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Modular Data: The Algebraic Combinatorics of Conformal Field Theory

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Abstract. This paper is primarily intended as an introduction for mathematicians to some of the rich algebraic combinatorics arising in for instance conformal field theory (CFT). It tries to refine, modernise, and bridge the gap between papers [6] and [55]. Our paper is essentially self-contained, apart from some of the background motivation (Section 1) and examples (Section 3) which are included to give the reader a sense of the context. Detailed proofs will appear elsewhere. The theory is still a work-in-progress, and emphasis is given here to several open questions and problems.

Keywords: fusion ring, modular data, conformal field theory, affine Kac-Moody algebra

1. Introduction

In Segal's axioms of CFT [105], any Riemann surface with boundary is assigned a certain linear homomorphism. Roughly speaking, Borcherds [21] and Frenkel-Lepowsky-Meurman [53] axiomatised this data corresponding to a sphere with 3 disks removed, and the result is called a vertex operator algebra. Here we do the same with the data corresponding to a torus (and to a lesser extent a cylinder). The result is considerably simpler, as we shall see.

Moonshine in its more general sense involves the assignment of modular (automorphic) functions or forms to certain algebraic structures, e.g. theta functions to lattices, or vector-valued Jacobi forms to affine algebras, or Hauptmoduls to the Monster. This paper explores an important facet of Moonshine theory: the associated modular group representation. From this perspective, *Monstrous* Moonshine [22] is maximally uninteresting: the corresponding representation is completely trivial!

Let's focus now on the former context. It is unfortunate but unavoidable that this introductory section contains many terms most readers will find unfamiliar. This section is motivational, supplying some of the background physical context, and many of the terms here will be mathematically addressed in later sections. It is intended to be skimmed.

A rational conformal field theory (RCFT) has two vertex operator algebras (VOAs) $\mathcal{V}$, $\mathcal{V}'$. For simplicity we will take them to be isomorphic (otherwise the RCFT is called 'heterotic'). The VOA $\mathcal{V}$ will have finitely many irreducible modules A. Consider their (normalised) characters

$$\mathrm{ch}_A(\tau) = q^{-c/24}\,\mathrm{Tr}_A q^{L_0} \tag{1.1}$$

where c is the rank of the VOA and $q = e^{2\pi i\tau}$, for τ in the upper half-plane $\mathbb{H}$. A VOA $\mathcal{V}$ is (among other things) a vector space with a grading given by the eigenspaces of the operator L_0; (1.1) defines the character to be obtained from the induced L_0-grading on the $\mathcal{V}$-modules A. These characters yield a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ of the torus, given by its familiar action on $\mathbb{H}$ via fractional linear transformations. In particular, we can define matrices S and T by

$$\mathrm{ch}_A(-1/\tau) = \sum_B S_{AB}\,\mathrm{ch}_B(\tau), \quad \mathrm{ch}_A(\tau+1) = \sum_B T_{AB}\,\mathrm{ch}_B(\tau); \tag{1.2a}$$

this representation sends

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T. \tag{1.2b}$$

We call this representation the *modular data* of the RCFT. It has some interesting properties, as we shall see. For example, in Monstrous Moonshine the relevant VOA is the Moonshine module $V^\natural$. There is only one irreducible module of $V^\natural$, namely itself, and its character $j(\tau) - 744$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$.

Incidentally, there is in RCFT and related areas a (projective) representation of each mapping class group—see e.g. [3, 5, 60, 95, 110] and references therein. These groups play the role of modular group, for any Riemann surface. Their representations coming from e.g. RCFT are still poorly understood, and certainly deserve more attention, but in this paper we will consider only $\mathrm{SL}_2(\mathbb{Z})$ (i.e. the unpunctured torus).

Strictly speaking we need linear independence of our characters, which means considering the '1-point functions'

$$\mathrm{ch}_A(\tau, u) = q^{-c/24}\,\mathrm{Tr}_A(q^{L_0}\,o(u))$$

—this is why $\mathrm{SL}_2(\mathbb{Z})$ and not $\mathrm{PSL}_2(\mathbb{Z})$ arises here — but for simplicity we will ignore this technicality in the following.

In physical parlance, the two VOAs are the (right- and left-moving) algebras of (chiral) observables. The observables operate on the space $\mathcal{H}$ of physical states of the theory; i.e. $\mathcal{H}$ carries a representation of $\mathcal{V} \otimes \mathcal{V}$. The irreducible modules $A \otimes A'$ of $\mathcal{V} \otimes \mathcal{V}$ in $\mathcal{H}$ are labelled by the *primary fields*—special states $|\phi, \phi'\rangle$ in $\mathcal{H}$ which play the role of highest weight vectors. More precisely, the primary field will be a vertex operator $Y(\phi, z)$ and the ground state $|\phi\rangle$ will be the state created by the primary field at time $t = -\infty$: $|\phi\rangle = \lim_{z\to 0} Y(\phi, z)|0\rangle$. The VOA $\mathcal{V}$ acting on the (chiral) primary field $|\phi\rangle$ generates the module $A = A_\phi$ (and similarly for ϕ'). The characters ch_A form a basis for the vector space of 0-point 1-loop conformal blocks (see (3.7) with $g = 1, t = 0$).

Modular data is a fundamental ingredient of the RCFT. It appears for instance in Verlinde's formula (2.1), which gives (by definition) the structure constants for what is called the fusion ring. It also constrains the torus partition function $\mathcal{Z}$:

$$\mathcal{Z}(\tau) = q^{-c/24}\,\bar{q}^{-c/24}\,\mathrm{Tr}_{\mathcal{H}}\,q^{L_0}\,\bar{q}^{L'_0} \tag{1.3a}$$

where $\bar{q}$ is the complex conjugate of q. Now as mentioned above, $\mathcal{H}$ has the decomposition

$$\mathcal{H} = \oplus_{A,B} M_{AB}\, A \otimes B \tag{1.3b}$$

into $\mathcal{V}$-modules, where the M_{AB} are multiplicities, and so

$$\mathcal{Z}(\tau) = \sum_{A,B} M_{AB}\, \mathrm{ch}_A(\tau)\, \overline{\mathrm{ch}_B(\tau)} \tag{1.3c}$$

Physically, $\mathcal{Z}$ is the 1-loop vacuum-to-vacuum amplitude of the closed string (or rather, the amplitude would be $\int \mathcal{Z}(\tau)\, d\tau$). 'Amplitudes' are the fundamental numerical quantities in quantum theories, from which the probabilities are obtained; it is through probabilities that the theory makes contact with experiment. In Segal's formalism, the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is assigned the homomorphism $\mathbb{C} \to \mathbb{C}$ corresponding to multiplication by $\mathcal{Z}(\tau)$. We will see in Section 5 that $\mathcal{Z}$ must be invariant under the action (1.2a) of the modular group $\mathrm{SL}_2(\mathbb{Z})$, and so we call it (or equivalently its matrix M of multiplicities) a *modular invariant*.

Another elementary but fundamental quantity is the 1-loop vacuum-to-vacuum amplitude $\mathcal{Z}_{\alpha\beta}$ of the open string, to whose ends are attached 'boundary states' $|\alpha\rangle$, $|\beta\rangle$—this cylindrical partition function looks like

$$\mathcal{Z}_{\alpha\beta}(t) = \sum_{A} \mathcal{N}_{A\alpha}^{\beta}\, \mathrm{ch}_A(\mathrm{i}t) \tag{1.4}$$

where these multiplicities $\mathcal{N}_{A\alpha}^{\beta}$ have something to do with Verlinde's formula (2.1). These functions $\mathcal{Z}_{\alpha\beta}$ (or equivalently their matrices $(\mathcal{N}_A)_{\alpha\beta} = \mathcal{N}_{A\alpha}^{\beta}$ of coefficients) are called *fusion graphs* or NIM-*reps*, for reasons that will be explained in Section 5.

We define modular invariants and NIM-reps axiomatically in Section 5. Classifying them is essentially the same as classifying (boundary) RCFTs, and is an interesting and accessible challenge. All of this will be explained more thoroughly and rigourously in the course of this paper.

In this paper we survey the basic theory and examples of modular data and fusion rings. In our context, modular data is much more fundamental as it contains much more information. Basic (combinatorial) things to do with modular data are to construct and classify them and their associated modular invariants and NIM-reps. Certainly, we are still missing key ideas here, and in part this paper is a call for help. We sketch the basic theory of modular invariants and NIM-reps. Finally, we specialise to the modular data associated to affine Kac-Moody algebras, and discuss what is known about their modular invariant and NIM-rep classifications. A familiarity with RCFT is not needed to read this paper (apart from this introduction!).

The mathematics of CFT is extremely rich, but what isn't always appreciated is how much of it is combinatorial. This paper certainly doesn't exhaust all of this combinatorial content—for this, the reader should study the Moore-Seiberg data [95] (for a mathematical treatment, see especially [5]). In this paper we focus on the most accessible, and probably most important, part of this, namely those aspects related to $\mathrm{SL}_2(\mathbb{Z})$ and fusion rings.

The theory of fusion rings in its purest form is the study of the algebraic consequences of requiring structure constants to obey the constraints of positivity and integrality, as well as imposing some sort of self-duality condition identifying the ring with its dual. But one of the thoughts running through this note is that we don't know yet its correct definition (nor, more importantly, that of modular data). In the next section is given the most standard definition, but surely it can be improved. How to determine the correct definition is clear: we probe it from the 'inside'—i.e. with strange examples which we probably want to call modular data—and also from the 'outside'—i.e. with examples probably too dangerous to include in the fold. Some of these critical examples will be described in the following sections.

Notational Remarks: Throughout the paper we let $\mathbb{Z}_{\geq}$ denote the nonnegative integers, and $\bar{x}$ denote the complex conjugate of x. The transpose of a matrix A will be written A^t.

2. Modular data and fusion rings

The most basic structure considered in this paper is that of modular data; the particular variant studied here—and the most common one in the literature—is given in Definition 1. But there are alternatives, and a natural general one is given by **MD1**$'$, **MD2**$'$, **MD3**, and **MD4**. In the more limited context of e.g. RCFT, axioms **MD1**, **MD2**$'$, and **MD3**-**MD6** are more appropriate.

Definition 1 Let Φ be a finite set of labels, one of which—we will denote it 0 and call it the 'identity'—is distinguished. By *modular data* we mean matrices $S = (S_{ab})_{a,b\in\Phi}$, $T = (T_{ab})_{a,b\in\Phi}$ of complex numbers such that:

MD1. S is unitary and symmetric, and T is diagonal and of finite order: i.e. $T^N = I$ for some N;
MD2. $S_{0a} > 0$ for all $a \in \Phi$;
MD3. $S^2 = (ST)^3$;
MD4. The numbers defined by

$$N_{ab}^c = \sum_{d\in\Phi} \frac{S_{ad}\, S_{bd}\, \overline{S_{cd}}}{S_{0d}} \tag{2.1}$$

are in $\mathbb{Z}_{\geq}$.

The matrix S is more important than T. The name 'modular data' is chosen because S and T give a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$—as **MD3** strongly hints and as we will see in Section 4. Trying to remain consistent with the terminology of RCFT, we will call (2.1) 'Verlinde's formula', the N_{ab}^c 'fusion coefficients', and the $a \in \Phi$ 'primaries'. The distinguished primary '0' is called the 'identity' because of its role in the associated fusion ring, defined below. A possible fifth axiom will be proposed shortly, and later we will propose refinements to **MD1** and **MD2**, as well as a possible 6th axiom, but in this paper we will limit ourselves to the consequences of **MD1**–**MD4**.

Modular data arises directly in many places in math—some of these will be reviewed in the next section. In many of these interpretations, there is for each primary $a \in \Phi$ a function (a 'character') $\chi_a : \mathbb{H} \to \mathbb{C}$ which yields the matrices S and T as in (1.2a). Also, in many examples, to each triple $a, b, c \in \Phi$ we get a vector space $\mathcal{H}_{ab}^c$ (an 'intertwiner space' or 'multiplicity module') with $\dim(\mathcal{H}_{ab}^c) = N_{ab}^c$, and with natural isomorphisms between $\mathcal{H}_{ab}^c$, $\mathcal{H}_{ba}^c$, etc. In many of these examples, we have '6j-symbols', i.e. for any 6-tuple $a, b, c, d, e, f \in \Phi$ we have a homomorphism $\{\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\}$ from $\mathcal{H}_{cd}^e \otimes \mathcal{H}_{ab}^c$ to $\mathcal{H}_{af}^e \otimes \mathcal{H}_{bd}^f$ obeying several conditions (see e.g. [110, 47] for a general treatment). Classically, 6j-symbols explicitly described the change between the two natural bases of the tensor product $(L_\lambda \otimes L_\mu) \otimes L_\nu \cong L_\lambda \otimes (L_\mu \otimes L_\nu)$ of modules of a Lie group, and our 6j-symbols are their natural extension to e.g. quantum groups. Characters, intertwiner spaces, and 6j-symbols don't play any role in this paper.

If **MD2** looks unnatural, think of it in the following way. It is easy to show (using **MD1** and **MD4** and Perron-Frobenius theory [75]) that some column of S is nowhere 0 and of constant phase (i.e. $\mathrm{Arg}(S_{\updownarrow b})$ is constant for some $b \in \Phi$); **MD2** tells us that it is the 0 column, and that the phase is 0 (so these entries are positive). The ratios S_{a0}/S_{00} are sometimes called *q(uantum)-dimensions* (see (4.2b) below).

If **MD4** looks peculiar, think of it in the following way. For each $a \in \Phi$, define matrices N_a by $(N_a)_{bc} = N_{ab}^c$. These are usually called *fusion matrices*. Then **MD4** tells us these N_a's are simultaneously diagonalised by S, with eigenvalues S_{ad}/S_{0d}.

The key to modular data is Eq. (2.1). It should look familiar from the character theory of finite groups: Let G be any finite group, let $K_1, \ldots, K_h$ be the conjugacy classes of G, and write k_i for the formal sum $\sum_{g \in K_i} g$. These k_i's form a basis for the centre of the group algebra $\mathbb{C}G$ of G. If we write

$$k_i\, k_j = \sum_{\ell} c_{ij\ell} k_\ell$$

then the structure constants $c_{ij\ell}$ are nonnegative integers, and we obtain

$$c_{ij\ell} = \frac{\|K_i\|\,\|K_j\|\,\|K_\ell\|}{\|G\|} \sum_{\chi \in \mathrm{Irr}\, G} \frac{\chi(g_i)\,\chi(g_j)\,\overline{\chi(g_\ell)}}{\chi(e)}$$

where $g_i \in K_i$. This resembles (2.1), with S_{ab} replaced with $S_{i,\chi} = \chi(g_i)$ and the identity 0 replaced with the group identity e. This formal relation between finite groups and Verlinde's formula seems to have first been noticed in [89]; we will return to it later this section.

The matrix T is fairly poorly constrained by **MD1**–**MD4**. There are however many other independent properties which the modular data coming from RCFT is known to obey. The most important of these is that, for any $a \in \Phi$, the quantity

$$\sum_{b,c \in \Phi} N_{bc}^a S_{0b} S_{0c} T_{cc}^2 T_{bb}^{-2}$$

lies in $\{0, \pm 1\}$ (this doesn't follow from **MD1**–**MD4**) and plays the role of the Frobenius–Schur indicator here [9]. Another axiom, also obeyed by any RCFT [36], is sometimes introduced, though it also won't be adopted here:

MD5. For all choices $a, b, c, d \in \Phi$,

$$\left(T_{aa}T_{bb}T_{cc}T_{dd}T_{00}^{-1}\right)^{N_{abcd}} = \prod_{e\in\Phi} T_{ee}^{N_{abcd,e}}$$

where

$$N_{abcd} := \sum_{e\in\Phi} N_{ab}^{e}\, N_{ce}^{d}\ , \quad N_{abcd,e} := N_{ab}^{e}N_{ce}^{d} + N_{bc}^{e}N_{ae}^{d} + N_{ac}^{e}N_{be}^{d}$$

From **MD5** it can be proved that T has finite order (take $a = b = c = d$), so admitting **MD5** permits us to remove that statement from **MD1**. But it doesn't have any other interesting consequences that this author knows—though perhaps it will be useful in proving the Congruence Subgroup Property given below, or give us some finiteness result.

