



Kostka-Foulkes Polynomials Cyclage Graphs and Charge Statistic for the Root System C_n

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Abstract. We establish a Morris type recurrence formula for the root system C_n . Next we introduce cyclage graphs for the corresponding Kashiwara-Nakashima’s tableaux and use them to define a charge statistic. Finally we conjecture that this charge may be used to compute the Kostka-Foulkes polynomials for type C_n .

Keywords: crystal graphs, cyclage graphs, Kostka-Foulkes polynomials

1. Introduction

The multiplicity $K_{\lambda,\mu}$ of the weight μ in the irreducible finite dimensional representation $V(\lambda)$ of the simple Lie algebra g can be written in terms of the ordinary Kostant’s partition function \mathcal{P} defined from the equality:

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1-x^\alpha)} = \sum_{\beta} \mathcal{P}(\beta)x^\beta$$

where β runs on the set of nonnegative integral combinations of positive roots of g . Thus $\mathcal{P}(\beta)$ is the number of ways the weight β can be expressed as a sum of positive roots. Then we have

$$K_{\lambda,\mu} = \sum_{\sigma \in W} (-1)^{l(\sigma)} \mathcal{P}(\sigma(\lambda + \rho) - (\mu + \rho))$$

where W is the Weyl group of g .

There exists a q -analogue $K_{\lambda,\mu}(q)$ of $K_{\lambda,\mu}$ obtained by substituting the ordinary Kostant’s partition function \mathcal{P} by its q -analogue \mathcal{P}_q satisfying

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1-qx^\alpha)} = \sum_{\beta} \mathcal{P}_q(\beta)x^\beta.$$

So we have

$$K_{\lambda,\mu}(q) = \sum_{\sigma \in W} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda + \rho) - (\mu + \rho)).$$

As shown by Lusztig [17] $K_{\lambda,\mu}(q)$ is a polynomial in q with non negative integer coefficients. These polynomials naturally appear in the classical theory of Hall-Littlewood polynomials. They coincide with the Kostka-Foulkes polynomials that is, with the entries of the transition matrix between the basis of Hall-Littlewood polynomials and the basis of Schur functions [18]. Many interpretations of the Kostka-Foulkes polynomials exist. For example, they appear in the filtrations of weight spaces by the kernels of powers of a regular nilpotent element, and degree in harmonic polynomials [1, 2, 6]. We recover them in the expansion of the Hall Littlewood polynomials in terms of the affine Hecke algebra (see [19]).

For type A_{n-1} the positivity of the Kostka-Foulkes Polynomials can also be proved by a purely combinatorial method. Recall that for any partitions λ and μ with n parts the number of semi-standard tableaux of shape λ and weight μ is equal to the multiplicity of the weight μ in the finite dimensional irreducible module of $U_q(SL_n)$ with highest weight λ . In [14] Lascoux and Schützenberger have introduced a beautiful statistic ch_A on standard tableaux called the charge and, by using Morris recurrence formula, have proved the equality

$$K_{\lambda,\mu}(q) = \sum_{T \in ST(\mu)_\lambda} q^{ch_A(T)} \tag{1}$$

where $ST(\mu)_\lambda$ is the set of semi-standard tableaux of shape λ and weight μ . Set $\mathcal{A}_n = \{1 < \dots < n\}$. The charge may be defined by endowing $ST(\mu)$ the set of semi-standard tableaux of weight μ with a structure of graph defined from Lascoux-Schützenberger's plactic monoid. Recall that the plactic monoid is the quotient set of \mathcal{A}_n^* the free monoid on \mathcal{A}_n by the Knuth relations

$$abx \equiv \begin{cases} bax & \text{if } a < x \leq b \\ axb & \text{if } x \leq a < b \end{cases}.$$

For any tableau T we denote by $w(T)$ the column reading of T that is, the word obtained by reading the columns of T from right to left and from top to bottom. The cyclage graph structure on $ST(\mu)$ can be defined as follows. We draw an arrow $T \rightarrow T'$ between the two tableaux T and T' of $ST(\mu)$, if and only if there exists u in \mathcal{A}_n^* and $x \neq 1 \in \mathcal{A}_n$ such that $w(T) \equiv xu$ and $w(T') \equiv ux$. Then we say that T' is a cocyclage of T . The essential tool to define this graph structure is the insertion algorithm for the semi-standard tableaux. The cyclage graph $ST(\mu)$ contains a unique row tableau L_μ which can not be obtained as the cocyclage of another tableau of $ST(\mu)$. Let T_μ be the unique semi-standard tableau of shape μ belonging to $ST(\mu)$. Then there is no cocyclage of T_μ . For any $T \in ST(\mu)$ all the paths joining L_μ to T have the same length. This length is called the cocharge of T and denoted $coch_A(T)$. Similarly, all the paths joining T to T_μ have the same length called the charge of T . The maximal value of ch_A is $\|\mu\| = ch_A(L_\lambda) = \sum_i (i-1)\mu_i$. Moreover the charge and the cocharge satisfy the equality $ch_A(T) = \|\mu\| - coch_A(T)$ for any $T \in ST(\mu)$.

The initial cocyclage of the tableau T of reading $w(T) = xu$ with $x \neq 1$ is obtained by inserting x in the sub-tableau of T of reading u . Every tableau $T \in ST(\mu)$ can be related to T_μ by a sequence of initial cocyclages. So it is enough to consider initial cocyclages to define ch_A .

The charge of T can also be defined directly from $w(T)$ when μ is a dominant weight. Moreover it can be characterized in terms of the geometry of the crystal graph $B(\lambda)$ associated to $V(\lambda)$ [12].

In this article we restrict ourselves to the root system C_n . Our aim is two folds. First we establish Morris type recurrence formula for type C_n and use it to give explicit formulae for Kostka-Foulkes polynomials when λ is a row partition or a column partition of height 2. Next we introduce a cyclage graph structure and a notion of charge for type C_n . For any dominant weight λ , let $V(\lambda)$ be the finite dimensional irreducible $U_q(sp_{2n})$ -module with highest weight λ . In Kashiwara and Nakashima [11] have given a combinatorial description of $B(\lambda)$ the crystal graph of $V(\lambda)$ in terms of symplectic tableaux analogous to the semi-standard tableaux for type C_n . From the plactic monoid and the insertion algorithm described in [15] it is natural to try to obtain cyclage graphs for symplectic tableaux. Nevertheless the situation is more complex than for type A_{n-1} . First we have to restrict the possible cocyclage operations to the initial cocyclage to avoid loops in our cyclage graphs. Moreover if we use the complete insertion algorithm for type C_n , the number of boxes of the cocyclage of a tableau T may be strictly less than that of T due to the contraction relation in the plactic monoid. The cyclage graphs obtained by this mean seem to be not relevant to define a charge related to the Kostka-Foulkes polynomials. To overcome this problem we will execute the insertion algorithm without this contraction relation and consider that the symplectic tableaux are filled by letters of the totally ordered alphabet

$$C_n = \{\bar{n} < \dots < \bar{1} < 1 < \dots < n\}$$

which can be naturally embedded in the infinite alphabet

$$C_\infty = \{\dots < \bar{n} < \dots < \bar{1} < 1 < \dots < n < \dots\}.$$

Our convention for the alphabet C_n is not identical to that of [11] to dispose of a natural infinite extension of C_n . Denote by $\mathbf{ST}(n)$ the set of symplectic tableaux defined on C_n . If $T \in \mathbf{ST}(n)$ the initial cocyclage (without contraction) of T does not belong to $\mathbf{ST}(n)$ in general but belongs to $\mathbf{ST}(n + 1)$. So it is natural to consider the cyclage graph structure of $\mathbf{ST} = \bigcup_{n \geq 1} \mathbf{ST}(n)$.

Let μ be a dominant weight for the root system C_n . With our convention μ may be identified with the partition $(\mu_{\bar{n}}, \dots, \mu_{\bar{1}})$. We are going to endow $\mathbf{ST}(\mu)$ the subset of \mathbf{ST} containing the symplectic tableaux T such that for any $m \geq 1$, the number of letters \bar{k} in T minus the number of letters k is $\mu_{\bar{k}}$ if $k \leq n$, 0 otherwise with a structure of cyclage graph. This structure is more complex than for type A_n . In particular $\mathbf{ST}(\mu)$ decomposes into connected components. These components can be isomorphic and do not necessarily contain a row tableau. We define on $\mathbf{ST}(\mu)$ a charge statistic ch_n . Many computations allows us to conjecture that an analogue to (1) exists for type C_n with the charge ch_n . However it seems to be impossible to derive it from our Morris type recurrence formula.

In Section 1 we recall the Background on Kostka-Foulkes polynomials and crystal basis theory that we need in the sequel. We also summarize the basic properties of the insertion algorithm introduced in [15]. Section 2 is devoted to the Morris type recurrence formula for

type C_n and its applications. In Section 3 we define the cyclage graph structure on $\mathbf{ST}(\mu)$ and give some of its properties. Finally we introduced ch_n in Section 4 and conjecture that it permits to compute the Kostka-Foulkes polynomials for type C_n .

2. Background

2.1. Kostka-Foulkes polynomials for type C_n

We choose to label the Dynkin diagram of sp_{2n} by

$$\overset{0}{\circ} \Rightarrow \overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} - \overset{4}{\circ} - \dots - \overset{n-1}{\circ} . \tag{2}$$

The weight lattice P_n of C_n can be identified with \mathbb{Z}^n equipped with the orthonormal basis $\varepsilon_{\bar{i}}, i = 1, \dots, n$. We take for the simple roots

$$\alpha_0 = 2\varepsilon_{\bar{1}} \quad \text{and} \quad \alpha_i = \varepsilon_{\overline{i+1}} - \varepsilon_{\bar{i}}, i = 1, \dots, n - 1.$$

Then the set of positive roots of sp_{2n} is

$$R_n^+ = \{\varepsilon_{\bar{i}} - \varepsilon_{\bar{j}}, \varepsilon_{\bar{i}} + \varepsilon_{\bar{j}} \text{ with } 1 \leq j < i \leq n\} \cup \{2\varepsilon_{\bar{i}} \text{ with } 1 \leq i \leq n\}.$$

Denote by P_n^+ the set of dominant weights of sp_{2n} . Write $\Lambda_0, \dots, \Lambda_{n-1}$ for the fundamentals weights. Then we have $\Lambda_i = \varepsilon_{\bar{n}} + \dots + \varepsilon_{\overline{i+1}}, 0 \leq i \leq n - 1$. Consider $\lambda \in P_n^+$ and set $\lambda = \sum_{i=0}^{n-1} \hat{\lambda}_i \Lambda_i$ with $\hat{\lambda}_i \in \mathbb{N}$. The dominant weight λ is characterized by the partition $(\lambda_{\bar{n}}, \dots, \lambda_{\bar{1}})$ where $\lambda_{\bar{i}} = \hat{\lambda}_0 + \dots + \hat{\lambda}_{i-1}, i = 1, \dots, n$. In the sequel we will identify λ and $(\lambda_{\bar{n}}, \dots, \lambda_{\bar{1}})$ by setting $\lambda = (\lambda_{\bar{n}}, \dots, \lambda_{\bar{1}})$. Then $\lambda = \lambda_{\bar{1}}\varepsilon_{\bar{1}} + \dots + \lambda_{\bar{n}}\varepsilon_{\bar{n}}$ that is, the $\lambda_{\bar{i}}$'s are the coordinates of λ on the basis $(\varepsilon_{\bar{n}}, \dots, \varepsilon_{\bar{1}})$. Let ρ be the half sum of positive roots. We have $\rho = (n, n - 1, \dots, 1)$. For any $\lambda \in P_n^+$, set $|\lambda| = \lambda_{\bar{n}} + \dots + \lambda_{\bar{1}}$.

The Weyl group W_n of sp_{2n} can be regarded as the sub group of the permutation group of $C_n = \{\bar{n}, \dots, \bar{2}, \bar{1}, 1, 2, \dots, n\}$ generated by $s_i = (i, i + 1)(\bar{i}, \bar{i} + \bar{1}), i = 1, \dots, n - 1$ and $s_0 = (1, \bar{1})$ where for $a, b \in C_n$ with $a \neq b, (a, b)$ is the simple transposition which switches a and b . Note that any $\sigma \in W_n$ verifies $\sigma(\bar{i}) = \overline{\sigma(i)}$ for $i \in \{1, \dots, n\}$. We denote by l the length function corresponding to the set of generators $s_i, i = 0, \dots, n - 1$. The action of $\sigma \in W_n$ on $\beta = (\beta_{\bar{n}}, \dots, \beta_{\bar{1}}) \in P_n$ is given by

$$\sigma \cdot (\beta_{\bar{n}}, \dots, \beta_{\bar{1}}) = (\beta_{\bar{n}}^\sigma, \dots, \beta_{\bar{1}}^\sigma)$$

where $\beta_{\bar{i}}^\sigma = \beta_{\sigma(\bar{i})}$ if $\sigma(\bar{i}) \in \{\bar{1}, \dots, \bar{n}\}$ and $\beta_{\bar{i}}^\sigma = -\beta_{\sigma(i)}$ otherwise.

