Journal of Algebraic Combinatorics, 21, 137–146, 2005

Katriel's Operators for Products of Conjugacy Classes of $\mathfrak{S}_n$

ALAIN GOUPIL* goupil@lacim.uqam.ca
LaCIM, Université du Québec à Montréal

DOMINIQUE POULALHON Dominique.Poulalhon@liafa.jussieu.fr
LIAFA, Université Paris 7, France

GILLES SCHAEFFER† Gilles.Schaeffer@lix.polytechnique.fr
LIX, École Polytechnique, France

Received January 24, 2003; Revised March 1, 2004; Accepted March 8, 2004

Abstract. We define a family of differential operators indexed with fixed point free partitions. When these differential operators act on normalized power sum symmetric functions $q_\lambda(x)$, the coefficients in the decomposition of this action in the basis $q_\lambda(x)$ are precisely those of the decomposition of products of corresponding conjugacy classes of the symmetric group $\mathfrak{S}_n$. The existence of such operators provides a rigorous definition of Katriel's elementary operator representation of conjugacy classes and allows to prove the conjectures he made on their properties.

Keywords: conjugacy classes, symmetric functions, operator, structure constants

Notational conventions

In the whole text, λ, μ, ν stand for partitions, with ℓ_i, m_i, n_i denoting the multiplicities of their parts. Permutations are denoted ρ, σ, τ. Finally recall that $c(\rho)$ is the cycle type of a permutation ρ and that $\pi(\lambda)$ is the canonical permutation of cycle type λ.

1. Introduction

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of weight n and length $\ell(\lambda) = k$, i.e. a finite non increasing sequence of k positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ summing up to n. We write $\lambda \vdash n$ or $|\lambda| = n$, and $\lambda = 1^{\ell_1} 2^{\ell_2} \dots n^{\ell_n}$ when ℓ_i parts of λ are equal to i ($i = 1 \dots n$). Given a permutation $\sigma \in \mathfrak{S}_n$ we denote by $c(\sigma)$ its cycle type, that is the partition giving the length of its cycles. Conversely, given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, the canonical permutation $\pi(\lambda)$

*Work partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.
†Work partially supported by EC's Research Training Network *Algebraic Combinatorics in Europe* (grant HPRN-CT-2001-00272).

of cycle type λ is the permutation with cycles $(\lambda_1 + \cdots + \lambda_{i-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_i)$ for $1 \leq i \leq k$. Classically, to a partition λ, one associates the conjugacy class $\mathcal{C}_\lambda$, that is the set of permutations of cycle type λ, and following [15] we write $z_\lambda = 1^{\ell_1}\ell_1!2^{\ell_2}\ell_2!\ldots n^{\ell_n}\ell_n!$, so that $|\mathcal{C}_\lambda| = n!/z_\lambda$.

Let $\mathbb{Q}[\mathfrak{S}_n]$ be the group algebra of the symmetric group over the field $\mathbb{Q}$ of rational numbers and let $\mathcal{Z}_n$ be the center of this group algebra. The formal sum $K_\lambda = \sum_{\sigma\in\mathcal{C}_\lambda}\sigma$ of the permutations in a conjugacy class $\mathcal{C}_\lambda$ belongs to $\mathcal{Z}_n$, and the set $\{K_\lambda\}_{\lambda\vdash n}$ of these formal sums forms a linear basis for the center $\mathcal{Z}_n$.

Here we consider K_λ or $\mathcal{C}_\lambda$ as an operator acting on $\mathcal{Z}_n$ by multiplication. The multiplicative structure of $\mathcal{Z}_n$ has been extensively studied in terms of *connexion coefficients* [5, and ref. therein], also called *structure constants* [7, 17, and ref. therein]. These coefficients are defined for all triples of partitions (λ, μ, ν) of n by

$$K_\lambda \cdot K_\mu = \sum_\nu c^\nu_{\lambda\mu} K_\nu. \tag{1}$$

In a set of conjectures presented at the conference FPSAC'98 [13] and derived from previous weaker conjectures [10, 12, and ref. therein], Katriel looks for expressions of the conjugacy classes as sums of some loosely defined *elementary* operators. Many examples of explicit expressions of conjugacy classes indexed by small partitions in terms of these elementary operators are given in [13] and conjectures are made on the form of the coefficients. In particular, Katriel requires his expressions to depend only on the reduced cycle type: the *reduced* partition of a partition $\lambda = 1^{\ell_1}2^{\ell_2}\ldots k^{\ell_k}$ is the partition $\bar{\lambda} = 2^{\ell_2}\ldots k^{\ell_k}$, and the reduced cycle type of a permutation is defined accordingly. A partition is *reduced* if it is equal to its reduced partition, that is, if it contains no part equal to 1.

