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# A Formula for $N$-Row Macdonald Polynomials

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**Abstract.** We derive a formula for the $n$-row Macdonald polynomials with the coefficients presented both combinatorically and in terms of very-well-poised hypergeometric series.

**Keywords:** Macdonald polynomials, symmetric functions, hypergeometric series

## 1. Introduction

Denote the ring of symmetric functions over the field $F$ as $\Lambda_F$ and let $\Lambda_F^n$ denote its $n$th graded space. The space $\Lambda_F^n$ consists of all symmetric functions of total degree $n \in Z$, indexed by the partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ for which $\sum_i \lambda_i = n$. There are four $Z$-bases and one $Q$-basis of $\Lambda^n$. The $Q$-basis consists of the power sum symmetric functions $p_n = \sum_i x_i^n$, where $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$, and the four $Z$-bases are: The monomial symmetric functions $m_\lambda = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\lambda_1} \ldots x_{i_k}^{\lambda_k}$, the elementary symmetric functions $e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \ldots x_{i_n}$, where $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$, the complete symmetric functions $h_\lambda = \sum_{i_1 \leq \ldots \leq i_k} x_{i_1}^{\lambda_1} \ldots x_{i_k}^{\lambda_k}$, and the Schur functions $s_\lambda(x_1, \ldots, x_k) = \det(x_i^{\lambda_j + k - j})_{1 \leq i, j \leq k} / \det(x_i^{k-j})$.

Let $H = Q(q, t)$ be the field of rational functions in $q$ and $t$. In 1988, Macdonald introduced a new class of two-parameter symmetric functions $P_\lambda(q, t)$, over the ring $\Lambda_H$, which generalize several classes of symmetric functions. In particular, taking $q = t$ we obtain the Schur functions, setting $t = 1$ we have the monomial symmetric functions, and letting $q = 0$ gives the Hall-Littlewood functions.

We know from [4] that the $(P_\lambda)$ are a basis of $\Lambda_H^n$. Further, with respect to the scalar product:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \prod_i i^{m_i} m_i! \prod_{j=1}^{l(\lambda)} \frac{1 - q^{\lambda_j}}{1 - t^{\lambda_j}}$$

we have that

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \text{if} \quad \lambda \neq \mu,$$

where $m_i$ denotes the multiplicity of $i$ in $\lambda$ and $l(\lambda)$ denote the length of $\lambda$. We also know that for each $\lambda$, there exists a unique $P_\lambda(q,t)$ such that:

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu \quad \text{where} \quad c_{\lambda\mu} \in Q(q,t).$$

Define:

$$Q_\lambda = \frac{P_\lambda}{\langle P_\lambda, P_\lambda \rangle}.$$

Then, the bases $(P_\lambda)$ and $(Q_\lambda)$ of $\Lambda_H^n$ are dual to each other, $\langle Q_\lambda, P_\mu \rangle = \delta_{\lambda,\mu}$, and from [4], for $\lambda = (n)$:

$$Q_{(n)} = \sum_{|\lambda|=n} \prod_i \frac{1}{i^{m_i} m_i!} \prod_{j=1}^{l(\lambda)} \frac{1-t^{\lambda_j}}{1-q^{\lambda_j}} p_\lambda.$$

For partitions of the form $\lambda = (\lambda_1, \ldots, \lambda_k)$, we will derive a formula for the Macdonald polynomials $Q_\lambda$ with coefficients presented first combinatorically in Section 3 and then as very-well-poised hypergeometric series in section 5. A formula for $Q_{(\lambda_1,\lambda_2)}$ can be found in [2] and following the completion of this work, the author discovered the paper [3] which gives a formula, without proof, for $Q_{(\lambda_1,\lambda_2,\lambda_3)}$.

## 2. Preliminaries

Let $\lambda, \mu$ be partitions such that $\mu \subset \lambda$; the diagram of $\mu$ being contained in the diagram of $\lambda$. For compactness, we will denote the skew diagram $\lambda - \mu$ as $\lambda\backslash\mu$. Moreover, we will view the diagram of a partition with row one being the largest and row $n$ begin the smallest (English notation).

The diagram of $\lambda\backslash\mu$ is said to be a *horizontal r-strip* if the number of blocks contained in $\lambda\backslash\mu$ equals $r$ and, of the remaining $r$ blocks, there is at most one in each column of $\lambda\backslash\mu$.

For each block $d$ found in the diagram of $\lambda$, let:

$$b_\lambda(d) = \begin{cases} \dfrac{(1-q^{e(d)}t^{s(d)+1})}{(1-q^{e(d)+1}t^{s(d)})} & \text{if } d \in \lambda \\ 1 & \text{if } d \notin \lambda, \end{cases} \tag{1}$$

where $e(d)$ denotes the number of blocks to the east of $d$ and $s(d)$ denotes the number of blocks to the south of $d$ in the diagram of $\lambda$ [4].

Let $R_{\lambda\backslash\mu}$ denote the union of the rows and let $C_{\lambda\backslash\mu}$ denote the union of the columns which intersect the diagram of $\lambda\backslash\mu$.

Using a Pieri-type formula [4], for $\lambda$, $\mu$ such that $\mu \subset \lambda$ is a horizontal $r$-strip, we have:

$$Q_\mu Q_r = \sum_\lambda \prod_d \frac{b_\mu(d)}{b_\lambda(d)} Q_\lambda \tag{2}$$

for $d \in R_{\lambda\backslash\mu} - C_{\lambda\backslash\mu}$.

Let $r = (r_1, \ldots, r_n)$ and $r' = (r_1, \ldots, r_{n-1})$ be partitions with the length of row $i$ equal to $r_i$. In order to derive a formula for $Q_{(r_1,\ldots,r_n)}$, we begin with $Q_{(r_1,\ldots,r_{n-1})}$.

To utilize (2), we must obtain all partitions $\omega = (w_1, \ldots, w_n)$ such that $r' \subset \omega$ and $\omega\backslash r'$ is a horizontal $r_n$-strip. Denote this set as $\Omega$.

## 3. Construction of $\Omega$

Begin with the desired partition $r' = (r_1, \ldots, r_{n-1})$. For all $\omega \in \Omega$, $\omega\backslash r'$ is a horizontal $r_n$-strip and therefore we must add exactly $r_n$ blocks to $r' = (r_1, \ldots, r_{n-1})$. To this end, we strategically enlarge the length of the rows, in some cases creating an $n$th row, by adding the appropriate number of blocks to row $r_i$ , and below row $r_{n-1}$, for all $i$.

Let $j_i^0$ denote the number of blocks added to row $r_i$. We may add a maximum of $r_n$ blocks to row $r_1$ and thus:

$$j_1^0 = 0, \ldots, r_n.$$

Consequently, the number of blocks added to all other rows of $r'$ is first dependent on $j_1^0$. Second, beginning with row $r_1$ and adding blocks to $r'$ in order of row succession, we see that the number of blocks which can be added to row $r_i$ is depedent upon the number of blocks added to rows $r_k$ for $1 \leq k < i$. Finally, since we may add at most one block anywhere in $r'$ in order to create any new column which appears in $\omega$, the number of blocks added to row $r_m$ is dependent on the difference between its length and the length of the preceeding row, $(r_{m-1} - r_m)$.

Assimilating this, for $m \in \{1, \ldots, (n-1)\}$:

$$j_m^0 = \begin{cases} 0, \ldots, \left(r_n - \sum_{s=1}^{m-1} j_s^0\right) & \text{if } \left(r_n - \sum_{s=1}^{m-1} j_s^0\right) \leq (r_{m-1} - r_m) \\ 0, \ldots, (r_{m-1} - r_m) & \text{if } \left(r_n - \sum_{s=1}^{m-1} j_s^0\right) > (r_{m-1} - r_m). \end{cases}$$

Lastly, for row $w_n \in \omega$, where $0 \leq w_n \leq r_n$, set:

$$w_n = r_n - \sum_{m=1}^{(n-1)} j_m^0.$$

Taking all possible combinations of $j_m^0$, beginning with $m = 1$ and ending with $m = (n-1)$, we generate $\Omega = \{\omega = (w_1, \ldots, w_n)\}$.

**Example 3.1** Let $r' = (5, 4, 3)$ and $r_4 = 2$. Then, $j_1^0 = \{0, 1, 2\}$, $j_2^0 = \{0, 1\}$, and $j_3^0 = \{0, 1\}$. Thus: $\Omega = \{(5, 4, 3, 2), (5, 4, 4, 1), (5, 5, 3, 1), (6, 4, 3, 1), (5, 5, 4), (6, 4, 4), (6, 5, 3), (7, 4, 3)\}$.