Intimately related to modular data are the fusion rings. There is no standard terminology here and this does occassionally cause confusion; we suggest the following as being unambiguous and yet close to most treatments in the literature.

Definition 2 A *fusion algebra* $A = \mathcal{F}(\beta, N)$ is an associative commutative $\mathbb{Q}$-algebra A with unity 1, together with a finite basis $\beta = \{x_0, x_1, \ldots, x_n\}$ with $x_0 = 1$, such that:

F1. The structure constants $N_{ab}^{c} \in \mathbb{Q}$, defined by $x_a x_b = \sum_{c=0}^{n} N_{ab}^{c} x_c$, are all nonnegative;
F2. There is a ring endomorphism $x \mapsto x^*$ stabilising the basis β (write $x_a^* = x_{a^*}$);
F3. $N_{ab}^{0} = \delta_{b,a^*}$;
F4. There is a symmetric unitary matrix S, $S = S^t$, such that Verlinde's formula (2.1) holds for all $a, b, c \in \Phi := \{0, 1, \ldots, n\}$.

We usually will be interested in the 'fusion coefficients' N_{ab}^{c} being (nonnegative) integers. In this case it will usually be convenient to consider the $\mathbb{Z}$-span of β. The resulting free $\mathbb{Z}$-module with basis β and structure constants N_{ab}^{c} will be called a *fusion ring*. In those rare situations where we are interested more generally in the scalars being e.g. real or complex, i.e. when A is an $\mathbb{R}$- or $\mathbb{C}$-algebra, we will speak of $\mathbb{R}$-*fusion algebras* and $\mathbb{C}$-*fusion algebras*, respectively (of course positivity **F1** requires in all cases that the N_{ab}^{c} be real).

If the algebra A obeys only **F1**–**F3** we'll call it a *generalised fusion algebra*. We will see shortly that given any generalised fusion algebra, there is a unitary matrix S such that (2.1) holds $\forall a, b, c \in \Phi$, so the content of the important **F4** is that this matrix S can be chosen to be symmetric. We will see later that algebraically this is a self-duality condition.

RCFT is much more interested in fusion rings than generalised fusion algebra, and the remainder of the paper after this section will specialise to them. However, generalised fusion algebras do appear in RCFT and so perhaps deserve more attention there. For instance, the subrings of fusion rings will typically be generalised fusion rings—e.g. consider the subring spanned by the 'even' primaries $\{0, 2, \ldots, k\}$ of the affine algebra $A_1^{(1)}$ at even level k (see (3.5) below). Also, the 'classifying algebra' in boundary conformal field theory [17] can be a generalised fusion ring. Much more general fusion-like rings arise naturally in subfactors (see Example 6 below) and nonrational logarithmic CFT (see e.g. [57, 66]) so there is a much broader theory here to be developed, and of course the people to do this are algebraic combinatorists.

That $x \mapsto x^*$ is an involution is clear from **F3** and commutativity of A. Axiom **F3** and associativity of A imply $N_{ab}^c = N_{ac^*}^{b^*}$ (a.k.a. Frobenius reciprocity or Poincaré duality); hence the numbers $N_{abc} := N_{ab}^{c^*}$ will be symmetric in a, b, c. Axiom **F3** is equivalent to the existence on A of a linear functional 'Tr' for which β is orthonormal: $\mathrm{Tr}(x_a x_b^*) = \delta_{a,b}$ $\forall a, b \in \beta$. Then $N_{ab}^c = \mathrm{Tr}(x_a x_b x_c^*)$.

As an abstract algebra, A is not very interesting: in particular, because A is commutative and associative, the fusion matrices $(N_a)_{bc} = N_{ab}^c$ pairwise commute; because of **F2**, $(N_a)^t = N_{a^*}$. Thus they are normal and can be simultaneously diagonalised. Hence A is semisimple, and will be isomorphic to a direct sum of number fields (see Example 7 below). For example, the fusion algebra for $A_1^{(1)}$ level k (see (3.5c)) is isomorphic to $\oplus_d \mathbb{Q}[\cos(\pi \frac{d}{k+2})]$, where d runs over all divisors of $2(k+2)$ in the interval $1 \leq d < k+2$. Likewise, the $\mathbb{C}$-fusion algebra $A \otimes_{\mathbb{Q}} \mathbb{C}$ is isomorphic as a $\mathbb{C}$-algebra to $\mathbb{C}^{\|\beta\|}$ with operations defined component-wise. Of course what is important for fusion rings is that they have a preferred basis β.

(Generalised) fusion algebras are closely related to association schemes and C-algebras (as first noted in [34, 35], and independently in [6]), hypergroups [115], table algebras, etc. That is to say, their axiomatic systems are similar. In particular, a generalised fusion algebra is a table algebra [2], with structure constants in $\mathbb{Q}$ and normalised appropriately; a fusion algebra obeys in addition a strong self-duality. However, the exploration of an axiomatic system is influenced not merely by its intrinsic nature (i.e. its formal list of axioms and their logical consequences), but also by what are perceived by the local research community to be its characteristic examples. There always is a context to math; the development of formal structure is directed by its implicit context. The prototypical examples of a table algebra are the space of class functions of a finite group or the centre of the group algebra, while that of modular data corresponds to the $\mathrm{SL}_2(\mathbb{Z})$ representation associated to an affine Kac-Moody algebra at level $k \in \mathbb{Z}_{\geq}$ (Example 2 below). Nevertheless it can be expected that techniques and questions from one of these areas can be profitably carried over to the other, and solidifying that bridge is this paper's *raison d'être*. Surely, implicit or explicit in the literature on e.g. table algebras, there are results which to CFT people would be new and interesting. This paper tries to explain the relevant CFT language, and describe questions conformal field theorists would find natural and important.

To give one interesting disparity, the commutative association schemes have been classified up to 23 vertices [79], while modular data is known for only 3 primaries [27] (and that proof assumes additional axioms)! In fact we still don't have a finiteness theorem: for a given cardinality $\|\Phi\|$, are there only finitely many possible modular data? (We know there are infinitely many fusion rings in each dimension >1.)

Consider for the next several paragraphs that A is a generalised $\mathbb{C}$-fusion algebra (tensoring by $\mathbb{C}$ if necessary). Our treatment will roughly follow that of Kawada's C-algebras as given in [8]. The fusion matrices N_a are linearly independent, by **F3**. Let $\underline{y_i}$, for $0 \leq i \leq n$, be a basis of common eigenvectors, with eigenvalues $\ell_i(a)$. Normalise all vectors $\underline{y_i}$ to have unit length (there remains an ambiguity of phase which we will fix below), and let $\underline{y_0}$ be the Perron-Frobenius one—since $\sum_a N_a > 0$ here, we can choose $\underline{y_0}$ to be strictly positive. Let S be the matrix whose ith column is $\underline{y_i}$, and L the eigenmatrix $L_{ai} = \ell_i(a)$. Then S is unitary and L is invertible. Note that for each i, the map $a \mapsto \ell_i(a)$ defines a linear representation of A. That means that each column of L will be a common eigenvector of all N_a, with eigenvalue $\ell_i(a)$, and hence must equal a scalar multiple of the ith column of S (see the BASIC FACT in Section 4). Note that each $L_{0i} = 1$; therefore each S_{0i} will be nonzero and we may uniquely determine S (up to the ordering of the columns) by demanding that each $S_{0i} > 0$. Then $L_{ai} = S_{ai}/S_{0i}$. Therefore we get (2.1).

Note though that the rows of S are indexed by the basis indices $\Phi := \{0, 1, \ldots, n\}$, but its columns are indexed by the eigenvectors. Like the character table of a group, although S is a square matrix it is not (for *generalised* fusion algebras) 'truly square'. This simple observation will be valuable for the paragraph after Proposition 1.

The involution $a \mapsto a^*$ in **F2** appears in the matrix $C_l := SS^t$: $(C_l)_{ab} = \delta_{b,a^*}$. The matrix $C_r := S^t S$ is also an order 2 permutation, and

$$\overline{S_{ai}} = S_{C_l a,i} = S_{a,C_r i} \tag{1.2}$$

For a proof of those statements, see (4.4) below.

Let $\hat{A}$ be the set of all linear maps of $A \otimes_{\mathbb{Q}} \mathbb{C}$ into $\mathbb{C}$, equivalently the set of all maps $\Phi \to \mathbb{C}$. $\hat{A}$ has the structure of an $(n+1)$-dimensional commutative algebra over $\mathbb{C}$, using the product $(fg)(x_a) = f(x_a)\, g(x_a)$. A basis $\hat{\beta}$ of $\hat{A}$ consists of the functions $a \mapsto \frac{S_{ai}}{S_{a0}}$, for each $0 \leq i \leq n$—denote this function $\hat{i}$. The resulting structure constants are

$$\hat{N}_{\hat{i}\hat{j}}^{\hat{k}} = \sum_{a \in \Phi} \frac{S_{ai}\, S_{aj}\, \overline{S_{ak}}}{S_{a0}} =: \hat{N}_{ij}^{k} \tag{1.3}$$

In other words, we have replaced S in (2.1) with S^t. It is easy to verify that $\hat{A} = \mathcal{F}(\hat{\beta}, \hat{N})$ obeys all axioms of a generalised $\mathbb{C}$-fusion algebra, except possibly that some structure constants $\hat{N}_{ij}^{k}$ may be negative. They will all necessarily be real, however. We call $\hat{A} = \mathcal{F}(\hat{\beta}, \hat{N})$ the *dual* of $A = \mathcal{F}(\Phi, N)$. Note that $\hat{\hat{A}}$ can always be naturally identified with the original generalised $\mathbb{C}$-fusion algebra A.

We call $A = \mathcal{F}(\beta, N)$ *self-dual* if $\hat{A} = \mathcal{F}(\hat{\beta}, \hat{N})$ is isomorphic as a generalised $\mathbb{C}$-fusion algebra to A—equivalently, if there is a bijection $\iota : \beta \to \hat{\beta}$ such that $N_{ab}^{c} = \hat{N}_{\iota a,\iota b}^{\iota c}$ (see the definition of 'fusion-isomorphism' in Section 4).

Proposition 1. *Given any generalised $\mathbb{C}$-fusion algebra $A = \mathcal{F}(\beta, N)$, there is a unique (up to ordering of the columns) unitary matrix S obeying* (2.1) *and all S_{0i} and S_{a0} are* *positive. The generalised fusion algebra $A = \mathcal{F}(\beta, N)$ is self-dual iff the corresponding matrix S obeys*

$$S_{a,\iota' b} = S_{b,\iota a} \qquad \text{for all } a, b \in \beta \tag{1.4}$$

for some bijections $\iota, \iota' : \beta \to \hat{\beta}$.

What this tells us is that there isn't a natural algebraic interpretation for our precise condition $S = S^t$ in **MD1**; this study of (generalised) fusion algebras strongly suggests that the definition of modular data (and fusion algebra) be extended to the more general setting where '$S = S^t$' is replaced with (2.4). Fortunately, all properties of fusion rings extend naturally to this new setting. But what should T look like then? A priori this isn't so clear. But requiring the existence of a representation of $\mathrm{SL}_2(\mathbb{Z})$ really forces matters. In particular note that, when S is not symmetric, the matrices S and T themselves cannot be expected to give a natural representation of any group (modular or otherwise) since for instance the expression S^2 really isn't sensible—S is not 'truly square'. Write P and Q for the matrices $P_{a,i} = \delta_{i,\iota a}$ and $Q_{a,i} = \delta_{i,\iota' a}$, and let m be the order of the permutation $\iota^{-1} \circ \iota'$. Then for any k, $\tilde{S} = SQ^t(PQ^t)^k$ *is* 'truly square' and its square $\tilde{S}^2 = C_l(PQ^t)^k$ is a permutation matrix, where C_l is as in (2.2). We also want $\tilde{S}^4 = I$, which requires $m = 2k+1$ or $m = 4k+2$. In either of those cases, $\tilde{T} = TP^t(QP^t)^k$ defines with $\tilde{S}$ a representation of $\mathrm{SL}_2(\mathbb{Z})$ provided $TS^tTS^tT = S(Q^tP)^{2k+1}$. (When 4 divides m, the best we will get in general will be a representation of some extension of $\mathrm{SL}_2(\mathbb{Z})$.) But S is only determined by the generalised fusion algebra up to permutation of the columns, so we may as well replace it with $\tilde{S}$. Do likewise with T. So it seems that we can and should replace **MD1** with:

MD1$'$. S is unitary, $S^t = SP$ where P is a permutation matrix of order a power of 2, and T is diagonal and of finite order;

and leave **MD2**–**MD4** intact. That simple change seems to provide the natural generalisation of modular data to any self-dual generalised fusion ring. Let m be the order of P; then $m = 1$ recovers modular data, $m \leq 2$ yields a representation of $\mathrm{SL}_2(\mathbb{Z})$, and $m > 2$ yields a representation of a central extension of $\mathrm{SL}_2(\mathbb{Z})$. In the interests of notational simplicity we will adopt in later sections the standard **MD1** rather than the new **MD1$'$**, although everything we discuss below has an analogue for this more general setting.

If we don't require an $\mathrm{SL}_2(\mathbb{Z})$ representation, then of course we get much more freedom. It is very unclear though what T should look like when the generalised fusion ring is not self-dual, which probably indicates that the definition of fusion algebra should include some self-duality constraint. This is of course the attitude we adopt, although in the mathematical literature it is unfortunately common to ignore it, and this difference can cause confusion.

Incidentally, the natural appearance of a self-duality constraint here perhaps should not be surprising in hindsight. Drinfeld's 'quantum double' construction has analogues in several contexts related to RCFT, and is a way of generating algebraic structures which possess modular data (see examples next section). It always involves combining a given (inadequate) algebraic structure with its dual in an appropriate way. A general categorical interpretation of quantum double is the *centre construction*, described for instance in [88]; it assigns to a tensor category a braided tensor category. It would be interesting to interpret this construction at the more base level of fusion algebra—it could supply a general way for obtaining self-dual generalised fusion algebras from non-self-dual ones.

In Example 4 of Section 3 we will propose a further generalisation of modular data. In this paper however, we will restrict for convenience to the consequences of the standard axioms **MD1**–**MD4**.

In any case, a fusion ring is completely equivalent to a unitary and symmetric matrix S obeying **MD2** and satisfying **MD4**. This special case of Proposition 1 was known to Bannai and Zuber. More generally, $\iota^{-1}\circ\iota'$ will define a *fusion-automorphism* of a self-dual generalised fusion algebra $A = \mathcal{F}(\beta, N)$. Note that an unfortunate choice of matrix S in [6] led to an inaccurate conclusion there regarding fusion rings and Verlinde's formula (2.1). In fact, Verlinde's formula will hold with a unitary matrix S obeying $S_{0i} > 0$, even if we drop nonnegativity **F1**.

Proposition 1 shows that although (2.1) looks mysterious, it is quite canonical, and that the depth of Verlinde's formula lies in the interpretation given to S and N (for instance (1.2a) and $N_{ab}^{c} = \dim(\mathcal{H}_{ab}^{c})$) within the given context.

The two-dimensional generalised fusion algebras $\mathcal{F}(\{0, 1\}, N)$ are classified by their value of $r = N_{11}^{1}$—there is a unique fusion ring for every $r \in \mathbb{Q}$, $r \geq 0$. All are self-dual in the strong sense, and so are in fact fusion algebras. A diagonal unitary matrix T satisfying **MD3** exists, iff $0 \leq r \leq \frac{2}{\sqrt{3}}$. However, T will in addition be of finite order, i.e. S and T will constitute modular data, iff $r = 0$ (realised e.g. by the affine algebras $A_1^{(1)}$ and $E_7^{(1)}$ at level 1) or $r = 1$ (realised e.g. by affine algebras $G_2^{(1)}$ and $F_4^{(1)}$ at level 1). Both $r = 0, 1$ have six possibilities for the matrix T (T can always be multiplied by a third root of unity). All 12 sets of modular data with two primaries can be realised by affine algebras (see Example 2 below). This seeming omnipresence of the affine algebras is an accident of small numbers of primaries; even when $\|\beta\| = 3$ we find non-affine algebra modular data. The fusion algebras given here can be regarded as a deformation interpolating between e.g. the $A_1^{(1)}$ and $G_2^{(1)}$ level 1 fusion rings; similar deformations are typical in higher dimensions. For example in 3-dimensions, the $A_2^{(1)}$ level 1 fusion ring lies in a family of fusion algebras parametrised by the Pythagorean triples.