In order to define rigorously Katriel's elementary operators we use a representation of the action of conjugacy classes on $\mathcal{Z}_n$ by an action of differential operators on the space of symmetric functions. Once stated in this form (Definitions 3 and 4) the various observations of Katriel on this representation are relatively easy to prove: our main result (Theorem 1) completely settles the conjectures of [10, 12, 13]. Our approach is reminiscent of Goulden and Jackson's use of differential operators in slightly different context (see [5] and ref. therein). Since these results were presented at the 12th International conference on *Formal Power Series and Algebraic Combinatorics* [8], Lascoux and Thibon have shown that the differential operators that we introduce in an elementary way can also be constructed at a more algebraic level using Gaussian integrals of complex square matrices and vertex operators [14].

Apart from considering the operators themselves, as we do here, Katriel also formulated conjectures about their eigenvalues, that is the central characters of the symmetric group [9, 11]. Remarkably, this approach also leads to a representation of the structure constants $c^\nu_{\lambda\mu}$ but this time as linearization constants for a new basis of non homogeneous symmetric functions [2]. Finally it should be mentioned that similar ideas can also be applied to the calculus of inner tensor, or Kronecker product, of irreducible representations of the symmetric group, with interesting combinatorial consequences ([6] and [4]).

2. Symmetric functions

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of indeterminates and let $\Lambda = \Lambda_{\mathbb{Q}}[\mathbf{x}]$ be the ring of symmetric functions in $x_1, x_2, \ldots$ over the field $\mathbb{Q}$ of rational numbers. The power sums symmetric functions $p_\lambda(\mathbf{x})$ are defined by

$$p_r(\mathbf{x}) = x_1^r + x_2^r + \cdots,$$
$$p_\lambda(\mathbf{x}) = p_{\lambda_1}(\mathbf{x})\; p_{\lambda_2}(\mathbf{x})\; \ldots\; p_{\lambda_k}(\mathbf{x}).$$

The ordinary scalar product $<,>$ on Λ is defined on the linear basis $\{p_\lambda\}_\lambda$ by:

$$\forall\, \lambda, \mu \qquad < p_\lambda, p_\mu >= z_\lambda \delta_{\lambda,\mu}.$$

where $\delta_{\lambda,\mu}$ is the Kronecker delta. We need the differential operators $p_\lambda{}^\perp$ known in the literature as Hammond's operators (see [15] or [16]) obtained from the following definition:

Definition 1 For any symmetric function $f \in \Lambda$, let $f^\perp$ be the adjoint operator to the multiplication by f in λ with respect to the scalar product $<,>$:

$$\forall\, g, h \in \Lambda \qquad < fg, h > \;=\; < g, f^\perp h > .$$

In particular the operator $p_\lambda{}^\perp$ is conveniently described as a differential operator on Λ:

$$p_\lambda{}^\perp = \lambda_1\lambda_2\ldots\lambda_k \,\frac{\partial^k}{\partial p_{\lambda_1}\partial p_{\lambda_2}\ldots\partial p_{\lambda_k}}.$$

The use of such operators in relation with connexion coefficients is not new and can be found for instance in [5]. We are interested in representing the multiplication by a conjugacy class as an action of an operator on the space of symmetric functions. More precisely, we consider the normalized power sums functions $q_\lambda = p_\lambda/z_\lambda$ with the property that $q = \{q_\lambda\}_\lambda$ is an orthogonal basis of Λ, dual to $\{p_\lambda\}_\lambda$. Given a partition λ, we look for an operator G_λ acting on the basis q and satisfying the condition

$$\forall \mu, \nu \vdash |\lambda|, \qquad G_\lambda \cdot q_\mu\,[q_\nu] \;=\; (K_\lambda \cdot K_\mu)\,[K_\nu]\,.$$

where $f\,[g]$ means the coefficient of g in f. Here is a trivial way to do this:

Definition 2 Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n. For any fixed permutation $\rho \in \mathcal{C}_\lambda$, define the operator $G_\lambda : \Lambda \to \Lambda$ by

$$G_\lambda = \frac{1}{z_\lambda}\sum_{\sigma\in\mathfrak{S}_n} p_{c(\rho\sigma)}p_{c(\sigma)}{}^\perp \;=\; \sum_{\mu,\nu\vdash n}\frac{c_{\lambda\mu}^{\nu}}{z_\mu}p_\nu p_\mu{}^\perp. \tag{2}$$

From this definition and the orthogonality relation $p_\lambda^\perp \cdot p_\mu = z_\lambda \delta_{\lambda,\mu}$ for partitions of the same weight, it is immediate that for any partitions μ, λ of n

$$G_\lambda \cdot q_\mu = \sum_{\nu \vdash n} c_{\lambda\mu}^\nu q_\nu.$$

The operators G_λ are not very interesting because their definition uses the structure constants $c_{\lambda\mu}^\nu$ which they are meant to produce; however they provide an easy introduction to what we mean by representing the multiplication in $\mathcal{Z}_n$ by the action of an operator on symmetric functions.