## 4. Construction of the coefficients

For the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define:

$$v_i = \{p \text{ where } p = \max\{1, \ldots, n\} \text{ such that } \lambda_p \geq i\}.$$

**Proposition 4.1** *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with row $i$, $\lambda_i$. For each row $\lambda_m$ and each block $d_k \in \lambda_m$, $1 \leq k \leq \lambda_m$, numbered left to right,*

$$s(d_k) = (v_k - m).$$

**Proof:** Let $\lambda'$ denote the conjugate partition of $\lambda$, the partition whose diagram is the transpose of the diagram of $\lambda$. Let $d_k$ be the $k$th block of row $\lambda_m$. Then, $d_k$ is the $m$th block of the $k$th row of $\lambda'$. Since row $\lambda_m$ of $\lambda$ is equal to column $m$ of $\lambda'$, it follows that the number of blocks in row $\lambda'_m$ is equal to $p$ where:

$$p = \max\{1, \ldots, n\} \quad \text{such that} \quad \lambda_p \geq m.$$

This implies that the number of blocks to the east of $d_k$ is the diagram of $\lambda'$ is equal to $p - m$. Taking the transpose, it follows that the number of blocks south of $d_k$ in the diagram of $\lambda$ equals $p - m$. □

**Remark 4.1** As demonstrated by the proof of Proposition 4.1, for $\lambda = (\lambda_1, \ldots, \lambda_n)$, $v_i \equiv \lambda_i'$. As we shall see, the desirability of using $v_i$ rather than the conjugate partition is evidenced by the considerable facility that it gives to the implimentation of the principal formula. For example, when $\lambda = (6, 4, 2, 2)$ and $i = 3$, we may simply observe from $\lambda$ that $v_i = 2$.

**Proposition 4.2** *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with row $i$, $\lambda_i$. For each row $\lambda_m$, let $\{d_k\}$, $1 \leq k \leq \lambda_m$, denote the set of blocks which compose $\lambda_m$, numbered left to right. Then*:

$$b_{\lambda_m} = b_\lambda(\{d_k\}) = \prod_{i=0}^{\lambda_m - 1} \frac{\left(1 - q^{\lambda_m - (i+1)} t^{v_{(i+1)} - (m-1)}\right)}{\left(1 - q^{\lambda_m - i} t^{v_{(i+1)} - m}\right)}$$

**Proof:** For the product limits $\{i = 0, \ldots, (\lambda_m - 1)\}$, let block $k$, $d_k$, for $1 \leq k \leq \lambda_m$, correspond to $i = (k - 1)$. Then, $\{i = 0, \ldots, (\lambda_m - 1)\}$ corresponds directly to $\{d_k\}$, where $1 \leq k \leq \lambda_m$.

It is easily seen that the number of blocks east of block $d_k$ in the diagram of $\lambda$ is equal to $(\lambda_m - k)$. By Proposition 4.1, we know that the number of blocks south of $d_k$ in the diagram of $\lambda$ is equal to $(v_k - m)$. Therefore:

$$b_\lambda(d_k) = \frac{\left(1 - q^{\lambda_m - k}\, t^{v_{(k)}-(m-1)}\right)}{\left(1 - q^{\lambda_m - k + 1}\, t^{v_{(k)}-m}\right)}.$$

Shifting block $b_k$ to block $b_i$ where $i = (k-1)$, we have:

$$b_\lambda(d_k) = b_\lambda(d_i) = \frac{\left(1 - q^{\lambda_m-(i+1)} t^{v_{(i+1)}-(m-1)}\right)}{\left(1 - q^{\lambda_m - i} t^{v_{(i+1)}-m}\right)}.$$

Taking the product over $\{i = 0, \ldots, (\lambda_m - 1)\}$ to encompass $\{d_k\},\ 1 \le k \le \lambda_m$, yields the desired result. □

Let $\omega \in \Omega$. In order to utilize the Pieri-type formula, we must construct the required coefficent for each $Q_{(\omega)}$:

$$\prod_d \frac{b_{r'}(d)}{b_\omega(d)} \quad d \in R_{\omega\backslash r'} - C_{\omega\backslash r'}.$$

For $m \in \{1, \ldots, (n-1)\}$, we compute these coefficients according to row, $r_m$, and according to the number of blocks added to $r_m$, $j_m^0$, where $\omega_m = (r_m + j_m^0)$.

**Proposition 4.3** *Let $\omega \in \Omega$ and $r' = (r_1, \ldots, r_{n-1})$.*
*For row $w_m = (r_m + j_m^0)$, $1 \le m \le (n-1)$, such that $d \in w_m$ and $d \in R_{\omega\backslash r'} - C_{\omega\backslash r'}$,*

$$\prod_d \frac{b_{r_m}(d)}{b_{w_m}(d)} = T_{j_m^0,0} T_{j_m^0, j_k^0, 0} \quad m < k \le (n-1) \tag{3}$$

*where*

$$T_{j_m^0,0} = \begin{cases} \displaystyle\prod_{i=r_n - \sum_{s=1}^{n-1} j_s^0}^{r_m - 1} \frac{\left(1 - q^{r_m-(i+1)} t^{v_{(i+1)}-(m-1)}\right)\left(1 - q^{r_m + j_m^0 - i} t^{v_{(i+1)}-m}\right)}{\left(1 - q^{r_m - i} t^{v_{(i+1)}-m}\right)\left(1 - q^{r_m + j_m^0 - (i+1)} t^{v_{(i+1)}-(m-1)}\right)} & j_m^0 \ne 0 \\ 1 & j_m^0 = 0 \end{cases} \tag{4}$$

*and*

$$T_{j_m^0, j_k^0, 0} = \begin{cases} \displaystyle\prod_{i=r_k}^{r_k + j_k^0 - 1} \frac{(1 - q^{r_m - i} t^{k-m-1})(1 - q^{r_m + j_m^0 - (i+1)} t^{k-m})}{(1 - q^{r_m-(i+1)} t^{k-m})(1 - q^{r_m + j_m^0 - i} t^{k-m-1})} & j_m^0 \ne 0 \text{ and } j_k^0 \ne 0 \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

*restricting $v_{(i+1)}$ to $r'$; $v_{(i+1)} = \{p$* where $p = max\{1, \ldots (n-1)\}$ such that $r_p \ge (i+1)\}$.

**Proof:** First, note that within this proof, we use the expressions $b'_{r_m}$ and $b'_{w_m}$ to identify "incomplete stages" in the development of $b_{r_m}$ and $b_{w_m}$.

For row $r_m$, we include only the $d \in r_m$ for which $d \in R_{\omega\backslash r'} - C_{\omega\backslash r'}$. Since row $w_n = (r_n - \sum_{m=1}^{n-1} j_m^0)$, it follows that, for each row of $\omega$ and hence of $r'$, the blocks $\{d_1, \ldots, d_{(r_n - \sum_{m=1}^{n-1} j_m^0)}\}$ are not included in $R_{\omega\backslash r'} - C_{\omega\backslash r'}$. Therefore, by Proposition 4.2:

$$b'_{r_m} = \prod_{i = r_n - \sum_{m=1}^{n-1} j_m^0}^{r_m - 1} \frac{\left(1 - q^{r_m-(i+1)} t^{v_{(i+1)}-(m-1)}\right)}{\left(1 - q^{r_m - i} t^{v_{(i+1)}-m}\right)}. \tag{6}$$

Similarly, for row $w_m = (r_m + j_m^0)$:

$$b'_{w_m} = \prod_{i = r_n - \sum_{m=1}^{n-1} j_m^0}^{r_m - 1} \frac{\left(1 - q^{r_m + j_m^0 -(i+1)} t^{v_{(i+1)}-(m-1)}\right)}{\left(1 - q^{r_m + j_m^0 - i} t^{v_{(i+1)}-m}\right)} \tag{7}$$

where we restrict $v_{(i+1)}$ to $r'$ in order to exclude any blocks $d \notin r'$ which lie beneath row $r_m$.

