



The Terwilliger Algebra Associated with a Set of Vertices in a Distance-Regular Graph*

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Abstract. Let Γ be a distance-regular graph of diameter D . Let X denote the vertex set of Γ and let Y be a nonempty subset of X . We define an algebra $\mathcal{T} = \mathcal{T}(Y)$. This algebra is finite dimensional and semisimple. If Y consists of a single vertex then \mathcal{T} is the corresponding subconstituent algebra defined by P. Terwilliger. We investigate the irreducible \mathcal{T} -modules. We define endpoints and thin condition on irreducible \mathcal{T} -modules as a generalization of the case when Y consists of a single vertex. We determine when an irreducible module is thin. When the module is generated by the characteristic vector of Y , it is thin if and only if Y is a completely regular code of Γ . By considering a suitable subset Y , every irreducible $\mathcal{T}(x)$ -module of endpoint i can be regarded as an irreducible $\mathcal{T}(Y)$ -module of endpoint 0.

Keywords: distance-regular graph, association scheme, subconstituent algebra, Terwilliger algebra, tight graph, completely regular code

1. Introduction

Let Γ denote a distance-regular graph. In [45], P. Terwilliger introduced the subconstituent algebra (now often called the Terwilliger algebra) of Γ with respect to one of its vertices (the “base vertex”). Since then, these algebras have been investigated in many papers [3, 4, 6–14, 16–19, 22, 23, 26, 43, 45–50]. In this paper we generalize some of this work by replacing the base vertex with a base subset of vertices. To describe our results more precisely we recall some facts.

Let Γ denote a distance-regular graph with diameter D , vertex set X , and valency k . Let $\text{Mat}_X(\mathbb{C})$ denote the complex algebra of matrices whose rows and columns are indexed by X and whose entries are in the complex numbers \mathbb{C} . Let $A_i \in \text{Mat}_X(\mathbb{C})$ ($0 \leq i \leq D$) denote the i -th distance matrix of Γ . Then the adjacency matrix A ($= A_1$) has $D + 1$ distinct eigenvalues. Let $E_i \in \text{Mat}_X(\mathbb{C})$ ($0 \leq i \leq D$) denote the primitive idempotents of A . The matrix subalgebra \mathcal{M} generated by A is called the *Bose-Mesner algebra* of Γ . It is well-known that $\mathcal{M} = \text{Span}(A_0, A_1, \dots, A_D) = \text{Span}(E_0, E_1, \dots, E_D)$ [1, 2, 24]. Fix a base vertex $x \in X$. Let $E_i^* = E_i^*(x) \in \text{Mat}_X(\mathbb{C})$ denote the projection onto the i -th subconstituent of Γ . The *Terwilliger algebra* of Γ with respect to x is the matrix subalgebra $\mathcal{T} = \mathcal{T}(x)$ generated by A and $E_0^*, E_1^*, \dots, E_D^*$.

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The algebra \mathcal{T} is semisimple since it is generated by real symmetric matrices. Thus much of the structure of \mathcal{T} is determined by its modules. Let $V = \mathbf{C}^X$ denote the complex vector space consisting of column vectors whose entries are indexed by X . Let $W \subset V$ be an irreducible \mathcal{T} -module. Then W is the direct sum of the nonzero subspaces among $E_0^*W, E_1^*W, \dots, E_D^*W$. W is said to be thin if $\dim E_i^*W \leq 1$ for every i . The index $\min\{i \mid 0 \leq i \leq D, E_i^*W \neq 0\}$ is called the endpoint of W . There is only one irreducible module of endpoint 0 and it is always thin. Recently Terwilliger investigated thin modules of endpoint 1 and obtained an inequality involving the local eigenvalues and showed that the equality holds if and only if every irreducible module of endpoint 1 is thin [48, 49]. Tight distance-regular graphs defined in [31] have lots of interesting properties, one of which is that every irreducible module of endpoint 1 is thin and of dimension $D - 1$ [23]. Hence they are in the special class satisfying the equality mentioned above. For tight distance-regular graphs and related topics, see [27–30, 32, 33, 36–39, 44].

This paper is an attempt to generalize the results found in [23, 48, 49]. We first generalize the algebra \mathcal{T} by replacing the base vertex by a base subset. Indeed let Y be a nonempty subset of X . For $0 \leq i \leq D$ let $\Gamma_i(Y)$ denote the set of vertices in X which are at distance i from Y . Let $E_i^* = E_i^*(Y)$ denote the projection onto $\Gamma_i(Y)$. We let $\mathcal{T} = \mathcal{T}(Y)$ denote the subalgebra of $\text{Mat}_X(\mathbf{C})$ generated by $A, E_0^*, E_1^*, \dots, E_D^*$. Now we can define endpoints and the thin property similarly for irreducible \mathcal{T} -modules.

For a nonempty subset Y of X let $w(Y)$ denote the maximal distance of vertices of Y in Γ , which is called the *width* of Y . Let $\mathbf{v} \in E_0^*V$ be a nonzero vector. Then the \mathcal{T} -module $\mathcal{T}\mathbf{v}$ contains $\mathcal{M}\mathbf{v}$. Moreover, if $\mathcal{T}\mathbf{v}$ is a thin irreducible module, $\mathcal{T}\mathbf{v} = \mathcal{M}\mathbf{v}$.

We now state our main results, which will be proved in the body of the paper.

Our first result concerns the dimension of $\mathcal{M}\mathbf{v}$. Before we state our result we make a few comments. Observe that $\mathcal{M}\mathbf{v}$ has a basis consisting of the nonzero vectors among $E_0\mathbf{v}, E_1\mathbf{v}, \dots, E_D\mathbf{v}$. Therefore,

$$\dim \mathcal{M}\mathbf{v} = |\{i \mid E_i\mathbf{v} \neq \mathbf{0}, i \in \{0, 1, \dots, D\}\}|.$$

Theorem 1.1 *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter D , and let Y denote a nonempty subset of X . Let \mathbf{v} be a nonzero vector in $E_0^*(Y)V$. Let $\mathcal{T} = \mathcal{T}(Y)$. Then the following hold.*

- (i) $\dim \mathcal{M}\mathbf{v} \geq D - w(Y) + 1$.
- (ii) *Suppose $\dim \mathcal{M}\mathbf{v} = D - w(Y) + 1$. Then $\mathcal{M}\mathbf{v}$ is a thin irreducible \mathcal{T} -module.*

In order to state our next result we make a few more comments. Under the assumption of Theorem 1.1, let $\dim \mathcal{M}\mathbf{v} = r + 1$. Since A_i is expressed as a polynomial in A of degree i , $\mathcal{M}\mathbf{v}$ has another basis $A_i\mathbf{v}$ ($0 \leq i \leq r$). Applying Gram-Schmidt to this basis, we find $\mathcal{M}\mathbf{v}$ has an orthogonal basis $g_i(A)\mathbf{v}$ ($0 \leq i \leq r$), where $g_i(t) \in \mathbf{R}[t]$ is a polynomial with degree exactly i with leading coefficient $(c_1c_2 \dots c_i)^{-1}$. We compute $\|g_i(A)\mathbf{v}\|^2$ in terms of the intersection numbers and the scalars

$$\frac{\langle \mathbf{v}, A_i\mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}, \quad \text{for } i \in \{0, 1, \dots, w(Y)\}.$$

We show that for $0 \leq j \leq r$,

$$\|E_i^* A_i \mathbf{v}\|^2 \leq \|g_i(A) \mathbf{v}\|^2. \quad (1)$$

Concerning the case of equality we have the following.

Theorem 1.2 *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter D , and let Y denote a nonempty subset of X . Let \mathbf{v} be a nonzero vector in $E_0^*(Y)V$ and define $r = \dim \mathcal{M}\mathbf{v} - 1$. Let $\mathcal{T} = \mathcal{T}(Y)$. Then the following are equivalent.*

- (i) $\mathcal{T}\mathbf{v}$ is a thin irreducible \mathcal{T} -module.
- (ii) Equality holds in (1) for $i \in \{0, 1, \dots, r\}$.
- (iii) Equality holds in (1) for $i = r$.

When \mathbf{v} is the characteristic vector of Y , we show $\mathcal{M}\mathbf{v}$ is an irreducible \mathcal{T} -module if and only if Y is a completely regular code. See Proposition 7.2. Hence our attempt is to provide a new tool to study codes by considering not only the module generated by the characteristic vector of a code but also those generated by nonzero vectors whose supports lie in the code. For algebraic characterizations of completely regular codes, see [2, 20, 21, 24, 25, 34, 35, 41].

Let W be an irreducible $\mathcal{T}(x)$ -module of endpoint i . Then the same subspace W becomes an irreducible $\mathcal{T}(Y)$ -module of endpoint 0 by setting $Y = \Gamma_i(x)$. Hence by considering a suitable subset Y , every irreducible $\mathcal{T}(x)$ -module can be regarded as an irreducible $\mathcal{T}(Y)$ -module of endpoint 0. Thus it is advantageous to choose a base subset rather than a base vertex. In fact the results in [23, 48, 49] are those on $\mathcal{T}(\Gamma_1(x))$ -modules. Since $w(\Gamma_1(x)) = 2$, our results are generalization to the case with arbitrary width. We believe that our results set a foundation to investigate $\mathcal{T}(x)$ -modules of endpoint larger than one as well.

The paper is organized as follows. The next two sections are for preliminaries. Section 2 is for orthogonal polynomials and Section 3 is for distance-regular graphs. The readers who are familiar with distance-regular graphs may skip these two sections and come back when quoted. In Section 4, we define Terwilliger algebras and their modules with respect to a base subset, and prove some basic results. Section 5 deals with the properties of change of base subsets. In Section 6, we discuss the structures of thin modules. In Section 7, we study the primary module. In Section 8, we collect basic properties of local eigenvalues. In Sections 9 and 10, we prove our main theorems. In Section 11, we give description of the case when the width $w(Y)$ of Y is at most 2. This contains the case when $Y = \Gamma_1(x)$, which was studied in [23, 48, 49].

2. Preliminaries: Orthogonal polynomials

In this section we will review some of the basic properties of orthogonal polynomials. Orthogonal polynomials frequently arise in connection with distance-regular graphs [1, 2, 24], and they play a fundamental role in our work. Thus we take a moment to recall some basic properties of orthogonal polynomials (most of this material is found in Szegő's book [40], see also [5, 15, 24]). To make these results more directly applicable to our

situation, we specialize them to the orthogonal polynomials with associated weight functions having only a finite number of points of increase.

Definition 2.1 Let $X = \{x_0, x_1, \dots, x_N\}$ be the set of $N + 1$ distinct real numbers. Let w be a real-valued function on the set X such that $w(x) > 0$ for all $x \in X$. Let $\mathbf{R}[t]$ be the set of polynomials in t with real coefficients. For $f(t), g(t) \in \mathbf{R}[t]$, let

$$\langle f(t), g(t) \rangle_w = \sum_{x \in X} f(x)g(x)w(x) = \sum_{i=0}^N f(x_i)g(x_i)w(x_i).$$

Then $\langle \cdot, \cdot \rangle_w$ defines an inner product on $\mathbf{R}[t]$. A system of polynomials $\{p_0(t), p_1(t), \dots, p_n(t)\} \subset \mathbf{R}[t]$ is called a system of *orthogonal polynomials* on X associated with the *weight function* w , if they are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_w$ and that $\deg(p_i(t)) = i$. A system of orthogonal polynomials $\{p_0(t), p_1(t), \dots, p_n(t)\} \subset \mathbf{R}[t]$ is called *orthonormal* if $\langle p_i(t), p_i(t) \rangle_w = 1$ for every $i \in \{0, 1, \dots, n\}$.

If $\{p_0(t), p_1(t), \dots, p_n(t)\} \subset \mathbf{R}[t]$ is a system of orthogonal polynomials, then every polynomial $p(t) \in \mathbf{R}[t]$ of degree at most m with $m \leq n$ can be written as an \mathbf{R} -linear combination of $p_0(t), p_1(t), \dots, p_m(t)$ and $\langle p(t), p_i(t) \rangle_w = 0$ for every $i \in \{m+1, \dots, n\}$.

Proposition 2.1 ([40]) Let $X = \{x_0, x_1, \dots, x_N\}$ be the set of $N + 1$ distinct real numbers. Let w be a real-valued function on the set X such that $w(x) > 0$ for all $x \in X$. Let $f_0(t) = 1, f_i(t) \in \mathbf{R}[t]$ such that $\deg f_i(t) = i$ for $i \in \{1, \dots, N\}$.

- (i) Let $c_{i,j} = \langle f_i(t), f_j(t) \rangle_w$ for every $i, j \in \{0, 1, \dots, N\}$. Then for every $n \in \{0, 1, \dots, N\}$ the value of the following determinant is positive.

$$D_n = \begin{vmatrix} c_{0,0} & c_{0,1} & c_{0,2} & \cdots & c_{0,n} \\ c_{1,0} & c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n-1,0} & c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n} \\ c_{n,0} & c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{vmatrix}.$$

- (ii) Let $\{p_0(t), p_1(t), \dots, p_N(t)\} \subset \mathbf{R}[t]$ be a system of polynomials defined as follows.

$$p_0(t) = 1,$$

and for $n \in \{1, 2, \dots, N\}$

$$p_n(t) = (D_{n-1})^{-1} \begin{vmatrix} c_{0,0} & c_{0,1} & c_{0,2} & \cdots & c_{0,n} \\ c_{1,0} & c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n-1,0} & c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n} \\ f_0(t) & f_1(t) & f_2(t) & \cdots & f_n(t) \end{vmatrix}.$$

Then for each $n \in \{0, 1, \dots, N\}$, $p_n(t)$ is an \mathbf{R} -linear combination of $f_0(t), f_1(t), \dots, f_n(t)$ and the coefficient of $f_n(t)$ is 1. Moreover, $\{p_0(t), p_1(t), \dots, p_N(t)\}$ is a system of orthogonal polynomials satisfying the following.

$$\langle p_i(t), p_j(t) \rangle_w = \delta_{i,j} D_i.$$

(iii) Suppose $\{q_0(t), q_1(t), \dots, q_N(t)\}$ is a system of orthogonal polynomials on X associated with a weight function w . Assume the following.