Classifying modular data and fusion rings for small sets of primaries, or at least obtaining new explicit families beyond Examples 1–3 given next section, is perhaps the most vital challenge in the theory.

3. Examples of modular data and fusion rings

We can find (2.1), if not modular data in its full splendor, in a wide variety of contexts. In this section we sketch several of these. Historically for the subject, Example 2 has been the most important. As with the introductory section, this presentation cannot be self-contained and should be treated as an annotated guide to the literature. So don't be concerned if most of these examples aren't familiar—Section 4 is largely independent of them.

Example 1 (Lattices) See [29] for the essentials of lattice theory.

Let Λ be an even lattice—i.e. Λ is the $\mathbb{Z}$-span of a basis of $\mathbb{R}^n$, with the property that $x \cdot y \in \mathbb{Z}$ and $x \cdot x \in 2\mathbb{Z}$ for all $x, y \in \Lambda$. Its dual Λ^* consists of all vectors w in $\mathbb{R}^n$ whose dot product $w \cdot x$ with any $x \in \Lambda$ is an integer. So we have $\Lambda \subset \Lambda^*$. Let $\Phi = \Lambda^*/\Lambda$ be the cosets. The cardinality of Φ is finite, given by the determinant $|\Lambda|$ of Λ (which equals the volume-squared of any fundamental region). The dot products $a \cdot b$ and norms $a \cdot a$ for the classes $[a], [b] \in \Phi$ are well-defined (mod 1) and (mod 2), respectively. Define matrices by

$$S_{[a],[b]} = \frac{1}{\sqrt{|\Lambda|}} e^{2\pi \mathrm{i} a \cdot b} \tag{3.1a}$$

$$T_{[a],[a]} = e^{\pi \mathrm{i} a \cdot a - n\pi \mathrm{i}/12} \tag{3.1b}$$

The simplest special case is $\Lambda = \sqrt{N}\mathbb{Z}$ for any even number N, where $\Lambda^* = \frac{1}{\sqrt{N}}\mathbb{Z}$ and $|\Lambda| = N$. Then Φ can be identified with $\{0, 1, \ldots, N-1\}$, and $a \cdot b$ is given by ab/N, so (3.1a) becomes the finite Fourier transform.

For any such lattice Λ, this defines modular data. Note that the $\mathrm{SL}_2(\mathbb{Z})$-representation is essentially a Weil representation of $\mathrm{SL}_2(\mathbb{Z}/|\Lambda|\mathbb{Z})$, and that it is realised in the sense of (1.2) by characters $\mathrm{ch}_{[a]}$ given by theta functions divided by $\eta(\tau)^n$. The identity '0' here is $[0] = \Lambda$. The fusion coefficients $N_{[a],[b]}^{[c]}$ equal the Kronecker delta $\delta_{[c],[a+b]}$, so the product in the fusion ring is given by addition in Λ^*/Λ. From our point of view, this lattice example is too trivial to be interesting.

When Λ is merely integral (i.e. some norms $x \cdot x$ are odd), we don't have modular data: T^2 (but not T) is defined by (3.1b), and we get a representation of $\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \rangle$, an index-3 subgroup of $\mathrm{SL}_2(\mathbb{Z})$. However, nothing essential is lost, so the definition of modular data should be broadened to include at minimum all these integral lattice examples.

Example 2 (Kac-Moody algebras) See [84, 87] for the basics of Kac-Moody algebras.

The source of some of the most interesting modular data are the affine nontwisted Kac-Moody algebras $X_r^{(1)}$. The simplest way to construct affine algebras is to let X_r be any finite-dimensional simple (more generally, reductive) Lie algebra. Its loop algebra is the set of all formal series $\sum_{\ell \in \mathbb{Z}} t^\ell a_\ell$, where t is an indeterminant, $a_\ell \in X_r$ and all but finitely many a_ℓ are 0. This is a Lie algebra, using the obvious bracket, and is infinite-dimensional. The affine algebra $X_r^{(1)}$ is simply a certain central extension of the loop algebra. (As usual, the central extension is taken in order to get a rich supply of representations.)

The representation theory of $X_r^{(1)}$ is analogous to that of X_r. We are interested in the so-called integral highest weight representations. These are partitioned into finite families parametrised by the level $k \in \mathbb{Z}_{\geq}$. Write $P_+^k(X_r^{(1)})$ for the set of finitely many level k highest weights $\lambda = \lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r$, $\lambda_i \geq 0$, where Λ_i are the fundamental weights. For example, $P_+^k(A_r^{(1)})$ consists of the $\binom{k+r}{r}$ such λ, which obey $\lambda_0 + \lambda_1 + \cdots + \lambda_r = k$.

The $X_r^{(1)}$-character $\chi_\lambda(\tau)$ associated to highest weight λ is given by a graded trace, as in (1.1). Thanks mostly to the structure and action of the affine Weyl group on the Cartan subalgebra of $X_r^{(1)}$, the character χ_λ is essentially a lattice theta function, and so transforms nicely under the modular group $\mathrm{SL}_2(\mathbb{Z})$. In fact, for fixed algebra $X_r^{(1)}$ and level $k \in \mathbb{Z}_{\geq}$, these χ_λ define a representation of $\mathrm{SL}_2(\mathbb{Z})$, exactly as in (1.2) above, and the matrices S and T constitute modular data. The 'identity' is $0 = k\Lambda_0$, and the set of 'primaries' is the highest weights $\Phi = P_+^k(X_r^{(1)})$. The matrix T is related to the values of the second Casimir of X_r, and S to characters evaluated at elements of finite order in the Lie group corresponding to X_r:

$$T_{\lambda\mu} = \alpha \exp\left[\frac{\pi \mathrm{i}(\lambda+\rho \mid \lambda+\rho)}{\kappa}\right]\delta_{\lambda,\mu} \tag{3.2a}$$

$$S_{\mu\nu} = \alpha' \sum_{w\in\overline{W}} \det(w) \exp\left[-2\pi\mathrm{i}\frac{(w(\mu+\rho) \mid \nu+\rho)}{\kappa}\right] \tag{3.2b}$$

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \mathrm{ch}_{\overline{\lambda}}\left(\exp\left[-2\pi\mathrm{i}\,\frac{(\overline{\lambda} \mid \overline{\mu}+\rho)}{\kappa}\right]\right) \tag{3.2c}$$

The numbers $\alpha, \alpha' \in \mathbb{C}$ are normalisation constants whose precise values are unimportant here, and are given in Theorem 13.8 of [84]. The inner product in (3.2) is the usual Killing form, ρ is the Weyl vector $\sum_i \Lambda_i$, and $\kappa = k + h^\vee$, where $h^\vee$ is the dual Coxeter number ($= r+1$ for $A_r^{(1)}$). The (finite) Weyl group $\bar{W}$ of X_r acts on the affine weights $\mu = \sum_i \mu_i \Lambda_i$ by fixing Λ_0. Here, $\bar{\lambda}$ denotes the projection $\lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r$, and '$\mathrm{ch}_{\overline{\lambda}}$' is a finite-dimensional Lie group character.

The combinatorics of Lie group characters at elements of finite order, i.e. the ratios (3.2c), is quite rich. For example, in [83] they are used to prove quadratic reciprocity, while [94] uses them for instance in a fast algorithm for computing tensor product decompositions in Lie groups.

The fusion coefficients $N_{\lambda\mu}^\nu$, defined by (2.1), are essentially the tensor product multiplicities $T_{\lambda\mu}^\nu := \mathrm{mult}_{\overline{\lambda}\otimes\overline{\mu}}(\overline{\nu})$ for X_r (e.g. the Littlewood-Richardson coefficients for A_r), except 'folded' in a way depending on k. This is seen explicitly by the Kac-Walton formula [84 p. 288, 113,61]:

$$N_{\lambda\mu}^\nu = \sum_{w\in W} \det(w)\, T_{\lambda\mu}^{w.\nu}, \tag{3.3}$$

where $w.\gamma := w(\gamma+\rho) - \rho$ and W is the affine Weyl group of $X_r^{(1)}$ (the dependence on k arises through this action of W). The proof of (3.3) follows quickly from (3.2c).

The fusion ring R here is isomorphic to $\mathrm{Ch}(X_\ell)/\mathcal{I}_k$, where $\mathrm{Ch}(X_\ell)$ is the character ring of X_ℓ (which is isomorphic as an algebra to the polynomial algebra in ℓ variables), and where $\mathcal{I}_k$ is its ideal generated by the characters of the 'level $k+1$' weights (for $X_\ell = A_\ell$, these consist of all $\overline{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ obeying $\lambda_1 + \cdots + \lambda_\ell = k+1$). In important recent work, [52] has expressed it using equivariant K-theory.

Equation (3.3) has the flaw that, although it is manifest that the $N_{\lambda\mu}^\nu$ will be integral, it is not clear why they are positive. A big open challenge here is to find a combinatorial rule, e.g. in the spirit of the well-known Littlewood-Richardson rule, for the affine fusions. Three preliminary steps in this direction are [104, 109, 50]. A general combinatorial rule has been conjectured in [24] for $A_\ell^{(1)}$, but it is complicated even for $A_1^{(1)}$.

Identical numbers $N_{\lambda\mu}^{\nu}$ appear in several other contexts. For instance, Finkelberg [51] proved that the affine fusion ring is isomorphic to the K-ring of Kazhdan-Lusztig's category $\widetilde{\mathcal{O}}_{-k}$ of level $-k$ integrable highest weight $X_r^{(1)}$-modules, and to Gelfand-Kazhdan's category $\widetilde{\mathcal{O}}_q$ coming from finite-dimensional modules of the quantum group $U_q(X_r)$ specialised to the root of unity $q = \exp[\mathrm{i}\pi/m\kappa]$ for appropriate choice of $m \in \{1, 2, 3\}$. Because of these isomorphisms, we know that the $N_{\lambda\mu}^{\nu}$ do indeed lie in $\mathbb{Z}_{\geq}$, for any affine algebra. We also know [54] that they increase with k, with limit $T_{\lambda\mu}^{\nu}$.

Also, they arise as dimensions of spaces of generalised theta functions [48], as tensor product coefficients in quantum groups [61] and Hecke algebras [78] at roots of 1 and Chevalley groups for $\mathbb{F}_p$ [76], and in quantum cohomology [116, 16].

For an explicit example, consider the simplest affine algebra ($A_1^{(1)}$) at level k. We may take $P_+^k = \{0, 1, \ldots, k\}$ (the value of λ_1), and then the S and T matrices and fusion coefficients are given by

$$S_{ab} = \sqrt{\frac{2}{k+2}} \sin\left(\pi \, \frac{(a+1)(b+1)}{k+2}\right) \tag{3.5a}$$

$$T_{aa} = \exp\left[\frac{\pi \mathrm{i}(a+1)^2}{2(k+2)} - \frac{\pi \mathrm{i}}{4}\right] \tag{3.5b}$$

$$N_{ab}^{c} = \begin{cases} 1 & \text{if } c \equiv a+b \text{ (mod 2) and } |a-b| \leq c \leq \min\{a+b, 2k-a-b\} \\ 0 & \text{otherwise} \end{cases} \tag{3.5c}$$

The only other affine algebras for which the fusions have been explicitly calculated for all levels k are $A_2^{(1)}$ [13] and $A_3^{(1)}$ [14], and their formulas are also surprisingly compact.

Incidentally, an analogous modular transformation matrix S to (3.2b) exists for the so-called *admissible representations* of $X_r^{(1)}$ at fractional level [86]. The matrix is symmetric, but has no column of constant phase and thus naively putting it into Verlinde's formula (2.1) will necessarily produce some negative numbers (it appears that they'll always be integers though). A legitimate fusion ring has been obtained for $A_1^{(1)}$ at fractional level $k = \frac{p}{q} - 2$ in other ways [4, 49]; it factorises into the product of the $A_{1,p-2}$ fusion ring with a fusion ring at 'level' $q-1$ associated to the rank 1 supersymmetric algebra osp(1|2). A similar theory should exist at least for the other $A_r^{(1)}$; initial steps for $A_2^{(1)}$ have been made in [62]. Serious doubt however on the relevance of these efforts has been cast by [63] and [91], and at this time things here are confused. Sorting this out, and generalising modular data to accommodate admissible representations, is a high priority.

Related roles for other Kac-Moody algebras are slowly being found. The *twisted* affine algebras play the same role for the NIM-reps of the modular data (3.2), as the untwisted affine algebras do for the fusion ring [17, 64]. *Lorentzian* Kac-Moody algebras have been proposed as the symmetries of 'M-theory', the conjectural 11-dimensional theory underlying superstrings (see e.g. [45]). String theories are also known to give rise to the so-called *Borcherds-Kac-Moody* algebras (see e.g. [80, 37]).

Example 3 (Finite groups) The relevant aspects of finite group theory are given in e.g. [82].

Let G be any finite group. Let Φ be the set of all pairs (a, χ), where the a are representatives of the conjugacy classes of G and χ is the character of an irreducible representation of the centraliser $C_G(a)$. (Recall that the conjugacy class of an element $a \in G$ consists of all elements of the form $g^{-1}ag$, and that the centraliser $C_G(a)$ is the set of all $g \in G$ commuting with a.) Put [38, 93]

$$S_{(a,\chi),(a',\chi')} = \frac{1}{\|C_G(a)\| \, \|C_G(a')\|} \sum_{g \in G(a,a')} \overline{\chi'(g^{-1}ag)} \, \overline{\chi(ga'g^{-1})} \tag{3.6a}$$

$$T_{(a,\chi),(a',\chi')} = \delta_{a,a'}\delta_{\chi,\chi'} \frac{\chi(a)}{\chi(e)} \tag{3.6b}$$

where $G(a, a') = \{g \in G \,|\, aga'g^{-1} = ga'g^{-1}a\}$, and $e \in G$ is the identity. For the 'identity' 0 take $(e, 1)$. Then (3.6) is modular data. See [32] for several explicit examples.

There are group-theoretic descriptions of the fusion coefficient $N_{(a,\chi),(b,\chi')}^{(c,\chi'')}$. That these fusion coefficients are nonnegative integers, follows for instance from Lusztig's interpretation of the corresponding fusion ring as the Grothendieck ring of equivariant vector bundles over G: Φ can be identified with the irreducible vector bundles.

This class of modular data played an important role in Lusztig's determination of irreducible characters of Chevalley groups. But there is a remarkable variety of contexts in which (3.6) appears (these are reviewed in [32]). For instance, modular data often has a Hopf algebra interpretation: just as the affine fusions are recovered from the quantum group $U_q(X_r)$, so are these finite group fusions recovered from the quantum-double of G.

This modular data is quite interesting for nonabelian G, and deserves more study. It behaves very differently than the affine data [32]. Conformal field theory explains how very general constructions (Goddard-Kent-Olive and orbifold) build up modular data from combinations of affine and finite group data—see e.g. [36]. Finite group modular data is known to distinguish all groups of order up to at least 127, although there are nonisomorphic groups of order $2^{15} \cdot 3^4 \cdot 5 \cdot 7$ which have indistinguishable modular data [46].

For a given finite group G, there doesn't appear to be a natural unique choice of characters $\mathrm{ch}_{(a,\chi)}$ realising this modular data in the sense of (1.2).

This modular data can be twisted [39] by a 3-cocycle $\alpha \in H^3(G, \mathbb{C}^\times)$, which plays the same role here that level did in Example 2. A further major generalisation of this finite group data will be discussed in Example 6 below, and of this cohomological twist α in the paragraph after Example 6.

Example 4 (RCFT, TFT.) See e.g. [36, 5], and [110], and references therein, for good surveys of 2-dimensional conformal and 3-dimensional topological field theories. In [55] can be found a survey of fusion rings in rational conformal field theory (RCFT).