Our aim is to define a more interesting family $H = \{H_{\bar\lambda}\}_{\bar\lambda}$ of operators, indexed with reduced partitions $\bar\lambda$ and satisfying

$$\forall\lambda, \quad \forall\mu, \nu \vdash |\lambda|, \qquad H_{\bar\lambda} \cdot q_\mu\,[q_\nu] \;=\; (K_\lambda \cdot K_\mu)\,[K_\nu]\,.$$

3. Restricted and extended permutations

In order to define our operators $H_{\bar\lambda}$, we need some elementary results on restricted permutations. For a subset S of $[n] = \{1, \ldots, n\}$, and a permutation $\sigma \in \mathfrak{S}_n$, let $\sigma_{|S}$ be the restriction to S obtained by removing all the elements of $[n]\backslash S$ from the disjoint-cycles presentation of σ. More formally $\sigma_{|S}$ is such that, for all $i \in S$, $\sigma_{|S}(i) = \sigma^{k_i}(i)$ where k_i is the least positive integer such that $\sigma^{k_i}(i) \in S$.

For the definition of the operators $H_{\bar\lambda}$ we shall need the following observation: If ρ, σ are two permutations of $\mathfrak{S}_n$ and $S \in [n]$ is such that $[n] \setminus S$ contains only fixed points of ρ, we have $(\rho\sigma)_{|S} = \rho_{|S}\sigma_{|S}$. Moreover $\rho\sigma$ can be recovered from $(\rho\sigma)_{|S}$ and σ by inserting in $(\rho\sigma)_{|S}$ after each $i \in S$ the block that separates i and $\rho_{|S}(i)$ in the presentation of σ as a product of disjoint cycles.

Example If $\rho = (1\ 2\ 3)$ and $\sigma = (1\ aaa\ 2\ bbb)\ (3\ ccc)$, then $\sigma_{|[3]} = (1\ 2)\ (3)$, $(\rho\sigma)_{|[3]} = (1\ 3)\ (2)$ and $\rho\sigma = (1\ aaa\ 3\ ccc)\ (2\ bbb)$.

Conversely, given a permutation σ_0 of $\mathfrak{S}_p$, we shall use two ways to extend σ_0 to a permutation of $\mathfrak{S}_n$. First, given a composition $i = (i_1, \ldots, i_p)$ with $i_j \geq 1$, we are interested in permutations obtained from σ_0 by inserting a block of size $i_j - 1$ after each point $j \in [p]$ to obtain a permutation of size $n = |i|$. Let $\sigma_0^{\uparrow i}$ denote one of these permutations.

Second, For $p \leq n$, any permutation σ_0 of $\mathfrak{S}_p$ can be extended naturally to a permutation $\sigma \in \mathfrak{S}_n$ by adding fixed points to σ_0: $\sigma(i) = i$ for $i > p$. We call σ the *natural extension* of σ_0 in $\mathfrak{S}_n$. In this way, $\sigma_0 \in \mathfrak{S}_p$ acts by left multiplication on $\mathfrak{S}_n$ through its natural extension σ. Observe that for any partition λ, the canonical permutation $\pi(\lambda)$ is the natural extension of the canonical permutation $\pi(\bar\lambda)$.

4. The operator $H_{\bar\lambda}$ and Katriel's notations

We give two equivalent definitions of the operators $H_{\bar\lambda}$.

Definition 3 Let $\bar{\lambda}$ be a reduced partition of weight p, and recall that $\pi(\bar{\lambda})$ denotes the canonical permutation of cycle type $\bar{\lambda}$. Then the operator $H_{\bar{\lambda}} : \Lambda \to \Lambda$ is defined by:

$$H_{\bar{\lambda}} = \frac{1}{z_{\bar{\lambda}}} \sum_{\sigma_0 \in \mathfrak{S}_p} \sum_{i_1,\ldots,i_p \geq 1} p_{c(\rho\sigma)} p^{\perp}_{c(\sigma)}, \quad \text{where} \begin{cases} \sigma = \sigma_0^{\uparrow i}, \\ \rho = \pi(\bar{\lambda} 1^{|i|-p}). \end{cases} \tag{3}$$

The fact that the cycle type of $\rho\sigma$ depends only on the composition $i = (i_1, \ldots, i_p)$, and not on the elements inserted in σ_0 to give σ, is a consequence of the previous discussion on restricted permutations.