If $j_m^0 = 0$, we do not include blocks from $r_m$ in (3), and thus from (6) and (7):

$$\frac{b'_{r_m}}{b'_{w_m}} = T_{j_m^0, 0}$$
$$= \begin{cases} \displaystyle\prod_{i=r_n - \sum_{s=1}^{n-1} j_s^0}^{r_m - 1} \frac{\left(1 - q^{r_m-(i+1)} t^{v_{(i+1)}-(m-1)}\right)\left(1 - q^{r_m + j_m^0 - i} t^{v_{(i+1)}-m}\right)}{\left(1 - q^{r_m - i} t^{v_{(i+1)}-m}\right)\left(1 - q^{r_m + j_m^0 -(i+1)} t^{v_{(i+1)}-(m-1)}\right)} & j_m^0 \neq 0 \\ 1 & j_m^0 = 0. \end{cases}$$

However, we do not include any $d \in w_m$ such that $d \in C_{\omega\backslash r'}$; therefore, we must get rid of any terms corresponding to these blocks in the quotient (4). These blocks, which generate the unwanted terms, directly correspond to the $j_k^0$ , for $(m+1) \le k \le (n-1)$ , which were added below them on row $r_k$ to create row $w_k$. In order for these terms to appear in (4), we must have $j_m^0 \neq 0$ and $j_k^0 \neq 0$. Therefore, to correct this in order to include only the $d \in w_m$ such that $d \in R_{\omega\backslash r'} - C_{\omega\backslash r'}$ , we multiply (4) by:

$$T_{j_m^0, j_k^0, 0} = \begin{cases} \displaystyle\prod_{i=r_k}^{r_k + j_k^0 - 1} \frac{\left(1 - q^{r_m - i} t^{k-m-1}\right)\left(1 - q^{r_m + j_m^0 -(i+1)} t^{k-m}\right)}{\left(1 - q^{r_m-(i+1)} t^{k-m}\right)\left(1 - q^{r_m + j_m^0 - i} t^{k-m-1}\right)} & j_m^0 \neq 0 \text{ and } j_k^0 \neq 0 \\ 1 & \text{otherwise,} \end{cases}$$

yeilding the desired result. □

## 5. Construction of the principal formula

Let $r = (r_1, \ldots, r_n)$ be a partition. For $l \in \{1, \ldots, (r_n - 1)\}$, let

$$j_1^l = 0, \ldots, \left( r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c \right).$$

And, for $m \in \{2, \ldots, (n-1)\}$, let

$$j_m^l = \begin{cases} 0, \ldots, \left( r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l \right) \\ \quad \text{if } \left( r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l \right) \le \left( r_{m-1} - r_m + \sum_{c=0}^{l-1} \left( j_{m-1}^c - j_m^c \right) \right) \\ 0, \ldots, \left( r_{m-1} - r_m + \sum_{c=0}^{l-1} \left( j_{m-1}^c - j_m^c \right) \right) \\ \quad \text{if } \left( r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l \right) > \left( r_{m-1} - r_m + \sum_{c=0}^{l-1} \left( j_{m-1}^c - j_m^c \right) \right). \end{cases}$$

**Remark 5.1** The purpose of the $j_i^l$, $1 \le i \le (n-1)$, is to allow us to systematically reduce row $w_n = (r_n - \sum_{i=1}^{n-1} j_i^0)$ to zero. To this end, we first must compute the $j_i^0$ in order to generate $\Omega$. Then, restricting the $j_i^l$ such that $\sum_{i=1}^{n-1} j_i^l \neq 0$, $0 < \sum_{i=1}^{n-1} j_i^l \le (r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c)$, we are able to achieve the desired result. Further, restrict the $j_i^0$ such that $\sum_{i=1}^{n-1} j_i^0 \neq 0$, excluding the original partition $r = (r_1, \ldots, r_n)$ from this reduction. We must now add at least one block to the partition $r'$, and to any subsequently created partitions (i.e., $\omega \in \Omega$); thus, the maximum number of times that we may need to repeat this reduction process is $r_n$, yielding $(r_n - \sum_{i=1}^{n-1} \sum_{l=0}^{r_n-1} j_i^l) = 0$. □

**Example 5.1** Building upon Example 3.1, for the partition $\omega = (6, 4, 3, 1)$, $j_1^0 = 1$, $j_2^0 = 0$, and $j_3^0 = 0$, we desire to reduce row $w_4$ to zero. Using the restriction given in Remark 5.1, we have $j_1^1 = \{0, 1\}$, $j_2^1 = \{0, 1\}$, $j_3^1 = \{0, 1\}$ where $0 < \sum_{m=1}^{3} j_m^1 \le 1$. Taking all possible combinations of the $j_m^1$ dictated by the $j_1^0 = 1$, $j_2^0 = 0$, and $j_3^0 = 0$, $1 \le m \le 3$, and applying them to $\omega$, yields the set of three-row partitions $\{(7, 4, 3), (6, 5, 3), (6, 4, 4)\}$. □

Given $j_m^l$, $1 \le m \le (n-1)$, (4) becomes:

$$T_{j_m^l, 0} = \begin{cases} \prod_{i=\left(r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l} j_s^c\right)}^{\left(r_m + \sum_{c=0}^{l-1} j_m^c - 1\right)} \\ \quad \times \dfrac{\left(1 - q^{r_m + \sum_{c=0}^{l-1} j_m^c - (i+1)} t^{v_{(i+1)} - (m-1)}\right)\left(1 - q^{r_m + \sum_{c=0}^{l} j_m^c - i} t^{v_{(i+1)} - m}\right)}{\left(1 - q^{r_m + \sum_{c=0}^{l-1} j_m^c - i} t^{v_{(i+1)} - m}\right)\left(1 - q^{r_m + \sum_{c=0}^{l} j_m^c - (i+1)} t^{v_{(i+1)} - (m-1)}\right)} & j_m^l \neq 0 \\ 1 & j_m^l = 0 \end{cases} \tag{8}$$

and, (5) becomes:

$$T_{j_m^l,j_k^l,0}=\begin{cases}\displaystyle\prod_{i=r_k+\sum_{c=0}^{l-1}j_k^c}^{r_k+\sum_{c=0}^{l}j_k^c-1} \\ \quad\times\dfrac{\left(1-q^{r_m+\sum_{c=0}^{l-1}j_m^c-i}t^{k-m-1}\right)\left(1-q^{r_m+\sum_{c=0}^{l}j_m^c-(i+1)}t^{k-m}\right)}{\left(1-q^{r_m+\sum_{c=0}^{l-1}j_m^c-(i+1)}t^{k-m}\right)\left(1-q^{r_m+\sum_{c=0}^{l}j_m^c-i}t^{k-m-1}\right)} & j_m^l\neq 0 \text{ and } j_k^l\neq 0\\ 1 & \text{otherwise}\end{cases} \tag{9}$$

where for $l=0, \sum_{c=0}^{l-1} j_m^c = 0$ and where $v_{(i+1)}$ is restricted to the partition $(r_1 + \sum_{c=0}^{l-1} j_1^c, \ldots, r_{n-1} + \sum_{c=0}^{l-1} j_{n-1}^c)$; $v_{(i+1)} = \{p$ where $p = \max\{1, \ldots, (n-1)\}$ such that $(r_p + \sum_{c=0}^{l-1} j_p^c) \geq (i+1)\}$.

**Theorem 5.1** *For the partition $r=(r_1,\ldots,r_n)$, $m \in \{1,\ldots,(n-1)\}$, and $l \in \{1,\ldots,(r_n-1)\}$, we have*

$$\begin{aligned}Q_{(r_1,\ldots,r_n)} &= Q_{(r_1,\ldots,r_{n-1})}Q_{(r_n)}\\ &+\sum_{j_m^0,j_m^l}\sum_{i=1}^{r_n-1}(-1)^i\prod_{l=0}^{i-1}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)\\ &\times Q_{\left(r_1+\sum_{l=0}^{i-1}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{i-1}j_{n-1}^l\right)}\,Q_{\left(r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{i-1}j_m^l\right)}\\ &+\sum_{\substack{j_m^0,j_m^l\\ \sum_{m=1}^{n-1}j_m^{r_n-1}\neq 0}}(-1)^{r_n}\prod_{l=0}^{r_n-1}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)\\ &\times Q_{\left(r_1+\sum_{l=0}^{r_n-1}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-1}j_{n-1}^l\right)}\end{aligned}$$

where $\sum_{j_m^0,j_m^l}$ signifies to sum over all possible combinations of the values of $j_1^0$ and $j_m^0$ such that $0<\sum_{m=2}^{n-1}(j_1^0+j_m^0)\leq r_n$ and all possible combinations of the values $j_m^l$ dictated by the $j_m^0$ such that $\sum_{m=1}^{n-1} j_m^l \neq 0$.

**Proof:** We want to show that:

$$\begin{aligned}&Q_{(r_1,\ldots,r_n)} - Q_{(r_1,\ldots,r_{n-1})}Q_{(r_n)}\\ &-\sum_{j_1^0,j_m^0}\sum_{i=1}^{r_n-1}(-1)^i\prod_{l=0}^{i-1}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)\\ &\times Q_{\left(r_1+\sum_{l=0}^{i-1}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{i-1}j_{n-1}^l\right)}\,Q_{\left(r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{i-1}j_m^l\right)}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{j_1^0, j_m^0 \\ \sum_{m=1}^{n-1} j_m^{r_n-1} \neq 0}} (-1)^{r_n} \prod_{l=0}^{r_n-1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \\
&\times Q_{\left(r_1 + \sum_{l=0}^{r_n-1} j_1^l, \ldots, r_{n-1} + \sum_{l=0}^{r_n-1} j_{n-1}^l\right)} = 0. \qquad (10)
\end{aligned}
$$

Using (2), for $\sum_{m=1}^{n-1} j_m^0 \neq 0$, we have:

$$
\begin{aligned}
Q_{(r_1,\ldots,r_{n-1})} Q_{(r_n)} = Q_{(r_1,\ldots,r_n)} + \sum_{j_1^0, j_m^0} \left( \prod_{m=1}^{n-1} T_{j_m^0,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_k^0, j_k^0, 0} \right) \\
\times Q_{\left(r_1 + j_1^0, \ldots, r_{n-1} + j_{n-1}^0, r_n - \sum_{m=1}^{n-1} j_m^0\right)}. \qquad (11)
\end{aligned}
$$

