 0$, or the points (l, j) with $j > 0$. Under the augmented lexicographic ordering, l of these points are identified: points of the form $(i, k) = (i+1, -k)$, for $i = 0, 1, \ldots, l-1$. Any particular (i_s, j_s) can only occupy one of (i, k) or $(i+1, -k)$, but not both. So there are truly $2kl + 1$ distinct classes in which the n points can fall. This confirms our interpretation of the order polynomial.

Now as before, we will split the solutions to the inequalities into distinct cases. Let $\pi(0) = \pi(n+1) = 0$, $i_0 = j_0 = 0$, $i_{n+1} = l$, and $j_{n+1} = 0$. Let $F = ((i_1, j_1), \ldots, (i_n, j_n))$ be any solution to (11). If $\pi(s) < \pi(s+1)$, then $(i_s, j_s) \le (i_{s+1}, j_{s+1})$, which falls into one of two mutually exclusive cases:

$$i_s \le i_{s+1} \quad \text{and } j_s \le j_{s+1}, \text{ or} \tag{12}$$
$$i_s < i_{s+1} \quad \text{and } j_s > j_{s+1}. \tag{13}$$

If $\pi(s) > \pi(s+1)$, then $(i_s, j_s) < (i_{s+1}, j_{s+1})$, which we split as:

$$i_s \le i_{s+1} \quad \text{and } j_s < j_{s+1}, \text{ or} \tag{14}$$
$$i_s < i_{s+1} \quad \text{and } j_s \ge j_{s+1}, \tag{15}$$

also mutually exclusive. Define $I_F = \{s \in \{0, 1, \ldots, n\} \setminus \mathrm{aDes}(\pi) \mid j_s > j_{s+1}\} \cup \{s \in \mathrm{aDes}(\pi) \mid j_s \ge j_{s+1}\}$. Then I_F is the set of all s such that either (13) or (15) holds for F. Now for any $I \subset \{0, 1, \ldots, n\}$, let S_I be the set of all solutions F to (11) satisfying $I_F = I$. We have split the solutions of (11) into 2^{n+1} distinct cases indexed by all the different subsets I of $\{0, 1, \ldots, n\}$.

However, $S_\emptyset$ is empty, since

$$0 \le i_1 \le \cdots \le i_n \le l$$

yields

$$0 \le j_1 \le \cdots \le j_n \le 0 \quad \text{with } j_s < j_{s+1} \quad \text{if } s \in \mathrm{aDes}(\pi).$$

As discussed before, the augmented descent set of a signed permutation is never empty, so we would get $0 < 0$, a contradiction. At the other extreme, the set $S_{\{0,1,\ldots,n\}}$ has no solutions either. Here we get

$$0 < i_1 < \cdots < i_n < l$$

and consequently

$$0 \ge j_1 \ge \cdots \ge j_n \ge 0 \quad \text{with } j_s > j_{s+1} \quad \text{if } s \notin \mathrm{aDes}(\pi).$$

But $\mathrm{aDes}(\pi)$ cannot equal $\{0, 1, \ldots, n\}$, so we get the contradiction $0 > 0$.

Now let I be any nonempty, proper subset of $\{0, 1, \ldots, n\}$. Form the poset P_I by $\pi(s) >_{P_I} \pi(s+1)$ if $s \in I$, $\pi(s) <_{P_I} \pi(s+1)$ otherwise. The poset P_I looks like a zig-zag, labeled consecutively by $0 = \pi(0), \pi(1), \pi(2), \ldots, \pi(n), 0 = \pi(n+1)$ with downward zigs corresponding to the elements of I. Because I is neither empty nor full, we never have $0 <_{P_I} 0$, so P_I is a well-defined, nontrivial type B poset.