The operator $H_{\bar{\lambda}}$ is closely related to Katriel's bracket operators (which are not defined with complete rigor in his papers). A simple variation on his notation is:

$$\langle\!\langle i_1 + i_2; i_3 \mid i_1; i_2 + i_3 \rangle\!\rangle \quad \text{stands for} \quad \sum_{i_1,i_2,i_3 \geq 1} p_{[i_1+i_2,i_3]} p^{\perp}_{[i_1,i_2+i_3]},$$

where the brackets [,] denote multisets of integers (i.e. partitions up to reordering). A further simplification of this notation (even closer to Katriel's) is to replace each variable by its index and write sums as cycles:

$$\langle\!\langle (1,2)(3) \mid (1)(2,3) \rangle\!\rangle \quad \text{stands for} \quad \langle\!\langle i_1 + i_2; i_3 \mid i_1; i_2 + i_3 \rangle\!\rangle$$

Let us rewrite Definition 3 with this notation:

Definition 4 Let $\bar{\lambda}$ be a reduced partition of weight p, and recall that $\pi(\bar{\lambda})$ is the canonical partition of cycle type $\bar{\lambda}$. Then

$$H_{\bar{\lambda}} = \frac{1}{z_{\bar{\lambda}}} \sum_{\sigma_0 \in \mathfrak{S}_p} \langle\!\langle \pi(\bar{\lambda})\sigma_0 \mid \sigma_0 \rangle\!\rangle. \tag{4}$$

Finally, Katriel conjectured a symmetry in the coefficients, which allows the introduction of a last notation:

$$\langle P \mid Q \rangle \quad \text{stands for} \quad \langle\!\langle P \mid Q \rangle\!\rangle + \langle\!\langle Q \mid P \rangle\!\rangle$$

Examples We keep the intermediate notation which we find more descriptive.

$$\begin{aligned}
H_1 &= \langle\!\langle i_1; i_1 \rangle\!\rangle = \sum_{i_1 \geq 1} p_{i_1} p^{\perp}_{i_1} \\
H_2 &= \frac{1}{2}\langle\!\langle i_1; i_2 \mid i_1 + i_2 \rangle\!\rangle + \frac{1}{2}\langle\!\langle i_1 + i_2 \mid i_1; i_2 \rangle\!\rangle = \frac{1}{2}\langle i_1 + i_2 \mid i_1; i_2 \rangle \\
&= \frac{1}{2}\left(\sum_{i_1,i_2 \geq 1} p_{i_1,i_2} p^{\perp}_{i_1+i_2} + \sum_{i_1,i_2 \geq 1} p_{i_1+i_2} p^{\perp}_{i_1,i_2} \right) \\
&= \frac{1}{2} p_2 p^{\perp}_{11} + \frac{1}{2} p_{11} p^{\perp}_2 + p_3 p^{\perp}_{21} + p_{21} p^{\perp}_3 + p_4 p^{\perp}_{31} \frac{1}{2} p_4 p^{\perp}_{22} + \frac{1}{2} p_{22} p^{\perp}_4 + \cdots
\end{aligned}$$

$$\begin{aligned}
H_3 &= \frac{1}{3}\langle\langle i_1+i_2+i_3|i_1,i_2,i_3\rangle\rangle + \frac{1}{3}\langle\langle i_1,i_2,i_3|i_1+i_2+i_3\rangle\rangle + \frac{1}{3}\langle\langle i_1+i_2+i_3|i_1+i_2+i_3\rangle\rangle \\
&\quad + \frac{1}{3}\langle\langle i_1+i_2,i_3|i_1,i_2+i_3\rangle\rangle + \frac{1}{3}\langle\langle i_1+i_3,i_2|i_2,i_1+i_3\rangle\rangle + \frac{1}{3}\langle\langle i_2+i_3,i_1|i_3,i_1+i_2\rangle\rangle \\
&= \langle\langle i_1+i_2,i_3|i_1,i_2+i_3\rangle\rangle + \frac{1}{3}\langle i_1+i_2+i_3|i_1,i_2,i_3\rangle + \frac{1}{3}\langle\langle i_1+i_2+i_3|i_1+i_2+i_3\rangle\rangle \\
&= \frac{1}{3}\sum_{\substack{(i_1,i_2,i_3)\geq 1\\ n=i_1+i_2+i_3}} \left(p_n p^{\perp}_{(i_1,i_2,i_3)} + p_{(i_1,i_2,i_3)}p^{\perp}_n + p_n p^{\perp}_n + 3p_{(i_1+i_3,i_2)}p^{\perp}_{(i_1+i_2,i_3)}\right) \\
H_{2^2} &= \frac{1}{8}\langle i_1;i_2;i_3;i_4|i_1+i_2;i_3+i_4\rangle + \frac{1}{4}\langle\langle i_1+i_2;i_3;i_4|i_1;i_2;i_3+i_4\rangle\rangle + \frac{1}{4}\langle\langle i_1+i_2;i_3+i_4|i_1+i_3;i_2+i_4\rangle\rangle \\
&\quad + \langle\langle i_1+i_2+i_3;i_4|i_1;i_2+i_3+i_4\rangle\rangle + \frac{1}{2}\langle i_1+i_2+i_3+i_4|i_1+i_2;i_3;i_4\rangle + \frac{1}{4}\langle\langle i_1+i_2+i_3+i_4|i_1+i_2+i_3+i_4\rangle\rangle
\end{aligned}$$

