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The Terwilliger Algebra of a Distance-Regular Graph that Supports a Spin Model

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Abstract. Let Γ denote a distance-regular graph with vertex set X, diameter $D \geq 3$, valency $k \geq 3$, and assume Γ supports a spin model W. Write $W = \sum_{i=0}^{D} t_i A_i$ where A_i is the ith distance-matrix of Γ. To avoid degenerate situations we assume Γ is not a Hamming graph and $t_i \notin \{t_0, -t_0\}$ for $1 \leq i \leq D$. In an earlier paper Curtin and Nomura determined the intersection numbers of Γ in terms of D and two complex parameters η and q. We extend their results as follows. Fix any vertex $x \in X$ and let $T = T(x)$ denote the corresponding Terwilliger algebra. Let U denote an irreducible T-module with endpoint r and diameter d. We obtain the intersection numbers $c_i(U), b_i(U), a_i(U)$ as rational expressions involving r, d, D, η and q. We show that the isomorphism class of U as a T-module is determined by r and d. We present a recurrence that gives the multiplicities with which the irreducible T-modules appear in the standard module. We compute these multiplicites explicitly for the irreducible T-modules with endpoint at most 3. We prove that the parameter q is real and we show that if Γ is not bipartite, then $q > 0$ and η is real.

Keywords: distance-regular graph, spin model, Terwilliger algebra

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1. Introduction

Let Γ denote a distance-regular graph with vertex set X, diameter $D \geq 3$ and valency $k \geq 3$. Fix any $x \in X$ and let $T = T(x)$ denote the corresponding Terwilliger algebra (definitions appear below). T acts faithfully on the vector space $\mathbb{C}^X$ by matrix multiplication. Since T is semi-simple, $\mathbb{C}^X$ decomposes into a direct sum of irreducible T-modules.

Assume Γ supports a spin model W with Boltzmann weights $t_0, \ldots, t_D$. Write $W = \sum_{i=0}^{D} t_i A_i$ where A_i is the ith distance-matrix of Γ. To avoid degenerate situations we assume Γ is not a Hamming graph and $t_i \notin \{t_0, -t_0\}$ for $1 \leq i \leq D$. In an earlier paper Curtin and Nomura [8] determined the intersection numbers of Γ in terms of D and two complex parameters η and q. We extend their results as follows.

Let $U \subseteq \mathbb{C}^X$ denote an irreducible T-module with endpoint r and diameter d. By [7, 9] the module U is thin and dual thin. In this article, we describe the action of T on U using a set of scalars known as the *intersection numbers* of U. We obtain the intersection numbers as rational expressions involving r, d, D, η and q. We show that the isomorphism class of U as a T-module is determined by r and d.

Combining the above results, we find a recurrence that gives the multiplicities with which the irreducible T-modules occur in $\mathbb{C}^X$. Using this recurrence, we obtain explicit formulas for the multiplicities of the irreducible T-modules with endpoint at most 3.

The integrality and the nonnegativity of the multiplicities give us feasibility constraints on the parameters. We prove that the parameter q must be real. We also show that if Γ is not bipartite, then $q > 0$ and η is real. We hope that in future work, these constraints will contribute to a classification of the distance-regular graphs that support a spin model.

2. Preliminaries

We will use the following notation. Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices with entries in $\mathbb{C}$ whose rows and columns are indexed by X. We let $\mathbb{C}^X$ denote the $\mathbb{C}$-vector space of column vectors with entries in $\mathbb{C}$ whose coordinates are indexed by X. Observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on $\mathbb{C}^X$ by left multiplication. We endow $\mathbb{C}^X$ with the Hermitian inner product satisfying

$$\langle u, v\rangle = u^t \bar{v} \quad (u, v \in \mathbb{C}^X),$$

where u^t denotes the transpose of u, and $\bar{v}$ denotes the complex-conjugate of v. For each $y \in X$, let $\hat{y}$ denote the vector in $\mathbb{C}^X$ with a 1 in coordinate y, and 0 in all other coordinates. Observe that $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for $\mathbb{C}^X$.

We now recall the notion of a spin model.

Definition 2.1 [15] By a *symmetric spin model* on X we mean a symmetric matrix $W \in \mathrm{Mat}_X(\mathbb{C})$ with non-zero entries which satisfies the following equations for all $a, b, c \in X$:

$$\sum_{y\in X} W_{yb}(W_{yc})^{-1} = |X|\delta_{bc}, \tag{1}$$

$$\sum_{y\in X} W_{ya}W_{yb}(W_{yc})^{-1} = L\, W_{ab}(W_{ac})^{-1}(W_{cb})^{-1}, \tag{2}$$

for some $L \in \mathbb{R}$ such that $L^2 = |X|$.

For the rest of this paper, when we refer to a *spin model*, we mean a symmetric spin model.

Let W denote a spin model on X. We now recall the Nomura algebra of W. For $b, c \in X$, let $\mathbf{u}_{bc}$ denote the vector in $\mathbb{C}^X$ which has y-coordinate

$$(\mathbf{u}_{bc})_y = W_{yb}(W_{yc})^{-1} \quad (y \in X). \tag{3}$$

Define $N(W)$ to be the set of all matrices $B \in \mathrm{Mat}_X(\mathbb{C})$ such that for all $b, c \in X$, the vector $\mathbf{u}_{bc}$ is an eigenvector of B. Observe $N(W)$ is a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$. By [13, Proposition 9], $W \in N(W)$. We refer to $N(W)$ as the *Nomura algebra* of W.

3. Distance-regular graphs

Let Γ denote a finite, connected, undirected graph, without loops or multiple edges, with vertex set X, path-length distance function ∂, and diameter $D := \max\{\partial(y,z) \mid y, z \in X\}$. For each $y \in X$ and each integer i, set

$$\Gamma_i(y) := \{z \in X \mid \partial(y,z) = i\}.$$

We abbreviate $\Gamma(y) := \Gamma_1(y)$. Let k denote a nonnegative integer. We say Γ is *regular with valency* k whenever $|\Gamma(y)| = k$ for all $y \in X$. We say Γ is *distance-regular*, with *intersection numbers* $p^h_{ij}(0 \le h, i, j \le D)$, whenever for all integers $h, i, j (0 \le h, i, j \le D)$ and all $y, z \in X$ with $\partial(y,z) = h$,

$$|\Gamma_i(y) \cap \Gamma_j(z)| = p^h_{ij}.$$

Notice $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two.

For the remainder of this article we assume Γ is distance-regular with diameter $D \ge 3$. We abbreviate $c_i := p^i_{1i-1}$ $(1 \le i \le D)$, $a_i := p^i_{1i}$ $(0 \le i \le D)$, $b_i := p^i_{1i+1}$ $(0 \le i \le D-1)$, and $k_i := p^0_{ii}$ $(0 \le i \le D)$. We define $c_0 = 0$, $b_D = 0$. We observe $c_1 = 1$ and $a_0 = 0$. Note that Γ is regular with valency $k = k_1 = b_0$. Moreover

$$c_i + a_i + b_i = k \quad (0 \le i \le D). \tag{4}$$

We now recall the Bose-Mesner algebra of Γ. For each integer $i (0 \le i \le D)$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yz-entry

$$(A_i)_{yz} = \begin{cases} 1 & \text{if} \quad \partial(y,z) = i, \\ 0 & \text{if} \quad \partial(y,z) \ne i \end{cases} \qquad (y, z \in X).$$

We call A_i the *ith distance matrix* of Γ. Observe that $A_0 = I$, where I denotes the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$. Observe $A := A_1$ is the adjacency matrix of Γ. Furthermore, we have: (i) $A^t_i = A_i$ $(0 \le i \le D)$; (ii) $A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h$ $(0 \le i, j \le D)$; (iii) $A_0 + A_1 + \cdots + A_D = J$, where J denotes the all 1's matrix in $\mathrm{Mat}_X(\mathbb{C})$. These properties imply the matrices $A_0, A_1, \ldots, A_D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. Moreover, M is closed under the entry-wise matrix product $\circ$. We remark that for each i $(0 \le i \le D)$ the matrix A_i is a polynomial of degree exactly i in A. Moreover, A generates M. We call M the *Bose-Mesner algebra* of Γ.