As discussed earlier, a major source of modular data comes from RCFT (and string theory) and, more or less the same thing, 3-dimensional topological field theory (TFT).

In RCFT, the elements $a \in \Phi$ are called 'primary fields', and the privileged one '0' is called the 'vacuum state'. The entries of T are interpreted in RCFT to be $T_{aa} = \exp[2\pi\mathrm{i}(h_a - \frac{c}{24})]$, where c is the rank of the VOA or the 'central charge' of the RCFT, and h_a is the 'conformal weight' or L_0-eigenvalue of the primary field a. Eq. (2.1) is a special case of the so-called *Verlinde's formula* [112]:

$$V^{(g)}_{a^1\cdots a^t} = \sum_{b\in\Phi} (S_{0b})^{2(1-g)} \frac{S_{a^1b}}{S_{0b}} \cdots \frac{S_{a^tb}}{S_{0b}} \tag{3.7}$$

It arose first in RCFT as an extremely useful expression for the dimensions of the space of conformal blocks on a genus g surface with t punctures, labelled with primaries $a^i \in \Phi$—the fusions N_{ab}^c correspond to a sphere with 3 punctures. All the $V^{(g)}$'s are nonnegative integers iff all the N_{ab}^c's are. In RCFT, our unused axiom **MD5** is derived by applying Dehn twists to a sphere with 4 punctures to obtain an $N_{abcd} \times N_{abcd}$ matrix equation on the corresponding space of conformal blocks; **MD5** is the determinant of that Equation [111].

Example 1 corresponds to the string theory of n free bosons compactified on the torus $\mathbb{R}^n/\Lambda$. Example 2 corresponds to Wess-Zumino-Witten RCFT [77] where a closed string lives on a Lie group manifold. Example 3 corresponds to the untwisted sector in an orbifold of a holomorphic RCFT (a holomorphic theory has trivial modular data—e.g. a lattice theory when the lattice $\Lambda = \Lambda^*$ is self-dual) by G [38]. The RCFT interpretation of fractional level affine algebra modular data isn't understood yet, despite considerable effort (in [91] though it is suggested that they form a 'nonunitary quasi-rational conformal field theory').

An RCFT has a Hermitian inner product defined on its VOA modules A. If (as is usually assumed) this inner product is positive definite, the RCFT is called *unitary*; these are the standard and best-studied RCFT. The matrices S and T defined by (1.2a) will constitute modular data, provided the RCFT is unitary. When it is nonunitary, **MD2** won't be satisfied. For example, the '$c = c(7, 2) = -\frac{68}{7}$ nonunitary minimal model' has S and T, defined by (1.2a), given by

$$\begin{aligned} T &= \mathrm{diag}\{\exp[17\pi\mathrm{i}/21], \exp[5\pi\mathrm{i}/21], \exp[-\pi\mathrm{i}/21]\} \\ S &= \frac{2}{\sqrt{7}} \begin{pmatrix} \sin(2\pi/7) & -\sin(3\pi/7) & \sin(\pi/7) \\ -\sin(3\pi/7) & -\sin(\pi/7) & \sin(2\pi/7) \\ \sin(\pi/7) & \sin(2\pi/7) & \sin(3\pi/7) \end{pmatrix} \end{aligned} \tag{3.8a}$$

This is not modular data, since the first column is not strictly positive. However the 3rd column is. The nonunitary RCFTs tell us to replace **MD2** with

MD2′. For all $a \in \Phi$, S_{0a} is a nonzero real number. Moreover there is some $0' \in \Phi$ such that $S_{0'a} > 0$ for all $a \in \Phi$.

Incidentally, an S matrix which the proof of Proposition 1 in Section 2 would associate to that $c = -\frac{68}{7}$ minimal model is

$$S = \frac{2}{\sqrt{7}} \begin{pmatrix} \sin(\pi/7) & \sin(2\pi/7) & \sin(3\pi/7) \\ \sin(2\pi/7) & -\sin(3\pi/7) & \sin(\pi/7) \\ \sin(3\pi/7) & \sin(\pi/7) & -\sin(2\pi/7) \end{pmatrix} \tag{3.8b}$$

We can tell by looking at (3.8b) that it can't directly be given the familiar interpretation (1.2a). The reason is that any such matrix S must have a strictly positive eigenvector with eigenvalue 1: namely the eigenvector with ath component $\mathrm{ch}_a(\mathrm{i})$ ($\tau = \mathrm{i}$ corresponds to $q = e^{-2\pi} > 0$ and is fixed by $\tau \mapsto -1/\tau$; moreover the characters of VOAs converge at any $\tau \in \mathbb{H}$ [117]). Unlike the S in (3.8a), the S of (3.8b) has no such eigenvector. Thus we may find it convenient (especially in classification attempts) to introduce a new axiom:

MD6. S has a strictly positive eigenvector $\underline{x} > 0$ with eigenvalue 1.

Note that with the choice $T = \mathrm{diag}\{\exp[\pi \mathrm{i}/21], \exp[-17\pi\mathrm{i}/21], \exp[-5\pi\mathrm{i}/21]\}$, (3.8b) obeys **MD1**-**MD4**. Remarkably, all nonunitary RCFT known to this author behave similarly: their fusion rings can always be realised by modular data (although the interpretation (1.2a) typically will be lost).

Knot and link invariants in the 3-sphere S^3 (equivalently, $\mathbb{R}^3$) can be obtained from an R matrix and braid group representations—e.g. we have this with any quasitriangular Hopf algebra. The much richer structure of *topological field theory* (or, in category theoretic language, a *modular category* [110]) gives us link invariants in any closed 3-manifold, and with it modular data. In particular, the S entries correspond to the invariants of the Hopf link in S^3, T to the eigenvalues of the twist operation (Reidemeister 1, which won't act trivially here—strictly speaking, we have knotted ribbons, not strings), and the fusion coefficients to the invariants of 3 parallel circles $S^1 \times \{p_1, p_2, p_3\}$ in the manifold $S^1 \times S^2$. Link invariants are obtained for arbitrary closed 3-manifolds by performing Dehn surgery, transforming the manifold into S^3; the condition that the resulting invariants be well-defined, independent of the specific Dehn moves which get us to S^3, is essentially the statement that S and T form a representation of $\mathrm{SL}_2(\mathbb{Z})$. This is all discussed very clearly in [110]. For instance, we get S^3 knot invariants from the quantum group $U_q(X_r)$ with generic parameter, but to get modular data requires specialising q to a root of unity.

For extensions of this picture to representations of higher genus mapping class groups, see e.g. [5, 60] and references therein, but there is much more work to do here.

Example 5 (VOAs) See e.g. [53, 85] for the basic facts about VOAs; the review article [65] illustrates how VOAs naturally arise in CFT.

Another very general source of modular data comes from vertex operator algebras (VOAs), a rich algebraic structure first introduced by Borcherds [21]. In particular, let $\mathcal{V}$ be any 'rational' VOA (see e.g. [117]—actually, VOA theory is still sufficiently undeveloped that we don't yet have a generally accepted definition of rational VOA). Then $\mathcal{V}$ will have finitely many irreducible modules M, one of which can be identified with $\mathcal{V}$. Zhu [117] showed that their characters $\mathrm{ch}_M(\tau)$ transform nicely under $\mathrm{SL}_2(\mathbb{Z})$ (as in (1.2a)). Defining S and T in that way, and calling Φ the set of irreducible M and the 'identity' $0 = \mathcal{V}$, we get some of the properties of modular data.

A natural conjecture is that a large class (all?) of rational VOAs possess (some generalisation of) modular data. We know what the fusion coefficients mean (dimension of the space of intertwiners between the appropriate VOA modules), and what S and T should mean. We know that T is diagonal and of finite order, and that $S^2 = (ST)^3$ is an order-2 permutation matrix. A Holy Grail of VOA theory is to prove (a generalisation of) Verlinde's formula for a large class of rational VOAs. A problem is that we still don't know when (2.1) here is even defined (i.e. whether all $S_{0,M} \neq 0$). However, suppose $\mathcal{V}$ has the additional (natural) property that any irreducible module $M \neq \mathcal{V}$ has positive conformal weight h_M ($h_M - c/24$ is the smallest power of q in the Fourier expansion of the (normalised) character $\mathrm{ch}_M(\tau) = q^{-c/24} \sum_{n=0}^{\infty} a_n^M q^{n+h_M}$). This holds for instance in all VOAs associated to unitary RCFTs. Then consider the behaviour of $\mathrm{ch}_M(\tau)$ for $\tau \to 0$ along the positive imaginary axis: since each Fourier coefficient a_n^M is a nonnegative number, $\mathrm{ch}_M(\tau)$ will go to $+\infty$. But this is equivalent to considering the limit of $\sum_N S_{MN}\, \mathrm{ch}_N(\tau)$ as $\tau \to \mathrm{i}\infty$ along the positive imaginary axis. By hypothesis, this latter limit is dominated by $S_{M0}\, a_0^0 q^{-c/24}$, at least when $S_{M0} \neq 0$. So what we find is that, under this hypothesis, the 0-column of S consists of nonnegative real numbers (and also that the rank c is positive).

In this context, Example 1 corresponds to the VOA associated to the lattice Λ [40]. Example 2 is recovered by [54], who find a VOA structure on the highest weight $X_r^{(1)}$-module $L(k\Lambda_0)$; the other level k $X_r^{(1)}$-modules $M = L(\lambda)$ all have the structure of VOA modules of $\mathcal{V} := L(k\Lambda_0)$. Example 3 arises for example in the orbifold of a self-dual lattice VOA by a subgroup G of the automorphism group of Λ (see e.g. [43]). An interpretation of fractional level affine algebra data could be possible along the lines of [42], who did it for $A_1^{(1)}$ (but once again see [63, 91]).

Example 6 (Subfactors) See e.g. [47, 19, 20] for good reviews of the subfactor $\leftrightarrow$ CFT relation.

The final general source of modular data which we will discuss comes from subfactor theory. To start with, let $N \subset M$ be an inclusion of II_1 factors with finite Jones index $[M : N]$. Even though M and N will often be isomorphic as factors, Jones showed that there is rich combinatorics surrounding how N is embedded in M. Write $M_{-1} = N \subset M = M_0 \subset M_1 \subset \ldots$ for the tower arising from the 'basic construction'. Let Φ_M denote the set of equivalence classes of irreducible $M - M$ submodules of $\oplus_{n\geq 0}\, {}_M L^2(M_n)_M$, and Φ_N that for the irreducible $N - N$ submodules of $\oplus_{n\geq -1}\, {}_N L^2(M_n)_N$. Write $\mathcal{H}_{AB}^C$ for the intertwiner space $\mathrm{Hom}_{M-M}(C, A \otimes_M B)$. For any $A, B \in \Phi_M$, the Connes' relative tensor product $A \otimes_M B$ can be decomposed into a direct sum $\sum_{C\in\Phi_M} N_{AB}^C C$, where $N_{AB}^C = \dim \mathcal{H}_{AB}^C \in \mathbb{Z}_{\geq}$ are the multiplicities. The identity is the bimodule ${}_M L^2(M)_M$. Assume in addition that Φ_M is finite (i.e. that $N \subset M$ has 'finite depth'). Then all axioms of a fusion ring will be obeyed, except possibly commutativity: unfortunately in general $A \otimes_M B \ncong B \otimes_M A$.

We are interested in M and N being hyperfinite. An intricate subfactor invariant called a *paragroup* (see e.g. [97, 47]) can be formulated in terms of 6j-symbols and fusion rings [47], and resembles exactly solvable lattice models in statistical mechanics. One way to get modular data is by passing from $N \subset M$ to the asymptotic inclusion $\langle M, M' \cap M_\infty \rangle \subset M_\infty$; its paragroup will essentially be an RCFT. Asymptotic inclusion plays the role of quantum-double here, and corresponds physically to taking the continuum limit of the lattice model, yielding the CFT from the underlying statistical mechanical model. More recently [98], Ocneanu has significantly refined this construction, generalising 6j-symbols to what are called Ocneanu cells, and extending the context to subparagroups. His new cells have been interpreted by [99] in terms of Moore-Seiberg-Lewellen data [95, 92].

A very similar but simpler theory has been developed for type III factors (see e.g. the reviews [19, 20]). Bimodules now are equivalent to 'sectors', i.e. equivalence classes of endomorphisms $\lambda : N \to N$ (the corresponding subfactor is $\lambda(N) \subset N$). This use of endomorphisms is the key difference (and simplification) between the type II and type III fusion theories. Given $\lambda, \mu \in \mathrm{End}(N)$, we define $\langle \lambda, \mu \rangle$ to be the dimension of the vector space of intertwiners, i.e. all $t \in N$ such that $t\lambda(n) = \mu(n)t \ \forall n \in N$. The endomorphism $\lambda \in \mathrm{End}(N)$ is irreducible if $\langle \lambda, \lambda \rangle = 1$. Let $\Phi = {}_N\chi_N$ be a finite set of irreducible sectors. The fusion product is given by composition $\lambda \circ \mu$; addition can also be defined, and the fusion coefficient $N_{\lambda\mu}^{\nu}$ will then be the dimension $\langle \lambda\mu, \nu \rangle$. The 'identity' 0 is the identity id_N. Restricting to a finite set Φ of irreducible sectors, closed under fusion in the obvious way, the result is similar to a fusion ring, except again it is not necessarily commutative (after all, why should the compositions $\lambda \circ \mu$ and $\mu \circ \lambda$ be related). The missing ingredients are nondegenerate braidings $\epsilon^{\pm}(\lambda, \mu) \in \mathrm{Hom}(\lambda\mu, \mu\lambda)$, which say roughly that λ and μ nearly commute (the $\epsilon^{\pm}$ must also obey some compatibility conditions, e.g. the Yang-Baxter equations). Once we have a nondegenerate braiding, Rehren [100] proved that we will automatically have modular data.

We will return to subfactors in Section 5. It is probably too optimistic to hope to see in the subfactor picture to what the characters (1.1) correspond—different VOAs and RCFTs can correspond to the same subfactors. To give a simple example, the VOA associated to any self-dual lattice will correspond to the trivial subfactor $N = M$, where M is the unique hyperfinite II_1 factor. With this in mind, it would be interesting to find an S matrix arising here which violates axiom **MD6** given earlier, or the Congruence Subgroup Property of Section 4.

Jones and Wassermann have explicitly constructed the affine algebra subfactors (both type II and III) of Example 2, at least for $A_r^{(1)}$, and Wassermann and students Loke and Toledano Laredo later showed that they recover the affine algebra fusions (see e.g. [114] for a review). Also, to any subgroup-group pair $H < G$, we can obtain a subfactor $R \rtimes H \subset R \rtimes G$ of crossed products, where R is the type II_1 hyperfinite factor, and thus a (not necessarily commutative) fusion-like ring [90]. This subfactor $R \rtimes H \subset R \rtimes G$ can be thought of as giving a grouplike interpretation to G/H even when H is not normal. Sometimes it will have a braiding—e.g. the diagonal embedding $G < G \times G$ recovers the finite group data of Example 3. What is intriguing is that some other pairs $H < G$ probably also have a braiding, generalising Example 3. There is a general suspicion, due originally perhaps to Moore and Seiberg [95] and in the spirit of Tannaka-Krein duality, that RCFTs can always be constructed in standard ways (Goddard-Kent-Olive cosets and finite group orbifolds) from lattice and affine algebra models. These crossed product subfactors could conceivably provide reams of counterexamples, suggesting that the orbifold construction can be considerably generalised.

A uniform construction of the affine algebra and finite group modular data is provided in [39] where a 3-dimensional TFT is associated to any topological group G (G will be a compact Lie group in the affine case; G is given discrete topology in the finite case). There we see that the level k and twist α both play the same role, and are given by a cocycle in $H^3(G, \mathbb{C}^{\times})$. Crane-Yetter [33] are developing a theory of cohomological 'deformations' of modular data (more precisely, of modular categories). In [33] they discuss the infinitesimal deformations of tensor categories, where the objects are untouched but the arrows are deformed, though their ultimate interest would be in global deformations and in particular in specialising to the especially interesting ones—much as we deform the enveloping algebra $U(\mathfrak{g})$ to get the quantum group $U_q(\mathfrak{g})$ and then specialise to roots of unity to get e.g. modular data. Their work is still in preliminary stages and it probably needs to be generalised further (e.g. they don't seem to recover the level of affine algebras), but it looks very promising. Ultimately it can be hoped that some discrete H^3 group will be identified which parametrises the different quantum doubles of a given symmetric tensor category.