For a given $F \in S_I$, let $f : \pm[n] \to \pm[k]$ be defined by $f(\pi(s)) = j_s$ and $f(-s) = -f(s)$ for $s = 0, 1, \ldots, n$. We will show that f is an augmented P_I partition. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (12) tells us that $f(\pi(s)) = j_s \le j_{s+1} = f(\pi(s+1))$. If $\pi(s) <_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (14) tells us that $f(\pi(s)) = j_s < j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) < \pi(s+1)$ in $\mathbb{Z}$, then (13) gives us that $f(\pi(s)) = j_s > j_{s+1} = f(\pi(s+1))$. If $\pi(s) >_{P_I} \pi(s+1)$ and $\pi(s) > \pi(s+1)$ in $\mathbb{Z}$, then (15) gives us that $f(\pi(s)) = j_s \ge j_{s+1} = f(\pi(s+1))$. Since we required that $-k < j_s \le k$ if $\pi(s) < 0$ and $-k \le j_s < k$ if $\pi(s) > 0$, we have that for any particular solution in S_I, the j_s's can be thought of as an augmented P_I-partition. Conversely, any augmented P_I-partition f gives a solution in S_I since if $j_s = f(\pi(s))$, then $((i_1, j_1), \ldots, (i_n, j_n)) \in S_I$ if and only if $0 \le i_1 \le \cdots \le i_n \le l$ and $i_s < i_{s+1}$ for all $i \in I$. We can therefore turn our attention to counting augmented P_I-partitions.

Let $\sigma \in \mathcal{L}(P_I)$. Then we get for any σ-partition f,

$$0 \le f(\sigma(1)) \le f(\sigma(2)) \le \cdots \le f(\sigma(n)) \le k,$$

and $f(\sigma(s)) < f(\sigma(s+1))$ whenever $s \in \mathrm{aDes}(\sigma)$, where we take $f(\sigma(n+1)) = k$. The number of solutions to this set of inequalities is

$$\Omega_\sigma^{(a)}(k) = \binom{k + n - \mathrm{ades}(\sigma)}{n}.$$

Recall by Observation 3.1 that $\sigma^{-1}\pi(s) < \sigma^{-1}\pi(s+1)$ if $\pi(s) <_{P_I} \pi(s+1)$, i.e., if $s \notin I$. If $\pi(s) >_{P_I} \pi(s+1)$ then $\sigma^{-1}\pi(s) > \sigma^{-1}\pi(s+1)$ and $s \in I$. We get that $\mathrm{aDes}(\sigma^{-1}\pi) = I$ if and only if $\sigma \in \mathcal{L}(P_I)$. Set $\tau = \sigma^{-1}\pi$. The number of solutions to

$$0 \le i_1 \le \cdots \le i_n \le l \quad \text{and } i_s < i_{s+1} \quad \text{if } s \in \mathrm{aDes}(\tau)$$

is given by

$$\Omega_\tau(l) = \binom{l + n - \mathrm{ades}(\tau)}{n}.$$

Now for a given I, the number of solutions to (11) is

$$\sum_{\substack{\sigma\in\mathcal{L}(P_I)\\ \sigma\tau=\pi}} \binom{k + n - \mathrm{ades}(\sigma)}{n}\binom{l + n - \mathrm{ades}(\tau)}{n}.$$

Summing over all subsets $I \subset \{0, 1, \dots, n\}$, we can write the number of all solutions to (11) as

$$\sum_{\sigma\tau=\pi} \binom{k + n - \mathrm{ades}(\sigma)}{n}\binom{l + n - \mathrm{ades}(\tau)}{n},$$

and so the theorem is proved. □

The proof of Theorem 1.5 proceeds nearly identically, so we will omit unimportant details in the proof below.

Proof of Theorem 1.5: We equate coefficients and prove that

$$\binom{2kl + k + n - \mathrm{ades}(\pi)}{n} = \sum_{\sigma\tau=\pi} \binom{k + n - \mathrm{ades}(\sigma)}{n}\binom{l + n - \mathrm{des}(\tau)}{n}, \tag{16}$$

holds for any $\pi \in \mathfrak{B}_n$.