By [2, pp. 59, 64], M has a second basis $E_0, E_1, \ldots, E_D$ such that: (i) $E_0 = |X|^{-1}J$; (ii) $E^t_i = \bar{E}_i = E_i$ $(0 \le i \le D)$; (iii) $E_i E_j = \delta_{ij} E_i$ $(0 \le i, j \le D)$; (iv) $E_0 + E_1 + \cdots + E_D = I$. We refer to $E_0, E_1, \ldots, E_D$ as the *primitive idempotents* of Γ.

The graph Γ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for each integer i $(0 \le i \le D)$, the primitive idempotent E_i is a polynomial of degree exactly i in E_1, in the $\mathbb{C}$-algebra $(M, \circ)$.

We now recall the Terwilliger algebra. Fix any $x \in X$. For each integer $i (0 \leq i \leq D)$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy-entry

$$\left(E_i^*\right)_{yy} = \begin{cases} 1 & \text{if} \quad \partial(x, y) = i, \\ 0 & \text{if} \quad \partial(x, y) \neq i \end{cases} \qquad (y \in X).$$

We refer to E_i^* as the ith *dual idempotent of* Γ *with respect to* x. From the definition, the dual idempotents satisfy: (i) $E_i^{*t} = \overline{E_i^*} = E_i^*$ $(0 \leq i \leq D)$; (ii) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leq i, j \leq D)$; (iii) $E_0^* + E_1^* + \cdots + E_D^* = I$.

It follows that the matrices $E_0^*, \ldots, E_D^*$ form a basis for a subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra of* Γ *with respect to* x. Observe that M^* is commutative, since the dual idempotents are diagonal.

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *Terwilliger algebra of* Γ *with respect to* x. It is known that T is semisimple [17, Lemma 3.4].

4. Distance-regular graphs that support a spin model

Let Γ denote a distance-regular graph with vertex set X and diameter $D \geq 3$. Let W denote a spin model on X. Following [7, Definition 2.3] we say Γ *supports* W whenever $W \in M \subseteq N(W)$, where M denotes the Bose-Mesner algebra of Γ.

Assume Γ supports W. Then there exist complex scalars t_i $(0 \leq i \leq D)$ such that

$$W = \sum_{i=0}^{D} t_i A_i, \tag{5}$$

where $A_0, A_1, \ldots, A_D$ are the distance matrices of Γ. We refer to the scalars t_i as the *Boltzmann weights* of W. By (5) and since the entries of W are nonzero, we find $t_i \neq 0$ for $0 \leq i \leq D$. By (1) we have

$$W^{-1} = |X|^{-1} \sum_{i=0}^{D} t_i^{-1} A_i.$$

For $B \in M$, let $\Psi(B) \in \mathrm{Mat}_X(\mathbb{C})$ be the matrix with bc-entry defined by

$$B\mathbf{u}_{bc} = (\Psi(B))_{bc}\mathbf{u}_{bc} \quad (b, c \in X).$$

By [7, Lemma 2.2], there exists an ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents of Γ such that $\Psi(A_i) = |X| E_i$ $(0 \leq i \leq D)$. We refer to this as the *standard order* with respect to W. By [7, Lemma 2.10] Γ is Q-polynomial with respect to the standard order. Throughout this article, we will assume the primitive idempotents are labelled according to this standard ordering. For $0 \leq i \leq D$ let θ_i denote the eigenvalue of A associated with

E_i, so that

$$A = \sum_{i=0}^{D} \theta_i E_i. \tag{6}$$

Note that $\theta_0, \theta_1, \ldots, \theta_D$ are mutually distinct, since A generates M. Also $\theta_0 = k$ since $E_0 = |X|^{-1} J$. By [7, Lemma 2.7] we have

$$W = L \sum_{i=0}^{D} t_i^{-1} E_i, \tag{7}$$

$$W^{-1} = L^{-1} \sum_{i=0}^{D} t_i E_i. \tag{8}$$

In view of (7) we define $W^* = W^*(x)$ by

$$W^* = L \sum_{i=0}^{D} t_i^{-1} E_i^* \tag{9}$$

and observe

$$W^{*-1} = L^{-1} \sum_{i=0}^{D} t_i E_i^*. \tag{10}$$

We define $A^* = A^*(x)$ by

$$A^* = \sum_{i=0}^{D} \theta_i E_i^*. \tag{11}$$

We observe A^* generates M^* since $\theta_0, \ldots, \theta_D$ are mutually distinct.

5. The Terwilliger algebra when Γ supports a spin model

For the remainder of this article we will be concerned with distance-regular graphs which support a spin model, so we make the following definition.

Definition 5.1 Let Γ denote a distance-regular graph with vertex set X, diameter $D \geq 3$, and valency $k \geq 3$. Assume Γ supports a spin model W with Boltzmann weights $t_0, \ldots, t_D$. Write $W = \sum_{i=0}^{D} t_i A_i$ where A_i is the ith distance-matrix of Γ. To avoid degenerate situations we assume Γ is not a Hamming graph [4, p. 261] and that $t_i \notin \{t_0, -t_0\}$ for $1 \leq i \leq D$. We set $\eta_i = t_{i-1}^{-1} t_i$ $(1 \leq i \leq D)$ and let η_0, η_{D+1} denote indeterminates. We define $q := \eta_1^{-1}\eta_2$ and $\eta := \eta_1$. Note that q, η are nonzero. Let $E_0, E_1, \ldots, E_D$ denote the standard ordering of the primitive idempotents with respect to W. Fix any $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. (Where context allows, we also suppress reference to x for the individual matrices in T, e.g., $E_0^* = E_0^*(x)$, $W^* = W^*(x)$, etc). We abbreviate $V = \mathbb{C}^X$.

With reference to Definition 5.1, fix any integers $i, j (0 \leq i, j \leq D)$. By [17, Lemma 3.2] we have

$$E_i^* A E_j^* = 0 \quad \text{if } |i - j| > 1. \tag{12}$$

By [9, Theorems 4.1, 5.5] we have

$$E_i A^* E_j = 0 \quad \text{if } |i - j| > 1. \tag{13}$$

The next results describe how the matrices W and W^* are related.

Lemma 5.2 *With reference to Definition* 5.1*, we have*

$$W^* W E_i^* V \subseteq E_i V \quad (0 \leq i \leq D).$$

Proof: Let i be given. Pick any vertex $y \in X$ at distance $\partial(x, y) = i$. We show $W^* W \hat{y} \in E_i V$. Consider the vector $\mathbf{u}_{yx} \in V$ as defined in (3). We have $A\mathbf{u}_{yx} = \theta_i \mathbf{u}_{yx}$ by [8, Lemma 4.1] so $\mathbf{u}_{yx} \in E_i V$. From (5) and (9) we find $W^* W \hat{y} = L\, \mathbf{u}_{yx}$. By the above comments we have $W^* W \hat{y} \in E_i V$. The result follows. □

Theorem 5.3 *With reference to Definition* 5.1*, we have*

$$W A^* W^{-1} = W^{*-1} A W^*, \tag{14}$$
$$W^{-1} A^* W = W^* A W^{*-1}. \tag{15}$$

Proof: To prove (14) it suffices to show

$$W^* W A^* = A W^* W. \tag{16}$$

Pick any $y \in X$. We show $W^* W A^* \hat{y} = A W^* W \hat{y}$. Let $i := \partial(x, y)$. Observe $A^* \hat{y} = \theta_i \hat{y}$ by (11) so $W^* W A^* \hat{y} = \theta_i W^* W \hat{y}$. Observe $W^* W \hat{y} \in E_i V$ by Lemma 5.2 so $A W^* W \hat{y} = \theta_i W^* W \hat{y}$. From these comments we find $W^* W A^* \hat{y} = A W^* W \hat{y}$ and (16) follows. Rearranging (16) we get (14). By construction, each of A, A^*, W, W^* is symmetric. Taking the transpose in (14) we obtain (15). □

We finish this section with an aside. We will not use the following result, but we believe it is of independent interest.