Katriel's *global conjecture* in [13] is that K_λ "=" $H_{\bar\lambda}$. More formally, we shall prove the following theorem.

Theorem 1 (*Global Conjecture*) *Let* λ, μ *and* ν *be partitions of* n, *then*

$$H_{\bar\lambda}\cdot q_\mu\,[q_\nu] = (K_\lambda\cdot K_\mu)[K_\nu].$$

Observe that applying a permutation of indices in an elementary bracket operator does not change it. Collecting terms that are equivalent under relabelling of indices, one can form a sum over "distinct" elementary operators (see examples). The coefficient of each elementary operator is thus given an immediate interpretation, in accordance with the *central conjecture* of Katriel in [13].

Finally an immediate consequence of Definition 4 is that the expansions $H_{\bar\lambda} = \sum_{\mu,\nu} a^{\nu}_{\mu,\lambda}\times p_\nu p^{\perp}_\mu$, are symmetric in μ and ν. In terms of Katriel's notation, this proves that each non symmetric elementary operator $\langle\langle P\mid Q\rangle\rangle$ appears with the same coefficient as its mirror image $\langle\langle Q\mid P\rangle\rangle$, so that, as conjectured again by Katriel, the notation $\langle P\mid Q\rangle$ can be systematically used.

Proof (of Theorem 1): Let $p = |\bar\lambda|$ be the weight of $\bar\lambda$, $\rho_0 = \pi(\bar\lambda)$ the associated canonical permutation, and ρ its natural extension in $\mathfrak{S}_n$, so that $\rho = \pi(\bar\lambda 1^{n-p})$. On the one hand,

$$\begin{aligned}
K_\lambda\cdot K_\mu[K_\nu] &= \frac{|\mathcal{C}_\lambda|}{|\mathcal{C}_\nu|}\,K_\mu\cdot K_\nu[K_\lambda] = \frac{z_\nu}{z_\lambda}\,\mathrm{Card}\{(\sigma,\tau)\in\mathcal{C}_\mu\times\mathcal{C}_\nu\mid \rho\sigma=\tau\} \\
&= \frac{z_\nu}{z_\lambda}\,\mathrm{Card}\{\sigma\in\mathcal{C}_\mu\mid\rho\sigma\in\mathcal{C}_\nu\} \\
&= \frac{z_\nu}{z_\lambda}\sum_{\sigma_0\in\mathfrak{S}_p}\mathrm{Card}\{\sigma\in\mathcal{C}_\mu\mid\sigma_{|p}=\sigma_0,\ \rho\sigma\in\mathcal{C}_\nu\} \qquad (5)
\end{aligned}$$

Since the support of ρ, i.e. the set of elements moved by ρ, is included in $[p]$, the discussion of Section 3 applies: Assume that σ is of the form $\sigma'\sigma''$ where σ' is a permutation obtained from σ_0 by inserting a block of size $i_j - 1$ after each point $j \in [p]$ in the cycles of σ_0, and σ' and σ'' have disjoint support. Then $\rho\sigma = \tau'\sigma''$ where τ' is obtained by inserting the same blocks after the same points as before but in the permutation $\tau_0 = \rho_0\sigma_0$. In particular the cycle types μ', ν' and μ'' of σ', τ' and σ'' must satisfy $\mu = \mu' + \mu''$ and $\nu = \nu' + \mu''$ (where $+$ denotes the disjoint union of parts). In particular, μ'' is imposed by the choice of μ' or ν'.