We recognize the left-hand side of Eq. (16) as $\Omega_\pi^{(a)}(2kl+k)$, so we want to count augmented P-partitions $f : \pm[n] \to \pm X$, where X is a totally ordered set of order $2kl + k + 1$. We interpret this as the number of solutions, in the augmented lexicographic ordering, to

$$(0, 0) \le (i_1, j_1) \le (i_2, j_2) \le \cdots \le (i_n, j_n) \le (l, k), \tag{17}$$

where we have

- $0 \le i_s \le l$,
- $-k < j_s \le k$ if $\pi(s) < 0$,
- $-k \le j_s < k$ if $\pi(s) > 0$, and
- $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if $s \in \mathrm{aDes}(\pi)$.

With these restrictions, we split the solutions to (17) by our prior rules. Let $F = ((i_1, j_1), \dots, (i_n, j_n))$ be any particular solution. If $\pi(s) < \pi(s+1)$, then $(i_s, j_s) \le (i_{s+1}, j_{s+1})$, which falls into one of two mutually exclusive cases:

$$\begin{aligned} &i_s \le i_{s+1} \quad \text{and} \quad j_s \le j_{s+1}, \quad \text{or} \\ &i_s < i_{s+1} \quad \text{and} \quad j_s > j_{s+1}. \end{aligned}$$

If $\pi(s) > \pi(s+1)$, then $(i_s, j_s) < (i_{s+1}, j_{s+1})$, giving:

$$\begin{aligned} &i_s \le i_{s+1} \quad \text{and} \quad j_s < j_{s+1}, \quad \text{or} \\ &i_s < i_{s+1} \quad \text{and} \quad j_s \ge j_{s+1}, \end{aligned}$$

also mutually exclusive. With (i_n, j_n), there is only one case, depending on π. If $\pi(n) > 0$, then $(i_n, j_n) < (l, k)$ and $i_n \le l$ and $-k \le j_n < k$. Similarly, if $\pi(n) < 0$, then $(i_n, j_n) \le (l, k)$ and we have $i_n \le l$ and $-k < j_n \le k$. Define I_F and S_I as before. We get 2^n mutually exclusive sets S_I indexed by subsets $I \subset \{0, 1, \dots, n-1\}$ (these subsets will correspond to ordinary descent sets).

Now for any $I \subset \{0, 1, \dots, n-1\}$, define the $\mathfrak{B}_n$ poset P_I to be the poset given by $\pi(s) >_{P_I} \pi(s+1)$ if $s \in I$, and $\pi(s) <_{P_I} \pi(s+1)$ if $s \notin I$, for $s = 0, 1, \dots, n-1$. We form a zig-zag poset labeled consecutively by $\pi(0) = 0, \pi(1), \pi(2), \dots, \pi(n)$.

For a given solution $F \in S_I$, let $f : \pm[n] \to \pm[k]$ be defined by $f(\pi(s)) = j_s$ for $0 \le s \le n$, with $f(-s) = -f(s)$. It is not too difficult to check that f is an augmented P_I-partition, and that any augmented P_I-partition corresponds to a solution in S_I. Let $\sigma \in \mathcal{L}(P_I)$. Then for any σ-partition f we get

$$f(\sigma(0)) = 0 \le f(\sigma(1)) \le \cdots \le f(\sigma(n)) \le k,$$

with $f(\sigma(s)) < f(\sigma(s+1))$ whenever $s \in \mathrm{aDes}(\sigma)$. The number of solutions to this set of inequalities is