Corollary 5.4 [14] *With reference to Definition* 5.1*, we have*

$$WW^*W = W^*WW^*. \tag{17}$$

Proof: We begin with an observation. Comparing (5) and (9), we find

$$W^*_{zz} = L(W_{xz})^{-1} \quad (z \in X). \tag{18}$$

Now fix $a, b \in X$. The ab-entry of the left-hand side of (17) is

$$\begin{aligned}\sum_{z \in X} W_{az} W^*_{zz} W_{zb} &= L \sum_{z \in X} W_{az}(W_{xz})^{-1} W_{zb} \qquad \text{by (18)}\\ &= L^2 W_{ab}(W_{xa})^{-1}(W_{xb})^{-1} \quad \text{by (2)}\\ &= W^*_{aa} W_{ab} W^*_{bb}, \quad \text{by (18)}\end{aligned}$$

which equals the ab-entry of the right-hand side of (17). The result follows. □

6. The parameters of Γ

Let Γ be as in Definition 5.1. In [8] Curtin and Nomura determined the intersection numbers of Γ in terms of the diameter D and the scalars q and η. In this section we recall some of the details for future use.

Lemma 6.1 [8, Theorem 1.1] *With reference to Definition* 5.1*,*

$$\eta_i = q^{i-1}\eta \quad (1 \le i \le D). \tag{19}$$

Lemma 6.2 [8, Lemma 4.3] *With reference to Definition* 5.1*, for* $0 \le i \le D$*, both*

$$c_i \eta_i^{-1} + a_i + b_i \eta_{i+1} = \eta\theta_i, \tag{20}$$

$$c_i \eta_i + a_i + b_i \eta_{i+1}^{-1} = \eta^{-1}\theta_i. \tag{21}$$

Lemma 6.3 [8, Lemma 4.5] *With reference to Definition* 5.1*, for* $1 \le i \le D$ *the expressions*

$$\frac{\eta_i - \eta_i^{-1}}{\theta_{i-1} - \theta_i}$$

are independent of i.

Theorem 6.4 [8, Theorem 1.1] *With reference to Definition* 5.1*, there exists a nonzero scalar* $h \in \mathbb{C}$ *such that*

$$\theta_i = \theta_0 + h(1 - q^i)(1 - \eta^2 q^{i-1})q^{-i} \quad (0 \leq i \leq D). \tag{22}$$

Proof: First we claim $q \neq 1$. If $q = 1$, then by [8, Lemma 5.2] Γ has the parameters of a Hamming graph. So by [11] Γ is isomorphic to a Hamming graph or an Egawa graph. But by [7, Theorem 3.8], Γ is thin and so by [16], Γ is not an Egawa graph. By assumption Γ is not a Hamming graph. Therefore $q \neq 1$. Observe $\eta^2 \neq 1$ by Definition 5.1. Now there exists $h \in \mathbb{C}$ such that (22) holds for $i = 1$. Observe $h \neq 0$ since $\theta_1 \neq \theta_0$. Observe (22) holds for $i = 0$ since in this case both sides are equal to θ_0. Line (22) holds for $2 \leq i \leq D$ by induction and Lemma 6.3. □

Corollary 6.5 *With reference to Definition* 5.1*, the following hold.*

$$q^i \neq 1 \quad (1 \leq i \leq D), \tag{23}$$
$$q^i \eta^2 \neq 1 \quad (0 \leq i \leq 2D - 2). \tag{24}$$

Proof: Recall for $0 \leq i, j \leq D$, we have $\theta_i - \theta_j \neq 0$ if $i \neq j$. Evaluating $\theta_i - \theta_j$ using (22), we obtain (23) and (24). □

Theorem 6.6 [8, Theorem 1.1] *With reference to Definition* 5.1*, the intersection numbers of* Γ *are given by*

$$b_0 = \frac{h(1 - q^D)(q + \eta^3 q^D)}{q^{D+1}(\eta - 1)}, \tag{25}$$
$$b_i = \frac{hq^{i-D}(1 - q^{D-i})(1 - \eta^2 q^{i-1})(1 + \eta^3 q^{D+i-1})}{(\eta q^i - 1)(1 - \eta^2 q^{2i-1})} \quad (1 \leq i \leq D - 1), \tag{26}$$
$$c_i = \frac{h\eta q^{i-1-D}(1 - q^i)(1 + \eta q^{D-i})(1 - \eta^2 q^{D+i-1})}{(1 - \eta^2 q^{2i-1})(1 - \eta q^{i-1})} \quad (1 \leq i \leq D - 1), \tag{27}$$
$$c_D = \frac{h\eta(1 - q^D)(\eta + 1)}{q - \eta q^D}. \tag{28}$$

We remark the denominators in (25)–(28) *are nonzero by Corollary* 6.5.

Proof: To get (25), (28) combine (4), (20), (21) at $i = D$, using (19), (22) and $\theta_0 = k$. To obtain (26), (27) first eliminate a_i in (20), (21) using (4), and then solve the resulting linear system for b_i, c_i. Evaluate the result using (19), (22) and $\theta_0 = k$. □

Corollary 6.7 *With reference to Definition* 5.1*,*

$$q^i \eta^3 \neq -1 \quad (D - 1 \leq i \leq 2D - 2). \tag{29}$$

Proof: By (25), (26) and since $b_i \neq 0$ for $0 \leq i \leq D-1$. □

Theorem 6.8 [8, Theorem 1.1] *With reference to Definition* 5.1*, the scalar h is given by*

$$h = \frac{q^D(1-\eta^2 q)(\eta - 1)}{\eta(q-1)(1-\eta^2 q^D)(1+\eta q^{D-1})}. \tag{30}$$

We remark the denominator in (30) *is nonzero by Corollary* 6.5.

Proof: In (27) set $i = 1$, $c_1 = 1$ and solve for h. □

Corollary 6.9 *With reference to Definition* 5.1*,*

$$a_i = \frac{h(q^i-1)(q^D\eta-1)(q-\eta^2 q^i)(q^D\eta^2+q)}{q^{D+1}(\eta-1)(q^i\eta-q)(q^i\eta-1)} \quad (1 \leq i \leq D-1), \tag{31}$$

$$a_D = \frac{h(q^D-1)(q-\eta^2 q^D)(q^D\eta^2+q)}{q^{D+1}(\eta-1)(q^D\eta-q)}, \tag{32}$$

where h is from (30). *We remark the denominators in* (31), (32) *are nonzero by Corollary* 6.5.

Proof: Immediate by (4) and Theorem 6.6. □

Remark 6.10 Expressions (25)–(28) can be obtained from the type I fomulas in Leonard's theorem [2, p. 260] by setting $r_1 = -\eta/q$, $r_2 = -q^{D-2}\eta^3$, $r_3 = q^{-D-1}$, and $s = s^* = r_1^2$.

Let Γ denote a distance-regular graph with diameter $D \geq 3$. Recall Γ is *almost bipartite* whenever the intersection numbers satisfy $a_i = 0$ for $0 \leq i \leq D-1$ and $a_D \neq 0$.