Let Q_n^+ be the set of nonnegative integral linear combinations of positive roots. For any $\beta = (\beta_{\bar{n}}, \dots, \beta_{\bar{1}}) \in P_n$ we set $x^\beta = x_{\bar{n}}^{\beta_{\bar{n}}} \dots x_{\bar{1}}^{\beta_{\bar{1}}}$ where x_1, \dots, x_n are fixed indeterminates. The q -analogue \mathcal{P}_q of the Kostant function partition is defined by

$$\prod_{\alpha \in R_n^+} \frac{1}{1 - qx^\alpha} = \sum_{\beta \in Q_n^+} \mathcal{P}_q(\beta)x^\beta \quad \text{and} \quad \mathcal{P}_q(\beta) = 0 \quad \text{if} \quad \beta \notin Q_n^+.$$

Definition 2.1.1 Let $\lambda, \mu \in P_n^+$. The Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ is defined by

$$K_{\lambda,\mu}(q) = \sum_{\sigma \in W_n} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda + \rho) - (\mu + \rho)).$$

Let $\beta \in P_n$. We set

$$a_\beta = \sum_{\sigma \in W_n} (-1)^{l(\sigma)} (\sigma \cdot x^\beta)$$

where $\sigma \cdot x^\mu = x^{\sigma(\mu)}$. The Schur function s_β is defined by

$$s_\beta = \frac{a_{\beta+\rho}}{a_\rho}.$$

When $\lambda \in P_n^+$, s_λ is the Weyl character of $V(\lambda)$ the finite dimensional irreducible $U_q(sp_{2n})$ -module with highest weight λ . For any $\sigma \in W_n$, the dot action of σ on $\beta \in P_n$ is defined by $\sigma \circ \beta = \sigma \cdot (\beta + \rho) - \rho$. We have the following straightening law for the Schur functions. For any $\beta \in P_n$, $s_\beta = 0$ or there exists a unique $\lambda \in P_n^+$ such that $s_\beta = (-1)^{l(\sigma)} s_\lambda$ with $\sigma \in W_n$ and $\lambda = \sigma \circ \beta$. Set $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$ and write $\mathbb{K}[P_n]$ for the \mathbb{K} module generated by the $x^\beta, \beta \in P_n$. Set $\mathbb{K}[P_n]^{W_n} = \{f \in \mathbb{K}[P_n], \sigma \cdot f = f \text{ for any } \sigma \in W_n\}$. Then $\{s_\lambda\}$ is a basis of $\mathbb{K}[P_n]^{W_n}$.

To each positive root α , we associate the raising operator $R_\alpha : P_n \rightarrow P_n$ defined by

$$R_\alpha(\beta) = \alpha + \beta.$$

Given $\alpha_1, \dots, \alpha_p$ positive roots and $\beta \in P_n$, we set $(R_{\alpha_1} \dots R_{\alpha_p})s_\beta = s_{R_{\alpha_1} \dots R_{\alpha_p}(\beta)}$. Composing the action of raising operators on Schur function should be avoided in general. For example $(R_{\alpha_1} R_{\alpha_2})(s_\beta)$ is not necessarily equal to $(R_{\alpha_1})(R_{\alpha_2} s_\beta)$ (see example p 360 in [19]). For all $\beta \in P_n$, we define the Hall-Littlewood polynomial Q_β by

$$Q_\beta = \left(\prod_{\alpha \in R_n^+} \frac{1}{1 - q R_\alpha} \right) s_\beta$$

where $\frac{1}{1 - q R_\alpha} = \sum_{k=0}^{+\infty} q^k R_\alpha^k$.

Theorem 2.1.2 [19] For any $\lambda, \mu \in P_n^+$, $K_{\lambda,\mu}(q)$ is the coefficient of s_λ in Q_μ that is,

$$Q_\mu = \sum_{\lambda \in P_n^+} K_{\lambda,\mu}(q) s_\lambda.$$

When $n = 1$, the root system C_1 can be regarded as the root system A_1 and the Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ where λ and μ are partitions of length 1 satisfies

$$K_{\lambda,\mu}(q) = q^{(|\lambda| - |\mu|)/2}. \tag{3}$$

Lemma 2.1.3 Consider $\lambda, \mu \in P_n^+$ such that $\lambda_{\bar{n}} = \mu_{\bar{n}}$ and write $\lambda' = (\lambda_{\bar{n}-1}, \dots, \lambda_{\bar{1}})$, $\mu' = (\mu_{\bar{n}-1}, \dots, \mu_{\bar{1}})$. Then $K_{\lambda, \mu}(q) = K_{\lambda', \mu'}(q)$.

Proof: $K_{\lambda, \mu}(q)$ is the coefficient of s_λ in

$$\begin{aligned} & \prod_{\alpha \in R_n^+} \left(\frac{1}{1 - qR_\alpha} \right) s_\mu \\ &= \left(\prod_{1 \leq j \leq n} \frac{1}{1 - qR_{2\varepsilon_j}} \prod_{1 \leq j < i \leq n} \frac{1}{1 - qR_{\varepsilon_i - \varepsilon_j}} \prod_{1 \leq j < i \leq n} \frac{1}{1 - qR_{\varepsilon_i + \varepsilon_j}} \right) s_\mu \end{aligned}$$

but since $\lambda_{\bar{n}} = \mu_{\bar{n}}$ none of the raising operators involving $\varepsilon_{\bar{n}}$ contribute to $K_{\lambda, \mu}(q)$. Thus $K_{\lambda, \mu}(q) = K_{\lambda', \mu'}(q)$, the coefficient of $s_{\lambda'}$ in

$$\prod_{1 \leq j \leq n-1} \left(\frac{1}{1 - qR_{2\varepsilon_j}} \prod_{1 \leq j < i \leq n-1} \frac{1}{1 - qR_{\varepsilon_i - \varepsilon_j}} \prod_{1 \leq j < i \leq n-1} \frac{1}{1 - qR_{\varepsilon_i + \varepsilon_j}} \right) s_{\mu'}.$$

□

2.2. Crystal Graphs for Type C_n

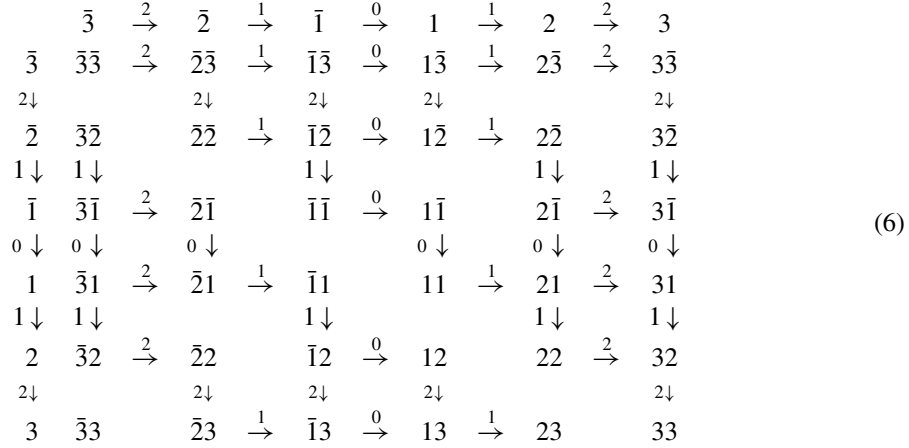
Recall that crystal graphs for the $U_q(sp_{2n})$ -modules are oriented colored graphs with colors $i \in \{0, \dots, n-1\}$. An arrow $a \xrightarrow{i} b$ means that $\tilde{f}_i(a) = b$ and $\tilde{e}_i(b) = a$ where \tilde{e}_i and \tilde{f}_i are the crystal graph operators (for a review of crystal bases and crystal graphs see [10]). A vertex $v^0 \in B$ satisfying $\tilde{e}_i(v^0) = 0$ for any $i \in \{0, \dots, n-1\}$ is called a highest weight vertex. The decomposition of V into its irreducible components is reflected into the decomposition of B into its connected components (see (6) below). Each connected component of B contains a unique highest weight vertex. The crystals graphs of two isomorphic irreducible components are isomorphic as oriented colored graphs. The action of \tilde{e}_i and \tilde{f}_i on $B \otimes B' = \{b \otimes b'; b \in B, b' \in B'\}$ is given by:

$$\tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i(u) \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v) \\ u \otimes \tilde{f}_i(v) & \text{if } \varphi_i(u) \leq \varepsilon_i(v) \end{cases} \tag{4}$$

and

$$\tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i(v) & \text{if } \varphi_i(u) < \varepsilon_i(v) \\ \tilde{e}_i(u) \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v) \end{cases} \tag{5}$$

where $\varepsilon_i(u) = \max\{k; \tilde{e}_i^k(u) \neq 0\}$ and $\varphi_i(u) = \max\{k; \tilde{f}_i^k(u) \neq 0\}$. The weight of the vertex u is defined by $\text{wt}(u) = \sum_{i=0}^{n-1} (\varphi_i(u) - \varepsilon_i(u))\Lambda_i$.



The 3 connected components of $B(\Lambda_2)^{\otimes 2}$ for $n = 3$.

The following lemma is a straightforward consequence of (4) and (5).

Lemma 2.2.1 *Let $u \otimes v \in B \otimes B' u \otimes v$ is a highest weight vertex of $B \otimes B'$ if and only if for any $i \in \{0, \dots, n-1\} \tilde{e}_i(u) = 0$ (i.e. u is of highest weight) and $\varepsilon_i(v) \leq \varphi_i(u)$.*

The Weyl group W_n acts on B by:

$$\begin{aligned}
 s_i(u) &= (\tilde{f}_i)^{\varphi_i(u) - \varepsilon_i(u)}(u) \quad \text{if } \varphi_i(u) - \varepsilon_i(u) \geq 0, \\
 s_i(u) &= (\tilde{e}_i)^{\varepsilon_i(u) - \varphi_i(u)}(u) \quad \text{if } \varphi_i(u) - \varepsilon_i(u) < 0.
 \end{aligned}
 \tag{7}$$

We have the equality $\text{wt}(\sigma(u)) = \sigma(\text{wt}(u))$ for any $\sigma \in W_n$ and $u \in B$. For any $\lambda \in P_n^+$, we denote by $B(\lambda)$ the crystal graph of $V(\lambda)$.

According to (2) we have

$$B(\Lambda_{n-1}) : \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \dots \rightarrow \bar{2} \xrightarrow{1} \bar{1} \xrightarrow{0} 1 \xrightarrow{1} 2 \dots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n.$$

Kashiwara-Nakashima's combinatorial description of the crystal graphs $B(\lambda)$ is based on the notion of symplectic tableaux analogous for type C_n to semi-standard tableaux.

We defined a total order on C_n by setting

$$C_n = \{\bar{n} < \dots < \bar{1} < 1 < \dots < n\}.$$

For any letter $x \in C_n$ we set $\bar{\bar{x}} = x$. Note that our convention for labelling the crystal graph of the vector representation are not those used by Kashiwara and Nakashima. To obtain the

original description of $B(\lambda)$ from that used in the sequel it suffices to change each letter $k \in \{1, \dots, n\}$ of \mathcal{C}_n into $n - k + 1$ and each letter $\bar{k} \in \{\bar{1}, \dots, \bar{n}\}$ into $n - k + 1$. The interest of this change of convention will appear in Sections 4 and 5.