- (a) The leading coefficient of $q_i(t)$ is the same as that of $f_i(t)$ for every $i \in \{0, 1, \dots, N\}$.
- (b) $\langle q_i(t), q_j(t) \rangle_w = \delta_{i,j} D_i$ for every $i, j \in \{0, 1, \dots, N\}$, where D_i is the determinant defined in (i).

Then $q_i(t) = p_i(t)$ for each $i \in \{0, 1, \dots, N\}$, where $p_i(t)$ is a polynomial defined in (ii).

Remark. By Lemma 2.1, if w is a weight function on the set X of size $N + 1$, then there exists a system of $N + 1$ orthogonal polynomials $\{p_0(t), p_1(t), \dots, p_N(t)\}$ associated with w , but no larger system.

Lemma 2.2 ([24, 40]) Let $\{f_0(t), f_1(t), \dots, f_N(t)\} \subset \mathbf{R}[t]$ be a system of orthogonal polynomials on the set X of cardinality $N + 1$ associated with the weight function w . Set $f_{N+1}(t) = \prod_{x \in X} (t - x)$. Then for each $n \in \{0, 1, \dots, N\}$, there exist real numbers β_{n-1}, α_n and γ_{n+1} satisfying the following relations.

$$t f_n(t) = \beta_{n-1} f_{n-1}(t) + \alpha_n f_n(t) + \gamma_{n+1} f_{n+1}(t), \tag{2}$$

and that

$$\frac{\langle f_{n+1}(t), f_{n+1}(t) \rangle_w}{\langle f_n(t), f_n(t) \rangle_w} = \frac{\beta_n}{\gamma_{n+1}}, \quad \text{for every } n \in \{0, 1, \dots, N - 1\}. \tag{3}$$

In particular, $\beta_n \gamma_{n+1} > 0$ for every $n \in \{0, 1, \dots, N - 1\}$.

Lemma 2.3 Let $\{p_0(t), p_1(t), \dots, p_N(t)\} \subset \mathbf{R}[t]$ be a system of orthogonal polynomials on X associated with the weight function w . Let $\rho(t) \in \mathbf{R}[t]$ such that $\rho(x) \geq 0$ for every $x \in X$. Suppose

$$\rho(t) = c(t - y_1)(t - y_2) \cdots (t - y_\ell) \quad (0 \neq c \in \mathbf{R}),$$

where y_1, y_2, \dots, y_ℓ are distinct elements of X . Let $D_{\rho,n}$ be a determinant of size $\ell \times \ell$ defined by the following.

$$D_{\rho,n} = \begin{vmatrix} p_n(y_1) & p_{n+1}(y_1) & \cdots & p_{n+\ell-1}(y_1) \\ p_n(y_2) & p_{n+1}(y_2) & \cdots & p_{n+\ell-1}(y_2) \\ \cdots & \cdots & \cdots & \cdots \\ p_n(y_\ell) & p_{n+1}(y_\ell) & \cdots & p_{n+\ell-1}(y_\ell) \end{vmatrix}. \tag{4}$$

Then $D_{\rho,n} \neq 0$ for every $n \in \{0, 1, \dots, N - \ell + 1\}$.

Proof: Suppose $D_{\rho,n} = 0$. Then there is a nontrivial linear combination of columns of the determinant. Let \mathbf{u}_i be the $(i + 1)$ -st column of the determinant (4) as follows:

$$(p_{n+i}(y_1), p_{n+i}(y_2), \dots, p_{n+i}(y_\ell)),$$

for $i = 0, 1, \dots, \ell - 1$. Suppose $\lambda_0, \lambda_1, \dots, \lambda_{\ell-1}$ are certain constants of real numbers, not all zero, such that

$$\lambda_0 \mathbf{u}_0 + \lambda_1 \mathbf{u}_1 + \dots + \lambda_{\ell-1} \mathbf{u}_{\ell-1} = \mathbf{0}. \quad (5)$$

and set

$$p(t) = \lambda_0 p_n(t) + \lambda_1 p_{n+1}(t) + \lambda_2 p_{n+2}(t) + \dots + \lambda_{\ell-1} p_{n+\ell-1}(t) \in \mathbf{R}[t].$$

Then the degree of $p(t)$ is at most $n + \ell - 1$. By (5), we have

$$p(y_i) = 0 \quad \text{for all } i \in \{1, 2, \dots, \ell\}.$$

Thus $\rho(t)$ divides $p(t)$, and there is a nonzero polynomial $q(t) \in \mathbf{R}[t]$ of degree at most $n - 1$ such that $p(t) = \rho(t)q(t)$. Since $p(t)$ is an \mathbf{R} -linear combination of $p_n(t), p_{n+1}(t), \dots, p_{n+\ell-1}(t)$, $p(t)$ is orthogonal to any polynomial in $\mathbf{R}[t]$ of degree at most $n - 1$. In particular,

$$\begin{aligned} 0 &= \langle p(t), q(t) \rangle_w = \langle \rho(t)q(t), q(t) \rangle_w = \sum_{x \in X} \rho(x)q(x)q(x)w(x) \\ &= \sum_{x \in X} |q(x)|^2 \rho(x)w(x). \end{aligned}$$

Since $\rho(x) \geq 0$ and $w(x) > 0$ for all $x \in X$, we have that

$$q(x) = 0 \quad \text{for every } x \in X \setminus \{y_1, y_2, \dots, y_\ell\}.$$

Hence if $N + 1 - \ell > n - 1$, we have $q(t) = 0$ as a polynomial, which is a contradiction. Thus $D_{\rho,n} \neq 0$ if $n \leq N + 1 - \ell$ as desired. \square

3. Preliminaries: Distance-regular graphs

In this section we recall some facts about distance-regular graphs. For the general theory of distance-regular graphs, we refer the reader to [1, 2, 24]. We shall follow the notation of [1] for the most part.

Let X denote a nonempty finite set. Let $\text{Mat}_X(\mathbf{C})$ denote the complex algebra consisting of all matrices whose rows and columns are indexed by X with complex entries. Let $V = \mathbf{C}^X$ denote the vector space over the complex number field consisting of column vectors whose coordinates are indexed by X with complex entries. We observe $\text{Mat}_X(\mathbf{C})$ acts on V by left

multiplication. We endow V with the Hermitean inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = {}^t\mathbf{u} \bar{\mathbf{v}} \quad (\mathbf{u}, \mathbf{v} \in V),$$

where ${}^t\mathbf{u}$ denotes the transpose of \mathbf{u} , and $\bar{\mathbf{v}}$ denotes the complex conjugate of \mathbf{v} . We abbreviate $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ for all $\mathbf{u} \in V$. For all $y \in X$, let \hat{y} denote the element of V with a 1 in the y -coordinate and 0 in all other coordinates.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R . For $x, y \in X$, let $\partial(x, y)$ denote the *distance* between x and y , that is the length of a shortest path connecting x and y . The *diameter* D is the maximal distance between vertices. The graph Γ is said to be *distance-regular* whenever for all integers $h, i, j \in \{0, 1, \dots, D\}$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{i,j}^h = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}| \quad (6)$$

is independent of x and y . These integers are called the *intersection numbers* of Γ . We abbreviate $c_i = p_{i-1,1}^i$ ($1 \leq i \leq D$), $a_i = p_{i,1}^i$ ($0 \leq i \leq D$), and $b_i = p_{i+1,1}^i$ ($0 \leq i \leq D-1$). For notational convenience, we define $c_0 = 0$ and $b_D = 0$.

For the rest of this paper we assume Γ is distance-regular with diameter D . By (6) and the triangular inequality,

$$p_{i,j}^h = 0 \quad \text{if } h+i < j, \ i+j < h, \ \text{or } j+h < i. \quad (7)$$

For a vertex $x \in X$ and an integer $i \in \{0, 1, \dots, D\}$, let

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\} \quad \text{and} \quad k_i = |\Gamma_i(x)|.$$

Then $k_i = p_{i,i}^0$ and the induced graph on $\Gamma_i(x)$ is an a_i -regular graph of size k_i , and $k = k_1 = c_1 + a_1 + b_1$ for every $i \in \{0, 1, \dots, D\}$. It is known that each $p_{i,j}^h$ can be written using c_i 's, a_i 's and b_i 's. We refer to $\{p_{i,j}^h \mid h, i, j \in \{0, 1, \dots, D\}\}$ as the set of *parameters* of Γ .

For $i \in \{0, 1, \dots, D\}$ let A_i denote the matrix in $\text{Mat}_X(\mathbb{C})$ whose (x, y) -entry is defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A_i is called the *i -th distance matrix* of Γ . For $i, j \in \{0, 1, \dots, D\}$ we have

$$A_i A_j = \sum_{h=0}^D p_{i,j}^h A_h.$$

In particular, by (7) we have for $i \in \{1, \dots, D\}$

$$A_i A_1 = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (8)$$

by setting $A_{D+1} = 0$ and $c_{D+1} = 1$. Observe that by (8), the linear span $\mathcal{M} = \text{Span}(A_0, A_1, \dots, A_D)$ is closed under multiplication and it is algebraically generated by $A = A_1$. \mathcal{M} is called the *Bose-Mesner algebra* of Γ . Since \mathcal{M} is commutative and generated by real symmetric matrices, it has a basis consisting of primitive idempotents. Let $E_0, E_1, E_2, \dots, E_D$ be the primitive idempotents. We write

$$A_i = \sum_{j=0}^D p_i(j) E_j \quad \text{for all } i \in \{0, 1, \dots, D\}, \quad (9)$$

and

$$E_i = \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j \quad \text{for all } i \in \{0, 1, \dots, D\}. \quad (10)$$

Set $m_i = q_i(0)$ and $\theta_i = p_1(i)$. Then $\theta_0, \theta_1, \dots, \theta_D$ are the distinct eigenvalues of $A = A_1$, and m_i is the multiplicity of θ_i in A . We order E_0, E_1, \dots, E_D so that

$$\theta_0 > \theta_1 > \dots > \theta_D.$$

Since Γ is a connected k -regular graph, its adjacency matrix has largest eigenvalue k with multiplicity 1. Hence $E_0 = \frac{1}{|X|} J$ in this ordering, where $J \in \text{Mat}_X(\mathbf{C})$ is the all 1's matrix. We use the following well-known formulas. For all $i, j \in \{0, 1, \dots, D\}$,

$$\frac{p_i(j)}{k_i} = \frac{q_j(i)}{m_j}, \quad (11)$$

$$q_j(1) = \frac{m_j \theta_j}{k}, \quad (12)$$

$$q_j(2) = \frac{m_j(\theta_j^2 - a_1 \theta_j - k)}{k b_1}. \quad (13)$$

Lemma 3.1 ([1, 23]) *The following hold.*

- (i) Let $v_0(t), v_1(t), \dots, v_D(t), v_{D+1}(t)$ denote polynomials in $\mathbf{R}[t]$ satisfying $v_0(t) = 1$ and for $i \in \{0, 1, \dots, D\}$,

$$t v_i(t) = b_{i-1} v_{i-1}(t) + a_i v_i(t) + c_{i+1} v_{i+1}(t), \quad (14)$$

where $b_{-1} = 0, c_{D+1} = 1$ and $v_{-1}(t) = 0$. Then for each integer $i \in \{0, 1, \dots, D+1\}$, the polynomial $v_i(t)$ has degree i , the leading coefficient $(c_1 c_2 \cdots c_i)^{-1}$, and $v_i(A_1) =$

A_i with $A_{D+1} = O$. Moreover,

$$c_1 c_2 \dots c_D v_{D+1}(t) = (t - \theta_0)(t - \theta_1) \dots (t - \theta_D), \quad (15)$$

and for $i, j \in \{0, 1, \dots, D\}$,

$$\sum_{h=0}^D v_i(\theta_h) v_j(\theta_h) m_h = \delta_{i,j} k_i |X|. \quad (16)$$

(ii) Let $p_0(t), p_1(t), \dots, p_D(t)$ denote polynomials in $\mathbf{R}[t]$ satisfying $p_0(t) = 1$ and for $i \in \{0, 1, \dots, D-1\}$,

$$t p_i(t) = b_i p_{i-1}(t) + (a_i - c_{i+1} + c_i) p_i(t) + c_{i+1} p_{i+1}(t), \quad (17)$$

where $p_{-1}(t) = 0$. Then for each integer $i \in \{0, 1, \dots, D\}$, $p_i(t) = v_0(t) + v_1(t) + \dots + v_i(t)$ and the polynomial $p_i(t)$ has degree i , the leading coefficient $(c_1 c_2 \dots c_i)^{-1}$. Moreover, $(t - \theta_0) p_D(t) = v_{D+1}(t)$ and for $i, j \in \{0, 1, \dots, D-1\}$

$$\sum_{h=1}^D p_i(\theta_h) p_j(\theta_h) (\theta_0 - \theta_h) m_h = \delta_{i,j} k_i b_i |X|. \quad (18)$$

Lemma 3.2 ([23, 48]) *Let θ be a real number such that $\theta \geq \theta_1$ or $\theta \leq \theta_D$. For every $\ell \in \{0, 1, \dots, D-1\}$, let*

$$g_\ell(t) = \sum_{h=0}^{\ell} \frac{b_\ell k_\ell}{b_h k_h} \frac{p_h(\theta)}{p_\ell(\theta)} p_h(t).$$

Then the following hold.