Observe now that given a composition $i = (i_1, \ldots, i_p)$, the exact composition of the blocks that are inserted in σ_0 to produce σ' has no influence on the resulting cycle types μ' and ν'. This allows to define the set $C(\lambda; \mu, \nu)$ of compositions such that the corresponding pair (μ', ν') satisfies $\mu = \mu' + \mu''$ and $\nu = \nu' + \mu''$ for some μ'' that depends on $(i_1, \ldots, i_p)$. Given the composition i, the number of ways to fill in the blocks is $\binom{n-p}{n-|i|}\cdot(|i| - p)!$ (choose the $|i| - p$ elements inserted and use a permutation to distribute them in the p blocks of size $i_1, \ldots, i_p$), and the number of ways to choose σ'' is $|\mathcal{C}_{\mu''}| = (n - |i|)!/z_{\mu''}$. Finally we have:

$$\begin{aligned}\mathrm{Card}\{\sigma \in \mathcal{C}_\mu \mid \sigma_{|p} = \sigma_0,\ \rho\sigma \in \mathcal{C}_\nu\}\\ = \sum_{i=(i_1,\ldots,i_p)\in C(\lambda;\mu,\nu)} \binom{n-p}{n-|i|} \cdot (|i| - p)! \cdot \frac{(n - |i|)!}{z_{\mu''}},\end{aligned} \tag{6}$$

where μ'' depends on $i = (i_1, \ldots, i_p)$ as before with $|i| = |\mu'| = |\nu'|$. Observing that $z_\lambda = z_{\bar\lambda}(n - p)!$, and simplifying we obtain from (5) and (6)

$$K_\lambda \cdot K_\mu\,[K_\nu] \;=\; \frac{z_\nu}{z_{\bar\lambda}} \sum_{\sigma_0\in\mathfrak{S}_p} \sum_{(i_1,\ldots,i_p)\in C(\lambda;\mu,\nu)} \frac{1}{z_{\mu''}}\,.$$

On the other hand, from Definition 3 we have

$$H_{\bar\lambda} \cdot q_\mu \;=\; \frac{1}{z_{\bar\lambda}} \sum_{\sigma_0\in\mathfrak{S}_p} \sum_{i_1,\ldots,i_p\geq 1} p_{c(\rho\sigma')} \frac{p^\perp_{c(\sigma')}p_\mu}{z_\mu} \quad \text{where } \begin{cases}\sigma' = \sigma_0^{\uparrow i},\\ \rho = \pi(\bar\lambda 1^{|i|-p}).\end{cases}$$

Taking $\mu' = c(\sigma')$ we observe that

$$\frac{\partial p_\mu}{\partial p_{\mu'}} = \begin{cases} 0 & \text{if } \mu' \not\subset \mu,\\ \dfrac{\ell_1!\ldots\ell_n!}{(\ell_1 - \ell_1')!\ldots(\ell_n - \ell_n')!}\, p_{\mu-\mu'} & \text{otherwise.}\end{cases}$$

Therefore, with $C(\mu)$ denoting the set of compositions $i = (i_1, \ldots, i_p)$ such that $\mu' = c(\sigma_0^{\uparrow i}) \subset \mu$, we have

$$H_{\bar\lambda} \cdot q_\mu \;=\; \frac{1}{z_{\bar\lambda}} \sum_{\sigma_0\in\mathfrak{S}_p} \sum_{(i_1,\ldots i_p)\in C(\mu)} \frac{p_{c(\rho\sigma')+\mu-c(\sigma')}}{z_{\mu-c(\sigma')}}, \quad \text{where } \begin{cases}\sigma' = \sigma_0^{\uparrow i},\\ \rho = \pi(\bar\lambda 1^{|i|-p}).\end{cases}$$

Finally, taking the coefficient of q_ν in the above identity, we obtain

$$H_{\bar\lambda} \cdot q_\mu \, [q_\nu] \;=\; \frac{z_\nu}{z_{\bar\lambda}} \sum_{\sigma_0 \in \mathfrak{S}_p} \; \sum_{(i_1,\ldots i_p) \in C(\lambda;\mu,\nu)} \frac{1}{z_{\mu - c(\sigma')}} \;=\; (K_\lambda \cdot K_\mu)[K_\nu].$$

□

5. Families of connexion coefficients

For a reduced partition $\bar\lambda$, let $K_{\bar\lambda}(n)$ be the sum in $\mathcal{Z}_n$ of all permutations with reduced cycle type $\bar\lambda$ if $n \geq |\bar\lambda|$, and 0 otherwise.