Incidentally, the fact that $H^3(G, \mathbb{C}^\times)$ is a group strongly suggests that it should be meaningful to compare the modular data for different cocycles—e.g. to fix the affine algebra and vary k. This idea still hasn't been seriously exploited (but e.g. see 'threshold level' in [13, 14]).

There are many examples of 'modular-like data'. These are interesting for probing the question of just what should be the definition of fusion algebra or modular data. Here is an intriguing example, inspired by (4.4) below.

Example 7 [72] (Number fields) A basic introduction to algebraic number theory is provided by e.g. [28].

Choose any finite normal extension $\mathbb{L}$ of $\mathbb{Q}$, and find any totally positive $\alpha \in \mathbb{L}$ with $\mathrm{Tr}(|\alpha|^2) = 1$ (total positivity will turn out to be necessary for **F1**). Now find any $\mathbb{Q}$-basis $x_1 = 1, x_2, \ldots, x_n$ of a subfield $\mathbb{K}$ of $\mathbb{L}$, where $n = \deg(\mathbb{K})$, the x_i being orthonormal with respect to the trace $\langle x, y\rangle_\alpha := \mathrm{Tr}(|\alpha|^2 x\bar{y})$ (orthonormality will guarantee **F3** to be satisfied). Let G denote the set of n distinct embeddings $\mathbb{K} \to \mathbb{C}$. Our construction requires complex conjugation to commute with all embeddings. Under these conditions $|\alpha|^{-2} = \sum_i |x_i|^2$. Then we get a fusion-like algebra with primaries $\beta = \{x_1, \ldots, x_n\}$, '$*$' given by complex conjugation, and structure constants $N_{ij}^k = \mathrm{Tr}(|\alpha|^2 x_i x_j \overline{x_k}) \in \mathbb{Q}$ given by ordinary multiplication and addition: $x_i x_j = \sum_k N_{ij}^k x_k$. Call the resulting fusion-like algebra $\mathbb{K}(\beta)$.

It is easy to see that all the properties of a generalised fusion algeba are satisfied, except possibly $N_{ij}^k \in \mathbb{Q}_\geq$. The fusion coefficients N_{ij}^k will be integers iff the $\mathbb{Z}$-span of the x_i form an 'order' of $\mathbb{K}$. We also find that the matrix $S_{ig} = g(\alpha x_i)$, for $g \in G$ (lift each g arbitrarily to $\mathbb{L}$), diagonalises these fusion matrices N_{x_i}. This matrix S is unitary, but (unless $\mathbb{K}$ is an abelian extension of $\mathbb{Q}$) the dual fusions $\hat{N}$ in (2.3) won't be rational.

Positivity **F1** requires one of the columns of S to be positive; permuting with g, we may require all basis elements $x_i > 0$. Hence $\mathbb{K}(\beta)$ will have a chance of being a generalised fusion algebra only when $\mathbb{K}$ is 'totally real'.

Incidentally this example is more general than it looks: it is easy to show

Proposition 2 Let A be a generalised fusion algebra which is isomorphic as a $\mathbb{Q}$-algebra to a field $\mathbb{K}$. Then A is isomorphic as a generalised fusion algebra to some $\mathbb{K}(\Phi)$.

More generally, recall that a generalised fusion algebra is isomorphic as a $\mathbb{Q}$-algebra to a direct sum of number fields. So an approach to studying (generalised) fusion algebras could be to study how they are built up from number fields. It would be very interesting to classify all generalised fusion algebras which are isomorphic as a $\mathbb{Q}$-algebra to a field. For example, take $\mathbb{K} = \mathbb{Q}[\sqrt{N}]$, where N is not a perfect square, and where also any prime divisor

$p \equiv -1 \pmod 4$ of N occurs with even multiplicity. Then we can find positive integers a, b such that $N = a^2 + b^2$. Take $\Phi = \{1, \frac{b}{a} + \frac{1}{a}\sqrt{N}\}$, then $\mathbb{K}(\Phi)$ is a fusion algebra with $N_{22}^2 = \frac{2b}{a}$. Note that this construction exhausts all 2-dimensional fusion algebras, except when $\sqrt{(N_{22}^2)^2 + 4}$ is rational, which corresponds to the $\mathbb{Q}$-algebra $\mathbb{Q} \oplus \mathbb{Q}$ (e.g. the fusion ring of affine algebra $A_1^{(1)}$ at level 1). For $N = 5$ and $a = 2$, we recover the fusion ring of affine algebra $F_4^{(1)}$ or $G_2^{(1)}$ at level 1.

4. Modular data: Basic theory

In this section we sketch the basic theory of modular data. Most of these results are elementary and can be proved quickly from the axioms; many appear in greater generality in the C-algebra/table algebra/... literature. For example, Eqs. (2.2) and (2.3) are in [18]. All of the following statements can be generalised to self-dual generalised fusion rings, and many to generalised fusion algebras, but we will focus on the case of greatest interest to RCFT: when $S = S^t$ and $N_{ab}^c \in \mathbb{Z}_{\geq}$.

It is important to reinterpret (2.1) in matrix form. For each $a \in \Phi$, define the *fusion matrix* N_a by

$$(N_a)_{b,c} = N_{ab}^c.$$

Then (2.1) says that the N_a are simultaneously diagonalised by S. More precisely, the bth column $S_{\updownarrow,b}$ of S is an eigenvector of each N_a, with eigenvalue $\frac{S_{ab}}{S_{0b}}$. Unitarity of S tells us: $\frac{S_{ab}}{S_{0b}} = \frac{S_{ac}}{S_{0c}}$ holds for all $a \in \Phi$, iff $b = c$. In other words:

Basic Fact. All simultaneous eigenspaces are of dimension 1, and are spanned by each column $S_{\updownarrow,b}$.

Take the complex conjugate of (2.1): we find that $\bar{S}$ also simultaneously diagonalises the fusion matrices N_a. Hence there is some permutation of Φ, which we will denote by C and call *conjugation*, and some complex numbers α_b, such that

$$\overline{S_{ab}} = \alpha_b\, S_{a,Cb}.$$

Unitarity forces each $|\alpha_b| = 1$. Looking at $a = 0$ and applying **MD2**, we see that the α_b must be positive. Hence

$$\overline{S_{ab}} = S_{a,Cb} = S_{Ca,b} \tag{4.1}$$

and so $C = S^2$. The conjugation C is trivial iff S is real. Note also that C, like complex conjugation, is an involution, and that $C_{00} = 1$. Some easy formulae are $N_0 = I$, $N_{ab}^0 = C_{ab}$, and $N_{Ca,Cb}^{Cc} = N_{ab}^c$. Because $C = S^2 = (ST)^3$, C commutes with both S and T: $S_{Ca,Cb} = S_{a,b}$ and $T_{Ca,Cb} = T_{a,b}$.

For example, in Example 1, $C[a] = [-a]$, while for $A_1^{(1)}$ the matrix S is real and so $C = I$. More generally, for the affine algebra $X_r^{(1)}$ the conjugation C corresponds to a symmetry of the Dynkin diagram of X_r. For finite groups (Example 3), C takes (a, χ) to

$(a^{-1}, \overline{\chi})$. In RCFT, C is called *charge-conjugation*; it's a symmetry in quantum field theory which interchanges particles with their antiparticles (and so reverses the sign of the charge, hence the name).

Because C is an involution, we know that the assignment (1.2b) defines a finite-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$, for any choice of modular data—hence the name. A surprising fact is that this representation usually (always?) seems to factor through a congruence subgroup. We'll return to this at the end of this section.

Perron-Frobenius theory, i.e. the spectral theory of nonnegative matrices (see e.g. [75]), has some immediate consequences. By **MD2** and our BASIC FACT, the Perron-Frobenius eigenvalue of N_a is $\frac{S_{a0}}{S_{00}}$; hence we obtain the important inequality

$$S_{a0}S_{0b} \geq |S_{ab}|\, S_{00}. \tag{4.2a}$$

Unitarity of S applied to (4.2a) forces

$$\min_{a\in\Phi} S_{a0} = S_{00}. \tag{4.2b}$$

In other words the *q-dimensions*, defined to be the ratios $\frac{S_{a0}}{S_{00}}$, are bounded below by 1. The name 'q-dimension' comes from quantum groups (and also affine algebras (3.2c)), where one finds a q-deformed Weyl dimension formula. In RCFT, $\frac{S_{a0}}{S_{00}} = \lim_{\tau\to 0^+ \mathrm{i}} \frac{\mathrm{ch}_a(\tau)}{\mathrm{ch}_0(\tau)}$. In the subfactor picture (Example 6), the Jones index is the square of the q-dimension.

Cauchy-Schwarz and unitarity, together with (4.2a), gives us the curious inequality

$$\sum_{e\in\Phi} N_{ac}^{e} N_{bd}^{e} \leq \frac{S_{a0}}{S_{00}}\frac{S_{b0}}{S_{00}} \tag{4.2c}$$

for all $a, b, c, d \in \Phi$. So for instance $N_{ab}^{c} \leq \min\{\frac{S_{a0}}{S_{00}}, \frac{S_{b0}}{S_{00}}, \frac{S_{c0}}{S_{00}}\}$. Equality holds in (4.2c) only if $S_{a0} = S_{b0} = S_{00}$ (i.e. only if a and b are *units*—see below), since it is only when a is a unit that equality in (4.2a) holds for all $b \in \Phi$. Other inequalities are possible, though perhaps not useful: e.g. Hölder gives us for all $a \in \Phi$ and $k, m = 1, 2, 3, \ldots$ the following bounds on traces of powers of fusion matrices:

$$\left(\mathrm{Tr}\left(N_a^{k}\right)\right)^{m} \leq \|\Phi\|^{m-1}\, \mathrm{Tr}\left(N_a^{km}\right) \tag{4.2d}$$

The equality (4.2b) suggests that we look at those primaries $a \in \Phi$ obeying the equality $S_{a0} = S_{00}$. Such primaries are called *simple-currents* in RCFT parlance (see e.g. [103, 36] and references therein), and in the table algebra literature are called *linear* [2], but the term we regard as more natural is *units*. To any unit $j \in \Phi$, there is a phase $\varphi_j : \Phi \to \mathbb{C}$ and a permutation J of Φ such that $j = J0$ and

$$S_{Ja,b} = \varphi_j(b)\, S_{a,b} \tag{4.3a}$$

$$T_{Ja,Ja}\overline{T_{aa}} = \overline{\varphi_j(a)}\, T_{jj}\overline{T_{00}} \tag{4.3b}$$

$$(T_{jj}\overline{T_{00}})^2 = \overline{\varphi_j(j)} \tag{4.3c}$$

Moreover, if J is of order n, then $\varphi_j(a)$ is an nth root of unity and $T_{Ja,Ja}\overline{T_{aa}}$ is a $2n$th root of 1; when n is odd, the latter will in fact be an nth root of 1. To reflect the physics heritage, the permutation J corresponding to a unit $j \in \Phi$ will be called a simple-current. The set of all simple-currents or units forms an abelian group (using composition of the permutations), called the *centre* of the modular data. Note that $CJ = J^{-1}C$, and $N_{Ja,J'b}^{JJ'c} = N_{ab}^c$ for any simple-currents J, J'.

For instance, for a lattice Λ, all $[a] \in \Phi$ are units, corresponding to permutation $J_{[a]}([b]) = [a+b]$ and phase $\varphi_{[a]}([b]) = e^{2\pi i a\cdot b}$. For the affine algebra $A_1^{(1)}$ at level k (recall (3.5)), there is precisely one nontrivial unit, namely $j = k$, corresponding to $J(a) = k - a$ and $\varphi_j(a) = (-1)^a$. More generally, to any affine algebra (except for $E_8^{(1)}$ at $k = 2$), the units correspond to symmetries of the extended Dynkin diagram. For $A_1^{(1)}$ this symmetry interchanges the 0th and 1st nodes, i.e. $J(\lambda_0\Lambda_0+\lambda_1\Lambda_1) = \lambda_1\Lambda_0+\lambda_0\Lambda_1$ (recall $a = \lambda_1$); for $A_r^{(1)}$ the centre is $\mathbb{Z}/(r+1)\mathbb{Z}$. In the finite group modular data, the units are the pairs (z, ψ) where z lies in the centre $Z(G)$ of G, and ψ is a dimension-1 character of G. It corresponds to simple-current $J_{(z,\psi)}(a, \chi) = (za, \psi\chi)$ and phase $\varphi_{(z,\psi)}(a, \chi) = \overline{\psi(a)}\,\overline{\chi(z)}/\chi(e)$. The centre of this modular data will thus be isomorphic to the direct product $Z(G) \times (G/G')$, where $G' = \langle ghg^{-1}h^{-1}\rangle$ is the commutator subgroup of G.

To see (4.3a), note first that (4.2a) tells us $S_{0b} \geq |S_{jb}|$ for any unit j, and any $b \in \Phi$. However, unitarity then forces $S_{0b} = |S_{jb}|$, i.e. (4.3a) holds for $a = 0$ (with $J0$ defined to be j), and some numbers $\varphi_j(b)$ with modulus 1. Putting this into (2.1), we get $N_jN_{Cj} = I$, the identity matrix. But the only nonnegative integer matrices whose inverses are also nonnegative integer matrices, are the permutation matrices. This defines the permutation J of Φ. Eq. (4.3a) now follows from Cauchy-Schwartz applied to

$$1 = N_{j,a}^{Ja} = \sum_{d\in\Phi} \varphi_j(d)\, S_{ad}\, \overline{S_{Ja,d}}$$

The reason $J \circ J' = J' \circ J$ is because the fusion matrices commute: $N_{J(J'0)} = N_{J0}N_{J'0} = N_{J'0}N_{J0} = N_{J'(J0)}$.

To see (4.3b), first write $(ST)^3 = C$ as $STS = \overline{T}S\overline{T}$, then use that and (4.3a) to show $(\overline{T}S\overline{T})_{Ja,0} = (\overline{T}S\overline{T})_{a,J0}$. To see (4.3c), use (4.3b) with $a = J^{-1}0$, together with the fact that C commutes with T. Note that $\varphi_j(j') = S_{j,j'}/S_{00} = \varphi_{j'}(j)$ and $\varphi_{JJ'0}(a) = \varphi_j(a)\,\varphi_{j'}(a)$, so $\varphi_{J^k0}(a) = (\varphi_j(a))^k$; from all these and (4.3b) we get that

$$1 = T_{J^n0,J^n0}\,\overline{T_{00}} = \overline{\varphi_j(j)}^{\,n(n-1)/2}(T_{jj}\overline{T_{00}})^n$$

Equations (4.2a) and (4.3b) also follow from the curious equation

$$\overline{S_{ab}}\,T_{aa}\,T_{bb}\,\overline{T_{00}} = \sum_{c\in\Phi} N_{ab}^c\,T_{cc}\,S_{c0}$$

which is derived from (2.1) and $STS = \overline{T}S\overline{T}$.

Simple-currents and units play an important role in the theory of modular data and fusion rings. One place they appear is gradings. By a *grading* on Φ we mean a map $\varphi : \Phi \to \mathbb{C}^\times$ with the property that if $N_{ab}^c \neq 0$ then $\varphi(c) = \varphi(a)\,\varphi(b)$. The phase φ_j coming from a unit is clearly a grading; a little more work [72] shows that any grading φ of Φ corresponds to a unit j in this way. The multiplicative group of gradings, and the group of simple-currents (the centre), are naturally isomorphic. (This is not true of generalised fusion algebras.)