(i) If $\theta = \theta_1$ or θ_D , then

$$(t - \theta) g_{D-1}(t) = p_D(t) = \frac{1}{c_1 c_2 \dots c_D} (t - \theta_1)(t - \theta_2) \dots (t - \theta_D).$$

(ii) $p_i(t) = g_i(t) - \frac{b_i}{c_i} \frac{p_{i-1}(\theta)}{p_i(\theta)} g_{i-1}(t)$ for every $i \in \{0, 1, \dots, D-1\}$.
 (iii) For every $\ell \in \{0, 1, \dots, D-2\}$

$$t g_\ell(t) = \beta_{\ell-1} g_{\ell-1}(t) + \alpha_\ell g_\ell(t) + \gamma_{\ell+1} g_{\ell+1}(t), \quad (19)$$

where

$$\begin{aligned} \beta_{\ell-1} &= \frac{b_\ell c_{\ell+1}}{c_\ell} \frac{p_{\ell-1}(\theta) p_{\ell+1}(\theta)}{(p_\ell(\theta))^2}, \\ \alpha_\ell &= a_\ell + c_\ell - c_{\ell+1} + \frac{p_{\ell-1}(\theta)}{p_\ell(\theta)} b_\ell - \frac{p_\ell(\theta)}{p_{\ell+1}(\theta)} b_{\ell+1}, \\ \gamma_{\ell+1} &= c_{\ell+1}. \end{aligned}$$

Moreover, if $\theta > \theta_1$ or $\theta < \theta_D$, then (19) holds for $\ell = D-1$ by setting $g_D(t) = p_D(t)$.

(iv) For every $i, j \in \{0, 1, \dots, D-1\}$

$$\sum_{h=1}^D g_i(\theta_h)g_j(\theta_h)(\theta - \theta_h)(\theta_0 - \theta_h)m_h = \delta_{i,j}c_{i+1} \frac{p_{i+1}(\theta)}{p_i(\theta)} b_i k_i |X|.$$

4. Terwilliger algebras and their modules

We now introduce the Terwilliger algebra of a distance-regular graph with respect to a subset of vertices.

Let Y be a nonempty subset of the vertex set X of a distance-regular graph Γ of diameter D . The number

$$w(Y) = \max\{\partial(x, y) \mid x, y \in Y\}$$

is called the *width* of Y in Γ . In particular, $w(X) = D$.

For $x \in X$, let

$$\partial(x, Y) = \partial(Y, x) = \min\{\partial(x, y) \mid y \in Y\}.$$

Set

$$\tau = \tau(Y) = \max\{\partial(x, Y) \mid x \in X\}.$$

The number τ is often called the *covering radius* of Y in X . For $i \in \{0, 1, \dots, \tau\}$, let

$$\Gamma_i(Y) = \{x \in X \mid \partial(x, Y) = i\},$$

and let $E_i^* = E_i^*(Y)$ denote the diagonal matrix in $\text{Mat}_X(\mathbf{C})$ with (x, x) -entry

$$(E_i^*)_{x,x} = \begin{cases} 1 & \text{if } x \in \Gamma_i(Y), \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

Throughout this paper, we adopt the convention that $E_i = 0$ and $E_j^* = 0$ for any integers i and j such that $i < 0$, $j < 0$, $i > D$ or $j > \tau$.

For a vector $\mathbf{v} = \sum_{x \in X} \alpha(x)\hat{x} \in V$ expressed as a linear combination of \hat{x} 's, $\text{supp}(\mathbf{v})$ denotes the *support* of \mathbf{v} , i.e., $\text{supp}(\mathbf{v}) = \{x \in X \mid \alpha(x) \neq 0\}$.

By definition, $E_i^*\mathbf{v} = \mathbf{v}$ if and only if $\text{supp}(\mathbf{v}) \subset \Gamma_i(Y)$. Moreover, for a subset Z of X , $\mathbf{1}_Z = \sum_{z \in Z} \hat{z}$ is called the *characteristic vector* of Z .

Lemma 4.1 For $h, i, j \in \{0, 1, \dots, D\}$ the following hold.

- (i) $E_i^*E_j^* = \delta_{i,j}E_i^*$.
- (ii) $E_0^* + E_1^* + \dots + E_D^* = I$.
- (iii) $(E_i^*)^2 = E_i^* = E_i^*$.

- (iv) $E_h^* A_i E_j^* \neq 0$ if and only if there exist vertices $x \in \Gamma_h(Y)$ and $y \in \Gamma_j(Y)$ such that $\partial(x, y) = i$.
- (v) If $E_h^* A_i E_0^* \neq 0$ then $h \leq i \leq h + w(Y)$.

Proof: Parts (i)–(iv) are immediate from the definition (20). For $x \in \Gamma_h(Y)$ and $y \in Y$, there exists $z \in Y$ such that $\partial(x, z) = h$ and $\partial(z, y) \leq w(Y)$. Hence $h \leq \partial(x, y) \leq h + w(Y)$. Thus (v) follows. \square

We have the decompositions

$$\begin{aligned} V &= E_0 V + E_1 V + \cdots + E_D V \text{ (orthogonal direct sum),} \\ &= E_0^* V + E_1^* V + \cdots + E_\tau^* V \text{ (orthogonal direct sum).} \end{aligned}$$

Definition 4.1 Let Γ be a distance-regular graph. Let X be the set of vertices and Y a nonempty subset of X such that $\tau = \tau(Y)$. Let $\mathcal{T} = \mathcal{T}(Y)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by the Bose-Mesner algebra \mathcal{M} and $E_0^*, E_1^*, \dots, E_\tau^*$. We call \mathcal{T} the *Terwilliger algebra* (or *subconstituent algebra*) of Γ with respect to Y .

Since $\mathcal{T} = \mathcal{T}(Y)$ is generated by symmetric real matrices, it is semisimple. The vector space V is called the *standard module* of \mathcal{T} . It is a fact that every \mathcal{T} -module is isomorphic to a submodule of V . Thus, throughout this paper, the term \mathcal{T} -module shall refer only to vector subspaces of V which are invariant under the action of \mathcal{T} by the usual matrix multiplication.

Definition 4.2 Let W be an irreducible $\mathcal{T}(Y)$ -module. W is said to be *thin* whenever

$$\dim E_i^* W \leq 1 \quad \text{for all } i \in \{0, 1, \dots, D\}.$$

W is said to be *dual-thin* whenever

$$\dim E_i W \leq 1 \quad \text{for all } i \in \{0, 1, \dots, D\}.$$

Definition 4.3 Let W be an irreducible $\mathcal{T}(Y)$ -module. The endpoint ν and diameter δ of W are the nonnegative integers defined by the following.

$$\nu = \min\{i \mid E_i^* W \neq 0\}, \quad \nu + \delta = \max\{i \mid E_i^* W \neq 0\}.$$

Lemma 4.2 ([45, Lemma 3.9]) *Let $\mathcal{T} = \mathcal{T}(Y)$ and let W denote an irreducible \mathcal{T} -module of endpoint ν and diameter δ . Then*

$$E_j^* W \neq 0 \quad \text{if and only if} \quad \nu \leq j \leq \nu + \delta.$$

Moreover, the following hold.

- (i) $AE_j^*W \subset E_{j-1}^*W + E_j^*W + E_{j+1}^*W$ for every $j \in \{0, 1, \dots, D\}$.
(ii) $E_i^*AE_j^*W \neq 0$ if $|i - j| = 1$ ($v \leq i, j \leq v + \delta$).
(iii) Suppose W is thin. Then for every $i \in \{0, 1, \dots, \delta\}$

$$E_v^*W + E_{v+1}^*W + \dots + E_{v+i}^*W = E_v^*W + AE_v^*W + \dots + A^i E_v^*W.$$

- (iv) Suppose W is thin. Then for $j \in \{0, 1, \dots, D\}$

$$E_j W = E_j E_v^* W.$$

Proof: Let $j \in \{0, 1, \dots, D\}$. By Lemma 4.1 (ii) and (iv), we have

$$\begin{aligned} AE_j^*W &= \sum_{i=0}^D E_i^* AE_j^*W \\ &= E_{j-1}^* AE_j^*W + E_j^* AE_j^*W + E_{j+1}^* AE_j^*W \subset E_{j-1}^*W + E_j^*W + E_{j+1}^*W. \end{aligned}$$

Hence we have (i).

Let $v' = \max\{i \mid E_i^*W \neq 0, \text{ and } E_{i-1}^*AE_i^*W = 0\}$. Let δ' be the least nonnegative integer satisfying the following.

$$E_{v'+\delta'}^*W \neq 0 \quad \text{and that} \quad E_{v'+\delta'+1}^*AE_{v'+\delta'}^*W = 0.$$

Let $W' = E_{v'}^*W + E_{v'+1}^*W + \dots + E_{v'+\delta'}^*W$. Then $AW' \subset W'$ by (i) in this lemma. Since $E_j^*W' \subset W'$ for every $j \in \{0, 1, \dots, D\}$, W' is a non-zero \mathcal{T} -invariant subspace of W . Since W is irreducible, we have $W = W'$. This shows $v = v'$ and $\delta = \delta'$.

Certainly,

$$E_{j+1}^*AE_j^*W \neq 0 \quad (v \leq j < v + \delta), \tag{21}$$

and

$$E_{j-1}^*AE_j^*W \neq 0 \quad (v < j \leq v + \delta).$$

Thus we have (ii). In particular, we have $E_j^*W \neq 0$ if and only if $v \leq j \leq v + \delta$.

If W is thin, the property (21) implies (iii). Thus we have $W = \mathcal{M}E_v^*W$. Multiplying both sides of this equation on the left by E_j gives (iv). \square

Lemma 4.3 *Let $\mathcal{T} = \mathcal{T}(Y)$ and let W be an irreducible \mathcal{T} -module. Then the following are equivalent.*

- (i) For some vector v , $W = \mathcal{M}v$.
(ii) W is dual thin.

Moreover, if W is thin, then W is dual-thin and the vector \mathbf{v} above can be taken from E_v^*W , where v is the endpoint of W .

Proof: Suppose $W = \mathcal{M}\mathbf{v}$ for some vector $\mathbf{v} \in V$. Then for every $i \in \{0, 1, \dots, D\}$,

$$E_i W = E_i \mathcal{M}\mathbf{v} \subset \text{Span}(E_i \mathbf{v}).$$

Hence W is dual thin.

Conversely, suppose that W is dual thin. Then $\dim E_i W \leq 1$. Hence we can choose \mathbf{v}_i so that $E_i W = \text{Span}(\mathbf{v}_i)$ for every $i \in \{0, 1, \dots, D\}$. Let $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \dots + \mathbf{v}_D$. Since $E_i \mathbf{v} = \mathbf{v}_i$, $E_i W \subset \mathcal{M}\mathbf{v}$ and $W \subset \mathcal{M}\mathbf{v}$. Therefore, $W = \mathcal{M}\mathbf{v}$.

If W is thin, then $\dim E_v^* W = 1$. Hence by Lemma 4.2 (iv) W is dual thin and the last assertion is obvious. \square

5. The significance of endpoint 0

Let Y be a nonempty subset of the vertex set X of Γ , and let $\mathcal{T} = \mathcal{T}(Y)$. Let W be a \mathcal{T} -module of endpoint v . In this section, we show that W can be regarded as a $\mathcal{T}(Y_v)$ -module of endpoint 0, where $Y_i = \Gamma_i(Y)$. Hence, to study a single irreducible \mathcal{T} -module W , it is sufficient to study one of endpoint zero with respect to a suitable base subset. We continue to use the notation $E_i^* = E_i^*(Y)$ and $\tau = \tau(Y)$.

Lemma 5.1 *The following hold.*

- (i) For each $x \in Y_j$, $\partial(x, Y_i) = j - i$ if $j \geq i$, and $i - j \leq \partial(x, Y_i)$ if $j < i$.
- (ii) Fix $i \in \{0, 1, \dots, D\}$. Then for each $j \in \{0, 1, \dots, D\}$,

$$E_j^*(Y_i) = E_{i+j}^* + E_j^*(Y_i)(E_0^* + E_1^* + \dots + E_{i-1}^*).$$

Proof: (i) is obvious, and (ii) follows from (i). \square

Lemma 5.2 *Let V be the standard module of \mathcal{T} . For $v \in \{0, 1, \dots, \tau\}$, let $V(v)$ denote the sum of irreducible \mathcal{T} -modules of endpoint v . Then*

$$V = V(0) + V(1) + \dots + V(\tau) \text{ (orthogonal direct sum).}$$

Let $\mathcal{I}(v)$ be the annihilator of $U(v) = V(v) + V(v+1) + \dots + V(\tau(Y))$:

$$\mathcal{I}(v) = \{M \in \mathcal{T} \mid M\mathbf{v} = \mathbf{0} \text{ for every } \mathbf{v} \in U(v)\}.$$

Then $\mathcal{T}/\mathcal{I}(v)$ is generated by the images of A and $E_v^*, E_{v+1}^*, \dots, E_\tau^*$. Moreover, if W is an irreducible \mathcal{T} -module of endpoint v then W is a $\mathcal{T}/\mathcal{I}(v)$ -module.