We identify the vertices of the crystal graph $G_n = \oplus_l B(\Lambda_{n-1})^{\otimes l}$ with the words on \mathcal{C}_n . For any $w \in G_n$ we have $\text{wt}(w) = d_{\bar{n}}\varepsilon_{\bar{n}} + d_{n-1}\varepsilon_{n-1} \cdots + d_{\bar{1}}\varepsilon_{\bar{1}}$ where for any $i = 1, \dots, n$ d_i is the number of letters \bar{i} of w minus the number of its letters i . Using Formulas (4) and (5) we obtain a simple rule to compute the action of \tilde{e}_i, \tilde{f}_i or s_i $i \in \{0, \dots, n-1\}$ on $w \in G_n$ that we will use in Section 4. Consider the subword w_i of w containing only the letters $\overline{i+1}, \bar{i}, i, i+1$ if $i \neq 0$ and the letters $\bar{1}, 1$ otherwise. Then encode in w_i each letter $\overline{i+1}$ or i by the symbol $+$ and each letter \bar{i} or $i+1$ by the symbol $-$. Because $\tilde{e}_i(\pm) = \tilde{f}_i(\pm) = 0$ in $B(\Lambda_{n-1}) \otimes B(\Lambda_{n-1})$ the factors of type \pm may be ignored in w_i . So we obtain a subword $w_i^{(1)}$ in which we can ignore all the factors \pm to construct a new subword $w_i^{(2)}$ etc... Finally we obtain a subword $\rho(w)$ of w of type

$$\rho(w) = -^r +^s.$$

Then we have the

Note 2.2.2

- When $r > 0$, $\tilde{e}_i(w)$ is obtained by changing the rightmost symbol $-$ of $\rho(w)$ into its corresponding symbol $+$ (i.e. $i+1$ into i and \bar{i} into $\overline{i+1}$ if $i \neq 0$, 1 into $\bar{1}$ otherwise) the others letters of w being unchanged. When $r = 0$, $\tilde{e}_i(w) = 0$.
- When $s > 0$, $\tilde{f}_i(w)$ is obtained by changing the leftmost symbol $+$ of $\rho(w)$ into their corresponding symbols $-$ (i.e. i into $i+1$ and $\overline{i+1}$ into \bar{i} if $i \neq 0$, $\bar{1}$ into 1 otherwise) the others letters of w being unchanged. When $s = 0$, $\tilde{f}_i(w) = 0$.
- When $r \geq s$, $s_i(w)$ is obtained by changing the $r - s$ rightmost symbols $-$ of $\rho(w)$ into its corresponding symbol $+$. When $r < s$, $s_i(w)$ is obtained by changing the $s - r$ leftmost symbols $+$ of $\rho(w)$ into their corresponding symbols $-$.

In the sequel we will need the following definitions and notation:

A column on \mathcal{C}_n is a Young diagram C of column shape filled from top to bottom by increasing letters of \mathcal{C}_n .

The height $h(C)$ of a column C is the number of its letters.

$\mathbf{C}(n, h)$ is the set of columns of height h on \mathcal{C}_n i.e. with letters in \mathcal{C}_n .

The reading of $C \in \mathbf{C}(n, h)$ is the word $w(C)$ of \mathcal{C}_n^* obtained by reading its letters from top to bottom.

The column C contains the pair (z, \bar{z}) when C contains the unbarred letter $z \geq 1$ and the barred letter $\bar{z} \leq \bar{1}$.

Let C_1 and C_2 be two columns. We will write $C_1 \leq C_2$ when $h(C_1) \geq h(C_2)$ and the rows of the tableau $C_1 C_2$ weakly increase.

Definition 2.2.3 Let C be a column on \mathcal{C}_n and $I_C = \{z_1 < \dots < z_r\}$ the set of unbarred letters z such that the pair (z, \bar{z}) occurs in C . The column C is n -admissible when there exists a set of unbarred letters $J_C = \{t_1 < \dots < t_r\} \subset \mathcal{C}_n$ such that:

- t_1 is the lowest letter of \mathcal{C}_n satisfying: $t_1 > z_1, t_1 \notin C$ and $\bar{t}_1 \notin C$,
- for $i = 2, \dots, r, t_i$ is the lowest letter of \mathcal{C}_n satisfying: $t_i > \max(t_{i-1}, z_i), t_i \notin C$ and $\bar{t}_i \notin C$.

In this case we write:

- rC for the column obtained from C by changing z_i into t_i for each letter $z_i \in I_C$,
- lC for the column obtained from C by changing \bar{z}_i into \bar{t}_i for each letter $z_i \in I_C$.

Consider $C = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$. Then C is not 4-admissible but is 5-admissible with $rC = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}$ and $lC = \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$.

As usually, we associate to each partition $\lambda = (\lambda_{\bar{n}}, \dots, \lambda_{\bar{1}})$ the Young diagram $Y(\lambda)$ whose i -th row has length $\lambda_{\overline{n-i+1}}$. By definition, a n -symplectic tableau T of shape λ is a filling of $Y(\lambda)$ by letters of \mathcal{C}_n satisfying the following conditions:

- the columns C_i of $T = C_1 \dots C_s$ are n -admissible,
- for $i = 1, \dots, s - 1 : rC_i \leq lC_{i+1}$.

The set of n -symplectic tableaux will be denoted $\mathbf{ST}(n)$.

If $T = C_1 C_2 \dots C_r \in \mathbf{ST}(n)$, the reading of T is the word $w(T) = w(C_r) \dots w(C_2)w(C_1)$.

From [11] we deduce the

Theorem 2.2.4

- (i) *The vertices of $B(\Lambda_p)$ $p = 0, \dots, n - 1$ are in one-to-one correspondence with the readings of n -admissible columns of height $n - p$.*
- (ii) *The vertices of $B(\lambda)$ are in one-to-one correspondence with the readings of the n -symplectic tableaux of shape λ .*

More precisely Kashiwara and Nakashima realize $B(\lambda)$ into a tensor power $B(\Lambda_{n-1})^{\otimes l}$. Given $p = 0, \dots, n - 1, B(\Lambda_p)$ can then be identified with the connected component of G_n whose highest weight vertex is $b_{\bar{p}} = \bar{n}(\bar{n} - 1) \dots \bar{p} + 1$. In this identification, the vertices of $B(\Lambda_p)$ are the readings of the admissible columns of height $n - p$. If $\lambda = \sum_{p=0}^{n-1} \hat{\lambda}_p \Lambda_p, B(\lambda)$ is identified with the connected component whose highest weight vertex is $b_\lambda = b_{\bar{n}}^{\otimes \hat{\lambda}_n} \dots \otimes b_{\bar{1}}^{\otimes \hat{\lambda}_1} b_0^{\otimes \hat{\lambda}_0}$.

By identifying $U_q(sp_{2(n-1)})$ with the sub-algebra of $U_q(sp_{2n})$ generated by the Chevalley's generators e_i, f_i and $t_i, i = 0, \dots, n - 1$, we endow $B(\lambda)$ with a structure of crystal graph for type C_{n-1} . The decomposition of $B(\lambda)$ into its $U_q(sp_{2(n-1)})$ -connected components is obtained by erasing all the arrows of color $n - 1$.

2.3. *Insertion scheme for symplectic tableaux*

In [15] we have introduced an insertion scheme for symplectic tableaux analogous for type C_n to the bumping algorithm on Young tableaux. Now we are going to summarize the properties of this scheme that we shall need in Section 4.

Consider first a letter x and a column C . The insertion of the letter x in the n -admissible column C is denoted $x \rightarrow C$. If x is strictly greater than the greatest letter of C then $x \rightarrow C$ is the column obtained by adding a box containing x on bottom of C , that is, $x \rightarrow C = \begin{bmatrix} C \\ x \end{bmatrix}$. Now suppose that x is less than the greatest letter of C . Then $x \rightarrow C$ is a symplectic tableau of two columns defined recursively as follows:

if $C =$ contains only one column then $x \rightarrow \begin{bmatrix} a \\ \end{bmatrix} = \begin{bmatrix} x & a \end{bmatrix}$

if $C = \begin{bmatrix} a \\ b \end{bmatrix}$ contains two letters,

1. $x \rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a & b \\ x & \end{bmatrix}$ if $a < x \leq b$ and $b \neq \bar{a}$,
2. $x \rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x & a \\ b & \end{bmatrix}$ if $x \leq a < b$ and $b \neq \bar{x}$,
3. $x \rightarrow \begin{bmatrix} \bar{b} \\ b \end{bmatrix} = \begin{bmatrix} \bar{b}+1 & b+1 \\ x & \end{bmatrix}$ if $a = \bar{b}$ and $\bar{b} \leq x \leq b$,
4. $\bar{b} \rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \bar{b}-1 & a \\ b-1 & \end{bmatrix}$ if $x = \bar{b}$ and $\bar{b} < a < b$.

Consider a n -admissible column C of height $k \geq 3$ and suppose we have defined our insertion for the n -admissible columns of height $< k$. If $w(C) = a_1, \dots, a_{k-1}a_k$ and $x \rightarrow \begin{bmatrix} a_{k-1} \\ a_k \end{bmatrix} = \begin{bmatrix} \delta_{k-1} & y \\ d_k & \end{bmatrix}$, then we have $y > a_{k-2}$ and the column C' of reading $a_1 \cdots a_{k-2}y$ is n -admissible. Write

$$\delta_{k-1} \rightarrow C' = \begin{bmatrix} d_1 & z \\ \cdot & \\ \cdot & \\ d_{k-1} & \end{bmatrix}$$

and set

$$x \rightarrow C = \begin{bmatrix} d_1 & z \\ \cdot & \\ \cdot & \\ d_{k-1} & \\ d_k & \end{bmatrix} .$$

This can be pictured by

$$x \rightarrow \begin{array}{|c|} \hline a_1 \\ \hline \cdot \\ \hline a_{k-2} \\ \hline a_{k-1} \\ \hline a_k \\ \hline \end{array} = \begin{array}{|c|} \hline a_1 \\ \hline \cdot \\ \hline a_{k-2} \\ \hline a_{k-1} \\ \hline \end{array} \begin{array}{|c|} \hline x \\ \hline a_k \\ \hline \end{array} = \begin{array}{|c|} \hline a_1 \\ \hline \cdot \\ \hline a_{k-2} \\ \hline \delta_{k-1} \\ \hline \end{array} \begin{array}{|c|} \hline y \\ \hline \\ \hline \\ \hline \end{array} = \dots = \begin{array}{|c|} \hline d_1 \\ \hline \cdot \\ \hline \cdot \\ \hline d_{k-1} \\ \hline d_k \\ \hline \end{array} \begin{array}{|c|} \hline z \\ \hline \\ \hline \\ \hline \end{array} .$$

During each step we apply one of the transformations 1 to 4 above. We have proved in [15] that $x \rightarrow C$ is then a n -symplectic tableau with two columns respectively of height $h(C)$ and 1.

Example 2.3.1 Suppose $n = 5$.

$$\bullet \ 5 \rightarrow \begin{array}{|c|} \hline \bar{4} \\ \hline \bar{2} \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} = \begin{array}{|c|} \hline \bar{4} \\ \hline \bar{2} \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array} .$$

$$\bullet \ \bar{4} \rightarrow \begin{array}{|c|} \hline \bar{4} \\ \hline \bar{2} \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} = \begin{array}{|c|} \hline \bar{4} \\ \hline \bar{2} \\ \hline 2 \\ \hline 3 \\ \hline \bar{4} \\ \hline \bar{4} \\ \hline \end{array} = \begin{array}{|c|} \hline \bar{4} \\ \hline \bar{2} \\ \hline \bar{2} \\ \hline \bar{3} \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|} \hline \bar{4} \\ \hline \bar{2} \\ \hline \bar{2} \\ \hline \bar{2} \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|} \hline \bar{4} \\ \hline \bar{3} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \bar{2} \\ \hline \bar{3} \\ \hline \end{array} = \begin{array}{|c|} \hline \bar{4} \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \bar{2} \\ \hline \bar{3} \\ \hline \end{array} \begin{array}{|c|} \hline \bar{3} \\ \hline \\ \hline \\ \hline \end{array} .$$

Remarks

- (i) If C is n -admissible and $x \rightarrow C$ is a column, then this column is $(n + 1)$ -admissible but not necessarily n -admissible.
- (ii) From transformations 1 to 4 we obtain the plactic relations

$$\begin{aligned}
 abx &= \begin{cases} bax & \text{if } a < x \leq b \text{ and } b \neq \bar{a} \\ axb & \text{if } x \leq a < b \text{ and } b \neq \bar{x} \end{cases} \text{ and} \\
 \begin{cases} \bar{b}bx = (b + 1)(\overline{b + 1})x & \text{if } \bar{b} \leq x \leq b \\ abb = a(\overline{b - 1})(b - 1) & \text{if } \bar{b} < a < b \end{cases} & \quad (8)
 \end{aligned}$$

introduced in [15]. These relations are not sufficient to define a plactic monoid for type C_n . We need a contraction relation which permits to obtain a n -admissible column from a non n -admissible one. Let $C' = \begin{array}{|c|} \hline C \\ \hline x \\ \hline \end{array}$ be a non n -admissible column on C_n such that C is n -admissible and x a letter. In this case we can prove that there exists an unbarred letter z

maximal such that the pair (z, \bar{z}) occurs in C' and

$$\text{card}\{t \in C_n, |t| \geq z\} > n - z + 1.$$

Write D for the column obtained by erasing the pair (z, \bar{z}) in C' . Then D is n -admissible and the contraction relation is defined by

$$w(C') \equiv_n w(D) \tag{9}$$

In fact this last relation is not needed to define the cocyclage in Section 4.