$$\Omega_\sigma^{(a)}(k) = \binom{k + n - \mathrm{ades}(\sigma)}{n}.$$

We see that for $s = 0, 1, \dots, n-1$, $\sigma^{-1}\pi(s) < \sigma^{-1}\pi(s+1)$ if $\pi(s) <_{P_I} \pi(s+1)$, i.e., if $s \notin I$. Also, if $\pi(s) >_{P_I} \pi(s+1)$ then $\sigma^{-1}\pi(s) > \sigma^{-1}\pi(s+1)$ and $s \in I$. This time we get that $\mathrm{Des}(\sigma^{-1}\pi) = I$, an ordinary descent set, if and only if $\sigma \in \mathcal{L}_{P_I}$. Set $\tau = \sigma^{-1}\pi$. The number of solutions to

$$0 \le i_1 \le \cdots \le i_n \le l \quad \text{and} \quad i_s < i_{s+1} \text{ if } s \in \mathrm{Des}(\tau)$$

is given by

$$\Omega_\tau(l) = \binom{l + n - \mathrm{des}(\tau)}{n}.$$

We take the sum over all subsets I to show the number of solutions to (17) is

$$\sum_{\sigma\tau=\pi} \binom{k+n-\mathrm{ades}(\sigma)}{n}\binom{l+n-\mathrm{des}(\tau)}{n},$$

and the theorem is proved. □

4. Some q-analogs

In this section we give some q-analogs of our previous theorems. First we define the q-variant of the (type A) order polynomial, or q-order polynomial.[4] Let P be a poset with n elements, and take $X = \{0, 1, \ldots, k-1\}$. Then,

$$\Omega_P(q;k) = \sum_{f\in\mathcal{A}(P)} \left(\prod_{i=1}^{n} q^{f(i)}\right).$$

Let $n_q! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ and define the *q-binomial coefficient* $\binom{a}{b}_q$ in the natural way:

$$\binom{a}{b}_q = \frac{a_q!}{b_q!(a-b)_q!}$$

Another way to interpret the q-binomial coefficient is as the coefficient of $x^b y^{a-b}$ in $(x+y)^a$ where x and y "q-commute" via the relation $yx = qxy$. This interpretation is good for some purposes, but we will use a different point of view. We will define the q-multi-choose function $\left(\!\binom{a}{b}\!\right)_q = \binom{a+b-1}{b}_q$ as the following:

$$\left(\!\!\binom{a}{b}\!\!\right)_q = \sum_{0\le i_1\le\cdots\le i_b\le a-1} \left(\prod_{s=1}^{n} q^{i_s}\right) = \Omega_{id}(q;a),$$

where id is the totally ordered chain $1 <_P 2 <_P \cdots <_P b$, the identity permutation in $\mathfrak{S}_b$. In this case, the q-order polynomial counts certain integer partitions (so there is at least some overlap between P-partitions and the more familiar integer partitions). Specifically, the coefficient of q^r is the number of integer partitions of r with at most b parts, each of size at most $a-1$.[5]

Now we will obtain formulas for q-order polynomials for all permutations. For any permutation $\pi \in \mathfrak{S}_n$, the q-order polynomial may be expressed as

$$\Omega_\pi(q;k) = \sum_{\substack{0\le i_1\le\cdots\le i_n\le k-1\\ s\in\mathrm{Des}(\pi)\Rightarrow i_s<i_{s+1}}} \left(\prod_{s=1}^{n} q^{i_s}\right).$$

When we computed the ordinary order polynomial in Section 3.1 we only cared about the *number* of solutions, rather than the *set* of solutions to the inequalities

$$0 \leq i_1 \leq \cdots \leq i_n \leq k-1 \quad \text{and } i_s < i_{s+1} \quad \text{if } s \in \operatorname{Des}(\pi). \tag{18}$$

Since we only cared how many solutions there were and not what the solutions were, we could count solutions to a system where all the inequalities were weak. We will still follow the same basic procedure, but as we manipulate our system of inequalities we need to keep track of how we modify the set of solutions. The q-order polynomial will be seen to be simply a power of q (depending on π) times a q-binomial coefficient.