Substituting (11) into (10), we need only to show:

$$
\begin{aligned}
&\sum_{j_1^0, j_m^0} \left( \prod_{m=1}^{n-1} T_{j_m^0,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_k^0, j_k^0, 0} \right) Q_{\left(r_1 + j_1^0, \ldots, r_{n-1} + j_{n-1}^0, r_n - \sum_{m=1}^{n-1} j_m^0\right)} \\
&\quad + \sum_{j_1^0, j_m^0} \sum_{i=1}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \\
&\quad \times Q_{\left(r_1 + \sum_{l=0}^{i-1} j_1^l, \ldots, r_{n-1} + \sum_{l=0}^{i-1} j_{n-1}^l\right)} Q_{\left(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l\right)} \\
&\quad = \sum_{\substack{j_1^0, j_m^0 \\ \sum_{m=1}^{n-1} j_m^{r_n-1} \neq 0}} (-1)^{r_n-1} \prod_{l=0}^{r_n-1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \\
&\quad \times Q_{\left(r_1 + \sum_{l=0}^{r_n-1} j_1^l, \ldots, r_{n-1} + \sum_{l=0}^{r_n-1} j_{n-1}^l\right)}. \qquad (12)
\end{aligned}
$$

Note that each Pieri-formula expansion (via (2) and (11)) of

$$
\sum_{j_1^0, j_m^0} \left( \prod_{m=1}^{n-1} T_{j_m^0,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_k^0, j_k^0, 0} \right) Q_{\left(r_1 + j_1^0, \ldots, r_{n-1} + j_{n-1}^0, r_n - \sum_{m=1}^{n-1} j_m^0\right)} \qquad (13)
$$

yields two terms. To show our desired result, we will show that with each successive "numbered" Pieri-formula expansion of (13), the term which contains the product of two Macdonald polynomials cancells with the correspondingly numbered term $i$ in

$$
\begin{aligned}
&\sum_{j_1^0, j_m^0} \sum_{i=1}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \\
&\quad \times Q_{\left(r_1 + \sum_{l=0}^{i-1} j_1^l, \ldots, r_{n-1} + \sum_{l=0}^{i-1} j_{n-1}^l\right)} Q_{\left(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l\right)} \qquad (14)
\end{aligned}
$$

and that, in the final Pieri expansion of (13), we will be left only with

$$\sum_{\substack{j_1^0, j_m^0 \\ \sum_{m=1}^{n-1} j_m^{r_n-1} \neq 0}} (-1)^{r_n-1} \prod_{l=0}^{r_n-1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-1} j_1^l, \ldots, r_{n-1}+\sum_{l=0}^{r_n-1} j_{n-1}^l\right)}. \tag{15}$$

To this end, we will induct on $i$.

Consder $i = 1$. Completing the first Pieri expansion of (13), and computing $i = 1$ from (14), (12) becomes:

$$\sum_{j_1^0, j_m^0} \left( \prod_{m=1}^{n-1} T_{j_m^0,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_k^0, j_k^0, 0} \right) Q_{\left(r_1+j_1^0, \ldots, r_{n-1}+j_{n-1}^0\right)} \, Q_{\left(r_n - \sum_{m=1}^{n-1} j_m^0\right)}$$
$$- \sum_{j_1^0, j_m^0} \prod_{l=0}^{1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{1} j_1^l, \ldots, r_{n-1}+\sum_{l=0}^{1} j_{n-1}^l, r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{1} j_m^l\right)}$$
$$- \sum_{j_1^0, j_m^0} \left( \left( \prod_{m=1}^{n-1} T_{j_m^0,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^0, j_k^0, 0} \right) \right) Q_{\left(r_1+j_1^0, \ldots, r_{n-1}+j_{n-1}^0\right)} \, Q_{\left(r_n - \sum_{m=1}^{n-1} j_m^0\right)}$$
$$+ \sum_{j_1^0, j_m^0} \sum_{i=2}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{i-1} j_1^l, \ldots, r_{n-1}+\sum_{l=0}^{i-1} j_{n-1}^l\right)} \, Q_{\left(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l\right)}$$
$$= - \sum_{j_1^0, j_m^0} \prod_{l=0}^{1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{1} j_1^l, \ldots, r_{n-1}+\sum_{l=0}^{1} j_{n-1}^l, r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{1} j_m^l\right)}$$
$$+ \sum_{j_1^0, j_m^0} \sum_{i=2}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left( \left( \prod_{m=1}^{n-1} T_{j_m^l,0} \right) \left( \prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{i-1} j_1^l, \ldots, r_{n-1}+\sum_{l=0}^{i-1} j_{n-1}^l\right)} \, Q_{\left(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l\right)}.$$

Therefore, in the first Pieri expansion of (13), the term containing the product of two Macdonald polynomials cancelled with the correspondingly numbered $i = 1$ in (14), as desired.

Assume that with each further "numbered" Pieri expansion of (13), up to $(r_n - 2)$, the term containing a product of two Macdonald polynomials cancells with the correspondingly numbered $i$ in (14). We will show the result for $(r_n - 1)$.

For $i = (r_n - 2)$, we have that (12) equals:

$$\sum_{j_1^0,j_m^0}(-1)^{r_n-3}\prod_{l=0}^{r_n-3}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-3}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-3}j_{n-1}^l\right)}Q_{\left(r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-3}j_m^l\right)}$$
$$+\sum_{j_1^0,j_m^0}(-1)^{r_n-2}\prod_{l=0}^{r_n-2}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-2}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-2}j_{n-1}^l,r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-2}j_m^l\right)}$$
$$+\sum_{j_1^0,j_m^0}(-1)^{r_n-2}\prod_{l=0}^{r_n-3}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-3}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-3}j_{n-1}^l\right)}Q_{\left(r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-3}j_m^l\right)}$$
$$+\sum_{j_1^0,j_m^0}(-1)^{r_n-1}\prod_{l=0}^{r_n-2}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-2}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-2}j_{n-1}^l\right)}Q_{\left(r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-2}j_m^l\right)}$$
$$=\sum_{j_1^0,j_m^0}(-1)^{r_n-2}\prod_{l=0}^{r_n-2}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-2}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-2}j_{n-1}^l,r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-2}j_m^l\right)}$$
$$+\sum_{j_1^0,j_m^0}(-1)^{r_n-1}\prod_{l=0}^{r_n-2}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-2}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-2}j_{n-1}^l\right)}$$
$$\times Q_{\left(r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-2}j_m^l\right)}.$$

Performing the Pieri expansion on (13) a final time, (12) is equal to:

$$\sum_{j_1^0,j_m^0}(-1)^{r_n-2}\prod_{l=0}^{r_n-2}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-2}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-2}j_{n-1}^l\right)}Q_{\left(r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-2}j_m^l\right)}$$
$$+\sum_{j_1^0,j_m^0}(-1)^{r_n-1}\prod_{l=0}^{r_n-1}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,j_k^l,0}\right)\right)$$
$$\times Q_{\left(r_1+\sum_{l=0}^{r_n-1}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-1}j_{n-1}^l\right)}$$

$$
\begin{aligned}
&+\sum_{j_1^0, j_m^0}(-1)^{r_n-1}\prod_{l=0}^{r_n-2}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,\,j_k^l,0}\right)\right)\\
&\times Q_{\left(r_1+\sum_{l=0}^{r_n-2}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-2}j_{n-1}^l\right)}Q_{\left(r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-2}j_m^l\right)}\\
= &\sum_{\substack{j_1^0, j_m^0\\ \sum_{m=1}^{n-1}j_m^{r_n-1}\neq 0}}(-1)^{r_n-1}\prod_{l=0}^{r_n-1}\left(\left(\prod_{m=1}^{n-1}T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{n-1}T_{j_m^l,\,j_k^l,0}\right)\right)\\
&\times Q_{\left(r_1+\sum_{l=0}^{r_n-1}j_1^l,\ldots,r_{n-1}+\sum_{l=0}^{r_n-1}j_{n-1}^l\right)}.
\end{aligned}
$$

Since, by definition of $j_m^l$, we must have

$$
r_n-\sum_{m=1}^{n-1}\sum_{l=0}^{r_n-1}j_m^l = 0.
$$

□

**Example 5.2** We shall build upon Example 3.1 to calculate the expansion of the Macdonald polynomial $Q_{(5,4,3,2)}$.

Given the nature of the partition $r = (5, 4, 3, 2)$, it is fitting and instructive to use the following notation:

$$
\begin{aligned}
&\prod_{m=1}^{n-1}T_{j_m^l,0}\equiv T^l_{(j_1^l,\ldots,j^l_{(n-1)})}\\
&\prod_{k=m+1}^{n-1}T^l_{j_m^l,j_k^l,0}\equiv T^{ll}_{(0,\ldots,j^l_{m+1},\ldots,j^l_{(n-1)})}\qquad j_m^l,\, j_{m+i}^l\neq 0,\qquad 1\leq i\leq (n-m-1)
\end{aligned}
$$