Definition 2.3.2 The congruence obtained by identifying the words of C_n^* which are equal up to relations (8) is denoted \equiv .

(iii): When $x \rightarrow C = \boxed{C' \mid y}$ is a tableau of two columns we have

$$h(C) = h(C') \quad \text{and} \quad C' \leq C. \tag{10}$$

Now we can define the insertion $x \rightarrow T$ of the letter x in the n -symplectic tableau T . Write $T = C_1 \cdots C_r$ where $C_i, i = 1, \dots, r$ are the n -admissible columns of T . If x is strictly greater to the greatest letter of C_1 then $x \rightarrow T$ is the tableau obtained by adding a box containing x on bottom of C_1 . Then $x \rightarrow T$ belongs to $\mathbf{ST}(n+1)$ but not to $\mathbf{ST}(n)$ in general since its first column may be non n -admissible. Otherwise write $x \rightarrow C = \boxed{C'_1 \mid y}$ where C'_1 is an admissible column of height $h(C_1)$ and y a letter. Then $x \rightarrow T = C'_1(y \rightarrow C_2 \cdots C_r)$ that is, $x \rightarrow T$ is the juxtaposition of C'_1 with the tableau obtained by inserting y in the tableau $C_2 \cdots C_r$. In this case $x \rightarrow T$ is a n -symplectic tableau [15].

Example 2.3.3 Suppose $n = 3$. Then

$$2 \rightarrow \left(1 \rightarrow \begin{array}{|c|c|c|} \hline \bar{1} & 1 & 2 \\ \hline 1 & 2 & \\ \hline 3 & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline \bar{2} & 1 & 2 & 2 \\ \hline 1 & 2 & & \\ \hline 2 & 3 & & \\ \hline \end{array}.$$

Remarks

- (i) The insertion scheme described above do not suffice to define a complete insertion algorithm (that is such that $x \rightarrow T \in \mathbf{ST}(n)$ if $T \in \mathbf{ST}(n)$ and $x \in C_n$) for the n -symplectic tableaux since the first column C' of $x \rightarrow T$ may be not n -admissible when x is greater than the greatest letter of C_1 . To obtain a complete insertion algorithm we have to apply relation (9) to C' . This give a column D of reading $x_1 \cdots x_p$. Finally we compute successively the insertions $x_p(\rightarrow x_{p-1} \cdots (x_1 \rightarrow C_2 \cdots C_r))$. In the sequel we only use insertion algorithm without the contraction relation (9).
- (ii) To each $w = x_1 \cdots x_r \in C_n^*$ of length r we can associate recursively a symplectic tableau $P(w)$ by setting $P(w) = \boxed{x_1}$ if $r = 1$ and $P(w) = x_r \rightarrow P(x_1 \cdots x_{r-1})$

otherwise. If $P(w)$ belongs to $\mathbf{ST}(m)$ with $m \geq n$, results of [15] implies that $P(w) \equiv_m w$. Moreover $P(w)$ is the unique m -symplectic tableau with this property. Denote by \sim_m the equivalence relation defined on the vertices of G_m by $w_1 \sim_m w_2$ if and only if w_1 and w_2 belong to the same connected component of G_m . Given two words w_1 and w_2 such that $P(w_1)$ and $P(w_2)$ belong $\mathbf{ST}(m)$ we have the equivalences

$$w_1 \equiv_m w_2 \Leftrightarrow P(w_1) = P(w_2) \Leftrightarrow P(w_1) \sim_m P(w_2). \tag{11}$$

Moreover we have for any $\sigma \in W_m$

$$P(\sigma(w)) = \sigma(P(w)). \tag{12}$$

- (iii) The insertion algorithm is reversible in the sense that if we know the tableau T' such that $x \rightarrow T = T'$ and the shape of T we can recover the tableau T and the letter x . This follows from the fact that the transformations 1 to 4 are reversible. More precisely, T' has one box more than T . Let y be the letter belonging to that box. Then if we apply transformations 1 to 4 from right to left starting from y , we recover T and x .
- (iv) In Section 4, we will need to find for a fixed tableau T' all the pairs (x, T) where T is a symplectic tableau and x a letter such that $x \rightarrow T = T'$. The outside corners of the tableau T' are the boxes c of T' such that there is no box down and to the right of c in T' . By (iii) the pairs (x, T) are obtained by applying the reverse insertion algorithm to the outside corners of T' .

Example 2.3.4 Suppose $n = 3$ and $T' = \begin{array}{|c|c|c|} \hline \bar{2} & 1 & 2 \\ \hline 1 & 2 & \\ \hline 3 & & \\ \hline \end{array}$. Then by applying reverse insertion algorithm to each outside corners of T' we obtain the pairs $(3, \begin{array}{|c|c|c|} \hline \bar{2} & 1 & 2 \\ \hline 1 & 2 & \\ \hline & & \\ \hline \end{array})$, $(1, \begin{array}{|c|c|c|} \hline \bar{1} & 1 & 2 \\ \hline 1 & 2 & \\ \hline & & \\ \hline \end{array})$ and $(1, \begin{array}{|c|c|} \hline \bar{1} & 1 \\ \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array})$.

3. Morris type recurrence formula

In this section we introduce a recurrence formula for computing Kostka polynomials analogous for type C_n to Morris recurrence formula. It allows to explain the Kostka polynomials for type C_n as combinations of Kostka polynomials for type C_{n-1} . We embed type C_{n-1} in type C_n by identifying $U_q(sp_{2(n-1)})$ with the sub-algebra of $U_q(sp_{2n})$ generated by the Chevalley operators e_i, f_i and $t_i, i = 0, \dots, n - 2$. The weight lattice P_{n-1} of $U_q(sp_{2n})$ is the \mathbb{Z} -lattice generated by the $\varepsilon_{\bar{i}}, i = 1, \dots, n - 1$ and $P_{n-1}^+ = P_n^+ \cap P_{n-1}$ is the set of dominant weights. The Weyl group W_{n-1} is the sub-group of W_n generated by the s_i ,

$i = 0, \dots, n-2$ and we have $R_{n-1}^+ = R_n \cap P_{n-1}$. Given any positive integer r , write $(r)_n$ for the row partition $(p, 0, \dots, 0)$ of length n . To obtain our recurrence formula we need to describe the decomposition $B(\gamma) \otimes B((r)_n)$ with $\gamma \in P_n^+$ and $r > 0$ an integer into its irreducible components. This is analogous for type C_n to Pieri rule.

3.1. Pieri rule for type C_n

Let $\gamma = (\gamma_{\bar{n}}, \dots, \gamma_{\bar{1}}) \in P_n^+$. By Theorem 2.2.4, the vertices of $B((r)_n)$ are the words

$$L = (n)^{k_n} \dots (2)^{k_2} (1)^{k_1} (\bar{1})^{k_{\bar{1}}} (\bar{2})^{k_{\bar{2}}} \dots (\bar{n})^{k_{\bar{n}}}$$

where $k_{\bar{i}}, k_i$ are positive integers, $(x)^k$ means that the letter x is repeated k times in L and $k_{\bar{1}} + \dots + k_{\bar{n}} + k_1 + \dots + k_n = r$. Let b_γ be the highest weight vertex of $B(\gamma)$.

Lemma 3.1.1 *$b_\gamma \otimes L$ is a highest weight vertex of $B(\gamma) \otimes B((r)_n)$ if and only if the following conditions holds:*

- (i) $\gamma_i - k_i \geq \gamma_{i-1}$ for $i = 2, \dots, n$ and $\gamma_{\bar{1}} - k_1 \geq 0$,
- (ii) $\gamma_i - k_i + k_{\bar{i}} \leq \gamma_{i+1} - k_{i+1}$ for $i = 1, \dots, n-1$.

Proof: By Lemma 2.2.1, $b_\gamma \otimes L$ is a highest weight vertex if and only if for any $m = 1, \dots, r$, each vertex $b_\gamma \otimes L_m$ (where L_m is the word obtained by reading the m leftmost letters of L) is a highest weight vertex. It means that $(\gamma_{\bar{n}} - k_n, \dots, \gamma_s - k_s, \gamma_{s-1}, \dots, \gamma_{\bar{1}})$ and $(\gamma_{\bar{n}} - k_n, \dots, \gamma_{t+1} - k_{t+1}, \gamma_t - k_t + k_{\bar{t}}, \dots, \gamma_{\bar{1}} + k_1 - k_{\bar{1}})$ are partitions respectively for $s = n, \dots, 1$ and $t = 1, \dots, n-1$. This is equivalent to the conditions

$$\begin{cases} \gamma_s - k_s \geq \gamma_{s-1} & \text{for } s = n, \dots, 2 \quad \text{and} \quad \gamma_{\bar{1}} - k_1 \geq 0 \\ \gamma_t - k_t + k_{\bar{t}} \leq \gamma_{t+1} - k_{t+1} & \text{for } t = 1, \dots, n-1 \end{cases}.$$

□

Corollary 3.1.2 $B(\gamma) \otimes B((r)_n) = \bigoplus \lambda \in P_n^+ B(\lambda)^{\oplus c_{\gamma,r}^\lambda}$ where $c_{\gamma,r}^\lambda$ is the number of vertices $L \in B((r)_n)$ such that

- (i) $k_{\bar{i}} - k_i = \lambda_{\bar{i}} - \gamma_i$ for $i = 1, \dots, n$,
- (ii) $\lambda_{\bar{i}} \leq \lambda_{i+1} - k_{i+1}$ for $i = 1, \dots, n-1$,
- (iii) $\lambda_{\bar{i}} - k_{\bar{i}} \geq \lambda_{i-1} + k_{i-1} - k_{i-1}$ for $i = 2, \dots, n$ and $\lambda_{\bar{1}} - k_1 \geq 0$.

Proof: The multiplicity $c_{\gamma,r}^\lambda$ is equal to the number of highest weight vertices $b_\gamma \otimes L \in B(\gamma) \otimes B((r)_n)$ of weight λ . The condition $\text{wt}(b_\gamma \otimes L) = \lambda$ is equivalent to

$$\gamma_i - k_i + k_{\bar{i}} = \lambda_{\bar{i}} \quad \text{for } i = 1, \dots, n$$

which gives (i). The assertions (ii) and (iii) are respectively obtained by replacing for any i , γ_i by $\lambda_i + k_i - k_i$ in assertions (ii) and (i) of the previous lemma. \square

Remarks

- (i) In the above corollary, $c_{\gamma,r}^\lambda$ is the number of ways of starting with γ , removing a horizontal strip to obtain a partition ν and then adding a horizontal strip to obtain λ .
- (ii) $B(\gamma) \otimes B((r)_n)$ is not multiplicity free in general.

3.2. *Recurrence formula*

Consider $\gamma \in P_n^+$ and r a positive integer. We set

$$(\gamma \otimes r)_n = \{\lambda \in P_n^+, c_{\gamma,r}^\lambda \neq 0\}$$

with $c_{\gamma,r}^\lambda$ defined as in Corollary 3.1.2.