Proof: Since $\{E_0^*, E_1^*, \dots, E_{v-1}^*\}$ is contained in the two sided ideal $\mathcal{I}(v)$ of \mathcal{T} , we have the assertion. \square

Proposition 5.3 *Let $U(\nu)$ be a \mathcal{T} -module defined in Lemma 5.2, and let W be a \mathcal{T} -submodule of $U(\nu)$. Then the following hold.*

- (i) *As linear transformations on W , $E_{i+\nu}^*$ and $E_i^*(Y_\nu)$ coincide for each $i \in \{0, 1, \dots, D - \nu\}$.*
- (ii) *W is a $\mathcal{T}(Y_\nu)$ -module of endpoint 0.*
- (iii) *W is an irreducible \mathcal{T} -module if and only if it is an irreducible $\mathcal{T}(Y_\nu)$ -module.*
- (iv) *W is a thin \mathcal{T} -module if and only if it is a thin $\mathcal{T}(Y_\nu)$ -module.*

Proof: Since $\{E_0^*, \dots, E_{\nu-1}^*\}$ is contained in the annihilator of $U(\nu)$ and W is contained in $U(\nu)$, (i) follows from Lemma 5.1. Now (ii), (iii), and (iv) follow immediately. \square

We end this section by a lemma related to another type of base change.

Lemma 5.4 *Suppose $Z \subset Y \subset X$. Let $Z_i = \{x \in X \mid \partial(x, Z) = i\}$. Then the following hold.*

- (i) *For every $x \in X$, $\partial(x, Z) \geq \partial(x, Y)$.*
- (ii) *$Z_i \subset Y_0 \cup Y_1 \cup \dots \cup Y_i$, and $Y_i \subset Z_i \cup Z_{i+1} \cup \dots \cup Z_D$ for every $i \in \{0, 1, \dots, D\}$.*
- (iii) *$E_h^*(Y)E_\ell^*(Z) = 0$ for every $h, \ell \in \{0, 1, \dots, D\}$ such that $h > \ell$.*
- (iv) *For every $i \in \{0, 1, \dots, D\}$,*

$$\begin{aligned} E_i^*(Z) &= E_0^*(Y)E_i^*(Z) + \dots + E_i^*(Y)E_i^*(Z), \quad \text{and} \\ E_i^*(Y) &= E_i^*(Z)E_i^*(Y) + \dots + E_D^*(Z)E_i^*(Y). \end{aligned}$$

Proof: (i) and (ii) are clear. (iii) and (iv) follow from (i) and (ii). \square

6. Thin modules

For the rest of the paper, we fix a nonempty subset Y of the vertex set X of Γ unless otherwise noted. Let $\mathcal{T} = \mathcal{T}(Y)$ denote the Terwilliger algebra of Γ with respect to Y , and $E_i^* = E_i^*(Y)$. Recall that \mathcal{T} is generated by the Bose-Mesner algebra \mathcal{M} of Γ and E_i^* ($i \in \{0, 1, \dots, D\}$) as an algebra.

Definition 6.1 Let $\nu \in V$ be a nonzero vector such that $\text{supp}(\nu) \subset Y \subset X$, i.e., $\mathbf{0} \neq \nu \in E_0^*V$.

- (i) Let $t = t(\nu) = \max\{i \mid E_i^*A_i\nu \neq \mathbf{0}\}$.
- (ii) Let $r = r(\nu) = |\{i \mid E_i\nu \neq \mathbf{0}, i \in \{0, 1, \dots, D\}\}| - 1$.

Remarks When ν is the characteristic vector of Y , $t(\nu)$ coincides with $\tau(Y)$. The number $r(\nu)$ is called the *dual degree*. See [2, Chapter 11].

Lemma 6.1 *Let $\mathbf{v} \in E_0^*V$ be a nonzero vector. Let $t = t(\mathbf{v})$ and $r = r(\mathbf{v})$. Then the following hold.*

(i) $E_i^*A_i\mathbf{v} \neq \mathbf{0}$ for every $i \in \{0, 1, \dots, t\}$. In particular,

$$t + 1 = |\{i \mid E_i^*A_i\mathbf{v} \neq \mathbf{0}\}|.$$

(ii) $\dim \mathcal{M}\mathbf{v} = r + 1$ and

$$\mathcal{M}\mathbf{v} = \text{Span}(\mathbf{v}, A\mathbf{v}, \dots, A^r\mathbf{v}) = \text{Span}(A_0\mathbf{v}, A_1\mathbf{v}, \dots, A_r\mathbf{v}).$$

(iii) $t(\mathbf{v}) \leq r(\mathbf{v})$.

Proof: (i) By definition $E_t^*A_t\mathbf{v} \neq \mathbf{0}$. Let $i \in \{0, 1, \dots, t\}$. Then

$$\mathbf{0} \neq p_{t-i}^t E_t^*A_t\mathbf{v} = E_t^*A_{t-i}A_i\mathbf{v} = E_t^*A_{t-i}E_i^*A_i\mathbf{v}.$$

Hence $E_i^*A_i\mathbf{v} \neq \mathbf{0}$. By the choice of t , we have that $t + 1 = |\{i \mid E_i^*A_i\mathbf{v} \neq \mathbf{0}\}|$.

(ii) Observe that $\mathcal{M}\mathbf{v} = \text{Span}(E_0\mathbf{v}, E_1\mathbf{v}, \dots, E_D\mathbf{v})$ and that nonzero vectors among $E_0\mathbf{v}, E_1\mathbf{v}, \dots, E_D\mathbf{v}$ are mutually orthogonal. Since nonzero mutually orthogonal vectors are linearly independent, $\dim \mathcal{M}\mathbf{v} = r + 1$ by the definition of $r(\mathbf{v})$. Since \mathcal{M} is algebraically generated by the adjacency matrix A , $\mathcal{M}\mathbf{v} = \mathcal{C}[A]\mathbf{v}$. Now $\mathcal{M}\mathbf{v} = \text{Span}(\mathbf{v}, A\mathbf{v}, \dots, A^r\mathbf{v})$ follows as the polynomial ring over a field is a principal ideal domain. The last equality follows immediately.

(iii) By (ii) above and Lemma 4.1 (v), for every $h > r$,

$$E_h^*A_h\mathbf{v} \in E_h^*\mathcal{M}\mathbf{v} = \text{Span}(E_h^*A_0\mathbf{v}, E_h^*A_1\mathbf{v}, \dots, E_h^*A_r\mathbf{v}) = \mathbf{0}.$$

Hence $t(\mathbf{v}) \leq r(\mathbf{v})$. □

Proposition 6.2 *Let $\mathbf{v} \in E_0^*V$ be a nonzero vector. Let $t = t(\mathbf{v})$. Then the following are equivalent.*

- (i) $\mathcal{T}\mathbf{v}$ is a thin irreducible \mathcal{T} -module.
- (ii) $\dim E_i^*\mathcal{M}\mathbf{v} \leq 1$ for every $i \in \{0, 1, \dots, D\}$.
- (iii) $E_i^*A_{i+1}\mathbf{v}, E_i^*A_{i+2}\mathbf{v} \in \text{Span}(E_i^*A_i\mathbf{v})$ for every $i \in \{0, 1, \dots, t\}$.
- (iv) $\text{Span}(E_0^*\mathbf{v}, E_1^*A_1\mathbf{v}, \dots, E_t^*A_t\mathbf{v})$ is A -invariant.

Suppose the equivalent conditions (i)–(iv) hold. Then

$$\mathcal{T}\mathbf{v} = \mathcal{M}\mathbf{v} = \text{Span}(E_0^*\mathbf{v}, E_1^*A_1\mathbf{v}, \dots, E_t^*A_t\mathbf{v}).$$

In particular, $r(\mathbf{v}) = t(\mathbf{v})$ and $\mathcal{T}\mathbf{v}$ is of dimension $t + 1$.

Proof: Below we shall frequently use the fact that $E_i^*A_j\mathbf{v} = \mathbf{0}$ whenever $i > j$.

Let $\mathbf{v}_i = E_i^*A_i\mathbf{v}$ for all $i \in \{0, 1, \dots, D\}$. We assume that $\mathbf{v}_j = \mathbf{0}$ for all $j < 0$ or $j > D$.

(i) \rightarrow (ii) Since $\mathcal{M} \subset \mathcal{T}$, if $\mathcal{T}\mathbf{v}$ is thin irreducible, then for $i \in \{0, 1, \dots, D\}$ we have

$$\dim E_i^* \mathcal{M}\mathbf{v} \leq \dim E_i^* \mathcal{T}\mathbf{v} \leq 1.$$

(ii) \rightarrow (iii) This is obvious by the definition of $t(\mathbf{v})$ and Lemma 6.1 (i).

(iii) \rightarrow (iv) Let $t = t(\mathbf{v})$ and $W_0 = \text{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_t)$. By definition, W_0 is invariant under the actions of $E_0^*, E_1^*, \dots, E_D^*$. Moreover, by our assumption,

$$E_i^* A_{i+1}\mathbf{v}, E_i^* A_{i+2}\mathbf{v} \in \text{Span}(\mathbf{v}_i) \quad \text{for every } i \in \{0, 1, \dots, t\}. \quad (22)$$

We claim that W_0 is A -invariant. Let $0 \leq j \leq t$ and assume that for $h < j$ $Av_h \in W_0$, or equivalently,

$$E_{h+1}^* Av_h, E_h^* Av_h, E_{h-1}^* Av_h \in W_0. \quad (23)$$

Then we have by Lemma 4.2 (i), and (22), (23),

$$\begin{aligned} Av_j &= AE_j^* A_j \mathbf{v} \\ &= E_{j+1}^* AE_j^* A_j \mathbf{v} + E_j^* AE_j^* A_j \mathbf{v} + E_{j-1}^* AE_j^* A_j \mathbf{v} \\ &= E_{j+1}^* AA_j \mathbf{v} + E_j^* AA_j \mathbf{v} - E_j^* AE_{j-1}^* A_j \mathbf{v} \\ &\quad + E_{j-1}^* AA_j \mathbf{v} - E_{j-1}^* AE_{j-1}^* A_j \mathbf{v} - E_{j-1}^* AE_{j-2}^* A_j \mathbf{v} \\ &= c_{j+1} E_{j+1}^* A_{j+1} \mathbf{v} + c_{j+1} E_j^* A_{j+1} \mathbf{v} + a_j E_j^* A_j \mathbf{v} - E_j^* AE_{j-1}^* A_j \mathbf{v} \\ &\quad + c_{j+1} E_{j-1}^* A_{j+1} \mathbf{v} + a_j E_{j-1}^* A_j \mathbf{v} + b_{j-1} E_{j-1}^* A_{j-1} \mathbf{v} \\ &\quad - E_{j-1}^* AE_{j-1}^* A_j \mathbf{v} - E_{j-1}^* AE_{j-2}^* A_j \mathbf{v} \\ &\in \text{Span}(\mathbf{v}_{j+1}, \mathbf{v}_j, \mathbf{v}_{j-1}, E_j^* Av_{j-1}, E_{j-1}^* Av_{j-1}, E_{j-1}^* Av_{j-2}) \\ &\subset W_0. \end{aligned}$$

Now by induction, W_0 is invariant under the action of A , and hence W_0 is \mathcal{T} -invariant. Since $\mathbf{v} \in W_0$, $\mathcal{T}\mathbf{v} = W_0$ and we have (iv).

(iv) \rightarrow (i) Let $W = \text{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_D)$. Then $\mathbf{v}_0 = \mathbf{v} \neq \mathbf{0}$ and $W \subset \mathcal{T}\mathbf{v}$. By definition, W is invariant under the actions of $E_0^*, E_1^*, \dots, E_D^*$. Moreover, by our assumption, W is invariant under the action of A . Hence $W = \mathcal{T}\mathbf{v}$. Let W_0 be an irreducible subconstituent of W such that $E_0^* W_0 \neq \mathbf{0}$. Since $E_0^* W_0 = \text{Span}(\mathbf{v}_0)$, $\mathbf{v} = \mathbf{v}_0 \in W_0$ and $W = \mathcal{T}\mathbf{v} \subset W_0 \subset W$. Hence W is irreducible. Moreover, since $E_i^* W \subset \text{Span}(\mathbf{v}_i)$, W is thin.

By (iv), $\mathcal{T}\mathbf{v} = \mathcal{M}\mathbf{v} = \text{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_t)$, which is of dimension $t + 1$ by the definition of $t(\mathbf{v})$. By Lemma 6.1 (ii), $\dim \mathcal{M}\mathbf{v} = r(\mathbf{v}) + 1$. Hence $r(\mathbf{v}) = t(\mathbf{v})$. The last assertion follows from Lemma 4.3. \square

7. The primary module

In this section, we summarize the results concerning the module generated by the characteristic vector of Y .

Let $\mathbf{1}$ be the all 1's vector of V . For every i , let

$$\mathbf{1}_i = \sum_{x \in \Gamma_i(Y)} \hat{x}.$$

We have $\mathbf{1} = \mathbf{1}_0 + \mathbf{1}_1 + \cdots + \mathbf{1}_\tau$, where $\tau = \tau(Y)$. Note that $\mathbf{1}_i$ is the characteristic vector of $\Gamma_i(Y)$. In particular, $\mathbf{1}_0$ is the characteristic vector of the base subset Y .

Lemma 7.1 *Let W be an irreducible module of \mathcal{T} . Then the following are equivalent.*

- (i) $E_0 W \neq 0$, i.e., $\mathbf{1} \in W$.
- (ii) $\mathbf{1}_0 \in W$.
- (iii) $\mathbf{1}_i \in W$ for all $i \in \{0, 1, \dots, \tau\}$.

Proof: Since $E_i^* \mathbf{1} = \mathbf{1}_i$, (i) \leftrightarrow (iii) and (i) \rightarrow (ii) are immediate. Suppose (ii) holds. Then $(|Y|/|X|)\mathbf{1} = E_0 \mathbf{1}_0 \in W$ and (i) holds. \square

The irreducible module satisfying the conditions in the previous lemma is called the *primary module* of \mathcal{T} .