For $2 \leq m \leq 3$, we have:

$$
j_1^0 = \{0, 1, 2\}\quad j_1^l=\left\{0,\ldots,\left(2-\sum_{s=1}^{3}\sum_{c=0}^{l-1}j_s^c\right)\right\},
$$

$$
j_m^0=\begin{cases}0,\ldots,\left(2-\sum_{s=1}^{m-1}j_s^0\right) & \text{if } \left(2-\sum_{s=1}^{m-1}j_s^0\right)\leq (r_{m-1}-r_m)\\[2ex] 0,\ldots,(r_{m-1}-r_m) & \text{if } \left(2-\sum_{s=1}^{m-1}j_s^0\right) > (r_{m-1}-r_m),\end{cases}
$$

$$j_m^l = \begin{cases} 0, \ldots, \left(2 - \sum_{s=1}^{3}\sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l\right) \\ \quad \text{if } \left(2 - \sum_{s=1}^{3}\sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l\right) \le \left(r_{m-1} - r_m + \sum_{c=0}^{l-1}\left(j_{m-1}^c - j_m^c\right)\right) \\ 0, \ldots, \left(r_{m-1} - r_m + \sum_{c=0}^{l-1}\left(j_{m-1}^c - j_m^c\right)\right) \\ \quad \text{if } \left(2 - \sum_{s=1}^{3}\sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l\right) > \left(r_{m-1} - r_m + \sum_{c=0}^{l-1}\left(j_{m-1}^c - j_m^c\right)\right). \end{cases}$$

We begin with $j_m^0$ since these values determine the subsequent choices for $j_m^l$. We have:

$$\begin{array}{ll} j_1^0 = \{0,1,2\} & j_1^1 = \{0,1\} \\ j_2^0 = \{0,1\} & j_2^1 = \{0,1\} \\ j_3^0 = \{0,1\} & j_3^1 = \{0,1\}. \end{array}$$

Taking all possible combinations of the $j_m^l$ , $l = \{0, 1\}$ , such that $0 < \sum_{m=1}^{3} j_m^0 \le 2$ and $0 < \sum_{m=1}^{3} j_m^1 \le 1$ , we have:

$$\begin{aligned} Q_{(5,4,3,2)} = {} & Q_{(5,4,3)}Q_{(2)} - \sum_{j_1^0, j_m^0} \left(\prod_{m=1}^{3} T_{j_m^0,0}\right)\left(\prod_{k=m+1}^{3} T_{j_m^0, j_k^0, 0}\right) \\ & \times Q_{\left(5+j_1^0, 4+j_2^0, 3+j_3^0\right)}\, Q_{\left(2-\sum_{m=1}^{3} j_m^0\right)} \\ & + \sum_{j_1^0, j_m^0} \prod_{l=0}^{1} \left(\left((\prod_{m=1}^{3} T_{j_m^l,0}\right)\left(\prod_{k=m+1}^{3} T_{j_m^l, j_k^l, 0}\right)\right) Q_{\left(5+\sum_{l=0}^{1} j_1^l, 4+\sum_{l=0}^{1} j_2^l, 3+\sum_{l=0}^{1} j_3^l\right)} \\ = {} & Q_{(5,4,3)}\, Q_{(2)} + \left(T^0_{(1,0,0)}\, T^1_{(1,0,0)} - T^0_{(2,0,0)}\right) Q_{(7,4,3)} + \left(T^0_{(1,0,0)}\, T^1_{(0,1,0)}\right. \\ & \left. + T^0_{(0,1,0)}\, T^1_{(1,0,0)} - T^0_{(1,1,0)}\, T^{00}_{(0,1,0)}\right) Q_{(6,5,3)} \\ & + \left(T^0_{(1,0,0)}\, T^1_{(0,0,1)} + T^0_{(0,0,1)}\, T^1_{(1,0,0)} - T^0_{(1,0,1)}\, T^{00}_{(0,0,1)}\right) Q_{(6,4,4)} \\ & + \left(T^0_{(0,1,0)}\, T^1_{(0,0,1)} + T^0_{(0,0,1)}\, T^1_{(0,1,0)} - T^0_{(0,1,1)}\, T^{00}_{(0,0,1)}\right) Q_{(5,5,4)} \\ & - T^0_{(1,0,0)}\, Q_{(6,4,3)}\, Q_{(1)} - T^0_{(0,1,0)}\, Q_{(5,5,3)}\, Q_{(1)} - T^0_{(0,0,1)}\, Q_{(5,4,4)}\, Q_{(1)}. \end{aligned}$$

□

## 6. Very-well-poised hypergeometric series

We may express the coefficients $T_{j_m^l}$ , for $m \in \{1, \ldots, (n-1)\}$ , and $l \in \{0, \ldots, r_{n-1}\}$ , in terms of very-well-poised hypergeometric series.

We will use the following notations.

$$\begin{aligned} (a;q)_0 &= 1 \\ (a;q)_n &= \prod_{i=0}^{n-1}(1 - aq^i) \end{aligned}$$

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$$
$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$
$$(a_1,\ldots,a_m;q)_n = (a_1;q)_n \cdots (a_m;q)_n$$
$$(a_1,\ldots,a_m;q)_\infty = (a_1;q)_\infty \cdots (a_m;q)_\infty$$

The *basic hypergeometric series* $\phi_{r+1,r}$ is of the form:

$$\left[\begin{matrix} a_1, & a_2 & ,\ldots, & a_{r+1} & \\ b_1, & b_2 & ,\ldots, & b_r & q, z \end{matrix}\right]$$

where

$$\phi_{r+1,r} = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_{r+1};q)_n}{(b_1,\ldots,b_r;q)_n(q;q)_n} z^n.$$

We say that $\phi_{r+1,r}$ is *very-well-poised*, denoted

$$W_{r+1,r} = (a_1; a_4, \ldots, a_{r+1}; q, z),$$

if

$$a_1 q = a_2 b_1 = \ldots = a_{r+1} b_r \quad \text{and} \quad a_2 = q a_1^{\frac{1}{2}},\ a_3 = -q a_1^{\frac{1}{2}}.$$

From [1], we have:

$$W_{6,5}\left(a; b, c, d; q, \frac{aq}{bcd}\right) = \frac{\left(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{cd}; q\right)_\infty}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{bcd}; q\right)_\infty}. \tag{16}$$

**The Coefficients $\mathbf{T_{j_m^0,0}}$ and $\mathbf{T_{j_m^0,j_k^0,0}}$**

We first compute the coefficient $T_{j_m^0}$ in terms of very-well-poised hypergeometric series.

Beginning with row $r_m$, we identify the “indentions” which lie “under” it in the diagram of $r'$. In other words, we identify all rows $r_i$, $m < i \leq (n-1)$, such that $r_i < r_{i+1}$.

For $k_1 \in \{1,\ldots,(n-m)\}$, let $r_{n-k_1}$ be the largest indexed row of $r'$ such that $r_{n-k_1} \leq r_m$.

**Proposition 6.1** *Suppose $r_{n-k_1} = r_m$. It follows that $k_1 = (n-m)$ and $r_m \equiv r_{n-1}$. Then*:

$$T_{j_m^0,0} = \begin{cases} W_{6,5}\left(q^{r_m - r_n + \sum_{s=1}^{n-1} j_s^0} t^{n-m-1}; qt^{-1}, q^{-j_m^0}, q^{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}; q, q^{j_m^0} t^{n-m}\right) & j_m^0 \neq 0 \\ 1 & j_m^0 = 0. \end{cases}$$

**Proof:** Let $r_{n-k_1} = r_m$; $k_1 = (n-m)$ and $r_m \equiv r_{n-1}$. There are two cases: $j_m = 0$ and $j_m^0 \neq 0$.

For $j_m^0 = 0$, we have that $T_{j_m^0,0} = 1$ by Proposition 4.3.