We can form a new system of inequalities that has the same number of solutions as (18), but in which every inequality is weak:

$$0 \leq i_1' \leq \cdots \leq i_n' \leq k-1-\operatorname{des}(\pi).$$

The bijection between these sets of solutions is given by $i_s' = i_s - a(s)$ where $a(s)$ is the number of descents to the left of s. Therefore the q-order polynomial is given by

$$\begin{aligned}
\Omega_\pi(q;k) &= \sum_{\substack{0\leq i_1\leq\cdots\leq i_n\leq k-1\\ s\in\operatorname{Des}(\pi)\Rightarrow i_s<i_{s+1}}} \left(\prod_{s=1}^{n} q^{i_s}\right)\\
&= \sum_{0\leq i_1'\leq\cdots\leq i_n'\leq k-1-\operatorname{des}(\pi)} \left(\prod_{s=1}^{n} q^{i_s'+a(s)}\right)\\
&= q^{\sum_{s=1}^{n} a(s)} \cdot \left(\sum_{0\leq i_1'\leq\cdots\leq i_n'\leq k-1-\operatorname{des}(\pi)} \left(\prod_{s=1}^{n} q^{i_s'}\right)\right).
\end{aligned}$$

The sum of all $a(s)$ can be expressed as $\sum_{s\in\operatorname{Des}(\pi)}(n-s)$, which is sometimes referred to as the *comajor index*, denoted $\operatorname{comaj}(\pi)$.[6] The rest of the sum is now recognizable as a q-binomial coefficient. In summary, we have

$$\Omega_\pi(q;k) = q^{\operatorname{comaj}(\pi)} \binom{k+n-1-\operatorname{des}(\pi)}{n}_q. \tag{19}$$

Now we will give a q-analog of Theorem 1.1. We would like to have a q-analog for Theorem 1.2 as well, but if such a theorem exists, it does not seem to be immediate given the method of proof in this paper. Recall that the proof of Theorem 1.2 hinged on the observation that cyclically permuting a permutation $\pi \in \mathfrak{S}_n$ leaves the number of cyclic descents unchanged: $\operatorname{cdes}(\pi) = \operatorname{cdes}(\pi\omega^i)$ for any $i = 0, 1, \ldots, n-1$. Unfortunately, cyclically permuting π has a more subtle effect on the comajor index.

Moving on, we define the q-version of the structure polynomial,

$$\phi(q;x) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{comaj}(\pi)} \binom{x+n-1-\mathrm{des}(\pi)}{n}_q \pi.$$

Theorem 4.1 *As polynomials in x and y (and q) with coefficients in the group algebra we have*

$$\phi(q;x)\phi(q^x;y) = \phi(q;xy).$$

Proof: The proof will follow nearly identical lines of reasoning as in the ordinary ($q = 1$) case. Here we sketch the proof with emphasis on the major differences. Again, we will decompose the coefficient of π:

$$q^{\mathrm{comaj}(\pi)} \binom{kl+n-1-\mathrm{des}(\pi)}{n}_q$$

By (19), we have that the coefficient of π is the order polynomial $\Omega_\pi(q;kl)$ so we will examine the π-partitions $f : [n] \to \{0, 1, \dots, l-1\} \times \{0, 1, \dots, k-1\}$. Notice that we are still mapping into a set with kl elements. As before we impose the lexicographic ordering on this image set. To ensure that we keep the proper powers of q, we think of the order polynomial now as:

$$\Omega_\pi(q;kl) = \sum_{\substack{(0,0)\le(i_1,j_1)\le\cdots\le(i_n,j_n)\le(l-1,k-1)\\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} \left(\prod_{s=1}^{n} q^{ki_s+j_s}\right).$$

We have given each point (i, j) the weight $ki + j$ so that the weight corresponds to the position of the point in the lexicographic ordering on $\{0, 1, \dots, l-1\} \times \{0, 1, \dots, k-1\}$. We now proceed exactly as in the proof of Theorem 1.1. See Figure 7.