For $z \in \Gamma_h(Y)$, let

$$\pi_{i,j}^h = \pi_{i,j}^h(z) = |\{y \in \Gamma_i(Y) \mid \partial(z, y) = j\}| = (E_h^* A_j \mathbf{1}_i)_z.$$

Recall that a subset Y of the vertex set X of a distance-regular graph is called a *completely regular code* if $\pi_{0,j}^h(z)$ is independent of the choice of $z \in \Gamma_h(Y)$, i.e., $E_h^* A_j \mathbf{1}_0 = \pi_{0,j}^h \mathbf{1}_h$ for all h and j . Using Proposition 6.2, we can easily prove the following.

Proposition 7.2 *Let $\mathbf{1}_0$ be the characteristic vector of the base subset Y of \mathcal{T} . Then the following are equivalent.*

- (i) Y is a completely regular code of Γ .
- (ii) $\mathcal{T}\mathbf{1}_0$ is a thin irreducible \mathcal{T} -module.
- (iii) $\mathcal{T}\mathbf{1}_0 = \text{Span}(\mathbf{1}_0, \mathbf{1}_1, \dots, \mathbf{1}_\tau)$.
- (iv) $E_h^* A_j \mathbf{1}_i = \pi_{i,j}^h \mathbf{1}_h$, i.e., $\pi_{i,j}^h(z)$ is independent of the choice of $z \in \Gamma_h(Y)$ for every $h, i, j \in \{0, 1, \dots, D\}$.
- (v) Let $\beta_j = \pi_{j+1,1}^j$, $\alpha_j = \pi_{j,1}^j$ and $\gamma_j = \pi_{j-1,1}^j$. Then for every $i \in \{0, 1, \dots, D\}$,

$$A\mathbf{1}_j = \beta_{j-1}\mathbf{1}_{j-1} + \alpha_j\mathbf{1}_j + \gamma_{j+1}\mathbf{1}_{j+1}.$$

Moreover, if one of these conditions are satisfied, then

$$\begin{aligned} \gamma_j |\Gamma_j(Y)| &= \beta_{j-1} |\Gamma_{j-1}(Y)| \quad \text{and} \\ |\Gamma_j(Y)| &= \frac{\beta_0 \beta_1 \cdots \beta_{j-1}}{\gamma_1 \gamma_2 \cdots \gamma_j} |Y|, \quad \text{for all } j \in \{0, 1, 2, \dots, \tau\}. \end{aligned}$$

Proof: Let $W = \mathcal{T}\mathbf{1}_0$ and $W_0 = \text{Span}(\mathbf{1}_0, \mathbf{1}_1, \dots, \mathbf{1}_\tau)$. (i) is equivalent to $\mathcal{M}\mathbf{1}_0 \subset W_0$. Hence Proposition 6.2 (i) (b) is satisfied. Thus (ii) holds. Other implications are also immediate. \square

Remarks

1. For definitions and algebraic characterizations of completely regular codes, see [2, Section 11.1] and [20, 21, 35].
2. If $Y = \{x\}$, then the primary module of $\mathcal{T}(Y)$ is always thin. It actually characterizes the property of a graph to be distance-regular with respect to a vertex x . See for example [42].

8. Local eigenvalues

Definition 8.1 For a nonzero vector $\mathbf{v} \in V = \mathcal{C}^X$, let

$$\eta^{(i)}(\mathbf{v}) = \frac{{}^t\mathbf{v}A_i\bar{\mathbf{v}}}{{}^t\mathbf{v}\bar{\mathbf{v}}}, \quad \text{for } i \in \{0, 1, \dots, D\}.$$

Let $\tilde{A}_i = E_0^*A_iE_0^*$ for $i \in \{0, 1, \dots, D\}$. Suppose $\text{supp}(\mathbf{v}) \subset Y$, i.e., $\mathbf{v} \in E_0^*V$. Since $\mathbf{v} = E_0^*\mathbf{v}$,

$${}^t\mathbf{v}A_i\bar{\mathbf{v}} = {}^t\mathbf{v}E_0^*A_iE_0^*\bar{\mathbf{v}} = {}^t\mathbf{v}\tilde{A}_i\bar{\mathbf{v}},$$

and $\eta^{(i)}(\mathbf{v}) = 0$ for every $i > w(Y)$ as $\tilde{A}_i = 0$. Moreover if \mathbf{v} is an eigenvector of \tilde{A}_i for some $i \in \{0, 1, \dots, D\}$, then $\eta^{(i)}(\mathbf{v})$ is the eigenvalue of \tilde{A}_i associated with the eigenvector \mathbf{v} as $\mathbf{v} = E_0^*\mathbf{v}$.

Lemma 8.1 *Let W be a thin irreducible \mathcal{T} -module of endpoint 0. Let \mathbf{v} be a nonzero vector in E_0^*W , and $\eta^{(i)} = \eta^{(i)}(\mathbf{v})$ for each i . Then $E_0^*W = \text{Span}(\mathbf{v})$ and \mathbf{v} is a common eigenvector of $\tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_D$. In particular, $\tilde{A}_i\mathbf{v} = \eta^{(i)}\mathbf{v}$ for every $i \in \{0, 1, \dots, D\}$.*

Proof: Since W is thin, $E_0^*W \neq 0$ is of dimension 1. Hence it is spanned by \mathbf{v} and we have the following.

$$\tilde{A}_i\mathbf{v} = E_0^*A_iE_0^*\mathbf{v} \in E_0^*W = \text{Span}(\mathbf{v}).$$

Hence we have the desired conclusion. \square

Lemma 8.2 *Let $\mathbf{v} \in E_0^*V$ be a nonzero vector, and let $v_j(t)$ ($j = 0, 1, \dots, D$) be the polynomials in Lemma 3.1. Then*

$$\frac{\|E_i\mathbf{v}\|^2}{\|\mathbf{v}\|^2} = \frac{1}{|X|} \sum_{j=0}^{w(Y)} \eta^{(j)}(\mathbf{v})q_i(j) = \frac{1}{|X|} \sum_{j=0}^{w(Y)} \frac{\eta^{(j)}(\mathbf{v})}{k_j} v_j(\theta_i)m_i.$$

Proof: Set $\eta^{(i)} = \eta^{(i)}(\mathbf{v})$. Since $v_j(\theta_i)$ is the eigenvalue of A_j on $E_i V$, $v_j(\theta_i) = p_j(i)$. See Lemma 3.1 (i), and (9). Hence by (11),

$$q_i(j) = \frac{m_i v_j(\theta_i)}{k_j}.$$

Now using (10) and the remark following Definition 8.1, we have

$$\begin{aligned} \|E_i \mathbf{v}\|^2 &= {}^t \mathbf{v} E_i \overline{E_i \mathbf{v}} \\ &= {}^t \mathbf{v} E_i \overline{\mathbf{v}} \\ &= {}^t \mathbf{v} \frac{1}{|X|} \sum_{j=0}^D q_i(j) A_j \overline{\mathbf{v}} \\ &= \frac{1}{|X|} \sum_{j=0}^D q_i(j) {}^t \mathbf{v} A_j \overline{\mathbf{v}} \\ &= \frac{1}{|X|} \sum_{j=0}^{w(Y)} \eta^{(j)} q_i(j) \|\mathbf{v}\|^2 \\ &= \frac{1}{|X|} \sum_{j=0}^{w(Y)} \frac{\eta^{(j)}}{k_j} v_j(\theta_i) m_i \|\mathbf{v}\|^2. \end{aligned}$$

□

9. The subspace $\mathcal{M}\mathbf{v}$

Definition 9.1 Let $\mathbf{v} \in V$ be a nonzero vector. Let

$$\rho_{\mathbf{v}}(t) = \frac{1}{|X|} \sum_{j=0}^D \eta^{(j)}(\mathbf{v}) \frac{v_j(t)}{k_j},$$

where $v_j(t)$ is a polynomial in $\mathbf{R}[t]$ of degree j such that $v_j(A) = A_j$. See Lemma 3.1.

Note that if in addition $\mathbf{v} \in E_0^* V$, i.e., $\text{supp}(\mathbf{v}) \subset Y$, then

$$\rho_{\mathbf{v}}(t) = \frac{1}{|X|} \sum_{j=0}^{w(Y)} \eta^{(j)}(\mathbf{v}) \frac{v_j(t)}{k_j},$$

and $\deg \rho_{\mathbf{v}}(t) \leq w(Y)$.

Let $\Theta = \{\theta_0, \theta_1, \theta_2, \dots, \theta_D\}$ and let m be a function on Θ such that $m(\theta_i) = m_i$. Then $\{v_0(t), v_1(t), \dots, v_D(t)\}$ is a system of orthogonal polynomials associated with the weight function m on Θ satisfying the following. See Lemma 3.1.

$$\langle v_i(t), v_j(t) \rangle_m = \sum_{\theta \in \Theta} v_i(\theta) v_j(\theta) m(\theta) = \delta_{i,j} |X| k_i \quad (24)$$

for every $i, j \in \{0, 1, 2, \dots, D\}$. Moreover, it follows from Lemma 8.2 that

$$\frac{\|E_i \mathbf{v}\|^2}{\|\mathbf{v}\|^2} = \rho_{\mathbf{v}}(\theta_i) m(\theta_i) \geq 0 \quad \text{for every } i \in \{0, 1, 2, \dots, D\}. \quad (25)$$

Proposition 9.1 *Let \mathbf{v} be a nonzero vector in $E_0^* V$. Then*

$$r(\mathbf{v}) = D - |\{\theta \in \Theta \mid \rho_{\mathbf{v}}(\theta) = 0\}| \geq D - \deg \rho_{\mathbf{v}}(t) \geq D - w(Y).$$

Proof: This is immediate from Definition 6.1 and (25). \square

Definition 9.2 A nonzero vector $\mathbf{v} \in E_0^* V$ is said to be *tight* with respect to Y , if $r(\mathbf{v}) = D - w(Y)$.

Theorem 9.2 *Let \mathbf{v} be a nonzero vector in $E_0^* V$. If \mathbf{v} is tight with respect to Y , then $\mathcal{M}\mathbf{v}$ is a thin irreducible \mathcal{T} -module. In particular, \mathbf{v} is a common eigenvector of $\widetilde{\mathcal{M}} = E_0^* \mathcal{M} E_0^*$.*

Proof: Let $w = w(Y)$ and $\rho(t) = \rho_{\mathbf{v}}(t)$. By (25) above, $\rho(\theta) \geq 0$ for every eigenvalue of the adjacency matrix A . Suppose $\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_w}$ are the roots of $\rho(t)$. Hence there is a nonzero constant c such that

$$\rho(t) = c(t - \theta_{i_1})(t - \theta_{i_2}) \cdots (t - \theta_{i_w}).$$

Then we have

$$E_{i_1} \mathbf{v} = E_{i_2} \mathbf{v} = \cdots = E_{i_w} \mathbf{v} = \mathbf{0}.$$

Hence for every $n \in \{0, 1, \dots, D - w\}$,

$$E_n^* E_{i_1} \mathbf{v} = E_n^* E_{i_2} \mathbf{v} = \cdots = E_n^* E_{i_w} \mathbf{v} = \mathbf{0}.$$

Thus for every $j \in \{1, 2, \dots, w\}$ we have

$$\begin{aligned} \mathbf{0} &= E_n^* E_{i_j} \mathbf{v} \\ &= \frac{m(\theta_{i_j})}{|X|} \sum_{h=0}^D \frac{v_h(\theta_{i_j})}{k_h} E_n^* A_h \mathbf{v} \\ &= \frac{m(\theta_{i_j})}{|X|} \sum_{h=0}^w \frac{v_{n+h}(\theta_{i_j})}{k_{n+h}} E_n^* A_{n+h} \mathbf{v}, \end{aligned}$$

where we used Lemma 4.1 (v).

Hence for each $j \in \{0, 1, \dots, w\}$, we have a linear equation

$$\sum_{h=1}^w v_{n+h}(\theta_{i_j}) \frac{1}{k_{n+h}} E_n^* A_{n+h} \mathbf{v} = -\frac{v_n(\theta_{i_j})}{k_n} E_n^* A_n \mathbf{v}.$$

By Lemma 2.3,

$$D_{\rho, n+1} = \begin{vmatrix} v_{n+1}(\theta_{i_1}) & v_{n+2}(\theta_{i_1}) & \cdots & v_{n+w}(\theta_{i_1}) \\ v_{n+1}(\theta_{i_2}) & v_{n+2}(\theta_{i_2}) & \cdots & v_{n+w}(\theta_{i_2}) \\ \dots & \dots & \dots & \dots \\ v_{n+1}(\theta_{i_w}) & v_{n+2}(\theta_{i_w}) & \cdots & v_{n+w}(\theta_{i_w}) \end{vmatrix} \neq 0.$$

Hence for every $h \in \{0, 1, \dots, w\}$, $\frac{1}{k_{n+h}} E_n^* A_{n+h} \mathbf{v}$, and therefore $E_n^* A_{n+h} \mathbf{v}$ is a scalar multiple of $E_n^* A_n \mathbf{v}$. Since $E_n^* A_{n+h} \mathbf{v} \neq \mathbf{0}$ only when $h \in \{0, 1, \dots, w\}$, we have shown the conditions in Proposition 6.2 (iii). Hence by the same proposition, $\mathcal{T} \mathbf{v}$ is a thin irreducible \mathcal{T} -module as desired. The last assertion follows directly from our proof above. See also Lemma 8.1. \square

Corollary 9.3 *Let Y be a subset of the vertex set X of a distance-regular graph. Suppose $w = w(Y)$ and for $i \in \{0, 1, \dots, D\}$ set*

$$\kappa_i = \frac{{}^t \mathbf{1}_0 A_i \mathbf{1}_0}{{}^t \mathbf{1}_0 \mathbf{1}_0} = \frac{1}{|Y|} |\{(y_1, y_2) \in Y \times Y \mid \partial(y_1, y_2) = i\}|,$$

where $\mathbf{1}_0$ is the characteristic vector of Y . Let $\rho(t) = \rho_{\mathbf{1}_0}(t)$. Then the following hold.