Theorem 3.2.1 *Let $\mu \in P_n^+$ with $n \geq 2$ and write $\mu = (\mu_{\bar{n}}, \mu')$ where $\mu_{\bar{n}}$ is the first part of μ and $\mu' = (\mu_{\bar{n}-1}, \dots, \mu_{\bar{1}}) \in P_{n-1}^+$. Then*

$$Q_\mu = \sum_{\gamma \in P_{n-1}^+} \sum_{r=0}^{+\infty} \sum_{m=0}^{+\infty} q^{m+r} \sum_{\lambda \in (\gamma \otimes r)_{n-1}} c_{\gamma,r}^\lambda K_{\lambda,\mu'}(q) s_{(\mu_{\bar{n}}+r+2m,\gamma)} \tag{13}$$

Proof: We start from $Q_\mu = (\prod_{\alpha \in R_n^+} \frac{1}{1-qR_\alpha}) s_\mu$. By Proposition 3.5 of [19] we can write

$$Q_\mu = \left(\prod_{\substack{\alpha \notin R_n^+ \\ \alpha \notin R_{n-1}^+}} \frac{1}{1-qR_\alpha} \right) \left[\left(\prod_{R_{n-1}^+} \frac{1}{1-qR_\alpha} \right) s_\mu \right].$$

Then by applying Theorem 2.1.2, we obtain

$$Q_\mu = \left(\prod_{\substack{\alpha \notin R_n^+ \\ \alpha \notin R_{n-1}^+}} \frac{1}{1-qR_\alpha} \right) \left(\sum_{\lambda \in P_{n-1}^+} K_{\lambda,\mu'}(q) s_{(\mu_{\bar{n}},\lambda)} \right). \tag{14}$$

Set $R_{\bar{i}} = R_{\varepsilon_{\bar{n}} - \varepsilon_{\bar{i}}}$ for $i = 1, \dots, n-1$ and $R_i = R_{\varepsilon_{\bar{n}} + \varepsilon_{\bar{i}}}$ for $i = 1, \dots, n$. Recall that for any $\beta \in P_{n-1}$, $R_{\bar{i}}(\beta) = \beta + \varepsilon_{\bar{n}} - \varepsilon_{\bar{i}}$ and $R_i(\beta) = \beta + \varepsilon_{\bar{n}} + \varepsilon_{\bar{i}}$. Then (14) implies

$$Q_\mu = \sum_{\lambda \in P_{n-1}^+} K_{\lambda, \mu'}(q) \left(\sum_{r=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{k_{\bar{1}} + \dots + k_{\bar{n-1}} + k_1 + \dots + k_{n-1} = r} q^{m+r} (R_n)^m (R_1)^{k_1} (R_{\bar{1}})^{k_{\bar{1}}} \dots (R_{n-1})^{k_{n-1}} (R_{\bar{n-1}})^{k_{\bar{n-1}}} S_{(\mu_{\bar{n}}, \lambda)} \right).$$

$$Q_\mu = \sum_{r=0}^{+\infty} \sum_{m=0}^{+\infty} q^{m+r} \sum_{\lambda \in P_{n-1}^+} K_{\lambda, \mu'}(q) \sum_{k_{\bar{1}} + \dots + k_{\bar{n-1}} + k_1 + \dots + k_{n-1} = r} S_{(\mu_{\bar{n}} + r + 2m, \lambda_{\bar{n-1}} + k_{n-1} - k_{\bar{n-1}}, \dots, \lambda_{\bar{1}} + k_1 - k_{\bar{1}})}.$$

Fix λ, m and r and consider

$$S = \sum_{k_{\bar{1}} + \dots + k_{\bar{n-1}} + k_1 + \dots + k_{n-1} = r} S_{(\mu_{\bar{n}} + r + 2m, \lambda_{\bar{n-1}} + k_{n-1} - k_{\bar{n-1}}, \dots, \lambda_{\bar{1}} + k_1 - k_{\bar{1}})}.$$

Set $\gamma = (\lambda_{\bar{n-1}} + k_{n-1} - k_{\bar{n-1}}, \dots, \lambda_{\bar{1}} + k_1 - k_{\bar{1}})$.

1. Suppose first that there exists $i \in \{1, \dots, n-2\}$ such that $\lambda_{\bar{i}} > \lambda_{\bar{i+1}} - k_{\bar{i+1}}$. Set $\tilde{\gamma} = s_i \circ \gamma$ that is

$$\tilde{\gamma} = s_i(\gamma_{\bar{n-1}} + n - 1, \dots, \gamma_{\bar{i+1}} + n - i + 1, \gamma_{\bar{i}} + n - i, \dots, \gamma_{\bar{1}} + 1) - (n - 1, \dots, 1).$$

Then $\gamma_s = \tilde{\gamma}_s$ for $s \neq i + 1, i$, $\tilde{\gamma}_{\bar{i+1}} = \gamma_{\bar{i}} - 1$ and $\tilde{\gamma}_{\bar{i}} = \gamma_{\bar{i+1}} + 1$ that is

$$\begin{cases} \tilde{\gamma}_{\bar{i+1}} = \lambda_{\bar{i}} + k_i - k_{\bar{i}} - 1 \\ \tilde{\gamma}_{\bar{i}} = \lambda_{\bar{i+1}} + k_{i+1} - k_{\bar{i+1}} + 1 \end{cases}.$$

Write $\tilde{k}_{i+1} = k_i$, $\tilde{k}_i = k_{i+1}$, $\tilde{k}_{\bar{i+1}} = \lambda_{\bar{i+1}} - \lambda_{\bar{i}} + k_{\bar{i}} + 1$ and $\tilde{k}_{\bar{i}} = \lambda_{\bar{i}} - \lambda_{\bar{i+1}} + k_{\bar{i+1}} - 1$. To make our notation homogeneous set $\tilde{k}_t = k_t$ for any $t \neq i, i + 1, \bar{i}, \bar{i} + 1$. Then $\lambda_{\bar{i}} > \lambda_{\bar{i+1}} - \tilde{k}_{\bar{i+1}}$. We have $\tilde{k}_{\bar{i+1}} \geq 0$ and $\tilde{k}_{\bar{i}} = \lambda_{\bar{i}} - \lambda_{\bar{i+1}} + k_{\bar{i+1}} - 1 \geq 0$ since $\lambda_{\bar{i}} > \lambda_{\bar{i+1}} - k_{\bar{i+1}}$. Moreover $\tilde{k}_{\bar{1}} + \dots + \tilde{k}_{\bar{n-1}} + \tilde{k}_1 + \dots + \tilde{k}_{n-1} = r$ and for any $s \in \{1, \dots, n-2\}$

$$\tilde{\gamma}_s = \lambda_{\bar{s}} + \tilde{k}_s - \tilde{k}_{\bar{s}}.$$

2. Suppose that $\lambda_{\bar{i}} \leq \lambda_{\bar{i}+1} - k_{\bar{i}+1}$ for all $i = 1, \dots, n-2$ and $\lambda_{\bar{1}} - k_{\bar{1}} < 0$. Set $\tilde{\gamma} = s_0 \circ \gamma$. Then $\gamma_{\bar{s}} = \tilde{\gamma}_{\bar{s}}$ for $s \neq 1$ and $\tilde{\gamma}_{\bar{1}} = -\lambda_{\bar{1}} - k_{\bar{1}} + k_1 - 2$. Write $\tilde{k}_{\bar{i}} = k_i$, $\tilde{k}_{\bar{i}} = k_{\bar{i}}$ for any $i = 2, \dots, n-1$ and set $\tilde{k}_{\bar{1}} = k_{\bar{1}} - \lambda_{\bar{1}} - 1$, $\tilde{k}_{\bar{1}} = k_1 + \lambda_{\bar{1}} + 1$. We have $\tilde{k}_{\bar{1}} \geq 0$, $\lambda_{\bar{i}} \leq \lambda_{\bar{i}+1} - \tilde{k}_{\bar{i}+1}$ for any $i = 1, \dots, n-2$ and $\lambda_{\bar{1}} - \tilde{k}_{\bar{1}} < 0$. Moreover $\tilde{k}_{\bar{1}} + \dots + \tilde{k}_{\bar{n-1}} + \tilde{k}_{\bar{1}} + \dots + \tilde{k}_{\bar{n-1}} = r$.
3. Now suppose that $\lambda_{\bar{s}} \leq \lambda_{\bar{s}+1} - k_{\bar{s}+1}$ for any $s \in \{1, \dots, n-2\}$, $\lambda_{\bar{1}} - k_{\bar{1}} \geq 0$ and there exists $i \in \{1, \dots, n-2\}$ such that $\lambda_{\bar{i}+1} - k_{\bar{i}+1} < \lambda_{\bar{i}} + k_i - k_{\bar{i}}$. Define $\tilde{\gamma} = s_i \circ \gamma$ as above. Set $\tilde{k}_{\bar{i}+1} = k_{\bar{i}+1}$, $\tilde{k}_{\bar{i}} = k_{\bar{i}}$, $\tilde{k}_{i+1} = \lambda_{\bar{i}} - \lambda_{\bar{i}+1} - k_{\bar{i}} + k_i + k_{\bar{i}+1} - 1$ and $\tilde{k}_i = (\lambda_{\bar{i}+1} - \lambda_{\bar{i}} - k_{\bar{i}+1}) + k_{i+1} + k_{\bar{i}} + 1$. Write $\tilde{k}_t = k_t$ for any $t \neq i, i+1, \bar{i}, \bar{i}+1$. We obtain $\tilde{k}_i \geq 0$ and $\tilde{k}_{i+1} \geq 0$ since $\lambda_{\bar{i}} \leq \lambda_{\bar{i}+1} - k_{\bar{i}+1}$ and $\lambda_{\bar{i}+1} - k_{\bar{i}+1} < \lambda_{\bar{i}} + k_i - k_{\bar{i}}$. Since $\tilde{k}_{\bar{s}} = k_{\bar{s}}$ for any $s = 1, \dots, n-1$, we have $\lambda_{\bar{s}} \leq \lambda_{\bar{s}+1} - \tilde{k}_{\bar{s}+1}$ for any $s \in \{1, \dots, n-2\}$ and $\lambda_{\bar{1}} - \tilde{k}_{\bar{1}} \geq 0$. Moreover the assertion $\lambda_{\bar{i}+1} - \tilde{k}_{\bar{i}+1} < \lambda_{\bar{i}} + \tilde{k}_i - \tilde{k}_{\bar{i}}$ holds since it is equivalent to $0 < k_{i+1} + 1$. Finally $\tilde{k}_{\bar{1}} + \dots + \tilde{k}_{\bar{n-1}} + \tilde{k}_{\bar{1}} + \dots + \tilde{k}_{\bar{n-1}} = r$ and for any $s \in \{1, \dots, n-2\}$

$$\tilde{\gamma}_{\bar{s}} = \lambda_{\bar{s}} + \tilde{k}_{\bar{s}} - \tilde{k}_{\bar{s}}.$$

Denote by E_1 , E_2 and E_3 the sets of multi-indices $(k_{\bar{1}}, \dots, k_{\bar{n-1}}, k_1, \dots, k_{n-1})$ such that $k_{\bar{1}} + \dots + k_{\bar{n-1}} + k_1 + \dots + k_{n-1} = r$ and satisfying respectively the assertions 1, 2, 3. Let χ be the map defined on $E_1 \cup E_2 \cup E_3$ by

$$\chi(\gamma) = \tilde{\gamma}.$$

Then by the above arguments χ is a bijection which verifies $\chi(E_i) = E_i$ for $i = 1, 2, 3$. Now the pairing $\gamma \leftrightarrow \tilde{\gamma}$ provides the cancellation of all the s_γ with $\gamma = (\lambda_{\bar{n-1}} + k_{n-1} - k_{\bar{n-1}}, \dots, \lambda_{\bar{1}} + k_1 - k_{\bar{1}})$ such that $(k_{\bar{1}}, \dots, k_{\bar{n}}, k_1, \dots, k_n) \in E_1 \cup E_2 \cup E_3$ appearing in S . Indeed $s_{(\mu_{\bar{n}}+r+2m, \gamma)} = -s_{(\mu_{\bar{n}}+r+2m, \tilde{\gamma})}$. By Corollary 3.1.2 it means that

$$S = \sum_{\substack{\gamma \in P_{n-1}^+ \\ \gamma \in (\gamma \oplus r)_{n-1}}} c_{\gamma, r}^\lambda s_{(\mu_{\bar{n}}+r+2m, \gamma)}$$

and the theorem is proved. □

Note that the theorem is also true for $n = 2$. In this case R_{n-1} is the set of positive roots of the root system A_1 .