Lemma 6.11 *With reference to Definition* 5.1*,* Γ *is not almost bipartite.*

Proof: By way of contradiction, suppose Γ is almost bipartite. By [9, Theorems 4.1, 5.5], θ_1 has multiplicity k. By [2, Example III.4(4)] there exists a bipartite antipodal 2-cover $\tilde{\Gamma}$ of Γ with valency k and diameter $2D+1$. By [4, Proposition 4.2.3] θ_1 is an eigenvalue of $\tilde{\Gamma}$ with multiplicity k, so $\tilde{\Gamma}$ is 2-homogeneous by [15, Proposition 3.3]. Now $\tilde{\Gamma}$ has diameter at most 5 by [15, Theorem 5.1], so Γ has diameter $D \leq 2$, contradicting our assumptions. □

7. The irreducible T-modules

With reference to Definition 5.1, by [17, Lemma 3.4] the space V decomposes into an orthogonal direct sum of irreducible T-modules. Let U denote an irreducible module for T. By [7, Theorem 3.8] and since Γ supports a spin model we have $\dim E_i^* U \leq 1$ for $0 \leq i \leq D$. By the *endpoint* of U we mean $\min\{i \mid 0 \leq i \leq D, E_i^* U \neq 0\}$. By the *dual* *endpoint* of U we mean $\min\{i | 0 \leq i \leq D, E_i U \neq 0\}$. By [9, Theorems 4.1, 5.5] we have $\dim E_i^* U = \dim E_i U$ for $0 \leq i \leq D$. It follows the endpoint of U is equal to the dual endpoint of U. By the *diameter* of U we mean one less than the dimension of U. Let r (resp. d) denote the endpoint (resp. diameter) of U. It follows by [17, Lemma 3.9] that for $0 \leq i \leq D$ we have

$$\begin{aligned} E_i^* U \neq 0 \quad &\text{if and only if} \quad r \leq i \leq r + d, \\ E_i U \neq 0 \quad &\text{if and only if} \quad r \leq i \leq r + d. \end{aligned}$$

By [5, Lemma 7.1] and since $r + d \leq D$ we have

$$(D - d)/2 \leq r \leq D - d. \tag{33}$$

Now let U denote an irreducible T-module with endpoint r and diameter d. We consider the action of A and A^* on U. Let u denote a nonzero vector in $E_r U$. By [17, Lemma 3.9] the sequence $E_{r+i}^* u$ $(0 \leq i \leq d)$ is a basis for U. Let $B = B(U)$ denote the matrix which represents the action of A with respect to this basis. By [17, Lemma 3.9] B is irreducible tridiagonal. We write

$$B(U) = \begin{pmatrix} a_0(U) & b_0(U) & & & \mathbf{0} \\ c_1(U) & a_1(U) & b_1(U) & & \\ & c_2(U) & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & b_{d-1}(U) \\ \mathbf{0} & & & c_d(U) & a_d(U) \end{pmatrix}. \tag{34}$$

We refer to the $c_i(U), a_i(U), b_i(U)$ as the *intersection numbers* of U. For notational convenience we define $c_0(U) = 0$, $b_d(U) = 0$. We will compute the scalars $a_i(U), b_i(U), c_i(U)$ in the next section. Note that since $B(U)$ is irreducible,

$$b_i(U) c_{i+1}(U) \neq 0 \quad (0 \leq i \leq d - 1). \tag{35}$$

We now consider a second basis for U. Let u^* denote a nonzero vector in $E_r^* U$. By [9, Theorems 4.1, 5.5] the sequence $E_{r+i} u^*$ $(0 \leq i \leq d)$ is a basis for U. Moreover the matrix $B(U)$ represents the action of A^* with respect to this basis.

There exists a unique irreducible T-module V_0 which has endpoint 0. The module V_0 has diameter D. We call V_0 the *primary T-module*. We remark that $a_i(V_0) = a_i$, $b_i(V_0) = b_i$, $c_i(V_0) = c_i$ for $0 \leq i \leq D$.

8. The scalars $a_i(U), b_i(U), c_i(U)$

With reference to Definition 5.1, let U denote an irreducible T-module. In this section we compute the scalars $a_i(U), b_i(U), c_i(U)$.

Lemma 8.1 *With reference to Definition* 5.1*, let U denote an irreducible T-module with endpoint r and diameter d. Then*

$$c_i(U) + a_i(U) + b_i(U) = \theta_r \quad (0 \le i \le d). \tag{36}$$

Proof: For nonzero $u \in E_r U$ we have $u = \sum_{i=0}^{d} E^*_{r+i} u$ and $Au = \theta_r u$. Combining these facts we get the result. □

Theorem 8.2 *With reference to Definition* 5.1*, let U denote an irreducible T-module with endpoint r and diameter d. Assume $d \ge 1$. Then for $0 \le i \le d$ we have*

$$c_i(U)\eta^{-1}_{r+i} + a_i(U) + b_i(U)\eta_{r+i+1} = a_0(U)(1 - \eta_{r+1}) + \theta_{r+i}\eta_{r+1}, \tag{37}$$
$$c_i(U)\eta_{r+i} + a_i(U) + b_i(U)\eta^{-1}_{r+i+1} = a_0(U)(1 - \eta^{-1}_{r+1}) + \theta_{r+i}\eta^{-1}_{r+1}. \tag{38}$$

Proof: To obtain (37), we use (15). Let u denote a nonzero vector in $E_r U$. First we claim

$$E^*_{r+i} E_{r+1} A^* u = (\theta_{r+i} - a_0(U)) E^*_{r+i} u \quad (0 \le i \le d). \tag{39}$$

To see (39), let i be given and observe

$$\begin{aligned}
\theta_{r+i} E^*_{r+i} u &= E^*_{r+i} A^* E_r u \quad \text{by (11)}\\
&= E^*_{r+i}\left(\sum_{i=0}^{D} E_i\right) A^* E_r u \quad \text{since } \sum_{i=0}^{D} E_i = I\\
&= E^*_{r+i}(E_r A^* E_r u + E_{r+1} A^* E_r u) \quad \text{by (13)}\\
&= a_0(U) E^*_{r+i} u + E^*_{r+i} E_{r+1} A^* u \quad \text{by (34),}
\end{aligned}$$

and (39) follows. Now we consider $E^*_{r+i} W^{-1} A^* W u$. On the one hand,

$$\begin{aligned}
E^*_{r+i} W^{-1} A^* W u &= E^*_{r+i} W^{-1} A^* t_r^{-1} E_r u && \text{by (7)}\\
&= t_r^{-1} E^*_{r+i}(t_r E_r A^* E_r u + t_{r+1} E_{r+1} A^* E_r u) && \text{by (8), (13)}\\
&= a_0(U) E^*_{r+i} u + \eta_{r+1} E^*_{r+i} E_{r+1} A^* u && \text{by (34)}\\
&= a_0(U) E^*_{r+i} u + \eta_{r+1}(\theta_{r+i} - a_0(U)) E^*_{r+i} u && \text{by (39).}
\end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned}
E^*_{r+i} W^{-1} A^* W u &= E^*_{r+i} W^* A W^{*-1} u \quad \text{by (15)}\\
&= L\, t^{-1}_{r+i} E^*_{r+i} A W^{*-1} u \quad \text{by (9)}\\
&= \big(\eta^{-1}_{r+i} E^*_{r+i} A E^*_{r+i-1} + E^*_{r+i} A E^*_{r+i}\\
&\quad + \eta_{r+i+1} E^*_{r+i} A E^*_{r+i+1}\big) u \quad \text{by (10), (12)}\\
&= \big(\eta^{-1}_{r+i} c_i(U) + a_i(U) + \eta_{r+i+1} b_i(U)\big) E^*_{r+i} u \quad \text{by (34).}
\end{aligned} \tag{41}$$

Equating (40), (41) and using $E^*_{r+i}u \neq 0$ we obtain (37). Line (38) is similarly obtained using (14). □

Remark 8.3 Referring to Theorem 8.2, suppose $U = V_0$. Setting $r = 0, d = D, a_0(U) = 0$ in (37), (38) we get (20), (21) respectively.