- (i) $\rho(\theta) \geq 0$ for every eigenvalue θ of the adjacency matrix of Γ , where $\rho(t)$ is the following polynomial.

$$\rho(t) = \frac{1}{|X|} \sum_{i=0}^w \frac{\kappa_i}{k_i} v_i(t).$$

- (ii) If $\rho(t)$ has w roots in the set of eigenvalues of the adjacency matrix of Γ , then Y is completely regular.

Proof: The assertions follow directly from Theorem 9.2. \square

Remarks Corollary 9.3 can be viewed as a generalization of the Hoffman bound of the size of a clique and the condition for equality, which can be derived easily by setting $w(Y) = 1$. See Lemma 11.1. Similar conditions were studied in [20, 21]. It would be interesting to further consider the case where $E_0^* V$ is spanned by tight vectors.

10. Thin conditions

In this section, we consider thin conditions. Let $r = r(\mathbf{v})$ and

$$\Theta(\mathbf{v}) = \{\theta \in \Theta \mid \rho_{\mathbf{v}}(\theta) \neq 0\}.$$

of cardinality $r + 1$. Then by Proposition 2.1, there is a system of orthogonal polynomials $\{g_0(t), g_1(t), \dots, g_r(t)\}$ associated with the weight function $\omega = \rho \cdot m$ on $\Theta(\mathbf{v})$ with $\rho = \rho_{\mathbf{v}}$. We may assume that the leading coefficient of $g_i(t)$ coincides with that of $v_i(t)$. Then this system of polynomials is unique and there exist $c_i^{(j)} \in \mathbf{R}$ for $j \in \{0, 1, \dots, i\}$ such that

$$g_i(t) = \sum_{j=0}^i c_i^{(j)} v_j(t), \quad (26)$$

with $c_i^{(i)} = 1$. Set

$$g_{r+1}(t) = \prod_{\theta \in \Theta(\mathbf{v})} (t - \theta).$$

Then for each $i \in \{0, 1, \dots, r\}$ there exist real numbers $\beta_{i-1}, \alpha_i, \gamma_{i+1}$ and $g_0(t), g_1(t), \dots, g_r(t), g_{r+1}(t)$ satisfy the following relations.

$$t g_i(t) = \beta_{i-1} g_{i-1}(t) + \alpha_i g_i(t) + \gamma_{i+1} g_{i+1}(t). \quad (27)$$

Moreover, $\beta_{i-1} \gamma_i > 0$ for $i \in \{1, 2, \dots, r\}$.

Theorem 10.1 *Let \mathbf{v} be a nonzero vector in E_0^*V , and let $\mathbf{u}_i = g_i(A)\mathbf{v}$ for $i \in \{0, 1, \dots, r+1\}$. Then the following hold.*

(i) *For every $i, j \in \{0, 1, \dots, r+1\}$,*

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \langle g_i(t), g_j(t) \rangle_{\omega} \langle \mathbf{v}, \mathbf{v} \rangle = \delta_{i,j} \langle g_i(t), g_i(t) \rangle_{\omega} \|\mathbf{v}\|^2,$$

and $\mathbf{u}_{r+1} = \mathbf{0}$.

(ii) *For every $i \in \{0, 1, \dots, r\}$,*

$$\|E_i^* A_i \mathbf{v}\|^2 \leq \|\mathbf{u}_i\|^2.$$

*Moreover, equality holds above if and only if $\mathbf{u}_i \in E_i^*V$. In this case, $\mathbf{u}_i = E_i^* A_i \mathbf{v}$.*

(iii) *The following are equivalent.*

- (a) *\mathcal{T} -module $\mathcal{T}\mathbf{v}$ is thin and irreducible.*
- (b) *Equality holds in (ii) for every $i \in \{0, 1, \dots, r\}$.*
- (c) *Equality holds in (ii) for $i = r$.*

Proof: (i) By (25), $\|E_h \mathbf{v}\|^2 = \rho(\theta_h)m(\theta_h)\|\mathbf{v}\|^2$. Hence

$$\begin{aligned}
\langle \mathbf{u}_i, \mathbf{u}_j \rangle &= \langle g_i(A)\mathbf{v}, g_j(A)\mathbf{v} \rangle \\
&= \left\langle g_i(A) \sum_{h=0}^D E_h \mathbf{v}, g_j(A) \sum_{h=0}^D E_h \mathbf{v} \right\rangle \\
&= \left\langle g_i(A) \sum_{\theta_h \in \Theta(\mathbf{v})} E_h \mathbf{v}, g_j(A) \sum_{\theta_h \in \Theta(\mathbf{v})} E_h \mathbf{v} \right\rangle \\
&= \left\langle \sum_{\theta_h \in \Theta(\mathbf{v})} g_i(\theta_h) E_h \mathbf{v}, \sum_{\theta_h \in \Theta(\mathbf{v})} g_j(\theta_h) E_h \mathbf{v} \right\rangle \\
&= \sum_{\theta_h \in \Theta(\mathbf{v})} g_i(\theta_h) g_j(\theta_h) \rho(\theta_h) m(\theta_h) \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \sum_{\theta_h \in \Theta(\mathbf{v})} g_i(\theta_h) g_j(\theta_h) \omega(\theta_h) \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \langle g_i(t), g_j(t) \rangle_{\omega} \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \delta_{i,j} \langle g_i(t), g_i(t) \rangle_{\omega} \|\mathbf{v}\|^2.
\end{aligned}$$

Since $g_{r+1}(t)$ is a zero function on $\Theta(\mathbf{v})$, $\mathbf{u}_{r+1} = \mathbf{0}$.

(ii) By the definition of $r = r(\mathbf{v})$, $\mathbf{u}_i \neq \mathbf{0}$ for $i \in \{0, 1, \dots, r\}$. Note that $g_i(t)$ is of degree i and the leading coefficient of $g_i(t)$ coincides with that of $v_i(t)$ by definition. Hence if $g_i(A)$ is expressed as a linear combination of A_0, A_1, \dots, A_D , then A_j with $j > i$ does not appear and the coefficient of A_i is 1. Hence $E_i^* g_i(A)\mathbf{v} = E_i^* A_i \mathbf{v}$ and

$$\mathbf{u}_i = g_i(A)\mathbf{v} = E_i^* A_i \mathbf{v} + \sum_{h=0}^{i-1} E_h^* g_i(A)\mathbf{v}.$$

Since the vector appearing in the right hand side of the equation are mutually orthogonal, $\|E_i^* A_i \mathbf{v}\|^2 \leq \|\mathbf{u}_i\|^2$. Hence the first assertion follows from (i). Moreover, $\mathbf{u}_i \in E_i^* V$ if and only if $\sum_{h=0}^{i-1} E_h^* g_i(A)\mathbf{v} = \mathbf{0}$. Thus we have (ii).

(iii) (a) \rightarrow (b) Suppose $W = \mathcal{T}\mathbf{v}$ is thin and irreducible. By Proposition 6.2 $r(\mathbf{v}) = t(\mathbf{v})$. Hence $W = \mathcal{M}\mathbf{v}$ and W is an orthogonal direct sum of $E_i^* W = \text{Span}(E_i^* A_i \mathbf{v})$. Set $\mathbf{v}_i = E_i^* A_i \mathbf{v}$ for $i = 0, 1, \dots, r$. Then $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_r\}$ forms a mutually orthogonal basis. On the other hand, since $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_r$ are mutually orthogonal and gives a basis of $\mathcal{M}\mathbf{v}$, we have the following.

$$\mathbf{v}_i \in E_i^* W \subset E_i^* (\text{Span}(\mathbf{u}_i, \mathbf{u}_{i+1}, \dots, \mathbf{u}_r) \cap \text{Span}(\mathbf{v}_{i+1}, \dots, \mathbf{v}_r)^\perp). \quad (28)$$

We prove by induction on i in reverse order that $\text{Span}(\mathbf{u}_i) = \text{Span}(\mathbf{v}_i)$. Suppose the assertion holds for $j > i$ for $i \in \{0, 1, \dots, r\}$. By the inductive hypothesis,

$$\text{Span}(\mathbf{u}_{i+1}, \dots, \mathbf{u}_r) = \text{Span}(\mathbf{v}_{i+1}, \dots, \mathbf{v}_r). \quad (29)$$

Hence by (28) and (29),

$$\begin{aligned} \mathbf{0} \neq \mathbf{v}_i &\in \text{Span}(E_i^* \mathbf{u}_i) \\ &\subset \text{Span}(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_{i-1})^\perp \cap \text{Span}(\mathbf{v}_{i+1}, \dots, \mathbf{v}_r)^\perp \cap W \\ &\subset \text{Span}(\mathbf{u}_i, \mathbf{u}_{i+1}, \dots, \mathbf{u}_r) \cap \text{Span}(\mathbf{u}_{i+1}, \dots, \mathbf{u}_r)^\perp \\ &\subset \text{Span}(\mathbf{u}_i). \end{aligned}$$

By (ii), we have the assertion.

(b) \rightarrow (c) This is obvious.

(c) \rightarrow (b) By (ii), we assume that $\mathbf{u}_r \in \text{Span}(E_r^* A_r \mathbf{v})$. Since $\mathbf{u}_{r+1} = \mathbf{0}$, by (27) we have

$$A \mathbf{u}_i = \beta_{i-1} \mathbf{u}_{i-1} + \alpha_i \mathbf{u}_i + \gamma_{i+1} \mathbf{u}_{i+1} \quad (30)$$

for every $i \in \{0, 1, \dots, r\}$. Since $\deg g_i(t) = i$,

$$\mathbf{u}_i \in E_0^* V + E_1^* V + \dots + E_i^* V$$

for every $i \in \{0, 1, \dots, r\}$. Suppose $\mathbf{u}_i \in E_i^* V$ for $i \leq r$. Then by (30)

$$\beta_{i-1} \mathbf{u}_{i-1} \in (E_{i-1}^* V + \dots + E_r^* V) \cap (E_0^* V + E_1^* V + \dots + E_{i-1}^* V) = E_{i-1}^* V.$$

Since $\beta_{i-1} \neq 0$ for $i \in \{1, 2, \dots, r\}$, we have $\mathbf{u}_{i-1} \in E_{i-1}^* V$. Thus we have (b) by induction.

(b) \rightarrow (a) Suppose $\mathbf{u}_i \in \text{Span}(\mathbf{v}_i)$ for every $i \in \{0, 1, \dots, r\}$, then $r(\mathbf{v}) = t(\mathbf{v})$ and

$$W = \text{Span}(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_r) = \text{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_t). \quad (31)$$

Hence W is A -invariant and it is also stable under the actions of $E_0^*, E_1^*, \dots, E_D^*$. Thus $W = \mathcal{T}\mathbf{v}$ and it is irreducible and thin by Proposition 6.2. \square

Corollary 10.2 *Let \mathbf{v} be a nonzero vector in $E_0^* V$. Let $r = r(\mathbf{v})$. Then the condition whether $\mathcal{T}\mathbf{v}$ is a thin irreducible \mathcal{T} -module is determined by $\eta^{(i)}(\mathbf{v})$, $i \in \{0, 1, \dots, D\}$, and $\|E_r^* A_r \mathbf{v}\|/\|\mathbf{v}\|$ together with the parameters of Γ .*

Proof: By Theorem 10.1, $\mathcal{T}\mathbf{v}$ is thin and irreducible if and only if

$$\|E_r^* A_r \mathbf{v}\|^2 / \|\mathbf{v}\|^2 = \langle g_r(t), g_r(t) \rangle_{\rho_m}.$$

Now the assertion follows immediately. \square

Corollary 10.3 *Let W_1 and W_2 are irreducible thin \mathcal{T} -modules. Then W_1 and W_2 are isomorphic if and only if the endpoints \mathbf{v} are same and $\rho_{\mathbf{w}_1}(t) = \rho_{\mathbf{w}_2}(t)$, where $E_{\mathbf{v}}^* W_1 = \text{Span}(\mathbf{w}_1)$ and $E_{\mathbf{v}}^* W_2 = \text{Span}(\mathbf{w}_2)$.*

Proof: By Proposition 5.3 we may assume that $\nu = 0$. Suppose $\rho_{\mathbf{w}_1}(t) = \rho_{\mathbf{w}_2}(t)$. Then the module structure is determined by the system of orthogonal polynomials defined above. Hence they are isomorphic.

Conversely, if W_1 and W_2 are isomorphic, $\eta^{(i)}(\mathbf{w}_1) = \eta^{(i)}(\mathbf{w}_2)$ for every $i \in \{0, 1, \dots, D\}$ by Lemma 8.1. Hence the polynomials $\rho_{\mathbf{w}_1}(t)$, $\rho_{\mathbf{w}_2}(t)$ defined by each vector are equal. \square

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2: Theorem 1.1 (i) is a direct consequence of Proposition 9.1 together with the fact that $\dim \mathcal{M}\nu = r(\nu) + 1$ proved in Lemma 6.1 (ii). Suppose $r(\nu) = D - w(Y)$. Then ν is tight and by Theorem 9.2 $\mathcal{T}\nu$ is a thin irreducible module.

Theorem 1.2 follows from Theorem 10.1 (iii). \square

The following is also obtained as an application of Theorem 10.1.