Corollary 3.2.2 *Let $v, \mu \in P_n^+$ such that $\mu_{\bar{n}} \geq v_{\bar{n-1}}$. Suppose $v_{\bar{n}} \geq \mu_{\bar{n}}$ (otherwise $K_{v, \mu}(q) = 0$) and set $l = v_{\bar{n}} - \mu_{\bar{n}}$, $v' = (v_{\bar{n-1}}, \dots, v_{\bar{1}})$. Then*

$$K_{v, \mu}(q) = \sum_{r+2m=l} q^{r+m} \sum_{\lambda \in (v' \otimes r)_{n-1}} c_{v', r}^\lambda K_{\lambda, \mu'}(q).$$

Proof: Let m, r be two integers such that $(\mu_{\bar{n}} + r + 2m, v') = v$. Then $l = r + 2m$. Consider $s_{(\mu_{\bar{n}} + r_* + 2m_*, \gamma)}$ and $s_{(\mu_{\bar{n}} + r + 2m, v')}$ appearing in (13). Suppose that there exists $\sigma \in W_n$ such that $(\mu_{\bar{n}} + r_* + 2m_*, \gamma) = \sigma \circ (\mu_{\bar{n}} + r + 2m, v') = \sigma(\mu_{\bar{n}} + r + 2m + n, v_{\bar{n}-1} + n - 1, \dots, v_{\bar{1}} + 1) - (n, \dots, 1)$. We can not have $\sigma(p) = \bar{n}$ with $p \in \{1, \dots, n\}$ otherwise $\mu_{\bar{n}} + r_* + 2m_* < 0$. Set $\sigma(\bar{p}) = \bar{n}$ with $p \in \{1, \dots, n\}$. If $p < n$ we must have

$$v_{\bar{p}} + p = \mu_{\bar{n}} + r_* + 2m_* + n.$$

Thus $v_{\bar{p}} = \mu_{\bar{n}} + r_* + 2m_* + n - p > \mu_{\bar{n}}$ which contradicts the hypothesis $\mu_{\bar{n}} \geq v_{\bar{n}-1}$. Hence $\sigma(\bar{n}) = \bar{n}$ and $r_* + 2m_* = l$ that is $\sigma \in W_{n-1}$. Moreover we have $\gamma = v'$ since $w \circ v' = \gamma$ and $v, \gamma \in P_{n-1}$. This proves that $s_{(\mu_{\bar{n}} + r + 2m, v')}$ can not be obtained by applying the straightening law for Schur functions on $s_{(\mu_{\bar{n}} + r_* + 2m_*, \gamma)}$ with $(\mu_{\bar{n}} + r_* + 2m_*, \gamma) \neq (\mu_{\bar{n}} + r + 2m, v')$. Then the corollary directly follows from (13) and Theorem 2.1.2. \square

Now suppose that $v = (p)_n = (p, 0, \dots, 0) \in P_n^+$. Then by Corollary 3.2.2 for any $\mu \in P_+^n$ we must have

$$K_{(p)_n, \mu}(q) = \sum_{r+2m=l} q^{r+m} K_{(r)_{n-1}, \mu'}(q) \quad (15)$$

with $l = p - \mu_{\bar{n}}$ and μ' defined as in statement of Theorem 3.2.1. This implies that $K_{(p)_n, \mu}(q)$ may be computed recursively. We are going to give an explicit formula for $K_{(p)_n, \mu}(q)$. The vertices of $B((p)_n)_\mu = \{L \in B(p\Lambda_{n-1}), \text{wt}(b) = \mu\}$ are the words

$$L = (n)^{k_n} \dots (2)^{k_2} (1)^{k_1} (\bar{1})^{k_{\bar{1}}} (\bar{2})^{k_{\bar{2}}} \dots (\bar{n})^{k_{\bar{n}}}$$

with $\mu_{\bar{i}} = k_{\bar{i}} - k_i$ for $i = 1, \dots, n$ and $k_{\bar{1}} + \dots + k_{\bar{n}} + k_1 + \dots + k_n = p$.

Proposition 3.2.3 *Let $p \geq 1$ be an integer. For any $\mu \in P_n^+$ we have*

$$K_{(p)_n, \mu}(q) = q^{f_n(\mu)} \sum_{L \in B((p)_n)_\mu} q^{\theta_n(L)}$$

where $f_n(\mu) = \sum_{i=1}^n (n-i)\mu_{\bar{i}}$ and $\theta_n(L) = \sum_{i=1}^n (2(n-i)+1)(k_{\bar{i}} - \mu_{\bar{i}})$.

Proof: We proceed by induction on n . Suppose $n = 1$ we have $f_1(\mu) = 0$. We can write $L = (1)^{p-k_{\bar{1}}} (\bar{1})^{k_{\bar{1}}}$ and $\mu_{\bar{1}} = 2k_{\bar{1}} - p$. Thus $\theta_1(L) = k_{\bar{1}} - \mu_{\bar{1}} = (p - \mu_{\bar{1}})/2$ and the proposition is true by (3).

Now suppose the proposition true for $n-1$. First note that $f_n(\mu) = f_{n-1}(\mu') + \sum_{i=1}^{n-1} \mu_{\bar{i}}$ where μ' is defined as in statement of Theorem 3.2.1. The set of vertices obtained by erasing the letters n and \bar{n} in $B((p)_n)_\mu$ is the disjoint union of the $B((r)_{n-1})_{\mu'}$ with $r \in \{0, \dots, l = p - \mu_{\bar{n}}\}$ since the number of letters n or \bar{n} belonging to a vertex $L \in B((p)_n)_\mu$ is a least equal to $\mu_{\bar{n}}$. Its reflects the decomposition of $B((p)_n)_\mu$ into its $U_q(sp_{2(n-1)})$ -connected components. Consider $L \in B((p)_n)_\mu$ and denote by L' the vertex obtained by erasing all the letters n and \bar{n} in L . Let r be such that $L' \in B((r)_{n-1})_{\mu'}$. Then $r \in \{0, \dots, l\}$ and $l-r$

is even since it is equal to the number of pairs (n, \bar{n}) erased in L . We set $l - r = 2m$. Then $k_{\bar{n}} = \mu_{\bar{n}} + m$.

We have

$$\theta_n(L) = \theta_{n-1}(L') + 2 \sum_{i=1}^{n-1} (k_{\bar{i}} - \mu_{\bar{i}}) + (k_{\bar{n}} - \mu_{\bar{n}}).$$

From the equality $r = \sum_{1 \leq i \leq n-1} \mu_{\bar{i}} + 2 \sum_{1 \leq i \leq n-1} (k_{\bar{i}} - \mu_{\bar{i}})$ we deduce

$$\theta_n(L) = \theta_{n-1}(L') + r - \sum_{i=1}^{n-1} \mu_{\bar{i}} + m.$$

Set

$$K = \sum_{L \in B((p)_n)_\mu} q^{\theta_n(L) + f_n(\mu)}.$$

Then by the above arguments

$$\begin{aligned} K &= \sum_{r+2m=l} \sum_{L' \in B((r)_{n-1})_{\mu'}} q^{\theta_{n-1}(L') + r - \sum_{i=1}^{n-1} \mu_{\bar{i}} + m + f_{n-1}(\mu') + \sum_{i=1}^{n-1} \mu_{\bar{i}}} \\ &= \sum_{r+2m=l} q^{r+m} \times \sum_{L' \in B((r)_{n-1})_{\mu'}} q^{\theta_{n-1}(L') + f_{n-1}(\mu')}. \end{aligned}$$

Thus we obtain by the induction hypothesis

$$K = \sum_{r+2m=l} q^{r+m} K_{(r)_{n-1}, \mu'}(q).$$

Finally $K = K_{(p)_n, \mu}(q)$ by (15). □

Corollary 3.2.4 Write $(1^2)_n$ for the partition of length n equal to $(1, 1, 0, \dots, 0)$. Then

$$K_{(1^2)_n, 0}(q) = \sum_{i=1}^{n-1} q^{2i}.$$

To prove this corollary we need the more general lemma above

Lemma 3.2.5 Write $(1^p)_n$ for the partition of length n $(1, \dots, 1, 0, \dots, 0)$ with $p \geq 2$ parts equal to 1. Then

$$K_{(1^p)_n, 0}(q) = (q - 1)K_{\gamma^p, 0}(q) + qK_{(1^p)_{n-1}, 0}(q) + qK_{(1^{p-2})_{n-1}, 0}(q)$$

where $\gamma^p = (2, 1, \dots, 1, 0, \dots, 0) \in P_{n-1}^+$ contains $p - 2$ parts equal to 1.

Proof: With $\mu = 0$ formula (13) becomes

$$Q_0 = \sum_{\gamma \in P_{n-1}^+} \sum_{r=0}^{+\infty} \sum_{m=0}^{+\infty} q^{m+r} \sum_{\lambda \in (\gamma \otimes r)_{n-1}} c_{\gamma,r}^\lambda K_{\lambda,0}(q) s_{(r+2m,\gamma)}.$$

By using the straightening law for Schur functions and Theorem 2.1.2 we have to find all the $(r + 2m, \gamma)$ such that there exists $\sigma \in W_{n-1}$ satisfying $\sigma \circ (r + 2m, \gamma) = (1^p)_n$ that is

$$\sigma(r + 2m + n, \gamma_{n-1} + n - 1, \dots, \gamma_1 + 1) = (n + 1, \dots, n - p + 2, n - p, \dots, 1). \tag{16}$$

We have $r + 2m + n \geq n$ hence $\sigma(\bar{n}) \in \{\bar{n}, \overline{n-1}\}$.

- (i) If $\sigma(\bar{n}) = \bar{n}$ then $r = 1$ and $m = 0$. For $k \notin \{1, n\}$, $\gamma_k + k > 1$ thus $\sigma(\bar{1}) = \bar{1}$. By a straightforward induction we obtain $\sigma(\bar{k}) = \bar{k}$ for $k \in \{1, \dots, n-p\}$. Moreover we have $\gamma_k + k \leq n$ for $k < n$. This implies that $\gamma_{n-1} \in \{0, 1\}$ since $\gamma_{n-1} + n - 1 \geq n - 1$. We can not have $\gamma_{n-1} = 0$ otherwise $\gamma_k = 0$ for any $k < n$ and the value n in the left hand side of (16) is not attained. Hence $\gamma_{n-1} = 1$ and $\sigma(\overline{n-1}) = \overline{n-1}$. By induction we can prove that $\gamma_{n-1} = \dots = \gamma_{n-p+1} = 1$ and $\sigma(\bar{k}) = \bar{k}$ for $k \in \{n-1, \dots, n-p+1\}$. It means that $\sigma = id$, $r = 1$, $m = 0$ and $\gamma = (1^{p-1})_{n-1}$.
- (ii) If $\sigma(\bar{n}) = \overline{n-1}$ then $R = m = 0$. By using similar arguments than above we obtain $\gamma_{n-1} = 2$, $\gamma_{n-2} = \dots = \gamma_{n-p+1} = 1$ and $\gamma_{n-p} = \dots = \gamma_1 = 0$. It means that $\sigma = s_n$ and $\gamma = \gamma^p$. Note that $s_{(0,\gamma^p)} = -s_{((1^p)_n)}$ since $s_n \circ (0, \gamma^p) = (1^p)_n$ and $l(s_n) = 1$.

Finally by Theorem 2.1.2 we must have

$$\begin{aligned} K_{(1^p)_n,0}(q) &= q \times \sum_{\lambda \in ((1^{p-1})_{n-1} \otimes 1)_{n-1}} c_{(1^{p-1})_{n-1},1}^\lambda K_{\lambda,0}(q) - K_{\gamma^p,0}(q) \\ &= (q - 1)K_{\gamma^p,0}(q) + qK_{(1^p)_{n-1},0} + qK_{(1^{p-2})_{n-1},0}. \end{aligned}$$

□

Proof: (of Corollary 3.2.4). We proceed by induction on n . For $n = 2$, $K_{(1^2),0}(q) = q^2$. Suppose the corollary true for $k < n$. Then by applying Lemma 3.2.5 we obtain

$$K_{(1^2)_n,0}(q) = (q - 1)K_{(2)_{n-1},0}(q) + qK_{(1^2)_{n-1},0}(q) + q.$$

It follows from Proposition 3.2.3 that

$$K_{(2)_{n-1},0}(q) = \sum_{i=1}^{n-1} q^{2i-1}.$$

Thus

$$\begin{aligned}
 K_{(1^2)_n,0}(q) &= (q-1) \sum_{i=1}^{n-1} q^{2i-1} + q \sum_{i=1}^{n-2} q^{2i} + q = \sum_{i=1}^{n-1} q^{2i} - \sum_{i=1}^{n-1} q^{2i-1} \\
 &\quad + \sum_{i=1}^{n-2} q^{2i+1} + q = \sum_{i=1}^{n-1} q^{2i}.
 \end{aligned}$$

□

Note that we can not deduce an explicit formula for $K_{(1^p)_n,0}(q)$ with $p > 2$ from the recurrence formula of Lemma 3.2.5 as we have done in Proposition 3.2.3 since we have no explicit formula for $K_{\gamma^p,0}(q)$ as soon as $p > 2$. Nevertheless we will give a conjectural general formula for $K_{(1^p)_n,0}(q)$ in Section 5.

4. Cyclage graphs for symplectic tableaux

Given a symplectic tableau $T \in \mathbf{ST}(n)$, we can factorize $w(T)$ in a unique way by setting $w(T) = xu$ where u is a word and x is a letter. It is easy to verify that u is also the reading of a symplectic tableau, say $T_* \in \mathbf{ST}(n)$. The initial cocyclage operation on T consists of the insertion $x \rightarrow T_*$. We are going to see that all the initial cocyclage operations are not relevant for defining a charge.