Lemma 8.4 *With reference to Definition* 5.1*, let U denote an irreducible T-module with endpoint r and diameter d. For $d = 0$ we have $a_0(U) = \theta_r$. For $d \geq 1$ we have*

$$a_0(U) = \frac{\theta_{r+d}\big(1 + \eta_{r+d}\eta^2_{r+1}\big) - \theta_r\eta_{r+1}(1 + \eta_{r+d})}{(1 - \eta_{r+1})(1 - \eta_{r+1}\eta_{r+d})}. \tag{42}$$

We remark the denominator in (42) *is nonzero by Lemma* 6.1 *and Corollary* 6.5.

Proof: Assume $d \geq 1$; otherwise the result is clear by (36). Now set $i = d$ in (37), (38) and evaluate the result using $b_d(U) = 0$, $a_d(U) = \theta_r - c_d(U)$. □

Theorem 8.5 *With reference to Definition* 5.1*, let U denote an irreducible T-module with endpoint r and diameter d. Then*

$$b_0(U) = \frac{h(1 - q^d)(q + \eta^3 q^{d+3r})}{q^{d+r+1}(\eta q^r - 1)}, \tag{43}$$

$$b_i(U) = \frac{hq^{i-r-d}(1 - q^{d-i})(1 - \eta^2 q^{2r+i-1})(1 + \eta^3 q^{d+3r+i-1})}{(\eta q^{r+i} - 1)(1 - \eta^2 q^{2r+2i-1})} \quad (1 \leq i \leq d-1), \tag{44}$$

$$c_i(U) = \frac{h\eta q^{i-1-d}(1 - q^i)(1 + \eta q^{r+d-i})(1 - \eta^2 q^{d+2r+i-1})}{(1 - \eta^2 q^{2r+2i-1})(1 - \eta q^{r+i-1})} \quad (1 \leq i \leq d-1), \tag{45}$$

$$c_d(U) = \frac{h\eta(1 - q^d)(1 + \eta q^r)}{q - \eta q^{r+d}}, \tag{46}$$

where h is from (30). *We remark the denominators in* (43)–(46) *are nonzero by Corollary 6.5.*

Proof: To see (43), first assume $d = 0$. Then the right-hand side is apparently zero and the left-hand side also vanishes since $b_d(U) = 0$. Next assume $d \geq 1$. Set $i = 0$ in (36), (37) and solve for $b_0(U)$ using $c_0(U) = 0$. We now have (43). Line (46) is obtained similarly.

It remains to prove (44) and (45). Assume $d \geq 2$ and fix any i $(1 \leq i \leq d-1)$. In (37) and (38), eliminate $a_i(U)$ on the left using (36) and eliminate $a_0(U)$ on the right using Lemma 8.4. Solve the resulting system for $b_i(U)$, $c_i(U)$. □

Remark 8.6 Expressions (43)–(46) can be obtained from the type I fomulas in Leonard's theorem [17, Theorem 2.1] by setting $r_1(U) = -\eta q^{r-1}$, $r_2(U) = -q^{d-2}\eta^3$, $r_3(U) = q^{-d-1}$, and $s(U) = s^*(U) = r_1(U)^2$.

Remark 8.7 Referring to Theorem 8.5, suppose $U = V_0$. Setting $r = 0$, $d = D$ in (43)–(46) we get (25)–(28) respectively.

Corollary 8.8 *With reference to Definition* 5.1, *let U, U' denote irreducible T-modules with respective endpoints r, r' and diameters d, d'. Then the following are equivalent.*

(i) *U and U' are isomorphic as T-modules.*
(ii) *$r = r'$ and $d = d'$.*
(iii) *$B(U) = B(U')$.*

Proof: (i)$\rightarrow$(ii). Let $\phi : U \rightarrow U'$ denote an isomorphism of T-modules. We recall ϕ is a vector space isomorphism such that $(C\phi - \phi C)U = 0$ for all $C \in T$. For $0 \leq i \leq D$ we have

$$E_i^* U = 0 \Leftrightarrow \phi(E_i^* U) = 0 \Leftrightarrow E_i^* \phi(U) = 0 \Leftrightarrow E_i^* U' = 0.$$

The result follows.
(ii)$\rightarrow$(iii). Observe that the entries in $B(U)$ and $B(U')$ agree by Theorem 8.5.
(iii)$\rightarrow$(i). Note $B(U)$ is a $d + 1$ by $d + 1$ matrix and $B(U')$ is $d' + 1$ by $d' + 1$, so $d = d'$. By (36), the row sums of $B(U)$ equal θ_r and the row sums of $B(U')$ equal $\theta_{r'}$. Hence $r = r'$, since the eigenvalues are distinct. Pick any nonzero $u \in E_r U$ and any nonzero $u' \in E_r U'$. Recall $E_r^* u, \ldots, E_{r+d}^* u$ is a basis for U and $E_r^* u', \ldots, E_{r+d}^* u'$ is a basis for U'. Moreover, $B(U) = B(U')$ is the matrix representing the action of A with respect to each of these bases. Let $\phi : U \rightarrow U'$ denote the vector space isomorphism such that

$$\phi : E_i^* u \mapsto E_i^* u' \quad (r \leq i \leq r + d).$$

We show ϕ is an isomorphism of T-modules. Since $A, E_0^*, E_1^*, \ldots, E_D^*$ generate T, it suffices to show

$$(A\phi - \phi A)U = 0, \tag{47}$$
$$(E_j^* \phi - \phi E_j^*)U = 0 \quad (0 \leq j \leq D). \tag{48}$$

Line (48) is clear from the construction, and (47) is immediate from the assumption $B(U) = B(U')$. □

9. The parameter q is in $\mathbb{R}$

In this section we show the scalar q from Definition 5.1 is contained in $\mathbb{R}$. Let us first consider the case when Γ is bipartite.

Lemma 9.1 *With reference to Definition* 5.1, *the following (i)–(iii) are equivalent.*

(i) $a_1 = 0$.
(ii) $q^{D-1}\eta^2 = -1$.
(iii) Γ *is bipartite.*

Proof: (i)→(ii). By way of contradiction, suppose $q^{D-1}\eta^2 \neq -1$. Evaluating the expression for a_1 given in (31) using Corollary 6.5, we find that $q^D\eta = 1$. So by Corollaries 6.5 and 6.9, $a_i = 0$ for $1 \leq i \leq D-1$ and $a_D \neq 0$. So Γ is almost bipartite, contradicting Lemma 6.11.

(ii)→(iii). Observe $a_i = 0$ for $0 \leq i \leq D$ by Corollary 6.9, so Γ is bipartite.
(iii)→(i). Clear. □

Lemma 9.2 *With reference to Definition* 5.1, *suppose* Γ *is bipartite. Then* $q \in \mathbb{R}$.

Proof: By [9, Theorems 4.1, 5.5], θ_1 has multiplicity k, so Γ is 2-homogeneous by [18, Theorem 1]. Now $q \in \mathbb{R}$ by [6, Corollary 8.3]. □

We now consider the case when Γ is not bipartite.

Lemma 9.3 *With reference to Definition* 5.1, *suppose* Γ *is not bipartite. Then the following hold.*

(i) $q^D\eta \neq 1$.
(ii) $p^i_{2,i-1} \neq 0$ *for* $2 \leq i \leq D$.