Proposition 10.4 *Let $\nu \in V = \mathbf{C}^X$ be a nonzero vector. Let $\text{supp}(\nu) \subset Z \subset Y \subset X$. Then the following hold.*

- (i) $E_i^*(Y)A_i\nu = E_i^*(Y)E_i^*(Z)A_i\nu$, and $\|E_i^*(Y)A_i\nu\| \leq \|E_i^*(Z)A_i\nu\|$ for every $i \in \{0, 1, \dots, D\}$. Equality holds if and only if $E_i^*(Y)A_i\nu = E_i^*(Z)A_i\nu$.
- (ii) If $W = \mathcal{T}(Y)\nu$ is a thin irreducible $\mathcal{T}(Y)$ -module, then W is a thin irreducible $\mathcal{T}(Z)$ -module.

Proof: (i) By Lemma 5.4 (iii), $E_i^*(Y)E_j^*(Z) = 0$ if $i > j$. Hence

$$\begin{aligned} E_i^*(Y)A_i\nu &= \sum_{j=0}^{i-1} E_i^*(Y)E_j^*(Z)A_i\nu + E_i^*(Y)E_i^*(Z)A_i\nu + \sum_{j=i+1}^D E_i^*(Y)E_j^*(Z)A_i\nu \\ &= E_i^*(Y)E_i^*(Z)A_i\nu. \end{aligned}$$

Therefore,

$$\begin{aligned} E_i^*(Z)A_i\nu &= (E_0^*(Y)E_i^*(Z) + \dots + E_i^*(Y)E_i^*(Z))A_i\nu \\ &= (E_0^*(Y)E_i^*(Z) + \dots + E_{i-1}^*(Y)E_i^*(Z))A_i\nu + E_i^*(Y)A_i\nu. \end{aligned}$$

Since the first term in the last expression is orthogonal to the last term, we have (i).

Since the right hand side of Theorem 10.1 does not depend on the choice of base subsets, for every $i \in \{0, 1, \dots, r(\nu)\}$ we have

$$\|E_i^*(Y)A_i\nu\|^2 \leq \|E_i^*(Z)A_i\nu\|^2 \leq \langle g_i(t), g_i(t) \rangle \|\nu\|^2,$$

by (i). Thus the assertion follows from Theorem 10.1 (iii). \square

11. Y with small width

In this section we study irreducible modules of endpoint 0 when $w(Y)$ is small. If $w(Y) = 0$, Y consists of a single vertex y . Then $\mathcal{T} = \mathcal{T}(\{y\})$ is the Terwilliger algebra of [45]. If \mathbf{v} is a vector in V with $\text{supp}(\mathbf{v}) \subset Y = \{y\}$, it is a scalar multiple of \hat{y} . Hence $\mathcal{T}\mathbf{v}$ is the primary module and it is always thin. See [45, Lemma 3.6]. Hence we now consider the cases $w(Y) = 1$ and $w(Y) = 2$. For $M \in \mathcal{M}$, let $\tilde{M} = E_0^* M E_0^*$ and $\tilde{\mathcal{M}} = E_0^* \mathcal{M} E_0^*$.

11.1. $w(Y) = 1$

Throughout this subsection assume that $w(Y) = 1$.

Lemma 11.1 *The following hold.*

- (i) $\tilde{\mathcal{M}} = \text{Span}(\tilde{A}_0, \tilde{A}_1) = \text{Span}(\tilde{I}, \tilde{J})$ is commutative.
- (ii) $\theta_D \geq -k/(|Y| - 1)$.
- (iii) Let \mathbf{v} be an eigenvector of \tilde{A} in E_0^*V associated with an eigenvalue η . Then

$$\frac{\|E_i \mathbf{v}\|^2}{\|\mathbf{v}\|^2} = \frac{m_i(k + \eta\theta_i)}{k|X|}$$

and one of the following holds.

- (a) $E_0 \mathbf{v} \neq \mathbf{0}$, $\eta = |Y| - 1$, and \mathbf{v} is a multiple of $\mathbf{1}_0$. Moreover, if $E_i \mathbf{v} = \mathbf{0}$ for some i , then $i = D$, $\theta_D \cdot \eta + k = 0$ and \mathbf{v} is tight.
- (b) $E_0 \mathbf{v} = \mathbf{0}$ and $\eta = -1$. In particular, \mathbf{v} is tight.

Proof: (i) is obvious.

Since $w(Y) = 1$, the formula in Lemma 8.2 yields

$$\frac{\|E_i \mathbf{v}\|^2}{\|\mathbf{v}\|^2} = \frac{1}{|X|} (q_i(0) + \eta q_i(1)) = \frac{m_i}{|X|} \left(1 + \eta \frac{\theta_i}{k}\right) = \frac{m_i(k + \eta^{(1)}(\mathbf{v})\theta_i)}{k|X|}$$

as $\eta^{(0)} = 1$, $\eta^{(1)} = \eta$.

Let $\mathbf{v} = \mathbf{1}_0$ be the characteristic vector of Y . Then $\eta = |Y| - 1$, and

$$0 \leq \frac{|X| \|E_i \mathbf{v}\|^2}{m_i \|\mathbf{v}\|^2} = 1 + \frac{(|Y| - 1)\theta_i}{k}.$$

Hence we have (ii). Moreover $E_i \mathbf{v} = \mathbf{0}$ if and only if $i = D$ and $\theta_D \cdot (|Y| - 1) + k = 0$. Now the rest of the assertions follow from Definition 9.2. \square

Lemma 11.2 *Let \mathbf{v} be an eigenvector of \tilde{A} in E_0^*V associated with an eigenvalue η . Then $\eta \neq 0$. Let $\rho(t) = \rho_{\mathbf{v}}(t)$ and let $\theta = -b_0/\eta$. Then $\theta = b_0 = k$ or $\theta \leq \theta_D$. Moreover, the following hold.*

- (i) $\rho(\theta_i) = \frac{1}{|X|} \left(1 + \frac{\eta}{k}\theta_i\right) = \frac{(\theta - \theta_i)}{\theta|X|} \geq 0$ for every $i \in \{0, 1, \dots, D\}$.

- (ii) $v_i(\theta) \neq 0$ for every $i \in \{0, 1, \dots, D\}$.
 (iii) For $\ell \in \{0, 1, \dots, D\}$, let $g_\ell(t) \in \mathbf{R}[t]$ defined by the following.

$$g_\ell(t) = \sum_{h=0}^{\ell} \frac{k_\ell v_h(\theta)}{k_h v_\ell(\theta)} v_h(t).$$

Then for every $i, j \in \{0, 1, \dots, D\}$ the following hold.

$$\sum_{h=0}^D g_i(\theta_h) g_j(\theta_h) \rho(\theta_h) m(\theta_h) = \delta_{i,j} k_i c_{i+1} \frac{v_{i+1}(\theta)}{\theta \cdot v_i(\theta)}.$$

- (iv) For every $\ell \in \{0, 1, \dots, D-1\}$, $g_0(t), g_1(t), \dots, g_D(t)$ satisfy the following three term recurrence.

$$t g_\ell(t) = \beta_{\ell-1} g_{\ell-1}(t) + \alpha_\ell g_\ell(t) + \gamma_{\ell+1} g_{\ell+1}(t), \quad (32)$$

where

$$\begin{aligned} \beta_{\ell-1} &= \frac{b_{\ell-1} c_{\ell+1} v_{\ell-1}(\theta) v_{\ell+1}(\theta)}{c_\ell (v_\ell(\theta))^2}, \\ \alpha_\ell &= a_\ell + \frac{v_{\ell-1}(\theta)}{v_\ell(\theta)} b_{\ell-1} - \frac{v_\ell(\theta)}{v_{\ell+1}(\theta)} b_\ell, \\ \gamma_{\ell+1} &= c_{\ell+1}. \end{aligned}$$

Moreover, if $\theta < \theta_D$, then (32) holds for $\ell = D$ by setting $g_{D+1}(t) = v_{D+1}(t)$.

Proof: Since $\eta = |Y| - 1$ or -1 , $\eta \neq 0$. Now (i) follows from Lemma 11.1 (iii).

The proofs of (ii), (iii), and (iv) are very similar to those of Lemma 3.2 and therefore omitted. See [23, 48] and [5, Section 1.7]. \square

Proposition 11.3 *Let \mathbf{v} be an eigenvector of \tilde{A} in E_0^*V such that $\tilde{A}\mathbf{v} = \eta\mathbf{v}$. Let $\mathbf{v}_i = E_i^* A_i \mathbf{v}$ for $i \in \{0, 1, \dots, D+1\}$. Then the following hold.*

- (i) *If $\eta = -1$, then the module $\mathcal{T}\mathbf{v}$ is always thin of dimension D . For $i \in \{0, 1, \dots, D-1\}$, $\mathbf{v}_i = p_i(A)\mathbf{v}$, where $p_0(t) = 1, p_1(t), p_2(t), \dots, p_D(t)$ is a system of orthogonal polynomials defined by*

$$t p_i(t) = b_i p_{i-1}(t) + (a_i - c_{i+1} + c_i) p_i(t) + c_{i+1} p_{i+1}(t) \quad (0 \leq i \leq D-1),$$

where $p_{-1}(t) = 0$. They satisfy $p_i(A_1) = A_0 + A_1 + \dots + A_i$. Moreover, $\mathcal{T}\mathbf{v} = \text{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{D-1})$ and

$$A\mathbf{v}_i = b_i \mathbf{v}_{i-1} + (a_i - c_{i+1} + c_i) \mathbf{v}_i + c_{i+1} \mathbf{v}_{i+1}, \quad (33)$$

for $i \in \{0, 1, \dots, D-1\}$ and $\mathbf{v}_D = \mathbf{0}$.

- (ii) *If $\eta = |Y| - 1$, $\dim \mathcal{M}\mathbf{v}$ is either D or $D+1$. Let $\theta = -b_0/\eta$. For $i \in \{0, 1, \dots, D\}$, let $\beta_{i-1}, \alpha_i, \gamma_{i+1}$, and $g_i(t)$ be as in Lemma 11.2. Then the following hold.*

- (a) *If $\dim \mathcal{M}\mathbf{v} = D$, then $\theta = \theta_D$, $|Y| = 1 - k/\theta_D$, and $\mathcal{T}\mathbf{v}$ is always thin. Moreover, for $i \in \{0, 1, \dots, D-1\}$ $\mathbf{v}_i = g_i(A)\mathbf{v}$, $\mathcal{T}\mathbf{v} = \text{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{D-1})$ and*

$$A\mathbf{v}_i = \beta_{i-1} \mathbf{v}_{i-1} + \alpha_i \mathbf{v}_i + \gamma_{i+1} \mathbf{v}_{i+1}$$

with $\mathbf{v}_{-1} = \mathbf{v}_D = \mathbf{0}$.

(b) If $\dim \mathcal{M}\mathbf{v} = D + 1$, then

$$|Y||\Gamma_D(Y)| \leq \frac{k_D v_{D+1}(\theta)}{\theta v_D(\theta)},$$

with $\theta = -k/(|Y|-1)$ and $\mathcal{T}\mathbf{v}$ is thin if and only if equality holds above. Moreover, for $i \in \{0, 1, \dots, D\}$ $\mathbf{v}_i = g_i(A)\mathbf{v}$, $\mathcal{T}\mathbf{v} = \text{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_D)$ and

$$A\mathbf{v}_i = \beta_{i-1}\mathbf{v}_{i-1} + \alpha_i\mathbf{v}_i + \gamma_{i+1}\mathbf{v}_{i+1}$$

with $\mathbf{v}_{-1} = \mathbf{v}_{D+1} = \mathbf{0}$.

Proof: (i) In this case $\theta = b_0 = k$. Hence $v_i(\theta) = k_i$ for every i . Moreover, $g_i(t)$ in Lemma 11.2 becomes $p_i(t)$ in Lemma 3.1 (ii). Hence the assertions follow from Lemma 3.1 (ii), Lemma 11.2 (iii), (iv), and Theorem 9.2 together with the fact in Lemma 11.1 (iii) (ii).

(ii) (a) follows similarly.

We need to consider the case when $\mathbf{v} = \mathbf{1}_0$ with $\dim \mathcal{M}\mathbf{v} = D + 1$. Since D is the diameter of Γ , $Y \subset \Gamma_D(x)$ for every $x \in \Gamma_D(Y)$. Hence

$$\|E_D^* A_D \mathbf{1}_0\|^2 = \|\mathbf{1}_D\|^2 = |Y|^2 |\Gamma_D(Y)|.$$

Since

$$\langle g_D(t), g_D(t) \rangle_{\rho_m} = k_D \frac{v_{D+1}(\theta)}{\theta \cdot v_D(\theta)},$$

we have the assertion by Theorem 10.1 (ii) and (iii). \square

11.2. $w(Y) = 2$

In this subsection we assume that $w(Y) = 2$. This is the case when $Y = \Gamma_1(x)$ for some vertex x .