For $j_m^0 \neq 0$, using properties of hypergeometric series and (16), (4) becomes:

$$
\begin{aligned}
T_{j_m^0,0} &= \prod_{i=r_n-\sum_{s=1}^{n-1} j_s^0}^{r_{n-1}-1} \frac{(1-q^{r_m-(i+1)}t^{n-m})(1-q^{r_m+j_m^0-i}t^{n-m-1})}{(1-q^{r_m-i}t^{n-m-1})(1-q^{r_m+j_m^0-(i+1)}t^{n-m})} \\
&= \frac{(q^{r_m-r_{n-1}}t^{n-m}, q^{r_m-r_{n-1}+j_m^0+1}t^{n-m-1}; q)_{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}}{(q^{r_m-r_{n-1}+1}t^{n-m-1}, q^{r_m-r_{n-1}+j_m^0}t^{n-m}; q)_{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}} \\
&= \frac{(q^{r_m-r_{n-1}}t^{n-m}, q^{r_m+j_m^0-r_{n-1}+1}t^{n-m-1}, q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0+1}t^{n-m-1}, q^{r_m+j_m^0-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m}; q)_\infty}{(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m}, q^{r_m+j_m^0-r_n+\sum_{s=1}^{n-1} j_s^0+1}t^{n-m-1}, q^{r_m-r_{n-1}+1}t^{n-m-1}, q^{r_m+j_m^0-r_{n-1}}t^{n-m}; q)_\infty} \\
&= W_{6,5}(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m-1}; qt^{-1}, q^{-j_m^0}, q^{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}; q, q^{r_m-r_{n-1}+j_m^0}t^{n-m}) \\
&= W_{6,5}(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m-1}; qt^{-1}, q^{-j_m^0}, q^{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}; q, q^{j_m^0}t^{n-m}).
\end{aligned}
$$

□

**Remark 6.1** Let $r_m = r_{n-k_1}$. It follows that for all $m < k \leq (n-1)$, we have $j_k^0 = 0$, and thus $T_{j_m^0,j_k^0,0} = 1$.

Consider the case when $r_{n-k_1} \neq r_m$. Let $r_{n-k_2}$ be the largest indexed row of $r' = (r_1, \ldots, r_{n-1})$ such that $r_{n-k_1} \neq r_{n-k_2}$. If $r_{n-k_2} \neq r_m$, let $r_{n-k_3}$ be the largest indexed row of $r'$ for which $r_{n-k_2} \neq r_{n-k_3}$, ect. Continue this process until one reaches row $r_{n-k_f}$ for which $r_{n-k_f} = r_m$, creating the chain:

$$r_{n-k_1} < r_{n-k_2} < \cdots < r_{n-k_f} = r_m$$

for $k_i \in \{1, \ldots, (n-m)\}$, $1 \leq i \leq f$.

**Proposition 6.2** *Suppose $r_{n-k_1} \neq r_m$. Then, (4) becomes:*

$$
T_{j_m^0,0} = \begin{cases}
W_{6,5}\big(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m-k_1}; qt^{-1}, q^{-j_m^0}, q^{r_{n-k_1}-r_n+\sum_{s=1}^{n-1} j_s^0}; \\
\quad q, q^{r_m-r_{n-k_1}+j_m^0}t^{n-m-k_1+1}\big) \cdot \prod_{h=2}^{f-1} W_{6,5}\big(q^{r_m-r_{n-k_{(h+1)}}+j_m^0}t^{n-k_{(h+1)}-m}; \\
\quad qt^{-1}, q^{j_m^0}, q^{r_{n-k_h}-r_{n-k_{(h+1)}}}; q, q^{r_m-r_{n-k_h}}t^{n-m-k_{(h+1)}+1}\big) & j_m^0 \neq 0 \\
1 & j_m^0 = 0.
\end{cases}
$$

**Proof:** Using our chain, we are able to identify the indentions under row $r_m$ in the diagram of $r'$; thus, we obtain all intervals of blocks $d \in r_m$ on which $s(d)$ changes. Using properties of hypergeometric series and (16), we decompose (4) as follows:

$$
\begin{aligned}
T_{j_m^0,0} &= \prod_{i=r_n-\sum_{s=1}^{n-1}}^{r_{n-k_1}-1} j_s^0 \frac{\left(1-q^{r_m-(i+1)}t^{n-m-k_1+1}\right)\left(1-q^{r_m+j_m^0-i}\,t^{n-m-k_1}\right)}{(1-q^{r_m-i}\,t^{n-m-k_1})(1-q^{r_m+j_m^0-(i+1)}t^{n-m-k_1+1})} \\
&\quad \cdot \prod_{h=2}^{f-1}\left(\prod_{i=r_{n-k_h}}^{r_{n-k_{(h+1)}}-1} \frac{\left(1-q^{r_m-(i+1)}t^{n-m-k_{(h+1)}+1}\right)\left(1-q^{r_m+j_m^0-i}\,t^{n-m-k_{(h+1)}}\right)}{\left(1-q^{r_m-i}\,t^{n-m-k_{(h+1)}}\right)\left(1-q^{r_m+j_m^0-(i+1)}t^{n-m-k_{(h+1)}+1}\right)}\right) \\
&= \frac{\left(q^{r_m-r_{n-k_1}}\,t^{n-m-k_1+1},\, q^{r_m+j_m^0-r_{n-k_1}+1}t^{n-m-k_1};q\right)_{r_{n-k_1}-r_n+\sum_{s=1}^{n-1} j_s^0}}{\left(q^{r_m+j_m^0-r_{n-k_1}}\,t^{n-m-k_1+1},\, q^{r_m-r_{n-k_1}+1}t^{n-m-k_1};q\right)_{r_{n-k_1}-r_n+\sum_{s=1}^{n-1} j_s^0}} \cdot \\
&\quad \prod_{h=2}^{f-1} \frac{\left(q^{r_m-r_{n-k_{(h+1)}}}\,t^{n-m-k_{(h+1)}+1},\, q^{r_m+j_m^0-r_{n-k_{(h+1)}}+1}t^{n-m-k_{(h+1)}};q\right)_{r_{n-k_{(h+1)}}-r_{n-k_h}}}{\left(q^{r_m-r_{n-k_{(h+1)}}+1}t^{n-m-k_{(h+1)}},\, q^{r_m-r_{n-k_{(h+1)}}}\,t^{n-m-k_{(h+1)}+1};q\right)_{r_{n-k_{(h+1)}}-r_{n-k_h}}} \\
&= \frac{\left(q^{r_m-r_{n-k_1}}\,t^{n-m-k_1+1},\, q^{r_m+j_m^0-r_{n-k_1}+1}t^{n-m-k_1},\, q^{r_m-r_n+j_m^0+\sum_{s=1}^{n-1} j_s^0}t^{n-m-k_1+1},\, q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0+1}t^{n-m-k_1};q\right)_\infty}{\left(q^{r_m+j_m^0-r_{n-k_1}}\,t^{n-m-k_1+1},\, q^{r_m+j_m^0-r_{n-k_1}}\,t^{n-m-k_1},\, q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}\,t^{n-m-k_1+1},\, q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0+j_m^0+1}t^{n-m-k_1};q\right)_\infty} \\
&\quad \cdot \prod_{h=2}^{f-1} \frac{P}{Q} \\
&= W_{6,5}\left(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m-k_1};qt^{-1},\, q^{-j_m^0},\, q^{r_{n-k_1}-r_n+\sum_{s=1}^{n-1} j_s^0};q,\, q^{r_m+j_m^0-r_{n-k_1}}\,t^{n-m-k_1+1}\right) \\
&\quad \cdot \prod_{h=2}^{f-1} W_{6,5}\left(q^{r_m-r_{n-k_{(h+1)}}+j_m^0}t^{n-k_{(h+1)}-m};qt^{-1},\, q^{j_m^0},\, q^{r_{n-k_h}-r_{n-k_{(h+1)}}};q,\, q^{r_m-r_{n-k_h}}\,t^{n-m-k_{(h+1)}+1}\right).
\end{aligned}
$$

Where:

$$
\begin{aligned}
P &= \left(q^{r_m-r_{n-k_{(h+1)}}}\,t^{n-m-k_{(h+1)}+1},\, q^{r_m+j_m^0-r_{n-k_{(h+1)}}+1}t^{n-m-k_{(h+1)}},\, q^{r_m-r_{n-k_h}+1}t^{n-m-k_{(h+1)}},\right. \\
&\qquad \left. q^{r_m-r_{n-k_h}+j_m^0}t^{n-m-k_{(h+1)}+1};q\right)_\infty \\
Q &= \left(q^{r_m-r_{n-k_{(h+1)}}+1}t^{n-m-k_{(h+1)}},\, q^{r_m+j_m^0-r_{n-k_{(h+1)}}}\,t^{n-m-k_{(h+1)}+1},\, q^{r_m-r_{n-k_h}}\,t^{n-m-k_{(h+1)}+1},\right. \\
&\qquad \left. q^{r_m-r_{n-k_h}+j_m^0+1}t^{n-m-k_{(h+1)}};q\right)_\infty.
\end{aligned}
$$

□

**Proposition 6.3** *For $r_{n-k_1} \neq r_m$, the coefficient $T_{j_m^0,j_k^0,0}$ becomes*:

$$
T_{j_m^0,j_k^0,0} = W_{6,5}\left(q^{r_m-r_k-j_k^0}t^{k-m-1};q^{-j_k^0},\, qt^{-1},\, q^{-j_m^0};q,\, q^{r_m-r_k+j_m^0}t^{k-m}\right).
$$