Let $\bar\lambda$, $\bar\mu$ be reduced partitions and define the coefficients $c^{\bar\nu}_{\bar\lambda\bar\mu}(n)$ by

$$K_{\bar\lambda}(n) \cdot K_{\bar\mu}(n) = \sum_{\bar\nu} c^{\bar\nu}_{\bar\lambda\bar\mu}(n) K_{\bar\nu}(n). \tag{7}$$

In [3] Farahat and Higman prove that these coefficients are polynomials in n. This also follows from Theorem 1: let k (resp. h) be the largest part of $\bar\mu$ (resp. $\bar\nu$), and apply the elementary operator $H_{\bar\lambda}$ to $q_{\bar\mu 1^{n-|\bar\mu|}}$; the non-zero contributions are of the form

$$p_\alpha p^{\perp}_\beta q_{\bar\mu 1^{n-|\bar\mu|}} \, [q_{\bar\nu 1^{n-|\bar\nu|}}]$$

where α and β are partitions of length $|\bar\lambda|$ having parts of size at most k and h respectively. There are finitely many such partitions and the contribution of this term is a polynomial in n of degree a_1, the number of 1's in α.

Theorem 1 is a generalization of this result in the sense that it proves that other families of coefficients are polynomials in n. For instance, for any reduced partition $\bar\lambda$ with even weight, the coefficient

$$\mathfrak{b}_{\bar\lambda}(n) = \; K_{\bar\lambda}(2n) \cdot K_{2^n}(2n) \, [K_{2^n}(2n)]$$

is a polynomial in n. The expression of H_{2^2} presented before gives

$$\mathfrak{b}_{2^2}(n) = \frac{1}{4} \;\; p_{2^2} p^{\perp}_{2^2} q_{2^n} \, [q_{2^n}] \;=\; n(n-1).$$

It should be observed that, while Theorem 1 yields a general proof that the coefficients $\mathfrak{b}_{\bar\lambda}(n)$ are polynomials, the actual computation of these polynomials is often easier by elementary techniques. For instance we claim the following.

Proposition 1 *For positive integers k and n such that k is even and $k \leq n/2$ we have*

$$\begin{aligned} \mathfrak{b}_{2^{2k}}(n) \;=\; H_{2^{2k}} q_{2^n}[q_{2^n}] &= \left(\frac{1}{2^{2k}(2k)!}\right) \frac{(2k)!}{k!} p_{2^{2k}} p^{\perp}_{2^{2k}} q_{2^n}[q_{2^n}] \\ &= K_{2^{2k}}(2n) \cdot K_{2^n}(2n)[K_{2^n}] = \binom{n}{2k} \frac{(2k)!}{k!} \end{aligned}$$

Proof: The proof is obtained with elementary counting techniques. The binomial coefficient $\binom{n}{2k}$ comes from choosing 2k transpositions from a set of n transpositions. Then we have to show that $K_{2^{2k}} \cdot K_{2^{2k}}[K_{2^{2k}}] = \frac{(2k)!}{k!}$. In the simple case $k = 1$ the two products that give the permutation $(1,2)(3,4)$ are

$$\begin{aligned}(1,2)(3,4) &= [(1,4)(2,3)][(1,3)(2,4)] \\ &= [(1,3)(2,4)][(1,4)(2,3)]\end{aligned} \tag{8}$$

For a larger k, to obtain the involution $(1,2)(3,4)\ldots(4k-1,4k)$ as a product of two fixed point free involutions, we have to partition the set of $2k$ transpositions $\{(1,2),(3,4),\ldots,(4k-1,4k)\}$ in k parts of two transpositions each. There are $\frac{(2k)!}{2^k k!}$ such partitions. For each set $\{(i_1,i_2),(i_3,i_4)\}$ of two transpositions in one partition, there exists 2 decompositions similar to the decompositions (8) so that for each partition we have 2^k decompositions. Since there is no other possible decomposition, the count is complete. $\square$

Observe that in Eq. (8) the products are commutative so that the set of involutions $\{id, [(1,2)(3,4)], [(1,4)(2,3)], [(1,3)(2,4)]\}$ involved in the decompositions forms a commutative subgroup isomorphic to the Klein group K_4. So the set of all transpositions involved in each decomposition of $(1,2)(3,4)\ldots(4k-1,4k)$ as a product of fixed point free involutions may be arranged in pairs to obtain the direct product of k disjoint copies of K_4.

As a last example, let us present an explicit expression for the coefficients $H_\ell[p_{(1^f,r)} p^{\perp}_{(1^j,m)}]$. From this, we recover a result of Boccara [1, Coroll. 6.15 and Th. 7.2], that gives coefficients $K_{(1^i,\ell)}K_{(1^j,m)}[K_{(1^f,r)}]$. Again, although Katriel's approach immediately yields the fact that these coefficients are polynomials in i, j, f for fixed ℓ, m, r, their actual computation amounts to reproducing Boccara analysis (which we thus omit here).