Proof: (i). By way of contradiction, suppose $q^D\eta = 1$. Then by (31), $a_i = 0$ for $1 \leq i \leq D-1$. Since Γ is not bipartite, $a_D \neq 0$ and Γ is almost bipartite, contradicting Lemma 6.11.
(ii). By [4, Lemma 4.1.7] we have

$$p^i_{2,i-1} = \frac{c_i}{c_2}(a_i + a_{i-1} - a_1) \quad (2 \leq i \leq D). \tag{49}$$

Evaluating (49) using (4) and Theorem 6.6, we have

$$p^i_{2,i-1} = \frac{c_i(1-q^D\eta)(q^i-q)(1-\eta^2q^3)(q-\eta^2q^i)(q+\eta^2q^D)}{\eta(1-\eta q^i)(q^2-\eta q^i)(1-q)(1-\eta^2q^{D+1})(q^2+\eta q^D)} \tag{50}$$

for $2 \leq i \leq D-1$ and

$$p^D_{2,D-1} = \frac{c_D(q^D-q)(1-\eta^2q^3)(q-\eta^2q^D)(q+\eta^2q^D)}{\eta(q^2-\eta q^D)(1-q)(1-\eta^2q^{D+1})(q^2+\eta q^D)}. \tag{51}$$

By (23), (24), Lemma 9.1, and part (i), no factor in the numerator of (50), (51) is zero. The result follows. □

Lemma 9.4 *With reference to Definition* 5.1*, suppose* Γ *is not bipartite. Fix an integer* i *with* $2 \leq i \leq D$ *and vertices* $y, z \in X$ *with* $\partial(x, y) = i - 1$, $\partial(x, z) = i$, *and* $\partial(y, z) = 2$. (*Such* y, z *exist by Lemma* 9.3(ii).) *Define* L_i, H_i *by*

$$L_i = |\Gamma_{i-1}(x) \cap \Gamma(y) \cap \Gamma(z)|, \tag{52}$$

$$H_i = |\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)|. \tag{53}$$

Then the following hold.

(i)

$$c_2 = L_i + H_i. \tag{54}$$

(ii)

$$L_i = c_2 \frac{(q^i\eta - 1)q}{(q^i\eta - q)(q+1)}, \qquad H_i = c_2 \frac{(q^i\eta - q^2)}{(q^i\eta - q)(q+1)}. \tag{55}$$

In particular, L_i *and* H_i *are constants independent of* x, y, z. *We remark the denominators in* (55) *are nonzero by Corollary* 6.5.

Proof: (i) Observe $L_i + H_i = |\Gamma(y) \cap \Gamma(z)| = c_2$.
(ii) First we claim

$$E^*_{i-1}AE^*_{i-1}AE^*_i = \xi_i E^*_{i-1}AE^*_iAE^*_i + \zeta_i E^*_{i-1}AE^*_i, \tag{56}$$

where

$$\xi_i = \frac{q^i\eta - 1}{q^{i-1}\eta - q}, \qquad \zeta_i = \frac{(q^D\eta - 1)(q\eta^2 - 1)(q^D\eta^2 + q)(q^{i-1}\eta + 1)}{\eta(q^{i-1}\eta - q)(q^D\eta + q)(q^D\eta^2 - 1)}.$$

To see this, let i be given. Let Δ denote the left-hand side of (56) minus the right-hand side of (56). We show $\Delta = 0$. In order to do so we show Δ vanishes on each irreducible T-module. Let U denote an irreducible T-module, and let r (resp. d) denote the endpoint (resp. diameter) of U. We assume $r < i \leq r + d$; otherwise $\Delta U = 0$ from the construction. Applying Theorem 8.5 we find

$$a_{i-r-1}(U)b_{i-r-1}(U) = \xi_i a_{i-r}(U)b_{i-r-1}(U) + \zeta_i b_{i-r-1}(U)$$

and it follows that $\Delta U = 0$. Recall V is a direct sum of irreducible T-modules. The element Δ vanishes on each of these modules so $\Delta = 0$. We now have (56).

We now consider the yz-entry of each term in (56). The yz-entry of $E^*_{i-1}AE^*_{i-1}AE^*_i$ is L_i. The yz-entry of $E^*_{i-1}AE^*_iAE^*_i$ is H_i. The yz-entry of $E^*_{i-1}AE^*_i$ is 0. Evaluating the

yz-entry in (56) using this we find

$$L_i = \xi_i H_i. \tag{57}$$

Observe $\xi_i \neq -1$ by (24) and Lemma 9.3(i). Solving (54), (57) using this we routinely obtain (55). □

Lemma 9.5 *With reference to Definition* 5.1*, suppose* Γ *is not bipartite. Let* L_i*,* H_i *be as in Lemma 9.4. Then the following hold.*

(i) $L_i \neq 0$ *and* $H_i \neq 0$ *for* $2 \leq i \leq D$.
(ii) $L_2 \leq L_3$.

Proof: (i) Clear by (24), Lemma 9.3(i), and (55).
(ii) By Lemma 9.3(ii) there exist $y, z \in X$ with $\partial(x, y) = 2$, $\partial(x, z) = 3$, and $\partial(y, z) = 2$. Since $L_3 \neq 0$, there exists a vertex $u \in \Gamma_2(x) \cap \Gamma(y) \cap \Gamma(z)$. Observe that any vertex w adjacent to both x and u has $\partial(z, w) = 2$, since $\partial(x, z) = 3$ and $\partial(u, z) = 1$. Therefore

$$L_2 = |\Gamma(x) \cap \Gamma(y) \cap \Gamma(u)| \leq |\Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)| = L_3,$$

as desired. □

Lemma 9.6 *With reference to Definition* 5.1*, suppose* Γ *is not bipartite. Let* $\beta = q + q^{-1}$ *and let* L_i*,* H_i *be as in Lemma* 9.4*. Then the following hold.*

(i) $(\beta + 2)L_2 H_3 = c_2^2$.
(ii) $\beta \in \mathbb{Q}$.
(iii) $\beta \geq 2$.
(iv) $q \in \mathbb{R}$ *and* $q > 0$.

Proof: (i) Compare L_2/c_2 and H_3/c_2 using (55).
(ii) Immediate by (i) and Lemma 9.5(i), since c_2, L_2, and H_3 are integers.
(iii) By (i), Lemma 9.4(i), and Lemma 9.5(ii),

$$\beta + 2 = \frac{c_2^2}{L_2 H_3} = \frac{c_2^2}{L_2(c_2 - L_3)} \geq \frac{c_2^2}{L_2(c_2 - L_2)} \geq 4.$$

(iv) Solving $\beta = q + q^{-1}$ for q using (iii), we find $q \in \mathbb{R}$ and $q > 0$. □

We finish this section with a comment.

Lemma 9.7 *With reference to Definition* 5.1*, suppose* Γ *is not bipartite. Then* $\eta \in \mathbb{R}$.

Proof: Let L_2 be as in Lemma 9.4 and define $\alpha = (q + 1)L_2/c_2 - q$. By Lemma 9.6(iv) we have $\alpha \in \mathbb{R}$. Evaluating α using Lemma 9.4 we obtain

$$\alpha = \frac{q - 1}{q\eta - 1}. \tag{58}$$

Recall $q \neq 1$ by (23) so $\alpha \neq 0$. Solving (58) for η we find $\eta = (\alpha + q - 1)/\alpha q$, so $\eta \in \mathbb{R}$ as desired. $\square$

10. Multiplicities of the irreducible T-modules

Let Γ be as in Definition 5.1. In this section, we compute the multiplicities with which the irreducible T-modules appear in the standard module V.