**Proof:** Using properties of hypergeometric series and (16), we have:

$$
\begin{aligned}
T_{j_m^0,j_k^0,0} &= \prod_{i=r_k}^{r_k+j_k^0-1} \frac{(1-q^{r_m-i}\,t^{k-m-1})\left(1-q^{r_m+j_m^0-(i+1)}t^{k-m}\right)}{(1-q^{r_m-(i+1)}t^{k-m})\left(1-q^{r_m+j_m^0-i}\,t^{k-m-1}\right)} \\
&= \frac{\left(q^{r_m-r_k-j_k^0-1}t^{k-m-1},\, q^{r_m-r_k+j_m^0-j_k^0}t^{k-m};q\right)_{j_k^0}}{\left(q^{r_m-r_k-j_k^0}t^{k-m},\, q^{r_m-r_k+j_m^0-j_k^0+1}t^{k-m-1};q\right)_{j_k^0}}
\end{aligned}
$$

$$= \frac{\left(q^{r_m-r_k-j_k^0+1}t^{k-m-1}, q^{r_m-r_k+j_m^0-j_k^0}t^{k-m}, q^{r_m-r_k}t^{k-m}, q^{r_m-r_k+j_m^0+1}t^{k-m-1}; q\right)_\infty}{\left(q^{r_m-r_k-j_k^0}t^{k-m}, q^{r_m-r_k+j_m^0-j_k^0+1}t^{k-m-1}, q^{r_m-r_k+1}t^{k-m-1}, q^{r_m-r_k+j_m^0}t^{k-m}; q\right)_\infty}$$
$$= W_{6,5}\left(q^{r_m-r_k-j_k^0}t^{k-m-1}; q^{-j_k^0}, qt^{-1}, q^{-j_m^0}; q, q^{r_m-r_k+j_m^0}t^{k-m}\right).$$

□

**The Coefficients $\mathbf{T_{j_m^l,0}}$ and $\mathbf{T_{j_m^l,j_k^l,0}}$.**
For $l = \{1, \ldots, (r_n - 1)\}$, we compute the coefficients $T_{j_m^l,0}$ and $T_{j_m^l,j_k^l,0}$ in terms of very-well-poised hypergeometric series. To this end, fix $l$ and carry out the procedure previously described to obtain the chain:

$$\left(r_{n-k_1} + \sum_{c=0}^{l-1} j_{n-k_1}^c\right) < \cdots < \left(r_{n-k_f} + \sum_{c=0}^{l-1} j_{n-k_f}^c\right) = \left(r_m + \sum_{c=0}^{l-1} j_m^c\right).$$

**Notation 6.1** Set:

$$\sum_{c=0}^{l-1} j_m^c \equiv J_m$$
$$\sum_{s=1}^{n-1}\sum_{c=0}^{l} j_s^c \equiv J_s$$
$$\sum_{c=0}^{l-1} j_{n-k_i}^c \equiv J_{n-k_i}$$

**Proposition 6.4** *Suppose that $(r_{n-k_1} + J_{n-k_1}) = (r_m + J_m)$. It follows that $(r_m + J_m) \equiv (r_{n-1} + J_{n-1})$ and* (8) *becomes*:

$$T_{j_m^l,0} = \begin{cases} W_{6,5}\left(q^{r_m-r_n+J_m+J_s}t^{n-m-k_1}; qt^{-1}, q^{-j_m^l}, q^{r_{n-k_1}+J_{n-k_1}-r_nJ_s};\right. \\ \quad \left. q, q^{r_m-r_{n-k_1}-J_{n-k_1}+J_m+j_m^l}t^{n-m-k_1}\right) \cdot \prod_{h=2}^{f-1} W_{6,5}\left(q^{r_m-r_{n-k_i}-J_{n-k_i}+J_m}t^{n-k_i-m+1};\right. \\ \quad \left. qt^{-1}, q^{-j_m^l}, q^{r_{n-k_{i+1}}-r_{n-k_i}-J_{n-k_i}+J_{n-k_{i+1}}}; q, q^{r_m-r_{n-k_{i+1}}-J_{n-k_{i+1}}+J_m+j_m^l}t^{n-m-k_i}\right) \\ \hfill j_m^l \neq 0 \\ 1 \hfill j_m^l = 0. \end{cases}$$

**Proof:** Let $(r_{n-k_1} + J_{n-k_1}) = (r_m + J_m) \Rightarrow (r_m + J_m) \equiv (r_{n-1} + J_{n-1})$. There are two cases: $j_m^l = 0$ and $j_m^l \neq 0$.
For $j_m^l = 0$, by Proposition 4.3, we have that $T_{j_m^l,0} = 1$.

For $j_m^l \neq 0$, using properties of hypergeometric series, we have:

$$\begin{aligned} T_{j_m^l,0} &= \prod_{i=r_n-J_s}^{r_{n-1}+J_{n-1}-1} \frac{\left(1-q^{r_m+J_m-(i+1)}t^{n-m}\right)\left(1-q^{r_m+J_m+j_m^l-i}t^{n-m-1}\right)}{\left(1-q^{r_m+J_m-i}t^{n-m-1}\right)\left(1-q^{r_m+J_m+j_m^l-(i+1)}t^{n-m}\right)} \\ &= \frac{\left(q^{r_m+J_m-r_{n-1}-J_{n-1}}t^{n-m}, q^{r_m+J_m+j_m^l-r_{n-1}-J_{n-1}+1}t^{n-m-1}; q\right)_{r_{n-1}+J_{n-1}-r_n+J_s}}{\left(q^{r_m+J_m-J_{n-1}-r_{n-1}+1}t^{n-m-1}, q^{r_m-r_{n-1}-J_{n-1}+J_m+j_m^l}t^{n-m}; q\right)_{r_{n-1}-r_n+J_s+J_{n-1}}} = \frac{P}{Q} \\ &= W_{6,5}\big(q^{r_m-r_n+J_s+J_m}t^{n-m-1}; qt^{-1}, q^{-j_m^l}, q^{r_{n-1}-r_n+J_s+J_{n-1}}; \\ &\qquad q, q^{r_m-r_{n-1}+J_m-J_{n-1}+j_m^l}t^{n-m}\big). \end{aligned}$$

Where:

$$\begin{aligned} P &= \big(q^{r_m-r_{n-1}+J_m-J_{n-1}}t^{n-m}, q^{r_m-r_{n-1}+J_m-J_{n-1}+j_m^l+1}t^{n-m-1}, q^{r_m-r_n+J_s+J_m+1}t^{n-m-1}, \\ &\qquad q^{r_m-r_n+J_s+J_m+j_m^l}t^{n-m}; q\big)_\infty \\ Q &= \big(q^{r_m-r_n+J_s+J_m}t^{n-m}, q^{r_m-r_n+J_s+J_m+j_m^l+1}t^{n-m-1}, q^{r_m-r_{n-1}+J_m-J_{n-1}}t^{n-m+1}, \\ &\qquad q^{r_m-r_{n-1}+J_m-J_{n-1}+j_m^l}t^{n-m}; q\big)_\infty. \end{aligned}$$

□

**Remark 6.2** Since $(r_m + J_m) \equiv (r_{n-1} + J_{n-1}) \Rightarrow j_k^l = 0$ for all $m < k \leq (n-1)$, we have that $T_{j_m^l, j_k^l, 0} = 1$.

**Proposition 6.5** *Suppose that* $(r_{n-k_1} + J_{n-k_1}) \neq (r_m + J_m)$. *Obtaining the chain*

$$\left(r_{n-k_1} + J_{n-k_1}\right) < \cdots < \left(r_{n-k_f} + J_{n-k_f}\right) = (r_m + J_m),$$

*we have that*:

$$T_{j_m^l,0} = \begin{cases} W_{6,5}\big(q^{r_m-r_n+J_s+J_m}t^{n-m-k_1}; qt^{-1}, q^{-j_m^l}, q^{r_{n-k_1}-r_n+J_{n-k_1}+J_s}; \\ \quad q, q^{r_m-r_{n-k_1}-J_{n-k_1}+J_m+j_m^l}t^{n-m-k_1+1}\big) \cdot \prod_{h=2}^{f-1} W_{6,5}\big(q^{r_m-r_{n-k_{(h+1)}}+J_m+j_m^l-J_{n-k_{(h+1)}}} \\ \quad t^{n-k_{(h+1)}-m}; qt^{-1}, q^{j_m^l}, q^{r_{n-k_h}-r_{n-k_{(h+1)}}+J_{n-k_h}-J_{n-k_{(h+1)}}}; \\ \quad q, q^{r_m-r_{n-k_h}-J_{n-k_h}+J_m}t^{n-m-k_{(h+1)}}\big) & j_m^l \neq 0 \\ 1 & j_m^l = 0. \end{cases}$$