Next, we will generalise the conjugation symmetry argument, to other Galois automorphisms. In particular, write $\mathbb{Q}[S]$ for the field generated over $\mathbb{Q}$ by all entries S_{ab}. Then for any Galois automorphism $\sigma \in \text{Gal}(\mathbb{Q}[S]/\mathbb{Q})$,

$$\sigma(S_{ab}) = \epsilon_\sigma(a)\, S_{\sigma a,b} = \epsilon_\sigma(b)\, S_{a,\sigma b} \tag{4.4}$$

for some permutation $c \mapsto \sigma c$ of Φ, and some signs $\epsilon_\sigma : \Phi \to \{\pm 1\}$. Moreover, the complex numbers S_{ab} will necessarily lie in the cyclotomic extension $\mathbb{Q}[\xi_n]$ of $\mathbb{Q}$, for some root of unity $\xi_n := \exp[2\pi \mathrm{i}/n]$. This follows from the Kronecker-Weber Theorem and the calculation from (4.4) that

$$\sigma\sigma' S_{ab} = \epsilon_\sigma(a)\, \epsilon_{\sigma'}(b)\, S_{\sigma a, \sigma' b} = \sigma' \sigma\, S_{ab}.$$

For a field extension $\mathbb{K}$ of $\mathbb{Q}$, Gal($\mathbb{K}/\mathbb{Q}$) denotes the automorphisms σ of $\mathbb{K}$ fixing all rationals. Recall that each automorphism $\sigma \in \text{Gal}(\mathbb{Q}[\xi_n]/\mathbb{Q})$ corresponds to an integer $1 \le \ell \le n$ coprime to n, acting by $\sigma(\xi_n) = \xi_n^\ell$. Note that Eq. (4.4) tells us the power $\sigma^{2\|\Phi\|!}$ will act trivially on each entry S_{ab}. In other words, the degree of the field extension $[\mathbb{Q}[S] : \mathbb{Q}]$ is bounded by (in fact divides) $2\|\Phi\|!$. This is perhaps the closest we have to a finiteness result for modular data (see however [10] which obtains a bound for n in terms of $\|\Phi\|$, for the modular data arising in RCFT).

In other incarnations, this Galois action appears in the $\chi(g) \mapsto \chi(g^\ell)$ symmetry of the character table of a finite group, and of the action of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on level N modular functions. Equation (4.4) was first shown in [30] and a related symmetry for commutative association schemes was found in [96]. The analogue of cyclotomy isn't known for association schemes. The reason is the additional 'self-duality' property of the fusion ring, i.e. the fact that $S = S^t$ or more generally (2.4).

Recall from Section 2 that a fusion ring $R = \mathcal{F}(\Phi, N)$ is isomorphic to a direct sum of number fields. The Galois orbits determine these fields. In particular, for any Galois orbit $[d]$ in Φ, let $\mathbb{K}_{[d]}$ denote the field generated by all numbers of the form $\frac{S_{ab}}{S_{0b}}$ for $a \in \Phi$ and $b \in [d]$. Then R is isomorphic as a $\mathbb{Q}$-algebra to the direct sum $\oplus_{[d]}\mathbb{K}_{[d]}$. We gave the $A_1^{(1)}$ level k example in Section 2.

The Galois action for the lattice modular data is simple: the Galois automorphisms $\sigma = \sigma_\ell$ correspond to integers ℓ coprime to the determinant $|\Lambda|$; σ_ℓ takes $[a]$ to $[\ell a]$, and all parities $\epsilon_\ell([a]) = +1$. The Galois action for the affine algebras is quite interesting (see e.g. [1]), and can be expressed geometrically using the action of the affine Weyl group on the weight lattice of X_r. Both $\epsilon_\ell(\lambda) = \pm 1$ will occur. For finite groups, σ_ℓ takes (a, χ) to $(a^\ell, \sigma_\ell \circ \chi)$, and again all $\epsilon_\ell(a, \chi) = +1$.

The presence of the Galois action (4.4) is an effective criterion (necessary and sufficient) on the matrix S for the numbers in (2.1) to be rational. It would be very desirable to find effective conditions on S such that the fusion coefficients are nonnegative, or integral. At present the best results along these lines are, respectively, the inequalities (4.2), and the fact that the ratios $\frac{S_{ab}}{S_{0b}}$ are algebraic integers (since they are eigenvalues of integer matrices). When there are units, then (4.3a) provides an additional strong constraint on nonnegativity.

As repeated throughout this paper, the classification of modular data and fusion rings is an important open question. A promising approach to this uses this Galois symmetry. If there are n primaries, the Galois permutations $a \mapsto \sigma a$ define an abelian subgroup of the symmetric group S_n. Up to equivalence, there are 4 of these for $n = 3$ and 8 for $n = 4$. For concreteness let us sketch the relatively difficult case $n = 3$ and the order-3 permutation group $\langle(012)\rangle$ in S_3. Because that subgroup contains no order-2 permutation fixing 0, the conjugation C is trivial, and S must be real and orthogonal. Choose any $\sigma \in \mathrm{Gal}(\mathbb{Q}[S]/\mathbb{Q})$ corresponding to the permutation (012) and write $\epsilon_i = \epsilon_\sigma(i)$ for $i = 0, 1, 2$. Hitting $\det(S) = \pm 1$ with σ tells us that $\epsilon_0\epsilon_1\epsilon_2 = +1$ and σ generates $\mathrm{Gal}(\mathbb{Q}[S]/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$. We can write

$$S = \begin{pmatrix} a & b & c \\ b & \epsilon_2 c & \epsilon_1 a \\ c & \epsilon_1 a & \epsilon_0 b \end{pmatrix}$$

where σ sends a, b, c to $\epsilon_0 b, \epsilon_1 c, \epsilon_2 a$, respectively. The invariants of this 3-dimensional $\mathbb{Z}/3\mathbb{Z}$-action are generated by $E_1 = a + \epsilon_0 b + \epsilon_2 c$, $E_2 = \epsilon_0 ab + \epsilon_1 bc + \epsilon_2 ac$, $E_3 = \epsilon_1 abc$, and $H = \epsilon_0 a^2 b + \epsilon_2 b^2 c + c^2 a$, with a syzygy quadratic in H. The value of E_1 is $\mathrm{Trace}(S) = \pm 1$, while orthogonality of S says $E_2 = 0$. Note that $m := \sum_i S_{1i}/S_{0i} = \epsilon_0 H/E_3$ is an integer; together with the syzygy, this fixes the values of all E_i and H in terms of m and the ϵ_j (e.g. we find $E_1 = -\epsilon_1$). The fusions, being rational, must be invariants of this $\mathbb{Z}/3\mathbb{Z}$ action; indeed we obtain

$$N_{11}^1 = m + \epsilon_0, \quad N_{11}^2 = -\epsilon_2, \quad N_{12}^2 = -\epsilon_0, \quad N_{22}^2 = -\epsilon_1 m - 2\epsilon_2$$

for the 4 undetermined fusion coefficients. Thus $\epsilon_0 = \epsilon_2 = -1 = -\epsilon_1$ and $m \in \{1, 2\}$, and we find that the only fusion rings realising this Galois automorphism is that of what we'll call $A_{1,5}/\langle J\rangle$. Once we know the possible fusion rings, the possible modular data can be quickly obtained from the known classification [108] of irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations in dimensions up to 5.

A very desirable property for modular data to possess is:

Congruence Subgroup Property [31] *Let N be the order of the matrix T, so $T^N = I$, and let ρ be the representation of $SL_2(\mathbb{Z})$ coming from the assignment* (1.2b). *Then ρ factors through the congruence subgroup*

$$\Gamma(N) := \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

and so (1.2b) *in fact defines a representation of the finite group $SL_2(\mathbb{Z}/N\mathbb{Z})$. Moreover, the characters* (1.1) *are modular functions for* $\Gamma(N)$. *The entries S_{ab} all lie in the cyclotomic field* $\mathbb{Q}[\exp(2\pi \mathrm{i}/N)]$, *and for any Galois automorphism σ_ℓ,*

$$T_{\sigma_\ell a, \sigma_\ell a} = T_{aa}^{\ell^2} \quad \forall a \in \Phi \tag{4.5}$$

For example, the modular data from Examples 1–3 in Section 3 all obey this property. In particular, affine algebra characters χ_λ are essentially lattice theta functions. It would be valuable to find examples of modular data which do *not* obey this property. For much more discussion, see [31, 10]. In those papers, considerable progress was made towards clarifying its role (and existence) in modular data. For example:

Proposition 3 [31] *Consider any modular data] Let N be the order of T, and suppose that N is either coprime to $p = 2$ or $p = 3$. Then the corresponding $SL_2(\mathbb{Z})$ representation factors through $\Gamma(N)$, provided* (4.5) *holds for $\ell = p$.*

In the remaining case, i.e. when 6 divides N, more conditions are needed; these are also given in [31]. Assuming some additional structure from RCFT, [10] recently established the congruence property ([31] had previously proved the $\Gamma(N)$ part when T has odd order). Though this is clearly an impressive feat, what it means in the more general context of modular data isn't clear: it is difficult to explicitly write down the additional axioms needed to supplement our definition of modular data, in order that the necessary calculations go through.

It is tempting to think that the congruence property is a good approach to verifying that rational VOA characters are modular functions. It also leads, via [44], to another promising approach to classifying modular data.

Let us conclude this section with some general remarks on the algebraic structure of the fusion ring. Whenever a structure is studied, of fundamental importance are the structure-preserving maps. It is through these maps that different examples of the structure can be compared. By a *fusion-homomorphism* π between fusion rings $\mathcal{F}(\Phi, N)$ and $\mathcal{F}(\Phi', N')$ we mean a ring homomorphism for which $\pi(\Phi) \subseteq \Phi'$. *Fusion-isomorphisms* and *fusion-automorphisms* are defined in the obvious ways. All fusion-isomorphisms between affine algebra fusion rings are known. Most of them are in fact fusion-automorphisms, and are constructed in simple ways from the symmetries of the Dynkin diagrams. Here are some basic general facts about fusion-homomorphisms:

Proposition 4 *Let $\pi : \Phi \to \Phi'$ be a fusion-homomorphism between any two fusion rings. Then*

(a) *$\pi 0 = 0'$ and $\pi(a^*) = \pi(a)^*$, and π takes units of Φ to units of Φ'.*
(b) *There exists a map $\pi' : \Phi' \to \Phi$ such that*

$$\frac{S'_{\pi a, b'}}{S'_{0', b'}} = \frac{S_{a, \pi' b'}}{S_{0, \pi' b'}} \quad \forall a \in \Phi,\ b' \in \Phi'$$

(c) *If $\pi a = \pi b$, then $b = Ja$ for some simple-current J. In addition, this J will obey $\pi(Jd) = \pi(d)$ for all $d \in \Phi$, and (provided J is nontrivial) there can be no J-fixed-points in Φ.*

(d) *If π is surjective, then $\pi' : \Phi' \to \Phi$ is an injective fusion-homomorphism, and*

$$S'_{\pi a,b'} = \sqrt{\|\ker(\pi)\|}\, S_{a,\pi' b'}$$

Part (a) follows from **F1** and **F3**. Part (b) follows because $\frac{S'_{\pi a,b'}}{S'_{0',b'}}$ is a 1-dimensional representation of the Φ' fusion ring. To get (c), consider $(\pi a)(\pi b)^* = \pi(ab^*)$. If f is a fixed-point of J in (c), count the multiplicity of the identity $0'$ in the fusion product $(\pi f) \cdot (\pi f)^*$. To see (d), apply (c) to

$$\sum_a \left| \frac{S'_{\pi a,b'}}{S'_{0',b'}} \right|^2 = \sum_a \left| \frac{S_{a,\pi' b'}}{S_{0,\pi' b'}} \right|^2$$

For example, fix any units $j, j' \in \Phi$ of equal order n. Then $a \mapsto J^{Q'(a)}a$ defines a fusion-endomorphism, where we write $\varphi_{j'}(a) = \exp[2\pi \mathrm{i}\, Q'(a)/n]$. It will be a fusion-automorphism iff $Q'(j) + 1$ is coprime to n. For another example, take any Galois automorphism σ for which $\sigma(S_{00}^2) = S_{00}^2$, or equivalently $\sigma 0 = J0$ for some simple-current J. Then $a \mapsto J\sigma a$ is a fusion-automorphism. For this Galois example $\pi' = \pi$, while for the simple-current one $\pi'(b) = J'^{Q(b)}b$. Part (d) doesn't hold for generalised fusion algebras.

The map π' of Proposition 4(b) won't in general be a fusion-homomorphism. E.g. consider the fusion-homomorphism $\pi : \{[0], [1]\} \to \{0, 1, \ldots, k\}$ between the fusion ring of the lattice $\Lambda = \sqrt{2}\mathbb{Z}$ and the fusion ring for $A_1^{(1)}$ level k, given by $\pi([0]) = 0, \pi([1]) = k$. Then π' is given by $\pi'(a) = [a]$.

In the context of RCFT, fusion-homomorphisms, to this author's knowledge, have been largely ignored. This has probably been a mistake; a challenge will be to find applications of these sorts of results to problems dearer to a conformal field theorist's heart. In particular, applications to the theory of NIM-reps should be easy to find.

5. Modular invariants and NIM-reps

In Section 3 we mentioned a natural question for algebraic combinatorists to address: finding the analogue of the Littlewood-Richardson rule for affine algebra fusions. Last section we underlined a more important potential contribution: the construction and classification of new families of fusion rings and, more important, modular data. In this section and the next, we describe a final topic, dear to e.g. CFT, to which algebraic combinatorics could make significant contributions: the study and classification of modular invariants and NIM-reps.

A *modular invariant* is a matrix M, rows and columns labeled by Φ, obeying:

MI1. $MS = SM$ and $MT = TM$;
MI2. $M_{ab} \in \mathbb{Z}_{\geq}$ for all $a, b \in \Phi$; and
MI3. $M_{00} = 1$.

As usual we write $\mathbb{Z}_{\geq}$ for the nonnegative integers. The simplest example of a modular invariant is of course the identity matrix $M = I$. Another example is conjugation C. All of the modular invariants for $A_1^{(1)}$ at level k are given below in (6.1).

Why are modular invariants interesting? Most importantly, they are central to the task of classifying RCFTs. The genus-1 'vacuum-to-vacuum amplitude'(=partition function) $\mathcal{Z}(\tau)$ of the theory looks like (1.3c). It assigns to the torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ the complex number $\mathcal{Z}(\tau)$. But the moduli space of conformally equivalent tori is the orbit space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, where the action is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau+b}{c\tau+d}$. Thus the partition function $\mathcal{Z}(\tau)$ must be invariant under this natural action of the modular group $\mathrm{SL}_2(\mathbb{Z})$, which gives us **MI1**. The coefficients M_{ab} count the primary fields $|\phi_a,\phi_b\rangle$ in the state space $\mathcal{H}$, i.e. the number of times the module $A_a \otimes A_b$ of left chiral algebra$\times$right chiral algebra, appears in $\mathcal{H}$. That gives us **MI2**. And the uniqueness of the vacuum $|0,0\rangle$ means **MI3**. That is to say, the coefficient matrix M of an RCFT partition function is a modular invariant. An RCFT is uniquely specified by the knowledge of its partition function, its (left and right) chiral algebras(=VOAs), and the so-called structure constants. Thus, an important fingerprint of the RCFT is its partition function $\mathcal{Z}$, i.e. its modular invariant M. From the related point of view of string theory, modular invariants describe the consistent 'first-quantised perturbative string theories'.

Another motivation for studying modular invariants is the extensions $\mathcal{V} \subset \mathcal{V}'$ of rational VOAs (similar remarks hold for braided subfactors). Let M_i and M'_j be the irreducible modules of $\mathcal{V}$ and $\mathcal{V}'$, respectively. Then each M'_j will be a $\mathcal{V}$-module. A rational VOA should have the complete reducibility property, so each M'_j should be expressible as a direct sum of M_i's—these are called the branching rules. As mentioned in Example 5, we would expect that the characters (1.1) of a rational VOA should yield (some form of) modular data via (1.2a). So the diagonal sum $\sum_j |\mathrm{ch}'_{M'_j}|^2$ should be invariant under the $\mathrm{SL}_2(\mathbb{Z})$-action; rewriting the $\mathrm{ch}'_{M'_j}$'s in terms of the ch_{M_i}'s via the branching rules yields a modular invariant for $\mathcal{V}$.