$$\begin{aligned}
\Omega_\pi(q;kl) &= \sum_{\substack{(0,0)\le(i_1,j_1)\le\cdots\le(i_n,j_n)\le(l-1,k-1)\\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} \left(\prod_{s=1}^{n} q^{ki_s+j_s}\right)\\
&= \sum_{I\subset[n-1]} \left(\sum_{\substack{0\le i_1\le\cdots\le i_n\le l-1\\ s\in I\Rightarrow i_s<i_s+1}} q^{ki_s}\right)\left(\sum_{\sigma\in\mathcal{L}(P_I)} \Omega_\sigma(q;k)\right)\\
&= \sum_{\sigma\tau=\pi} \Omega_\sigma(q;k)\Omega_\tau(q^k;l),
\end{aligned}$$

as desired. □

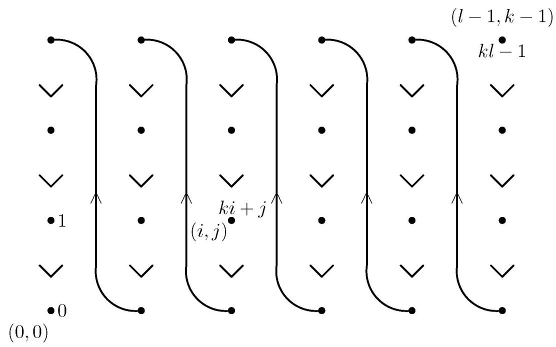

Figure 7. Weights in the lexicographic order.

In order to give a q-analog of Theorem 1.3, we define the q-order polynomial for a signed permutation $\pi \in \mathfrak{B}_n$ as

$$\Omega_\pi(q;k) = \sum_{\substack{0\le i_1\le\cdots\le i_n\le k\\ s\in\mathrm{Des}(\pi)\Rightarrow i_s<i_{s+1}}} \left(\prod_{s=1}^{n} q^{i_s}\right),$$

which as before can be rewritten,

$$\begin{aligned} &= \sum_{0\le i'_1\le\cdots\le i'_n\le k-\mathrm{des}(\pi)} \left(\prod_{s=1}^{n} q^{i'_s+a(s)}\right)\\ &= q^{\sum_{s=1}^{n} a(s)} \cdot \left(\sum_{0\le i'_1\le\cdots\le i'_n\le k-\mathrm{des}(\pi)} \left(\prod_{s=1}^{n} q^{i'_s}\right)\right)\\ &= q^{\mathrm{comaj}(\pi)} \binom{k+n-\mathrm{des}(\pi)}{n}_q. \end{aligned}$$

Now we define the type B q-structure polynomial,

$$\phi(q;x) = \sum_{\pi\in\mathfrak{B}_n} q^{\mathrm{comaj}(\pi)} \binom{x+n-\mathrm{des}(\pi)}{n}_q \pi.$$

Theorem 4.2 *The following relation holds as polynomials in x and y (and q) with coefficients in the group algebra of the hyperoctahedral group*:

$$\phi(q;x)\phi(q^{2x+1};y) = \phi(q;2xy+x+y),$$

or upon substituting

$$x \leftarrow (x-1)/2,$$
$$y \leftarrow (y-1)/2,$$

then

$$\phi(q;(x-1)/2)\phi(q^x;(y-1)/2) = \phi(q;(xy-1)/2).$$

Proof: We will omit most of the details, but the crucial step is to keep the proper exponent on q. We give to each point (i, j) the weight $(2k+1)i + j$ so that the weight corresponds to the position of the point in the lexicographic order on the set $\{0, 1, \ldots, l\} \times \{-k, \ldots, -1, 0, 1, \ldots, k\}$. The proof is outlined in two steps below. For any π and any pair of positive integers k, l,

$$\begin{aligned}\Omega_\pi(q; 2kl+k+l) &= \sum_{\substack{(0,0)\le(i_1,j_1)\le\cdots\le(i_n,j_n)\le(l,k)\\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} \left(\prod_{s=1}^n q^{(2k+1)i_s+j_s}\right)\\ &= \sum_{\sigma\tau=\pi} \Omega_\sigma(q;k)\Omega_\tau(q^{2k+1};l).\end{aligned}$$