Definition 10.1 With reference to Definition 5.1, fix a decomposition of the standard module V into an orthogonal direct sum of irreducible T-modules. For any integers $r, d (0 \leq r, d \leq D)$, we define $\mathrm{mult}(r, d)$ to be the number of irreducible modules in this decomposition which have endpoint r and diameter d. The scalar $\mathrm{mult}(r, d)$ is independent of the decomposition (cf. [10]).

Definition 10.2 With reference to Definition 5.1, define a set Υ by

$$\Upsilon := \{(r, d) \in \mathbb{Z}^2 \mid 0 \leq d \leq D, (D - d)/2 \leq r \leq D - d\}.$$

By (33), $\mathrm{mult}(r, d) = 0$ for all integers r, d such that $(r, d) \notin \Upsilon$. We define a partial order $\preceq$ on Υ by

$$(r, d) \preceq (r', d') \quad \text{if and only if} \quad r \leq r' \quad \text{and} \quad r' + d' \leq r + d.$$

Example 10.3 With reference to Definition 5.1, suppose $D = 7$. In Figure 10.4, we represent each element $(r, d) \in \Upsilon$ by a line segment beginning in column r and having length d. For any elements $a \in \Upsilon, b \in \Upsilon$, observe that $a \preceq b$ if and only if the line segment representing a extends the line segment representing b.

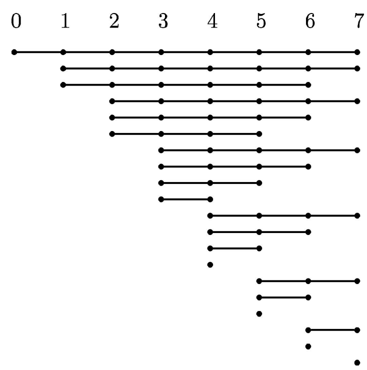

Figure 10.4 The set Υ when $D = 7$

In order to state our next theorem we need a bit of notation. With reference to Definition 5.1, for all $(r, d) \in \Upsilon$ we define

$$b_0(r,d) := \frac{h(1-q^d)(q+\eta^3 q^{d+3r})}{q^{d+r+1}(\eta q^r - 1)}, \tag{59}$$

$$b_i(r,d) := \frac{hq^{i-r-d}(1-q^{d-i})(1-\eta^2 q^{2r+i-1})(1+\eta^3 q^{d+3r+i-1})}{(\eta q^{r+i}-1)(1-\eta^2 q^{2r+2i-1})} \quad (1 \le i \le d-1), \tag{60}$$

$$c_i(r,d) := \frac{h\eta q^{i-1-d}(1-q^i)(1+\eta q^{r+d-i})(1-\eta^2 q^{d+2r+i-1})}{(1-\eta^2 q^{2r+2i-1})(1-\eta q^{r+i-1})} \quad (1 \le i \le d-1), \tag{61}$$

$$c_d(r,d) := \frac{h\eta(1-q^d)(1+\eta q^r)}{q-\eta q^{r+d}}, \tag{62}$$

where h is from (30). We also set $c_0(r,d) = b_d(r,d) = 0$. Observe that if U is any irreducible T-module with endpoint r and diameter d, then $(r,d) \in \Upsilon$ and $c_i(r,d) = c_i(U)$, $b_i(r,d) = b_i(U)$ $(0 \le i \le d)$. However, such a module need not exist.

We will use the fact that the scalars $b_i(r,d)$ and $c_i(r,d)$ from (59)–(62) are nonzero for every $(r,d) \in \Upsilon$. To establish this we need the following lemma.

Lemma 10.5 *With reference to Definition* 5.1*,*

$$q^i \eta^3 \ne -1 \quad (2D-1 \le i \le 3D-3). \tag{63}$$

Proof: By way of contradiction, let i be given such that $2D-1 \le i \le 3D-3$ and suppose

$$q^i \eta^3 = -1. \tag{64}$$

First suppose $a_1 = 0$. Then by Lemma 9.1 we have $q^{D-1}\eta^2 = -1$; combining this with (64) we find $q^{2i-3D+3} = -1$. Since q is real we must have $q = -1$, contradicting (23).

Now suppose $a_1 \ne 0$. Then $q, \eta \in \mathbb{R}$ and $q > 0$ by Lemmas 9.6 and 9.7. Recall $q \ne 1$ by (23). So either $q > 1$ or $0 < q < 1$. First consider the case $q > 1$. By (64) and since $2D-1 \le i \le 3D-3$ we have

$$-q^{-(2D-1)/3} \le \eta \le -q^{1-D}, \tag{65}$$

where $q^{-(2D-1)/3}$ denotes the real cube root of $q^{-(2D-1)}$. By (25), (30) we find

$$b_0 = \frac{(q^D-1)(1-\eta^2 q)(1+\eta^3 q^{D-1})}{\eta(1-q)(1+\eta q^{D-1})(1-\eta^2 q^D)}. \tag{66}$$

By (65) and since $q > 1$, each of the factors in the numerator of (66) is positive. By (24), (65) we have $1 + \eta q^{D-1} < 0$ so each of the first three factors in the denominator of (66) is negative. Similarly $1 - \eta^2 q^D > 0$. We conclude $b_0 < 0$, which is impossible. Now consider the case $0 < q < 1$. By (64) and since $2D - 1 \leq i \leq 3D - 3$ we have

$$-q^{1-D} \leq \eta \leq -q^{-(2D-1)/3}. \tag{67}$$

Again consider the expression for b_0 given in (66). By (67) and since $0 < q < 1$, each of the factors in the numerator is negative. In the denominator, clearly $\eta < 0$ and $1 - q > 0$. By (24), (67) we have $1 + \eta q^{D-1} > 0$ and $1 - \eta^2 q^D < 0$. We conclude $b_0 < 0$, which is impossible. □

Corollary 10.6 *With reference to Definition* 5.1*, fix any* $(r, d) \in \Upsilon$*. Then we have*

$$b_i(r, d) \neq 0 \quad 0 \leq i \leq d - 1, \tag{68}$$

$$c_i(r, d) \neq 0 \quad 1 \leq i \leq d. \tag{69}$$

Proof: Assume $d \geq 1$; otherwise there is nothing to prove. By the definition of Υ we have $(D - d)/2 \leq r \leq D - d$. In particular $r \leq D - 1$. Now consider the numerators in the expressions for $b_i(r, d)$ and $c_i(r, d)$ given in (59)–(62). Each of the factors in these numerators is nonzero by (23), (24), (29), and (63). □

The following theorem provides us with a recurrence relation satisfied by the multiplicities. In [5], this result was proved under the assumption that the graph is bipartite. However, the proof carries over without modification to the nonbipartite case.

Theorem 10.7 [5, Theorem 14.7] *With reference to Definition* 5.1*, fix any* $(r, d) \in \Upsilon$*. Then*

$$k_r \prod_{h=r}^{r+d-1} b_h c_{r+d-h} = \sum_{\substack{(i,j)\in\Upsilon \\ (i,j)\preceq(r,d)}} \mathrm{mult}(i, j) \prod_{h=r-i}^{r-i+d-1} b_h(i, j) c_{h+1}(i, j). \tag{70}$$

Remark 10.8 With reference to Definition 5.1, we can use Theorem 10.7 to recursively compute the multiplicities $\mathrm{mult}(r, d)$ for each $(r, d) \in \Upsilon$. Indeed, pick any $(r, d) \in \Upsilon$. Then (70) gives a linear equation in the variables $\{\mathrm{mult}(i, j) \mid (i, j) \in \Upsilon, (i, j) \preceq (r, d)\}$. In this equation, the coefficient of $\mathrm{mult}(r, d)$ is

$$\prod_{h=0}^{d-1} b_h(r, d) c_{h+1}(r, d). \tag{71}$$

By Corollary 10.6 the coefficient (71) is nonzero. Thus we can divide both sides of equation (70) by it to obtain $\mathrm{mult}(r, d)$ in terms of $\{\mathrm{mult}(i, j) \mid (i, j) \prec (r, d)\}$.