Proposition 2 *For positive integers ℓ, r, m, f, j such that $\ell \leq f + r = j + m$, $\ell - 1 \equiv f + j \bmod 2$ we have*:

$$H_\ell\big[p_{(1^f,r)} p^{\perp}_{(1^j,m)}\big] = \begin{cases} \dfrac{\binom{\ell-f-1}{j}\binom{\ell-j-1}{f}(r-j-1)!f!}{\binom{\ell-f-j+1}{2}(m+j-\ell)!} & \text{if } \ell - 1 \geq f + j \\ 0 & \text{otherwise} \end{cases}$$

Corollary 1 *For positive integers ℓ, m, r such that $\ell + m \geq r + 1$ and $\ell + m \equiv r + 1 \bmod 2$ we have:*

$$K_{(1^i,\ell)} \cdot K_{(1^j,m)}\big[K_{(1^f,r)}\big] = r \sum_{k=max\{0,\frac{j+f-\ell+1}{2}\}}^{min\{i,j,f\}} fkH_\ell\big[p_{(1^{f-k},r)} p^{\perp}_{(1^{j-k},m)}\big] = r \sum_{k=max\{0,\frac{j+f-\ell+1}{2}\}}^{min\{i,j,f\}}$$
$$\times \binom{f}{k} \frac{\binom{\ell-f+k-1}{j-k}\binom{\ell-j+k-1}{f-k}(r-j+k-1)!(f-k)!}{\binom{\ell-f-j+2k+1}{2}(i-k)!}$$

where i, j, f satisfy $i + \ell = j + m = f + r$.

Acknowledgments

We would like to thank Mike Zabrocki for his contribution through several discussions that have enriched our understanding of the subject. We also thank Jacob Katriel for explaining some of his conjectures to us.

References

1. G. Boccara, "Nombre de représentations d'une permutation comme produit de deux cycles de longueurs données," *Discrete Math.* **29** (1980), 105–134.
2. S. Corteel, A. Goupil, and G. Schaeffer, "Content evaluation and class symmetric functions," *Adv. in Math.* **188**(2), 315–336.
3. H.K. Farahat and G. Higman, "The centres of symmetric group rings," *Proc. Roy. Soc. London Ser. A* **250** (1959), 212–221.
4. A. Garsia and A. Goupil, "Explicit computation of characters's polynomials," (in preparation).
5. I.P. Goulden and D.M. Jackson, "Transitive factorizations in the symmetric group, and combinatorial aspects of singularity theory," preprint of Dept. of Combinatorics and optimization, University of Waterloo, March 1999.
6. A. Goupil and C. Chauve, "Combinatorial operators for Kronecker powers of representations of Sn," submitted, Seminaire Lotharingien de Combinatoire, March 2005.
7. A. Goupil and G. Schaeffer, "Factoring N-cycles and counting maps of given genus," *Europ. J. Combinatorics* **19** (1998), 819–834
8. A. Goupil, D. Poulalhon, and G. Schaeffer, Central Characters and Conjugacy Classes of the Symmetric Group or on some Conjectures of J. Katriel, Formal Power series and Algebraic combinatorics, (Moscow, 2000), Springer, Berlin 2000, pp. 238–249.
9. J. Katriel, "Some useful results concerning the representation theory of the symmetric group," *J. Phys. A* **24**(22) (1991), 5227–5234.
10. J. Katriel, "A conjecture concerning the evaluation of products of class-sums of the symmetric group.," in Groups '93 Galway/St. Andrews' conference, C.M. Campbell, et al. (Eds.), *Lond. Math. Soc. Lect. Note Ser.* **212** (1995), 322–332.
11. J. Katriel "Explicit expressions for the central characters of the symmetric group," *Discrete Applied Math.* **67**, (1996), 149–156.
12. J. Katriel, "Class-sum products in the symmetric group: Minimally connected reduced class coefficients," Combinatorics and physics (Marseille, 1995). *Math. Comput. Modelling* **26**(8–10) (1997), 161–167.
13. J. Katriel, "The class algebra of the symmetric group," in *FPSAC 10th Conference, Toronto*, 1998, pp. 401–410.
14. A. Lascoux and J.-Y. Thibon, "Vertex Operators and the class algebras of symmetric groups," *Zap. Nauchn. Sem. S. Petersburg. Otdel. Mat. Inst. Steklov. (POMI)* **283** (2001), *Teor. Predst. Din. Sist. Komb. i Algoritm. Metody* 156–177 **6** 261.
15. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Clarendon Press, Oxford, 1995.
16. P.A. MacMahon, *Combinatory Analysis*, vol. I, II, Chelsea, New York, 1960.
17. D. Poulalhon and G. Schaeffer, "Factorizations of large cycles in the symmetric group," *Discrete Math.*, **254**(1–3) (2002), 433–458.