□

The augmented version of the q-order polynomial gives us q-analogs for Theorem 1.4 and Theorem 1.5. For a signed permutation $\pi \in \mathfrak{B}_n$ we have

$$\Omega_\pi^{(a)}(q;k) = \sum_{\substack{0\le i_1\le\cdots\le i_n\le k\\ s\in\mathrm{aDes}(\pi)\Rightarrow i_s<i_{s+1}}} \left(\prod_{s=1}^n q^{i_s}\right) = q^{\mathrm{acomaj}(\pi)}\binom{k+n-\mathrm{ades}(\pi)}{n}_q,$$

where if $a(s)$ is the number of descents of π to the left of s, then the *augmented comajor index*, acomaj(π), is the sum over all s of the numbers $a(s)$. Define

$$\psi(q;x) = \sum_{\pi\in\mathfrak{B}_n} q^{\mathrm{acomaj}(\pi)}\binom{x+n-\mathrm{ades}(\pi)}{n}_q \pi.$$

Theorem 4.3 *As polynomials in x and y (and q) with coefficients in the group algebra of the hyperoctahedral group we have*

$$\psi(q;x)\psi(q^{2x};y) = \psi(q;2xy),$$

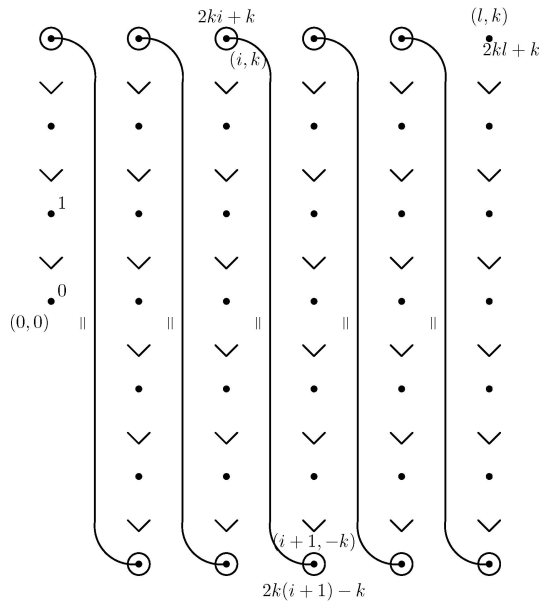

Figure 8. Weights in the augmented lexicographic order.

or upon substituting

$$x \leftarrow x/2,$$
$$y \leftarrow y/2,$$

then

$$\psi(q; x/2)\psi(q^x; y/2) = \psi(q; xy/2).$$

Proof: Here the key is to give the integer pairs (i, j) the proper weight in the augmented lexicographic ordering. If we take $2ki + j$ as the weight of the point (i, j) then we get that the points (i, k) and $(i + 1, -k)$ have the same weight. See Figure 8. So the weight does indeed correspond to the position of (i, j) in the augmented lexicographic ordering. Everything else follows as in the proof of Theorem 1.4. For any $\pi \in \mathfrak{B}_n$ and any pair of positive integers k, l,

$$\begin{aligned}\Omega_\pi^{(a)}(q; 2kl) &= \sum_{\substack{(0,0)\leq(i_1,j_1)\leq\cdots\leq(i_n,j_n)\leq(l,0)\\ s\in\mathrm{aDes}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} \left(\prod_{s=1}^{n} q^{2ki_s+j_s}\right)\\ &= \sum_{\sigma\tau=\pi} \Omega_\sigma^{(a)}(q; k)\Omega_\tau^{(a)}(q^{2k}; l).\end{aligned}$$