It follows from Paragraph 2.3 that the tableau obtained by cocycling a tableau $T \in \mathbf{ST}(n)$ does not belong to $\mathbf{ST}(n)$ in general but belongs to $\mathbf{ST}(n+1)$. To overcome this problem we are going to define our cocyclage operation directly on the complete symplectic tableaux set $\mathbf{ST} = \cup_{n \geq 1} \mathbf{ST}(n)$.

4.1. Cocyclage operation

Set $\mathcal{C}_\infty = \cup_{n \geq 1} \mathcal{C}_n$. Then \mathcal{C}_∞ is totally ordered by \leq . Given any $T \in \mathbf{ST}$ there exists an integer $m \geq 1$ such that $T \in \mathbf{ST}(m)$. Recall that $d_{\bar{i}}$ is the number of letters \bar{i} of T minus the number of letters i . For any weight $\mu \in P_n$, we will say that $T \in \mathbf{ST}$ is a tableau of weight μ if $T \in \mathbf{ST}(m)$ with $m \geq n$, $d_{\bar{i}} = 0$ for $i > n$ and $d_{\bar{k}} = \mu_{\bar{k}}$ for $k = 1, \dots, n$. For any $\mu \in P_n$, the set of tableaux of weight μ is denoted $\mathbf{ST}(\mu)$. If $T \in \mathbf{ST}(\mu)$, the number of letters k with $k > n$ which belong to T is equal to the number of letters \bar{k} .

Let $w \in \mathcal{C}_\infty^*$ and write $w = xu$ with x a letter and $u \in \mathcal{C}_\infty^*$. The cocyclage shift ξ is the map defined on \mathcal{C}_∞^* by

$$\xi(w) = ux.$$

Lemma 4.1.1 For any $n \geq 1$, $\sigma \in W_n$, and $w \in \mathcal{C}_\infty^*$, $\xi(\sigma(w)) = \sigma(\xi(w))$.

Proof: The proof is analogous to that of Proposition 5.6.1 of [16].

□

Consider a symplectic tableau $T = C_1 \cdots C_r \in \mathbf{ST}(m)$ with $r > 1$. We will say that the cocyclage operation is authorized for T if there is no letter $y \in \mathcal{C}_m$ such that $y \in C_i$ for all $i = 1, \dots, r$ and $\bar{y} \notin T$. Thus, since each column of T contains at most one letter \bar{y} , the cocyclage operation is not authorized for T when there exists an integer $p \in \{1, \dots, m\}$ such that $|d_{\bar{p}}|$ is equal to r . If the cocyclage operation is authorized for T , we write $w(T) = xw(T_*)$ where $T_* \in \mathbf{ST}(m)$ and $x \in \mathcal{C}_m$. and we set

$$U(T) = x \rightarrow T_*.$$

Remarks

- (i) $U(T)$ belongs to \mathbf{ST} and $\text{wt}(U(T)) = \text{wt}(T)$. More precisely, if $T \in \mathbf{ST}(m)$ then $U(T) \in \mathbf{ST}(m)$ if the heights of the first columns of T and $U(T)$ are equal, and otherwise $U(T) \in \mathbf{ST}(m + 1)$.
- (ii) If $\text{wt}(T) = 0$ then the cocyclage operation is always authorized.
- (iii) By convention there is no cocyclage operation on the columns.
- (iv) Suppose that T contains r columns. Then the number of columns of T_* is r or $r - 1$. Thus $U(T)$ contains at most $r + 1$ columns since its shape is obtained by adding one box to that of T_* . Moreover $U(T)$ contains $r + 1$ columns only if the height of its rightmost column is equal to 1.

Example 4.1.2 Consider the tableaux $T_1 = \begin{array}{|c|c|c|} \hline \bar{4} & \bar{3} & \bar{2} \\ \hline \bar{2} & \bar{2} & \bar{1} \\ \hline 2 & & \\ \hline \end{array}$, $T_2 = \begin{array}{|c|c|c|} \hline \bar{4} & \bar{3} & 4 \\ \hline \bar{2} & \bar{2} & \\ \hline 2 & & \\ \hline \end{array}$ and $T_3 = \begin{array}{|c|c|c|} \hline \bar{4} & \bar{3} & \bar{2} \\ \hline \bar{2} & \bar{2} & \bar{1} \\ \hline 2 & & 3 \\ \hline \end{array}$. Then the cocyclage operation is authorized for T_1 and T_2 but not in T_3 . For T_1 , we obtain

$$x_{1,*} = \bar{2}, T_{1,*} = \begin{array}{|c|c|c|} \hline \bar{4} & \bar{3} & \bar{1} \\ \hline \bar{2} & \bar{2} & \\ \hline 2 & & \\ \hline \end{array}$$

and

$$U(T_1) = \begin{array}{|c|c|c|} \hline \bar{4} & \bar{3} & \bar{1} \\ \hline \bar{3} & \bar{2} & \\ \hline \bar{2} & 3 & \\ \hline \end{array}.$$

Similarly we have

$$x_{2,*} = 4, T_{2,*} = \begin{array}{|c|c|} \hline \bar{4} & \bar{3} \\ \hline \bar{2} & \bar{2} \\ \hline 2 & \\ \hline \end{array}$$

and

$$U(T_2) = \begin{array}{|c|c|} \hline \overline{4} & \overline{3} \\ \hline \overline{2} & \overline{2} \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array}.$$

Lemma 4.1.3 *Suppose $T \in \mathbf{ST}(m)$ and consider $\sigma \in W_m$. Then the cocyclage operation is authorized for T if and only if it is authorized for $\sigma(T)$. In this case we have $U(\sigma(T)) = \sigma(U(T))$.*

Proof: The first statement follows from $\text{wt}(T) = \text{wt}(U(T))$ and the second statement follows from Lemma 4.1.1 and (12). □

4.2. *Cyclage graphs*

We endow the set \mathbf{ST} with a structure of graph by drawing an array $T \rightarrow T'$ if and only if the cocyclage operation is authorized on T and $U(T) = T'$. Write $\Gamma(T)$ for the connected component containing T . Let t be the translation operation on letters of \mathcal{C}_∞ defined by $t(k) = k + 1$ and $t(\overline{k}) = \overline{k + 1}$ for $k \geq 1$. We write $t(w)$ (resp. $t(T)$) for the word (resp. the tableau) obtained by applying t to each letter of $w \in \mathcal{C}_\infty^*$ (resp. to each letter of $T \in \mathbf{ST}$).

Lemma 4.2.1

- (i) *Suppose $T \in \mathbf{ST}(m)$. Then $\Gamma(T)$ and $\Gamma(\sigma(T))$ are isomorphic for any $\sigma \in W_m$.*
- (ii) *The cyclage graphs $\Gamma(T)$ and $\Gamma(t(T))$ are isomorphic.*
- (iii) *Suppose that $T_1 \neq T_2 \in \Gamma(T)$ are such that $U(T_1) = U(T_2) = T$. Then T_1 and T_2 have different shapes.*

Proof: Assertion (i) follows immediately from Lemma 4.1.3.

Let w_1 and w_2 be two words of \mathcal{C}_m . Then $w_1 \equiv_m w_2$ if and only if $t(w_1) \equiv_{m+1} t(w_2)$. This implies that $P(t(w)) = t(P(w))$ for any word $w \in \mathcal{C}_\infty$. Hence t commutes with U . Since t is a bijection, it is also an isomorphism between $\Gamma(T)$ and $\Gamma(t(T))$ which proves (ii).

Suppose that $T_1, T_2 \in \mathbf{ST}(m)$ have the same shape X . Write $w(T_1) = xw(R)$ and $w(T_2) = yw(S)$ with x, y two letters and R, S two symplectic tableaux. Then $w(R)x \equiv_{m+1} w(S)y$ since $P(w(R)x) = P(w(S)y) = T$. Then R and S have the same shape Y obtained by deleting one box in the rightmost column of X . The highest weight vertices of the connected components of G_{m+1} containing $w(R)x$ and $w(S)y$ may be respectively written $w(Y_0)x_0$ and $w(Y_0)y_0$ where $Y_0 \in \mathbf{ST}(m + 1)$ is the highest weight tableau of shape Y . The congruence $w(R)x \equiv_{m+1} w(S)y$ implies the congruence $w(Y_0)x_0 \equiv_{m+1} w(Y_0)y_0$. Thus we must have $\text{wt}(w(Y_0)x_0) = \text{wt}(w(Y_0)y_0)$. It means that $x_0 = y_0$. Hence $w(R)x$ and $w(S)y$ are congruent

and belong to the same connected component. This implies that $w(R)x = w(S)y$, thus $xw(R) = yw(S)$ and $T_1 = T_2$. So (iii) is proved. \square

Assertion (i) of the above lemma permits to restrict to the cyclage graphs $\Gamma(T)$ with $T \in \mathbf{ST}(\mu)$ and $\mu \in P_n^+$. Suppose first that $\mu = 0$. By remark (ii) above, the cocyclage operation is always authorized on symplectic tableaux of weight 0 with at least two columns. So we can define from T a sequence (T_n) of symplectic tableaux by setting $T_0 = T$ and $T_{k+1} = U(T_k)$ while T is not a column.

Proposition 4.2.2 *Let $T_0 \in \mathbf{ST}(0)$ and let $T_{k+1} = U(T_k)$. Then the sequence (T_n) is finite without repetition and there exists an integer e such that T_e is a column of weight 0.*

To prove this proposition we need two technical lemmas. Given two words $w_1, w_2 \in \mathcal{C}_\infty^*$, write $w_1 \triangleleft w_2$ if w_1 and w_2 can respectively be written $w_1 = u_1x_1v$ and $w_2 = u_2x_2v$ where $u_1, u_2, v \in \mathcal{C}_\infty^*$ and $x_1, x_2 \in \mathcal{C}_\infty$ satisfy $x_1 < x_2$. It means that \triangleleft is the inverse lexicographic order on words of \mathcal{C}_∞^* . For any symplectic tableau T with $r > 1$ columns, we denote by $N_r(T)$ the number of boxes belonging to the $r - 1$ rightmost columns of T .

Lemma 4.2.3 *Consider $\mu \in P_n^+$ and $\tau \in \mathbf{ST}(\mu)$ a tableau with $r > 1$ columns. Let T, T' two tableaux of $\Gamma(\tau)$ such that $T = U^{(i)}(\tau)$ and $T' = U^{(i+1)}(\tau)$ with $i \geq 0$ an integer. Then the following assertions hold.*

1. For any j , $S = U^{(j)}(\tau)$ contains at most $r + 1$ columns and if it contains $r + 1$ columns, the height of its rightmost column is equal to 1.
2. Suppose that T contains r or $r + 1$ columns and $N_r(T) = N_r(T')$. Then only one of the following situations can happen:
 - (i) T and T' contain r columns and their r -th columns have the same height.
 - (ii) T contains r columns and T' contains $r + 1$ columns.
 - (iii) T and T' contains $r + 1$ columns.
 - (iv) T contains $r + 1$ columns, T' contains r columns and the height of the last column C'_r of T' is equal to $h(C_r) + 1$.

Moreover in each case we can write $w(T) = x_*w(T_*)$ and $w(T') = x'_*w(T'_*)$ with $w(T'_*) \triangleleft w(T_*)$.

3. Suppose that T contains r or $r + 1$ columns and $N_r(T) \neq N_r(T')$. Then $N_r(T) > N_r(T')$.

Proof:

1. We proceed by induction on j starting from $j = 0$. If $S = U^{(j)}(\tau)$ contains at most r columns, then by Remark (iv) before Example 4.1.2 the assertion is true for $U(S)$. Now if $S = U^{(j)}(\tau)$ contains $r + 1$ columns, then S_* contains r columns and $U(S)$ contains either r columns either $r + 1$ columns with only one box in its rightmost column.

2. In case (i) the box which is added to T_* during the insertion $x_* \rightarrow T_*$ appears on the bottom of the last column $C_{r,*}$ (eventually empty) of T_* . This insertion can be written $x_* \rightarrow T_* = (x_* \rightarrow C_1 \cdots C_{r-1})C_{r,*} = (C'_1 \cdots C'_{r-1})(y \rightarrow C_{r,*})$ that is, x_* is first inserted in the sub-tableau composed of the $r - 1$ leftmost columns of T_* which gives a new tableau $C'_1 \cdots C'_{r-1}$ and a letter y . This letter is then inserted on the bottom of $C_{r,*}$. Suppose that there exists an integer $i \in \{1, \dots, r - 1\}$ such that $C'_i \neq C_i$. Then if we choose i minimal we have $w(C'_i) \triangleleft w(C_i)$ by (10) and finally $w(T'_*) \triangleleft w(T_*)$. Now if $C'_i = C_i$ for $i = 1, \dots, r - 1$ we have $y = x_*$ and $x_* \in C_i$, $\bar{x}_* \notin C_i$ for any $i = 1, \dots, r - 1$. So the letter x_* belongs to all the columns of T . Then x_* is a barred letter since $\mu \in P_n^+$ and $r > 1$. Moreover $\bar{x}_* \notin C_r^*$ for $x_* \rightarrow C_r^*$ is a column. Thus $\bar{x}_* \notin T$. This contradicts the fact that the cocyclage operation is authorized for T .