Theorem 10.9 *With reference to Definition* 5.1*, the following hold.*

(i) $\mathrm{mult}(0, D) = 1$.

(ii) $\mathrm{mult}(1, D-1) = -\dfrac{(\eta+1)(q^D-1)(q^{D-1}\eta^2+1)(q^D\eta^3+1)}{\eta(q-1)(q^{D-1}\eta+1)(q^{2D-1}\eta^3+1)}$.

(iii) $\mathrm{mult}(1, D-2) = \dfrac{q(\eta+1)(q^{D-1}-1)(q^D\eta-1)(q^{D-1}\eta^3+1)}{(q-1)(q^D\eta^2-1)(q^{2D-1}\eta^3+1)}$.

(iv) $\mathrm{mult}(2, D-2)$

$$= \frac{(\eta+1)(q^D-1)(q^{D-1}-1)(q^{2D-1}\eta^4-1)(q\eta+1)(q^{D-1}\eta^2+1)(q^{D-1}\eta^3+1)(q^{D+2}\eta^3+1)}{\eta^2(q^2-1)(q-1)(q^D\eta^2-1)(q^{D-1}\eta+1)(q^{D-2}\eta+1)(q^{2D-1}\eta^3+1)(q^{2D}\eta^3+1)}.$$

(v) $\mathrm{mult}(2, D-3)$

$$= -\frac{(\eta+q)(q^D-1)(q^{D-2}-1)(q^D\eta-1)(q\eta+1)(q^{D-1}\eta^2+1)(q^{D-1}\eta^3+1)(q^{D+1}\eta^3+1)}{\eta(q-1)^2(q^{D+1}\eta^2-1)(q^{D-2}\eta+1)(q^{2D-2}\eta^3+1)(q^{2D}\eta^3+1)}.$$

(vi) *Suppose* $D \geq 4$. *Then* $\mathrm{mult}(2, D-4)$

$$= \frac{q^2(\eta+1)(q^D-1)(q^{D-3}-1)(q^D\eta-1)(q\eta+1)(q^{2D-1}\eta^2-1)(q^{D-1}\eta^3+1)(q^D\eta^3+1)}{(q-1)(q^2-1)(q^{D+1}\eta^2-1)(q^{D-1}\eta+1)(q^{2D-2}\eta^3+1)(q^{2D-1}\eta^3+1)(q^D\eta^2-1)}.$$

(vii) $\mathrm{mult}(3, D-3)$

$$\begin{aligned}= &-\frac{(q^D-1)(q^{D-1}-1)(q^{D-2}-1)(q^{2D-1}\eta^4-1)}{(q-1)(q^2-1)(q^3-1)(q^{D+1}\eta^2-1)}\\ &\times\frac{(\eta+1)(q\eta+1)(q^2\eta+1)(q^D\eta^2+1)(q^{D-1}\eta^2+1)(q^D\eta^3+1)(q^{D+4}\eta^3+1)(q^{D-1}\eta^3+1)}{\eta^3(q^{D-1}\eta+1)(q^{D-2}\eta+1)(q^{D-3}\eta+1)(q^{2D-1}\eta^3+1)(q^{2D}\eta^3+1)(q^{2D+1}\eta^3+1)}.\end{aligned}$$

(viii) *Suppose* $D \geq 4$. *Then* $\mathrm{mult}(3, D-4)$

$$\begin{aligned}= &\frac{(q^D-1)(q^D\eta-1)(q^{D-1}-1)(q^{D-3}-1)(q^{2D-1}\eta^4-1)}{(q^D\eta^2-1)(q^{D+2}\eta^2-1)(q-1)^2(q^2-1)}\\ &\times\frac{(\eta+1)(q^D\eta^3+1)(\eta+q)(q^2\eta+1)(q^{D-1}\eta^3+1)(q^{D+3}\eta^3+1)(q^{D-1}\eta^2+1)}{\eta^2(q^{2D-1}\eta^3+1)(q^{2D-2}\eta^3+1)(q^{2D+1}\eta^3+1)(q^{D-3}\eta+1)(q^{D-2}\eta+1)}\end{aligned}$$

(ix) *Suppose* $D \geq 5$. *Then* $\mathrm{mult}(3, D-5)$

$$\begin{aligned}= &-\frac{q(q^D-1)(q^{D-1}-1)(q^{D-4}-1)(q^D\eta-1)(q^{2D-1}\eta^2-1)}{(q-1)^2(q^2-1)(q^{D+1}\eta^2-1)(q^{D+2}\eta^2-1)}\\ &\times\frac{(\eta+1)(q+\eta)(q^2\eta+1)(q^{D-1}\eta^2+1)(q^{D-1}\eta^3+1)(q^D\eta^3+1)(q^{D+2}\eta^3+1)}{\eta(q^{D-1}\eta+1)(q^{D-3}\eta+1)(q^{2D}\eta^3+1)(q^{2D-1}\eta^3+1)(q^{2D-3}\eta^3+1)}.\end{aligned}$$

(x) *Suppose* $D \geq 6$. *Then* mult$(3, D-6)$

$$= \frac{q^3(q^D-1)(q^{D-1}-1)(q^{D-5}-1)(q^D\eta-1)(q^{D-1}\eta-1)(q^{2D-1}\eta^2-1)}{(q-1)(q^2-1)(q^3-1)(q^D\eta^2-1)(q^{D+1}\eta^2-1)(q^{D+2}\eta^2-1)} \times \frac{(\eta+1)(q\eta+1)(q^2\eta+1)(q^{D-1}\eta^3+1)(q^D\eta^3+1)(q^{D+1}\eta^3+1)}{(q^{D-2}\eta+1)(q^{2D-1}\eta^3+1)(q^{2D-2}\eta^3+1)(q^{2D-3}\eta^3+1)}.$$

We remark the denominators in (i)–(x) *are nonzero by Corollaries* 6.5, 6.7, *and* (63).

Proof: Solve (70) recursively for the multiplicities, as outlined in Remark 10.8. □

We end this article with a conjecture. To state the conjecture, we use the following notation. For any integer $n \geq 0$ and any $a, q \in \mathbb{C}$, set

$$(a; q)_n := \prod_{i=1}^{n}(1 - aq^{i-1}).$$

Conjecture 10.10 With reference to Definition 5.1, pick any $(r, d) \in \Upsilon$. Then

mult(r, d)

$$= \frac{(q^{d+1}-1)(q^{2r+d-1}\eta^2-1)(q^{d+3r-2}\eta^3+1)(q^{2d+3r-1}\eta^3+1)(q^{r+d}\eta+1)(q^{r-1}\eta+1)}{(1-q^{D+1})(1+\eta^3q^{D+2r+d-1})(1+\eta^3q^{D+r-2})(1+\eta q^{t-1})(1+\eta q^D)(1-\eta^2q^{D+r-1})} \times \frac{q^t(-\eta^3q^{D-1};q)_r(-\eta q^{-1};q)_{r-t}(-\eta;q)_t(q^{D-r+2};q)_r(\eta^4q^{2D-2};q)_{r-t}(\eta^2q^{2D-t+1};q)_t}{(-\eta)^{r-t}(q;q)_{r-t}(q;q)_t(-\eta^3q^{2D-t-1};q)_r(\eta^2q^{D-1};q)_r(-\eta q^{D-r};q)_r}$$

where $t = D - r - d$. We remark the denominator is nonzero by Corollaries 6.5, 6.7, and (63).

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