Lemma 11.4 *Let $\mathbf{v} \in E_0^*V$ be a nonzero vector such that $E_0\mathbf{v} = \mathbf{0}$. Then*

$$\frac{\|E_i\mathbf{v}\|^2}{\|\mathbf{v}\|^2} = \frac{m_i(k - \theta_i)((1 + \eta^{(1)}(\mathbf{v}))(1 + \theta_i) + b_1)}{kb_1|X|}.$$

Proof: Set $\eta^{(i)} = \eta^{(i)}(\mathbf{v})$ and $\eta = \eta^{(1)}$. Since $E_0\mathbf{v} = \mathbf{0}$ and $w(Y) = 2$,

$$\eta^{(2)}\|\mathbf{v}\|^2 = {}^t\mathbf{v}\tilde{A}_2\bar{\mathbf{v}} = {}^t\mathbf{v}(\tilde{J} - \tilde{A}_1 - \tilde{A}_0)\bar{\mathbf{v}} = -(1 + \eta)\|\mathbf{v}\|^2. \quad (34)$$

Hence we have $\eta^{(2)} = -(1 + \eta)$ and by (12) and (13),

$$\begin{aligned}
& q_i(0) + \eta q_i(1) + \eta^{(2)} q_i(2) \\
&= m_i \left(1 + \eta \frac{\theta_i}{k} - (1 + \eta) \frac{\theta_i^2 - a_1 \theta_i - k}{kb_1} \right) \\
&= \frac{m_i}{kb_1} (kb_1 + \eta b_1 \theta_i - (1 + \eta)(\theta_i^2 - a_1 \theta_i - k)) \\
&= \frac{m_i}{kb_1} (kb_1 + \eta b_1 \theta_i - \theta_i^2 + a_1 \theta_i + k - \eta \theta_i^2 + a_1 \eta \theta_i + k\eta) \\
&= \frac{m_i}{kb_1} ((k - \theta_i)b_1 + (k - \theta_i)\theta_i + (k - \theta_i) + \eta(b_1 \theta_i - \theta_i^2 + a_1 \theta_i + k)) \\
&= \frac{m_i(k - \theta_i)((1 + \eta)(1 + \theta_i) + b_1)}{kb_1}.
\end{aligned}$$

Now the formula follows from Lemma 8.2. \square

Lemma 11.5 *Suppose the induced subgraph on Y is regular of valency κ and size $m = |Y|$, i.e., $\tilde{A}\mathbf{1}_0 = \kappa\mathbf{1}_0$. Let $\kappa = \eta_1 \geq \eta_2 \geq \dots \geq \eta_m$ be the eigenvalues of \tilde{A} on E_0^*V . Then the following hold.*

- (i) $\text{tr}(\tilde{A}) = \sum_{i=1}^m \eta_i = 0$.
- (ii) $\text{tr}(\tilde{A}^2) = \sum_{i=1}^m \eta_i^2 = m\eta_1$.
- (iii) *Suppose $w(Y) = 2 < D$, then*

$$-1 - \frac{b_1}{1 + \theta_D} \geq \eta_2 \geq \dots \geq \eta_m \geq -1 - \frac{b_1}{1 + \theta_1}.$$

Proof: (i) and (ii) are well-known.

Let \mathbf{v} be an eigenvector of \tilde{A} associated with an eigenvalue $\eta = \eta_i$ with $i \geq 2$. Since $\mathbf{1}_0$ is an eigenvector for $\kappa = \eta_1$, we may assume that \mathbf{v} is orthogonal to $\mathbf{1}_0$. Hence $E_0\mathbf{v} = \mathbf{0}$. By Lemma 11.4, we have

$$0 \leq \frac{\|E_i\mathbf{v}\|^2}{\|\mathbf{v}\|^2} = \frac{m_i(k - \theta_i)((1 + \eta)(1 + \theta_i) + b_1)}{kb_1|X|}.$$

Hence $(1 + \eta)(1 + \theta_i) + b_1 \geq 0$. It is well-known that $\theta_1 > -1 > \theta_D$. (See for example [24, Section 13.2] or [31, Lemmas 2.3 and 2.6].) Hence we have

$$-1 - \frac{b_1}{1 + \theta_D} \geq \eta \geq -1 - \frac{b_1}{1 + \theta_1}.$$

\square

Remark We remark that Lemma 11.5 (iii) is essentially proved in [31, 44].

Lemma 11.6 *The following hold.*

- (i) $\tilde{\mathcal{M}} = \text{Span}(\tilde{A}_0, \tilde{A}_1, \tilde{A}_2) = \text{Span}(\tilde{I}, \tilde{A}_1, \tilde{J})$. $\tilde{\mathcal{M}}$ is commutative if and only if the induced subgraph on Y is regular.
- (ii) Let \mathbf{v} be a common eigenvector of $\tilde{\mathcal{M}}$ in E_0^*V such that $\tilde{A}_1\mathbf{v} = \eta\mathbf{v}$. Then one of the following hold.
 - (a) $E_0\mathbf{v} \neq \mathbf{0}$, \mathbf{v} is a multiple of $\mathbf{1}_0$, and the induced subgraph on Y is regular of valency η . Moreover, if \mathbf{v} satisfies the tight condition, then $E_i\mathbf{v} = E_{i+1}\mathbf{v} = \mathbf{0}$ for some index $i \in \{1, \dots, D-1\}$.
 - (b) $E_0\mathbf{v} = \mathbf{0}$. Moreover, if \mathbf{v} satisfies the tight condition, then either $E_1\mathbf{v} = \mathbf{0}$ or $E_D\mathbf{v} = \mathbf{0}$.

Proof: (i) follows from the well-known fact that the adjacency matrix of a graph commutes with the all-one's matrix if and only if the graph is regular.

(ii) Let $\rho(t) = \rho_{\mathbf{v}}(t)$. Then the degree of $\rho(t)$ is at most 2 and $\rho(\theta_h) \geq 0$ for every $h \in \{0, 1, \dots, D\}$. Hence if $\rho(\theta_i) = \rho(\theta_j) = 0$ for some $i < j$ with $i, j \in \{0, 1, \dots, D\}$, then $\rho(t)$ is quadratic and either $j = i + 1$ or $i = 0$ and $j = D$. If $\rho(\theta_0) = 0$, then we have case (b). Now we have the assertions by Theorem 9.2. \square

Definition 11.1 For each element $z \in \mathbf{R} \setminus \{-1\}$ set

$$\tilde{z} = -1 - \frac{b_1}{1+z}.$$

Let $\{p_0(t), p_1(t), \dots, p_D(t)\}$ be the system of polynomials in Lemma 3.1. Let θ be a real number satisfying $\theta \geq \theta_1$ or $\theta \leq \theta_D$. For $\ell \in \{0, 1, \dots, D\}$, let $g_\ell(t)$ be the polynomial defined in Lemma 3.2.

Proposition 11.7 *Suppose the induced subgraph on Y is regular of valency κ . Let \mathbf{v} be an eigenvector of \tilde{A}_1 in E_0^*V such that $\tilde{A}\mathbf{v} = \eta\mathbf{v}$. Assume that $E_0\mathbf{v} = \mathbf{0}$. Then the following hold.*

- (i) *The following hold.*

$$\begin{aligned} \rho_{\mathbf{v}}(\theta_i)m(\theta_i) &= \frac{\|E_i\mathbf{v}\|^2}{\|\mathbf{v}\|^2} = \frac{m_i(k - \theta_i)((1 + \eta)(1 + \theta_i) + b_1)}{kb_1|X|} \\ &= \begin{cases} \frac{m_i(k - \theta_i)(\tilde{\eta} - \theta_i)}{k(1 + \tilde{\eta})|X|} & \text{for } \eta \neq -1, \\ \frac{m_i(k - \theta_i)}{k|X|} & \text{for } \eta = -1. \end{cases} \end{aligned}$$

- (ii) \mathbf{v} is tight with respect to Y , i.e., $r(\mathbf{v}) = D - 2$, if and only if $\eta = \tilde{\theta}_1$, or $\tilde{\theta}_D$. In this case $\mathcal{T}\mathbf{v}$ is a thin irreducible module of dimension $D - 1$. Moreover, if we set $\theta = \tilde{\eta}$ to

define polynomials $g_i(t)$'s, then $\mathbf{v}_i = E_i^* A_i \mathbf{v} = g_i(A) \mathbf{v}$ for $i \in \{0, 1, \dots, D-1\}$ with $\mathbf{v}_{D-1} = \mathbf{0}$. Moreover, for $i \in \{0, 1, \dots, D-2\}$,

$$A \mathbf{v}_i = \beta_{i-1} \mathbf{v}_{i-1} + \alpha_i \mathbf{v}_i + \gamma_{i+1} \mathbf{v}_{i+1}.$$

Proof: (i) This follows from the equation

$$(1 + \tilde{\eta})(1 + \eta) = -b_1.$$

(ii) $r(\mathbf{v}) + 1 = \dim \mathcal{M} \mathbf{v} = D - 1$ if and only if $\rho_{\mathbf{v}}(\theta_i) = 0$ for some $i \in \{1, 2, \dots, D\}$. It follows from Lemma 11.5 (iii) together with (i) above that it is equivalent to $\eta \in \{\tilde{\theta}_1, \tilde{\theta}_D\}$. In this case $\mathcal{T} \mathbf{v}$ is thin by Theorem 9.2. Since $\eta \neq -1$, $\{g_0(t), g_1(t), \dots, g_{D-2}(t)\}$ satisfies

$$\langle g_i(t), g_j(t) \rangle_{\omega} = \delta_{i,j} \frac{p_{i+1}(\tilde{\eta}) c_{i+1}}{p_i(\tilde{\eta})(\tilde{\eta} + 1)} \frac{k_i b_i}{k}$$

by Lemma 3.2 (iv) and (i) above.

Since the leading coefficient of $g_i(t)$ coincides with that of $v_i(t)$, we have the assertions by Lemma 3.2 and Theorem 10.1. The last three term recursion is obtained from Lemma 3.2 (iii). \square

Proposition 11.8 *Suppose the induced subgraph on Y is regular of valency κ . Let \mathbf{v} be an eigenvector of \tilde{A}_1 in $E_0^* V$ such that $\tilde{A} \mathbf{v} = \eta \mathbf{v}$. Assume that $E_0 \mathbf{v} = \mathbf{0}$. Let $\rho(t) = \rho_{\mathbf{v}}(t)$. If $\dim \mathcal{M} \mathbf{v} = D$, then the following hold.*

- (i) $\Theta(\mathbf{v}) = \{\theta_1, \theta_2, \dots, \theta_D\}$ and $\tilde{\eta} > \theta_1$ or $\tilde{\eta} < \theta_D$.
- (ii) If $\eta \neq -1$, then the system of polynomials $\{g_0(t), g_1(t), \dots, g_{D-1}(t)\}$ is the system of orthogonal polynomials associated with the weight function $\omega = \rho \cdot m$ on $\Theta(\mathbf{v})$ such that the leading coefficient of $g_{\ell}(t)$ coincides with that of $v_{\ell}(t)$ and $p_{\ell}(t)$. Moreover, the following hold.
 - (a) $\langle g_i(t), g_j(t) \rangle_{\omega} = \delta_{i,j} \frac{p_{i+1}(\tilde{\eta}) c_{i+1} k_i b_i}{p_i(\tilde{\eta})(\tilde{\eta} + 1) k}$.
 - (b) $\|E_i^* A_i \mathbf{v}\|^2 \leq \langle g_i(t), g_i(t) \rangle_{\omega} \|\mathbf{v}\|^2$ and the equality holds if and only if $\mathcal{T} \mathbf{v}$ is irreducible and thin.
 - (c) If $\mathcal{T} \mathbf{v}$ is an irreducible thin module, then $\mathbf{v}_i = E_i^* A_i \mathbf{v} = g_i(A) \mathbf{v}$ for $i \in \{0, 1, \dots, D\}$ with $\mathbf{v}_D = \mathbf{0}$. Moreover, for $i \in \{0, 1, \dots, D-1\}$,

$$A \mathbf{v}_i = \beta_{i-1} \mathbf{v}_{i-1} + \alpha_i \mathbf{v}_i + \gamma_{i+1} \mathbf{v}_{i+1}.$$

(iii) If $\eta = -1$, then the system of polynomials $\{p_0(t), p_1(t), \dots, p_{D-1}(t)\}$ is the system of orthogonal polynomials associated with the weight function $\omega = \rho \cdot m$ on $\Theta(\mathbf{v})$ such that the leading coefficient of $p_{\ell}(t)$ coincides with that of $v_{\ell}(t)$. Moreover, the following hold.

- (a) $\langle p_i(t), p_j(t) \rangle_{\omega} = \delta_{i,j} \frac{k_i b_i}{k}$.
- (b) $\|E_i^* A_i \mathbf{v}\|^2 \leq \langle p_i(t), p_i(t) \rangle_{\omega} \|\mathbf{v}\|^2$ and the equality holds if and only if $\mathcal{T} \mathbf{v}$ is irreducible and thin.

(c) If $\mathcal{T}\mathbf{v}$ is an irreducible thin module, then $\mathbf{v}_i = E_i^* A_i \mathbf{v} = p_i(A)\mathbf{v}$ for $i \in \{0, 1, \dots, D\}$ with $\mathbf{v}_D = \mathbf{0}$. Moreover, for $i \in \{0, 1, \dots, D-1\}$,

$$A\mathbf{v}_i = b_i \mathbf{v}_{i-1} + (a_i + c_i - c_{i+1})\mathbf{v}_i + c_{i+1}\mathbf{v}_{i+1}.$$

Proof: By Proposition 11.7 (i),

$$\rho_{\mathbf{v}}(\theta_i)m(\theta_i) = \begin{cases} \frac{m_i(k - \theta_i)(\tilde{\eta} - \theta_i)}{k(1 + \tilde{\eta})|X|} & \text{for } \eta \neq -1, \\ \frac{m_i(k - \theta_i)}{k|X|} & \text{for } \eta = -1. \end{cases}$$

Hence if $\eta \neq -1$, then $\{g_0(t), g_1(t), \dots, g_{D-1}(t)\}$ is a system of orthogonal polynomials with respect to the weight function $\omega = \rho \cdot m$, and if $\eta = -1$, then $\{p_0(t), p_1(t), \dots, p_{D-1}(t)\}$ is the one which plays the same role. Now the assertions are direct consequences of Lemma 3.2 and Theorem 10.1. \square

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