For instance, the VOA $L(\Lambda_0)'$ corresponding to the affine algebra $G_2^{(1)}$ level 1 contains the VOA $L(28\Lambda_0)$ corresponding to $A_1^{(1)}$ at level 28. We get the branching rules $L(\Lambda_0)' = L(0) \oplus L(10) \oplus L(18) \oplus L(28)$ and $L(\Lambda_2)' = L(6) \oplus L(12) \oplus L(16) \oplus L(22)$, where $L(\lambda_1) := L(\lambda)$. This corresponds to the $A_1^{(1)}$ level 28 modular invariant given below in (6.1f).

So knowing the modular invariants for some VOA $\mathcal{V}$ gives considerable information concerning its possible 'nice' extensions $\mathcal{V}'$. For instance, we are learning that the only finite 'rational' extensions of a generic affine VOA are those studied in [41] ('simple-current extensions') and whose modular data is conjecturally given in [58].

Another reason for studying modular invariants is that the answers are often surprising. Lists arising in math from complete classifications tend to be about as stale as phonebooks, but to give some samples:

- the $A_1^{(1)}$ modular invariants fall into the A-D-E metapattern;
- the $A_2^{(1)}$ modular invariants have connections with Jacobians of Fermat curves; and
- the $(U(1) \oplus \cdots \oplus U(1))^{(1)}$ modular invariants correspond to rational points on Grassmannians.

We will discuss this point a little more next section. These 'coincidences', presumably, have something to do with the nontrivial connections between RCFT and several areas of math, but it also is due to the beauty of the combinatorics of Lie characters evaluated at elements of finite order (3.2c).

In any case, in this section we will study the modular invariants corresponding to a given choice of modular data. For lattices, the classification is easy (use (5.2) below). For many finite groups, the classification typically will be hopeless—e.g. the quantum-double of the alternating group A_5, which has only 22 primaries, has a remarkably high number (8719) of modular invariants [7]. For affine algebra modular data, the classification of modular invariants seems to be just barely possible, and the answer is that (generically) the only modular invariants are constructed in straightforward ways from symmetries of the Dynkin diagrams.

Commutation with T is trivial to solve, since T is diagonal: it yields the selection rule

$$M_{ab} \neq 0 \;\Rightarrow\; T_{aa} = T_{bb} \tag{5.1}$$

This isn't as useful as it looks; commutation with S (or equivalently, the equation $SM\bar{S} = M$) is more subtle, but far more valuable.

An immediate observation is that there are only finitely many modular invariants associated to given modular data. This follows for instance from

$$1 = M_{00} = \sum_{a,b\in\Phi} S_{0a}\, M_{ab}\, S_{b0} \geq S_{00}^2 \sum_{a,b\in\Phi} M_{ab}$$

We will find that each basic symmetry of the S matrix yields a symmetry of the modular invariants, a selection rule telling us that certain entries of M must vanish, and a way to construct new modular invariants.

First consider simple-currents J, J'. Equation (4.3a) and positivity tell us

$$M_{J0,J'0} = \left| \sum_{c,d\in\Phi} \varphi_J(c)\, S_{0c}\, M_{cd}\, \overline{S_{d0}}\, \overline{\varphi_{J'}(d)} \right| \leq \sum_{c,d} S_{0c} M_{cd} S_{d0} = M_{00} = 1 \tag{5.2a}$$

Thus $M_{J0,J'0} \neq 0$ implies $M_{J0,J'0} = 1$, as well as the selection rule

$$M_{cd} \neq 0 \;\Rightarrow\; \varphi_J(c) = \varphi_{J'}(d) \tag{5.2b}$$

A similar calculation yields the symmetry

$$M_{J0,J'0} \neq 0 \;\Rightarrow\; M_{Ja,J'b} = M_{ab} \qquad \forall a, b \in \Phi \tag{5.2c}$$

The most useful application of simple-currents to modular invariants is to their construction. In particular, let J be a simple-current of order n. Then we learned in (4.3) that $\varphi_j(a)$ is an nth root of 1, and that $(T_{jj}\overline{T_{00}})^{2n} = 1$ and in fact $(T_{jj}\overline{T_{00}})^n = 1$ when n is odd. That is to say, we can find integers r_j and $Q_j(a)$ obeying

$$\varphi_j(a) = \exp\left[2\pi\mathrm{i}\,\frac{Q_j(a)}{n}\right], \quad T_{jj}\,\overline{T_{00}} = \exp\left[\pi\mathrm{i}\, r_j\,\frac{n-1}{n}\right]$$

For n odd, choose r_j to be even (by adding n to it if necessary). Now define the matrix $\mathcal{M}[J]$ by [103]

$$\mathcal{M}[J]_{ab} = \sum_{\ell=1}^{n} \delta_{J^\ell a,b}\, \delta\left(\frac{Q_j(a)}{n} + \frac{\ell}{2n} r_j\right) \tag{5.3}$$

where $\delta(x) = 1$ when $x \in \mathbb{Z}$ and is 0 otherwise. This matrix $\mathcal{M}[J]$ will be a modular invariant iff $(T_{jj}\overline{T_{00}})^n = 1$ (i.e. iff r_j is even), and a permutation matrix iff $T_{jj}\overline{T_{00}}$ is a *primitive* nth root of 1. When n is even, (4.3c) says $(T_{jj}\overline{T_{00}})^n = 1$ iff $\varphi_j(j)^{n/2} = 1$.

For instance, taking $J = id$ we get $\mathcal{M}[id] = I$. The affine algebra $A_1^{(1)}$ at level k has a simple-current with $r_j = k$ given by $Ja = k - a$; for even k the matrix $\mathcal{M}[J]$ is the modular invariant called $\mathcal{D}_{\frac{k}{2}+2}$ below in (6.1b), (6.1c).

Now look at the consequences of Galois. Applying the Galois automorphism σ to $M = SM\bar{S}$ yields from (4.4) and $M_{ab} \in \mathbb{Q}$ the fundamental equation

$$M_{ab} = \sum_{c,d\in\Phi} \epsilon_\sigma(a)\, S_{\sigma a,c}\, M_{cd}\, \overline{S_{d,\sigma b}}\, \epsilon_\sigma(b) = \epsilon_\sigma(a)\, \epsilon_\sigma(b)\, M_{\sigma a,\sigma b} \tag{5.4a}$$

Because $M_{ab} \geq 0$, we obtain the selection rule

$$M_{ab} \neq 0 \;\Rightarrow\; \epsilon_\sigma(a) = \epsilon_\sigma(b) \qquad \forall\sigma \tag{5.4b}$$

and the symmetry

$$M_{\sigma a,\sigma b} = M_{ab} \quad \forall\sigma \tag{5.4c}$$

Of all the Eqs. (5.2) and (5.4), (5.4b) is the most valuable. A way to construct modular invariants from Galois was first given in [56] but isn't useful for constructing affine algebra modular invariants and so won't be repeated here.

There are other very useful facts, which space prevents us from describing. For instance, we have the inequality

$$\sum_{b\in\Phi} S_{ab} M_{b0} \geq 0 \tag{5.5}$$

Perron-Frobenius tells us many things, e.g. that any modular invariant M obeying $M_{0a} = \delta_{0a}$ must be a permutation matrix. For affine algebra modular invariants, the Lie theory of the underlying finite-dimensional Lie algebra plays a crucial role, thanks largely to (3.2c).

Closely related to modular invariants is the notion of NIM-*rep* (short for 'nonnegative integer representation' [19, 20]) or equivalently *fusion graph*. These originally arose in two *a priori* unrelated contexts: the analysis, starting with Cardy's fundamental paper [26], of boundary RCFT; and Di Francesco–Zuber's largely empirical attempt [34, 35] to understand and generalise the A-D-E metapattern appearing in $A^{(1)}$ modular invariants, by attaching graphs to each conformal field theory.

A NIM-*rep* $\mathcal{N}$ is a nonnegative integer representation of the fusion ring, that is, an assignment $a \mapsto \mathcal{N}_a$ to each $a \in \Phi$ of a matrix $\mathcal{N}_a$ with nonnegative integer entries, obeying $\mathcal{N}_a\mathcal{N}_b = \sum_c N_{ab}^c \mathcal{N}_c$. In addition we require that $\mathcal{N}_0 = I$ and that transpose and conjugation be related by $\mathcal{N}_a^t = \mathcal{N}_{Ca}$, for all $a \in \Phi$.

Two obvious examples of NIM-reps are the fusion matrices, $a \mapsto N_a$, and their transposes $a \mapsto N_a^t$. The rows and columns of most NIM-reps however won't be labelled by Φ, in fact we will see that the dimension of the NIM-rep should equal the trace $\mathrm{Tr}(M)$ of some modular invariant.

Just as it is convenient to replace a Cartan matrix by its Dynkin diagram, so too is it convenient to realise $\mathcal{N}_a$ by a (directed multi-)graph: we put a node for each row/column, and draw $(\mathcal{N}_a)_{\alpha\beta}$ edges directed from α to β. We replace each pair of arrows $\alpha \to \beta$, $\beta \to \alpha$, with a single undirected edge connecting α and β. These graphs are called *fusion graphs*, and are often quite striking.

NIM-reps correspond in RCFT to the 1-loop vacuum-to-vacuum amplitude $\mathcal{Z}_{\alpha\beta}(t)$ of an open string, or equivalently of a cylinder whose edge circles are labelled by 'conformally invariant boundary states' $|\alpha\rangle, |\beta\rangle$ [26, 102, 59, 15]. In string theory these are called the 'Chan-Paton degrees-of-freedom' and are placed at the endpoints of open strings. The real variable $-\infty < t < \infty$ here is the modular parameter for the cylinder, and plays the same role here that $\tau \in \mathbb{H}$ plays in $\mathcal{Z}(\tau)$. In particular we get (1.4), where the matrices $(\mathcal{N}_a)_{\alpha\beta} = \mathcal{N}_{a\alpha}^{\beta}$ define a NIM-rep. These (finitely many) boundary states α are the indices for the rows and columns of each matrix $\mathcal{N}_a$. In the language of string theory, NIM-reps describe the possible 'D-branes' respecting the appropriate chiral algebra (VOA) symmetry.

Related to NIM-reps are what string theory calls the charges of symmetry-preserving D-branes living on Lie group manifolds (see e.g. [23]). Mathematically, these are nontrivial ring homomorphisms from the fusion ring into $\mathbb{Z}/m\mathbb{Z}$ for some m. Partition functions associated to nonorientable surfaces (especially the Möbius strip and Klein bottle) are also important in boundary RCFT or open string theory—see e.g. [102, 107]. We won't discuss these additional developments further in this paper.

By the usual arguments (see Section 4) we can simultaneously diagonalise all $\mathcal{N}_a$, and the eigenvalues of $\mathcal{N}_a$ will be S_{ab}/S_{0b} for b in some multi-set $\mathcal{E} = \mathcal{E}(\mathcal{N})$ (i.e. the elements of $\mathcal{E}$ come with multiplicities). This multi-set $\mathcal{E}$ depends only on $\mathcal{N}$ (i.e. is independent of $a \in \Phi$), and is called the *exponents* of the NIM-rep.

Two NIM-reps $\mathcal{N}, \mathcal{N}'$ are regarded as equivalent if there is a simultaneous permutation π of the rows and columns such that $\pi\mathcal{N}_a\pi^{-1} = \mathcal{N}_a'$ for all $a \in \Phi$. For example, the two NIM-reps given earlier are equivalent: $N_a^t = CN_aC^{-1}$. We write $\mathcal{N} = \mathcal{N}' \oplus \mathcal{N}''$, and call $\mathcal{N}$ *reducible*, if the matrices $\mathcal{N}_a$ can be simultaneously written as direct sums $\mathcal{N}_a = \mathcal{N}_a' \oplus \mathcal{N}_a''$. Necessarily, the summands $\mathcal{N}'$ and $\mathcal{N}''$ themselves will be NIM-reps. Irreducibility is equivalent to demanding that the identity 0 occurs in $\mathcal{E}(\mathcal{N})$ with multiplicity 1. We are interested in irreducible equivalence classes of NIM-reps—there will be only finitely many [70].

Two useful facts are: the Perron-Frobenius eigenvalue of $\mathcal{N}_a$ is the q-dimension $\frac{S_{a0}}{S_{00}}$ (we'll see this used next section); and for all $a \in \Phi$,

$$\sum_{b\in\mathcal{E}} \frac{S_{ab}}{S_{0b}} = \mathrm{Tr}\,(\mathcal{N}_a) \in \mathbb{Z}_{\geq} \tag{5.6}$$

The consequences of the simple-current and Galois symmetries are also important and are worked out in [70].

By the *exponents* of a modular invariant M we mean the multi-set $\mathcal{E}_M$ where $a \in \Phi$ appears with multiplicity M_{aa}. RCFT [26, 15] is thought to require that each modular invariant M have a companion NIM-rep $\mathcal{N}$ with the property that

$$\mathcal{E}_M = \mathcal{E}(\mathcal{N}) \tag{5.7}$$

So the size of the matrices $\mathcal{N}_a$, i.e. the dimension of the NIM-rep, should equal the trace $\mathrm{Tr}(M)$ of the modular invariant. For instance, the fusion matrix NIM-rep $a \mapsto N_a$ corresponds to the modular invariant $M = I$. However, there doesn't seem to be a general expression for the NIM-rep (if it exists) of the next simplest modular invariant, the conjugation $M = C$.

Incidentally, the inequality (5.6) is automatically obeyed by the exponents $\mathcal{E} = \mathcal{E}_M$ of any modular invariant M:

$$\sum_{b \in \mathcal{E}_M} \frac{S_{ab}}{S_{0b}} = \mathrm{Tr}(MD_a) = \mathrm{Tr}(\bar{S}SMD_a) = \mathrm{Tr}(MSD_a\bar{S}) = \mathrm{Tr}(MN_a) \in \mathbb{Z}_{\geq}$$

where D_a is the diagonal matrix with entries S_{ab}/S_{0b}.

Note that the NIM-rep definition depends on S, while a modular invariant also sees T. One consequence of this is the following. Suppose there is a primary $a \in \Phi$ such that

$$T_{bb} = T_{cc} \;\Rightarrow\; S_{ab}\,\overline{S_{ac}} \geq 0 \qquad \forall b, c \in \Phi \tag{5.8}$$

Then $M_{aa} = \sum_{b,c} S_{ab} M_{bc} \overline{S_{ac}} > 0$ and so $a \in \mathcal{E}_M$. It is thus natural to require of a NIM-rep $\mathcal{N}$ that any such primary $a \in \Phi$ must appear in $\mathcal{E}(\mathcal{N})$ with multiplicity ≥ 1, because otherwise no modular invariant M could be found obeying (5.7). We'll see an example next section.

An independent justification for studying modular invariants and NIM-reps comes from the subfactor picture (Example 6), where they appear very naturally [47, 98, 19, 20]. In this remarkable picture it is possible to interpret not only the diagonal entries of the modular invariant, as in (5.7), but in fact all entries [97,19,20] (this was already anticipated in [34, 35]). Extend the setting of Example 6 by considering a braided system of endomorphisms for a type III subfactor $N \subset M$. Here, the primaries $\Phi = {}_N\chi_N$ consist of irreducible endomorphisms of N, while the rows and columns of our NIM-rep will be indexed by irreducible homomorphisms $a \in {}_M\chi_N$, $a : N \to M$. The fusion-like ring of ${}_N\chi_N$ will be commutative, i.e. be a true fusion ring; that of ${}_M\chi_M$ however will generally be noncommutative. There is a simple expression [19, 20] for the corresponding modular invariant using 'α-induction' (a process of inducing an endomorphism from N to M using the braiding $\epsilon^{\pm}$): we get $M_{\lambda\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ where the dimension $\langle , \rangle$ is defined in Example 6. Then the (complexified) fusion algebra of ${}_M\chi_M$ will be isomorphic (as a complex algebra) to $\oplus_{\lambda,\mu}\mathrm{GL}_{M_{\lambda\mu}}(\mathbb{C})$. The NIM-rep is essentially α: $(\mathcal{N}_\lambda)_{a,b} = \langle b, \alpha_\lambda^{\pm} a \rangle$ (either choice of $\alpha^{\pm}$ gives the same matrix) [19, 20]. This NIM-rep arises as a natural action of ${}_M\chi_M$ on ${}_M\chi_N$. As these partition functions of tori and cylinders appear so nicely here, it is tempting to ask about other surfaces, especially the Möbius band and Klein bottle, which also play a basic role in boundary RCFT [102, 107].