□

Theorem 4.4 *As polynomials in x and y (and q) with coefficients in the group algebra of the hyperoctahedral group we have*

$$\psi(q;x)\phi(q^{2x};y) = \psi(q;2xy+x),$$

upon substituting

$$x \leftarrow x/2,$$
$$y \leftarrow (y-1)/2,$$

then

$$\psi(q;x/2)\phi(q^{x};(y-1)/2) = \psi(q;xy/2).$$

Proof: Because we exploit the augmented lexicographic order in the proof of Theorem 1.5 (the $q = 1$ case), we will use the same weighting scheme as in the proof of Theorem 4.3 for the points (i, j). We have for any $\pi \in \mathfrak{B}_n$ and any pair of positive integers k, l,

$$\begin{aligned}\Omega_\pi^{(a)}(q;2kl+k) &= \sum_{\substack{(0,0)\le(i_1,j_1)\le\cdots\le(i_n,j_n)\le(l,k)\\ s\in\mathrm{Des}(\pi)\Rightarrow(i_s,j_s)<(i_{s+1},j_{s+1})}} \left(\prod_{s=1}^{n} q^{2ki_s+j_s}\right)\\ &= \sum_{\sigma\tau=\pi} \Omega_\sigma^{(a)}(q;k)\Omega_\tau(q^{2k};l).\end{aligned}$$

□

5. Future work

We hope to be able to apply the method of P-partitions to study descent algebras more generally. In his thesis work, Vic Reiner [17] defined P-partitions purely in terms of the root system of a Coxeter group. His definition may provide a way to obtain a theorem that generalizes Theorems 1.1 and 1.3 to any Coxeter group.

Another avenue that we have already begun to investigate is the study of *peak algebras*. A *peak* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i-1) < \pi(i) > \pi(i+1)$. Kathryn Nyman [16] showed that the span of sums of permutations with the same set of peaks forms a subalgebra of the group algebra. Later, Aguiar, Bergeron, and Nyman [1] explored many properties of this and related peak algebras for the symmetric group, including connections to descent algebras of types B and D. A key tool in [16] was John Stembridge's *enriched* P-partitions [21]. In forthcoming work, we use Stembridge's tool to study subalgebras formed by the span of sums of permutations with the same *number* of peaks, obtaining results similar to the main theorems of this paper.

A meta-question for this area of research is: what is so special about descents (or peaks)? Why should this way of grouping permutations give rise to a subalgebra at all? In terms of permutation patterns, a descent is an occurrence of the pattern 21, and a peak is an occurrence of the pattern 132 or 231. What distinguishes descents and peaks from, say, grouping permutations according to instances of 1324 (which doesn't give a subalgebra)?

Notes

1. It will be clear from the context whether we are referring to the descent set of an ordinary permutation or that of a signed permutation.
2. Most generally, Cellini [6] uses the term "descent in zero" to represent this concept for any Weyl group. The term "augmented descent" was coined by Gessel, [13]. Conceptually, the name "type B cyclic descent" makes more sense, but in this paper we use the "augmented" label, primarily because it requires fewer characters to typeset.
3. We note that this definition varies from Richard Stanley's [20] in that our maps are order-preserving, whereas his are order reversing, i.e., $f(i) \geq f(j)$ if $i <_P j$. However, in [20] the maps of Definition 2.1 are called (perhaps misleadingly) *reverse* P-partitions.
4. Properly speaking, this q-analog of the order polynomial is not a polynomial in k. However, we will refer to it as the "q-order polynomial," even if it might be more appropriate to call it the "q-analog of the order polynomial."
5. It appears Fulman may have been aware of a q-analog of his shuffling results, though he never states q-analogs of his main theorems. See [10], Definitions 3 and 5.
6. The *major* index of a permutation is $\sum_{s\in \mathrm{Des}(\pi)} s$. Indeed, had we made our P-partitions order reversing, as in Stanley's original definition of a P-partition, we would have gotten $q^{\mathrm{maj}(\pi)}$ rather than comaj above.

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