**Proof:** For $(r_{n-k_1} + J_{n-k_1}) \neq (r_m + J_m)$, using properties of hypergeometric series, (8) becomes:

$$\begin{aligned} T_{j_m^l,0} =& \prod_{i=r_n-J_s}^{r_{n-k_1}+J_{n-k_1}-1} \frac{\left(1-q^{r_m+J_m-(i+1)}t^{n-m-k_1+1}\right)\left(1-q^{r_m+J_m+j_m^l-i}t^{n-m-k_1}\right)}{\left(1-q^{r_m+J_m-i}t^{n-m-k_1}\right)\left(1-q^{r_m+J_m+j_m^l-(i+1)}t^{n-m-k_1+1}\right)} \\ &\cdot \prod_{h=2}^{f-1}\left(\prod_{i=r_{n-k_h}+J_{n-k_h}}^{r_{n-k_{(h+1)}}+J_{n-k_{(h+1)}}-1} \frac{\left(1-q^{r_m+J_m-(i+1)}t^{n-m-k_{(h+1)}+1}\right)\left(1-q^{r_m+J_m+j_m^l-i}t^{n-m-k_{(h+1)}}\right)}{\left(1-q^{r_m+J_m-i}t^{n-m-k_{(h+1)}}\right)\left(1-q^{r_m+J_m+j_m^l-(i+1)}t^{n-m-k_{(h+1)}+1}\right)}\right) \end{aligned}$$

$$
\begin{aligned}
&=\frac{\left(q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}}t^{n-m-k_1+1},\,q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}+j_m^l+1}t^{n-m-k_1};q\right)_{r_{n-k_1}-r_n+J_s+J_{n-k_1}}}{\left(q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}+j_m^l}t^{n-m-k_1+1},\,q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}}t^{n-m-k_1};q\right)_{r_{n-k_1}-r_n+J_{n-k_1}+J_s}}\\
&\quad\cdot\prod_{h=2}^{f-1}\frac{P_1}{Q_1}\\
&=\frac{P_2}{Q_2}\cdot\prod_{h=2}^{f-1}\frac{P_3}{Q_3}\\
&=W_{6,5}\big(q^{r_m-r_n+J_m+J_s}t^{n-m-k_1};qt^{-1},\,q^{-j_m^l},\,q^{r_{n-k_1}+J_{n-k_1}-r_n+J_s};\\
&\quad q,\,q^{r_m-r_{n-k_1}-J_{n-k_1}+J_m+j_m^l}t^{n-m-k_1+1}\big).\\
&\quad\cdot\prod_{i=2}^{f-1}W_{6,5}\big(q^{r_m-r_{n-k_{(h+1)}}+J_m+j_m^l-J_{n-k_{(h+1)}}}t^{n-k_{(h+1)}-m};\\
&\quad qt^{-1},\,q^{j_m^l},\,q^{r_{n-k_h}-r_{n-k_{(h+1)}}+J_{n-k_h}-J_{n-k_{(h+1)}}};q,\,q^{r_m-r_{n-k_h}-J_{n-k_h}+J_m}t^{n-m-k_{(h+1)}}\big).
\end{aligned}
$$

Where:

$$
\begin{aligned}
P_1 &= \big(q^{r_m-r_{n-k_{(h+1)}}+J_m-J_{n-k_{(h+1)}}}t^{n-m-k_{(h+1)}+1},\\
&\qquad q^{r_m-r_{n-k_{(h+1)}}+J_m-J_{n-k_{(h+1)}}+j_m^l+1}t^{n-m-k_{(h+1)}};q\big)_{r_{n-k_{(h+1)}}-r_{n-k_h}+J_{n-k_{(h+1)}}-J_{n-k_h}}\\
Q_1 &= \big(q^{r_m-r_{n-k_{(h+1)}}+J_m-J_{n-k_{(h+1)}}+1}t^{n-m-k_{(h+1)}},\\
&\qquad q^{r_m-r_{n-k_{(h+1)}}+J_m-J_{n-k_{(h+1)}}+j_m^l}t^{n-m-k_{(h+1)}+1};q\big)_{r_{n-k_{(h+1)}}+J_{n-k_{(h+1)}}-r_{n-k_h}-J_{n-k_h}}\\
P_2 &= \big(q^{r_m+J_m-r_{n-k_1}-J_{n-k_1}}t^{n-m-k_1+1},\,q^{r_m+J_m+j_m^l-r_{n-k_1}-J_{n-k_1}+1}t^{n-m-k_1},\\
&\qquad q^{r_m-r_n+J_s+J_m+j_m^l}t^{n-m-k_1+1},\,q^{r_m-r_n+J_s+J_m+1}t^{n-m-k_1};q\big)_\infty\\
Q_2 &= \big(q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}+j_m^l}t^{n-m-k_1+1},\,q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}+j_m^l}t^{n-m-k_1},\\
&\qquad q^{r_m-r_n+J_s+J_m}t^{n-m-k_1+1},\,q^{r_m-r_s+J_n+J_m+j_m^l+1}t^{n-m-k_1};q\big)_\infty\\
P_3 &= \big(q^{r_m-r_{n-k_{(h+1)}}+J_m-J_{n-k_{(h+1)}}}t^{n-m-k_{(h+1)}+1},\,q^{r_m-r_{n-k_{(h+1)}}+J_m-J_{n-k_{(h+1)}}+j_m^l+1}t^{n-m-k_{(h+1)}},\\
&\qquad q^{r_m-r_{n-k_h}+J_{n-k_h}+J_m+1}t^{n-m-k_{(h+1)}},\,q^{r_m-r_{n-k_h}-J_{n-k_h}+J_m+j_m^l}t^{n-m-k_{(h+1)}+1};q\big)_\infty\\
Q_3 &= \big(q^{r_m-r_{n-k_{(h+1)}}-J_{n-k_{(h+1)}}+J_m+1}t^{n-m-k_{(h+1)}},\,q^{r_m+J_m+j_m^l-r_{n-k_{(h+1)}}-J_{n-k_{(h+1)}}}t^{n-m-k_{(h+1)}+1},\\
&\qquad q^{r_m-r_{n-k_h}+J_m-J_{n-k_h}}t^{n-m-k_{(h+1)}+1},\,q^{r_m-r_{n-k_h}-J_{n-k_h}+J_m+j_m^l+1}t^{n-m-k_{(h+1)}};q\big)_\infty.
\end{aligned}
$$

□

**Proposition 6.6** *For $T_{j_m^l,j_k^l,0}$, $m<k\le(n-1)$, we have:*

$$
T_{j_m^l,j_k^l,0}=\begin{cases}W_{6,5}\big(q^{r_m-r_k-J_k+J_m-j_k^l}t^{k-m-1};\\ \quad q^{-j_k^l},\,qt^{-1},\,q^{-j_m^l};q,\,q^{r_m-r_k+J_m+J_k+j_m^l}t^{k-m}\big) & j_m^l\neq 0 \text{ and } j_k^l\neq 0\\ 1 & \text{otherwise.}\end{cases}
$$

**Proof:** Using properties of hypergeometric series, (9) becomes:

$$
\begin{aligned}
T_{j^l_m, j^l_k, 0} &= \prod_{i=r_k+J_k}^{r_k+J_k+j^l_k-1} \frac{(1-q^{r_m+J_m-i}t^{k-m-1})\big(1-q^{r_m+J_m+j^l_m-(i+1)}t^{k-m}\big)}{\big(1-q^{r_m+J_m-(i+1)}t^{k-m}\big)\big(1-q^{r_m+J_m+j^l_m-i}t^{k-m-1}\big)} \\
&= \frac{\big(q^{r_m-r_k+J_m-J_k-j^l_k+1}t^{k-m-1}, q^{r_m-r_k+J_m-J_k+j^l_m-j^l_k}t^{k-m}; q\big)_{j^l_k}}{\big(q^{r_m-r_k-J_k+J_m-j^l_k}t^{k-m}, q^{r_m-r_k+J_m-J_k+j^l_m-j^l_k+1}t^{k-m-1}; q\big)_{j^l_k}} \\
&= \frac{P}{Q} \\
&= W_{6,5}\big(q^{r_m-r_k-J_k+J_m-j^l_k}t^{k-m-1}; q^{-j^l_k}, qt^{-1}, q^{-j^l_m}; q, q^{r_m-r_k+J_m+J_k+j^l_m}t^{k-m}\big).
\end{aligned}
$$

Where:

$$
\begin{aligned}
P &= \big(q^{r_m-r_k-J_k+J_m-j^l_k+1}t^{k-m-1}, q^{r_m-r_k+J_m-J_k+j^l_m-j^l_k}t^{k-m}, \\
&\quad q^{r_m-r_k-J_k+J_m}t^{k-m}, q^{r_m-r_k+J_m-J_k+j^l_m+1}t^{k-m-1}; q\big)_\infty \\
Q &= \big(q^{r_m-r_k-J_k+J_m-j^l_k}t^{k-m}, q^{r_m-r_k+J_m-J_k+j^l_m-j^l_k+1}t^{k-m-1}, \\
&\quad q^{r_m-r_k-J_k+J_m+1}t^{k-m-1}, q^{r_m-r_k+J_m-J_k+j^l_m}t^{k-m}; q\big)_\infty.
\end{aligned}
$$

□


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