We won't speak much more here about NIM-reps—see e.g. [34, 35, 15, 70] and references therein for more of the theory and classifications (and graphs!). What has happened in most of the classifications obtained thus far is that, at least for affine algebra modular data, there are slightly more NIM-reps than modular invariants, but otherwise their classifications match surprisingly well. For instance, as we'll see next section, the irreducible NIM-reps of $A_1^{(1)}$ have $\mathcal{N}_1$ equal to the incidence matrix of the A-D-E Dynkin graphs and tadpoles (line graphs with a loop at an end) [34, 35]—apart from the tadpole, this is in perfect agreement with the list of modular invariants for $A_1^{(1)}$ in (6.1)! However we know few NIM-rep classifications and this pleasant correspondence may break down as we go to higher rank and level. For instance, the complete list of NIM-reps for $A_2^{(1)}$ are known only for levels 1, 2, 3, and although the NIM-reps and modular invariants match up perfectly for levels 1 and 2, there are 8 NIM-reps at level 3 and only 4 modular invariants. This discrepancy may get worse as the level rises—clearly, more work along this line is needed. Moreover, there are some modular invariants which lack a corresponding NIM-rep. The simplest examples of modular invariants lacking NIM-reps occur for affine algebra $B_4^{(1)}$ at level 2, and the quantum-double of the symmetric group S_3 [70].

A tempting guess is that almost all of the enormous numbers of modular invariants associated to finite group modular data will likewise fail to have a corresponding NIM-rep. Recall that the Galois parities ϵ_ℓ for the finite group modular data are all $+1$, and hence the constraint (5.4b) becomes trivial. As a general rule, the number of modular invariants is inversely related to the severity (5.4b) possesses for that choice of modular data.

The moral of the story seems to be the following. The definition of modular invariants didn't come to us from God; it came to us from men like Witten, Cardy, ... The surprising thing is that so often their classification yields interesting answers. A modular invariant may not correspond to a CFT (we have infinitely many examples where it fails to), and the modular invariant may correspond to different CFTs (though all known examples of this are artificial, due to our characters depending on too few variables to distinguish the representations of the maximally extended VOAs). But—at least for most affine algebras and levels—it seems they're *usually* in one-to-one correspondence.

In any case, classifying modular invariants, and comparing their lists to those of NIM-reps, is a natural task and has led to interesting findings (see e.g. the review [118]).

6. Affine algebra modular invariant classifications

The most famous modular invariant classification was the first. In (3.5) we gave explicitly the modular data for the affine algebra $A_1^{(1)}$ at level k. Its complete list of modular invariants is [25] (using the simple-current $Ja = k - a$)

$$\mathcal{A}_{k+1} = \sum_{a=0}^{k} |\chi_a|^2 , \quad \forall k \geq 1 \tag{6.1a}$$

$$\mathcal{D}_{\frac{k}{2}+2} = \sum_{a=0}^{k} \chi_a \, \overline{\chi_{J^a a}} , \quad \text{whenever } \frac{k}{2} \text{ is odd} \tag{6.1b}$$

$$\mathcal{D}_{\frac{k}{2}+2} = |\chi_0 + \chi_{J0}|^2 + |\chi_2 + \chi_{J2}|^2 + \cdots + 2|\chi_{\frac{k}{2}}|^2\,, \quad \text{whenever } \frac{k}{2} \text{ is even} \tag{6.1c}$$

$$\mathcal{E}_6 = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2\,, \quad \text{for } k = 10 \tag{6.1d}$$

$$\begin{aligned}\mathcal{E}_7 = {} & |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 \\ & + \chi_8\,(\overline{\chi_2 + \chi_{14}}) + (\chi_2 + \chi_{14})\,\overline{\chi_8} + |\chi_8|^2\,, \quad \text{for } k = 16\end{aligned} \tag{6.1e}$$

$$\mathcal{E}_8 = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2, \quad \text{for } k = 28 \tag{6.1f}$$

Each of these is identified with a (finite) Dynkin diagram, in such a way that the Coxeter number h of the diagram equals $k+2$, and the *exponents* of the corresponding Lie algebra are given by $1 + \mathcal{E}_M$ (recall the definition of exponents $\mathcal{E}_M$ of a modular invariant, given at the end of last section). The exponents of the Lie algebra are the numbers m_i, where $4\sin^2(\pi\,\frac{m_i}{h})$ are the eigenvalues of the Cartan matrix. For instance, the Dynkin diagram D_8 has Coxeter number 14 and exponents $1, 3, 5, 7, 7$, while $\mathcal{D}_8$ occurs at level 12 and has exponents $\mathcal{E} = \{0, 2, 4, 6, 6\}$.

The A-D-E pattern appears in many places in math and mathematical physics [81, 106]: besides the simply-laced Lie algebras and $A_1^{(1)}$ modular invariants, these diagrams also classify simple singularities, finite subgroups of $SU_2(\mathbb{C})$, subfactors with Jones index <4, representations of quivers, etc. There seem to be two more-or-less inequivalent A-D-E patterns, one corresponding to the finite A-D-E diagrams, and the other corresponding to the affine (=extended) A-D-E diagrams. For instance, the modular invariants identify with the finite ones, while the finite subgroups of $SU_2(\mathbb{C})$ match with the affine ones. This suggests that a direct relation between e.g. the modular invariants and those finite subgroups could be a little forced. Patterns such as A-D-E are usually explained by identifying an underlying combinatorial fact which is responsible for its various incarnations. The A-D-E combinatorial fact is probably the classification of symmetric matrices over $\mathbb{Z}_{\geq}$, with zero diagonals, and with maximal eigenvalue <2 (for the finite diagrams) and $=2$ (for the affine ones). Perhaps the only A-D-E classification which still resists this 'explanation' is that of $A_1^{(1)}$ modular invariants. This is in spite of considerable effort (and some progress) by many people. The present state of affairs, and also a much simpler proof on the lines sketched in the previous section, is provided by [69].

Many other classes of affine algebras and levels have been classified. The main ones are: $A_2^{(1)}$, $(A_1 + A_1)^{(1)}$, and $(U(1) + \cdots + U(1))^{(1)}$, for all levels k; and $A_r^{(1)}$, $B_r^{(1)}$, $D_r^{(1)}$ for all ranks r, but with levels restricted to $k \leq 3$. See e.g. [68] for references to these results.

Has A-D-E been spotted in these other lists? No. However, a remarkable connection [101, 12] has been observed between the $A_2^{(1)}$ level k modular invariants, and the Jacobian of the Fermat curve $x^{k+3} + y^{k+3} + z^{k+3} = 0$. In particular, the $A_2^{(1)}$ Galois selection rule (5.4b) and the analysis of the simple factors in the Jacobian are essentially the same. This link between Fermat and $A_2^{(1)}$ is still unexplained, and how it extends to the other algebras, e.g. perhaps $A_r^{(1)}$ level k relates to $x_1^{k+r+1} + x_2^{k+r+1} + \cdots + x_{r+1}^{k+r+1} = 0$?, is still unclear. However, Batyrev [11] has suggested some possibilities involving toric geometry.

The third 'sample' listed last section (relating $(U(1)\oplus\cdots\oplus U(1))^{(1)}$ modular invariants to the Grassmannians) suggests a different link with geometry. The Grassmannian is essentially the moduli space of Narain compactifications of a (classical) string theory, so perhaps other families of modular invariants can be regarded as special points on other finite-dimensional moduli spaces.

Though there are no other appearances of A-D-E, there is a rather natural way to assign (multi-di)graphs to modular invariants, generalising the A-D-E pattern for $A_1^{(1)}$. Note first that we can classify the $A_1^{(1)}$ NIM-reps [34, 35]: $\mathcal{N}_1$ must be symmetric and have Perron-Frobenius eigenvalue $\frac{S_{10}}{S_{00}} = 2\cos(\frac{\pi}{k+2}) < 2$; thus the graph associated to $\mathcal{N}_1$ must be an A-D-E Dynkin diagram, or a tadpole. The tadpoles can be discarded, since they don't correspond via (5.7) to a modular invariant. Given $\mathcal{N}_1$, all other $\mathcal{N}_a$ can be recursively obtained using the special case $\mathcal{N}_1\mathcal{N}_i = \mathcal{N}_{i+1} + \mathcal{N}_{i-1}$ of (3.5c). The result is a NIM-rep.

In this way, we find that the Dynkin diagram which (6.1) assigned to a given $A_1^{(1)}$ modular invariant M is precisely the graph whose adjacency matrix equals the generator $\mathcal{N}_1$ of the unique NIM-rep compatible with M in the sense of (5.7). Likewise, we should assign to the modular invariants of e.g. $A_2^{(1)}$ the multi-digraph $\mathcal{N}_{\Lambda_1}$ generating the corresponding NIM-rep. The NIM-reps for $A_2^{(1)}$ are not yet classified, but at least one has been found for each M [34, 35, 98, 15, 19, 20].

There is a simple reason why the tadpole can't correspond to an $A_1^{(1)}$ modular invariant. Note that the unit $a = k$ satisfies (5.8), and thus will lie in any $\mathcal{E}_M$. However, k is not an exponent of the tadpole, and thus there can be no solution M in (5.7) for the choice $\mathcal{N} =$ tadpole.

By the way, *sub*modular invariants can usually be found for NIM-reps which lack a true modular invariant. For example, the seemingly extraneous n-vertex tadpole mentioned in the previous paragraph corresponds to the algebra $A_1^{(1)}$ at level $2n-1$, and the submodular invariant $M_{ab} = \delta_{b,J^a a}$. Perhaps a reasonable interpretation can be found by both the subfactor and boundary CFT camps for NIM-reps corresponding to matrices M commuting with certain small-index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Recall that we anticipated this thought at the end of Example 1.

Most of the modular invariant classification effort has been directed not at specific algebras and levels, but at the general argument. The major result obtained thus far is:

Theorem 5 ([67, 71, 74]) *Choose any affine algebra $X_r^{(1)}$ and level k. Let M be any modular invariant, obeying the constraint that the only primaries $a \in \Phi$ for which $M_{0a} \neq 0$ or $M_{a0} \neq 0$, are units. Then M lies on an explicit list.*

Note that, of the $A_1^{(1)}$ modular invariants, all but $\mathcal{E}_6$ and $\mathcal{E}_8$ obey the constraint of Theorem 5. That pattern seems to continue for the other algebras and levels: the list of modular invariants covered by Theorem 5 exhausts almost every modular invariant yet discovered.

There are very few *exceptional* modular invariants in the list of Theorem 5. Almost all of the modular invariants there are simple-current ones (5.3), and the product of these by the conjugation C (strictly speaking, any symmetry of the unextended Dynkin diagram can be used here in place of C).

Theorem 5 is important because, *for generic choice of algebra and level*, the various constraints we have on the 0-row and 0-column of a modular invariant (most importantly, Galois (5.4b), T (5.1), and the inequality (5.5)) force the condition of Theorem 5 to be satisfied.

Indeed, if we impose the full structure of Ocneanu cells [98] (this should be equivalent to saying that an RCFT exists with partition function given by M, although to this author's knowledge this hasn't been rigourously shown yet), we obtain Ocneanu's inequality [98]:

$$\sum_{\mu \in \text{clearing}} N_{\lambda, C\lambda}^{\mu} \, S_{\mu 0} \leq S_{\lambda 0} \tag{6.2}$$

where λ is any weight$\neq 0$ obeying $M_{\lambda 0} \neq 0$ with λ_0 as large as possible, and where 'clearing' is a subset of P_+^k close to 0: μ is in the clearing if $2(k-\mu_0) \leq k - \lambda_0$. The proof of this important inequality has not appeared in print yet. The left-side of (6.2) grows approximately quadratically with $S_{\lambda 0}/S_{00}$, while the right-side is only linear, so it tends to force $S_{\lambda 0}$ to be small; Eq. (5.1) on the other hand tends to force $S_{\lambda 0}$ to be large. This apparently implies (although the details of the argument have also not yet appeared in print) that, for fixed algebra $X_r^{(1)}$, there is a K (depending on the algebra) such that $\forall k > K$, the constraint of Theorem 5 will be obeyed! Thus, assuming these two announced claims, we obtain:

Corollary 6 *All possible modular invariants appearing in RCFT* (*or the subfactor interpretation*), *corresponding to any fixed choice of affine algebra* $X_r^{(1)}$, *and all sufficiently high levels*, *are known*.

In other words, what Corollary 6 tells us is that, apart from some low level exceptional modular invariants, all affine algebra modular invariants appearing in RCFT can be constructed in straightforward and known ways from the symmetries of the corresponding affine Dynkin diagram!

Theorem 5 has another consequence. It makes it relatively easy to find all modular invariants (using only conditions **MI1-MI3**) at 'small' levels, when the rank of the algebra isn't too large [73]. For example, all modular invariants for $E_8^{(1)}$ at all levels $k \leq 380$ can be determined. This isn't completely trivial: $E_8^{(1)}$ at $k = 380$ has over 10^{12} highest weights=primaries, so the S and M matrices have a number of entries approximately equal to Avogadro's number! And each of these entries of S, given by (3.2b), involves a sum of 10^9 complex numbers. The fact that we can reach such high levels isn't a sign of programming prowess, but rather to how close we are to a complete classification of these (unrestricted) affine algebra modular invariants. In [73] the modular invariants are given for all exceptional algebras, and the classical algebras of rank $\leq$6.

The big surprise here is how rare the affine algebra modular invariants are (for comparison, recall that there are over 8000 modular invariants for the finite group A_5). In the Table we've summarised the modular invariant classifications for various algebras of small rank. It describes the complete list of modular invariants for these algebras, when the level is sufficiently small (these limits are given in the Table). A very safe conjecture though is that the Table gives the complete classification for those algebras, for *all* levels k (at the time of writing, $E_7^{(1)}$ level 42 and $E_8^{(1)}$ level 90 still have not been eliminated). Our hope is that this Table (or more realistically, the paper [73] where more results are given and in more detail) will inspire someone to spot a new coincidence involving modular invariants and some other area of mathematics. For example, note in $A_1^{(1)}$ that the exceptionals appear at

Table Some affine algebra modular invariant classifications.

Algebra	# of series	Levels of exceptionals	Verified for:
$A_1^{(1)}$	k odd: 1 k even: 2	$k = 10, 16, 28$	$\forall k$
$A_2^{(1)}$	k arbitrary: 4	$k = 5, 9, 21$	$\forall k$
$C_2^{(1)}$	k arbitrary: 2	$k = 3, 7, 8, 12$	$k \leq 25\,000$
$G_2^{(1)}$	k arbitrary: 1	$k = 3, 4$	$k \leq 30\,000$
$A_3^{(1)}$	k odd: 2 k even: 4	$k = 4, 6, 8$	$k \leq 4000$
$B_3^{(1)}$	k arbitrary: 2	$k = 5, 8, 9$	$k \leq 3000$
$C_3^{(1)}$	k odd: 1 k even: 2	$k = 2, 4, 5$	$k \leq 4500$
$F_4^{(1)}$	k arbitrary: 1	$k = 3, 6, 9$	$k \leq 2000$
$E_6^{(1)}$	k arbitrary: 4	$k = 4, 6, 12$	$k \leq 500$
$E_7^{(1)}$	k odd: 1 k even: 2	$k = 3, 12, 18, (42?)$	$k \leq 400$
$E_8^{(1)}$	k arbitrary: 1	$k = 4, 30, (90?)$	$k \leq 380$

$k + 2 = 12, 18, 30$, which are the Coxeter numbers of E_6, E_7, E_8. Claude Itzykson noticed that the $A_2^{(1)}$ exceptionals occur at $k + 3 = 8, 12, 24$—all divisors of 24—and (inspired by the Fermat connection [101, 12]) found signs of these exceptionals in the Jacobian of $x^{24} + y^{24} + z^{24} = 0$. Can anyone spot any such pattern for the other algebras?

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