In case (ii) a new column of height 1 is added to the shape of T_* . The insertion can be written $x_* \rightarrow T_* = (x_* \rightarrow C_1 \cdots C_{r-1})C_{r,*} = (C'_1 \cdots C'_{r-1})(y \rightarrow C_{r,*}) = C'_1 \cdots C'_{r-1}C'_{r,*} \boxed{x_*}$ that is, x_* is inserted in the sub-tableau composed of the $r - 1$ leftmost columns of T_* which gives a new tableau $C'_1 \cdots C'_{r-1}$ and a letter y . This letter is then inserted in $C_{r,*}$ which gives the column $C'_{r,*}$ and the letter x'_* . If $y \neq x_*$, we terminate as in case (i). Otherwise we have $C'_i = C_i$ for any $i = 1, \dots, r - 1$. We can not have $x'_* = x_*$ since it would imply that $x_* \in C_{r,*}$ which is impossible since C_r can not contain two letters x_* . Thus $x'_* > x_*$, $w(C'_{r,*}) \triangleleft w(C_{r,*})$ and finally $w(T'_*) \triangleleft w(T_*)$.

Case (iii) is similar to case (i) with $h(C_r) = 1$.

In case (iv) C_{r+1} contains only the letter x_* and a new box appears on the bottom of the column C_r of T_* during the insertion $x_* \rightarrow T_*$. The insertion can be written $x_* \rightarrow T_* = x_* \rightarrow (C_1 \cdots C_r) = (C'_1 \cdots C'_{r-1})(y \rightarrow C_{r,*}) = C'_1 \cdots C'_{r-1}C'_{r,*}$, that is x_* is inserted in the sub-tableau composed of the $r - 1$ leftmost columns of T_* which gives the tableau $(C'_1 \cdots C'_{r-1})$ and the letter y . This letter is then inserted on the bottom of $C_{r,*}$ which gives the column $C'_{r,*}$. Suppose that $y = x_*$. We must have $x_* \in C_i$, and $\bar{x}_* \notin C_i$ for any $i = 1, \dots, r - 1$. Then $x_* = \bar{q}$ with $q \geq 1$ that is, is a barred letter as in (i). Moreover $\bar{x}_*, x_* \notin C_r$ because $x_* \rightarrow C_r$ is a column. Thus $d_{\bar{q}}(w(T)) = r$. Now T contains $r + 1$ columns, hence $T \neq \tau$. Let j minimal such that $R = U^{(i-j)}(\tau)$ contains r columns. Then $d_{\bar{q}}(w(R)) = d_{\bar{q}}(w(T)) = r$. Thus the cocyclage operation is not authorized in R and we obtain a contradiction. It means that $y \neq x_*$. So we can terminate as in case (i).

3. It is clear from the definition of the cocyclage operation. □

Lemma 4.2.4 *Let $T = C_1 \cdots C_p \in \mathbf{ST}(0)$ with $r > 1$ columns. Then there exists an integer k such that T_k has at most $r - 1$ columns. Moreover if k is minimal the sequence T_0, \dots, T_k is without repetition.*

Proof: Let $m \geq 1$ be an integer such that T_0, \dots, T_m have r or $r + 1$ columns and $N_r(T_i) = N_r(T)$ for any $i = 1, \dots, m$. Then by Lemma 4.2.3 we can write $w(T_i) =$

$x_{i,*}w(T_{i,*}), i = 0, \dots, m$ with

$$w(T_{m,*}) \triangleleft \dots \triangleleft w(T_{1,*}) \triangleleft w(T_{0,*}). \tag{17}$$

This implies that the sequence T_0, \dots, T_m is without repetition. Now suppose that $T \in \mathbf{ST}(n)$. Then, by Remark (i) after Lemma 4.1.1 and the fact that $N_r(T_i) = N_r(T)$ the height of the first column of any tableau $T_i, i = 0, \dots, m$ is always equal to that of T_0 . Thus $T_i \in \mathbf{ST}(n)$ for any $i = 0, \dots, m$. Denote by p the number of boxes in T_0 . Since the number of symplectic tableaux with p boxes belonging to $\mathbf{ST}(n)$ is finite there exists an integer s_1 minimal such that $N_r(T_{s_1}) = N_r(T) + 1$ or T_{s_1} has at most $r - 1$ columns. Then the sequence T_0, \dots, T_{s_1} is without repetition. If T_{s_1} has at most $r - 1$ columns we take $k = s_1$. Otherwise T_{s_1} has r or $r + 1$ columns and we can obtain similarly starting from T_{s_1} an integer s_2 minimal such that $N_p(T_{s_2}) = N_p(T_{s_1}) + 1$ or T_{s_2} has at most $r - 1$ columns. The sequence T_{s_1}, \dots, T_{s_2} is without repetition. Then the sequence T_0, \dots, T_{s_2} is also without repetition. Indeed a tableau T_i with $i \in \{0, \dots, s_1 - 1\}$ can not be equal to a tableau T_j with $j \in \{s_1, \dots, s_2 - 1\}$ since $N_r(T_i) \neq N_r(T_j)$. By induction we can construct $T_{s_{j+1}}$ from T_{s_j} while T_{s_j} has r or $r + 1$ columns, such that $N_r(T_{s_{j+1}}) = N_r(T_{s_j}) + 1$ and the sequence $T_0, \dots, T_{s_{j+1}}$ is without repetition. The procedure terminates since the number of boxes belonging to the columns r and $r + 1$ decreases by 1 to each step. So the lemma is proved. \square

Proof: (of Proposition 4.2.2): Let $r > 1$ be the number of columns of T . By Lemma 4.2.4, we can obtain from $T = T_0$ a tableau T_{k_1} with at most $r - 1$ columns and such that the sequence T_0, \dots, T_{k_1} is without repetition. If $r - 1 > 1$ we can obtain a tableau T_{k_2} from T_{k_1} with at most $r - 2$ columns and such that the sequence T_0, \dots, T_{k_2} is without repetition. We can define $T_{k_{s+1}}$ from T_{k_s} while $r - s > 1$ such that the sequence $T_0, \dots, T_{k_{s+1}}$ is without repetition. It is clear that the procedure terminates when $T_{k_s} = T_e$ is a column of weight 0. \square

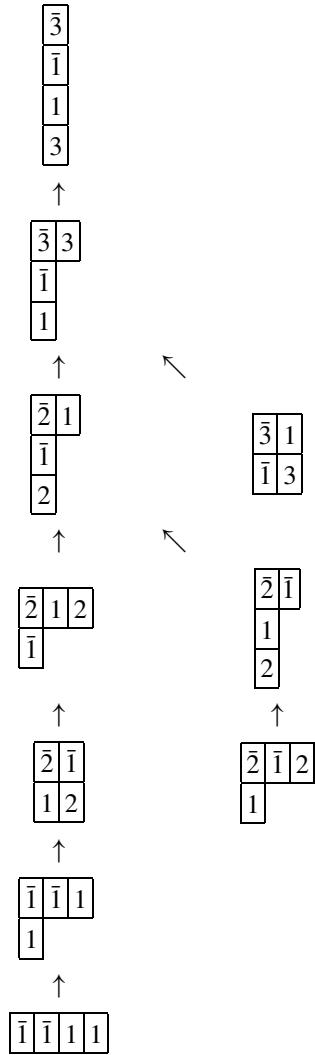
It follows from Proposition 4.2.2 that $\text{wt}(T_1) = \text{wt}(T_2) \not\Rightarrow \Gamma(T_1) = \Gamma(T_2)$ in general. For example all the columns of weight 0 occur in different connected components.

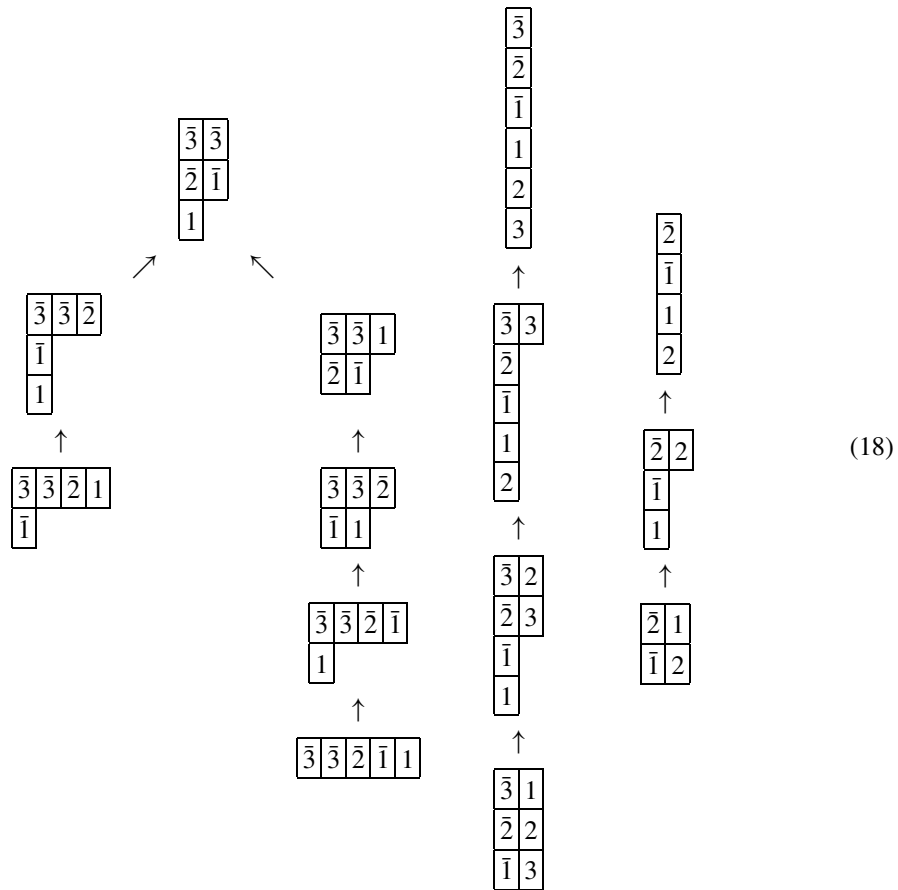
We give below

$$\Gamma \left(\begin{array}{|c|c|c|c|} \hline \bar{1} & \bar{1} & 1 & 1 \\ \hline \end{array} \right), \Gamma \left(\begin{array}{|c|c|c|c|} \hline \bar{3} & \bar{3} & \bar{2} & \bar{1} \\ \hline \end{array} \right), \Gamma \left(\begin{array}{|c|c|} \hline \bar{3} & 1 \\ \hline \bar{2} & 2 \\ \hline \bar{1} & 3 \\ \hline \end{array} \right)$$

and

$$\Gamma \left(\begin{array}{|c|c|} \hline \bar{2} & 1 \\ \hline \bar{1} & 2 \\ \hline \end{array} \right)$$





Remarks

- (i) Given $T' \in \mathbf{ST}$, it is possible to find the tableaux T (if there is any) such that $U(T) = T'$. To do this we find all the pairs (x, T_*) obtained by applying the reverse insertion algorithm on the outside corners of T' . By definition of U , the tableaux T are precisely those which verify $w(T) = xw(T_*)$ for a pair (x, T_*) . They are determined by the pairs (x, T_*) for which $xw(T_*)$ is the reading of a symplectic tableau. For example $T' = \begin{bmatrix} 2 & 1 \\ \bar{1} & 2 \end{bmatrix}$ has only one outside corner which gives $x = \bar{1}$ and $T_* = \begin{bmatrix} \bar{1} & 1 \\ 1 \end{bmatrix}$. There is no tableau T such that $U(T) = T'$ since $\bar{1}(1\bar{1}1)$ is not the reading of a symplectic tableau.
- (ii) In the definition of $U(T)$ we have restricted to the authorized cocyclages. For type A the cyclage graphs take also into account non initial cyclages. When T is of dominant evaluation that is $\text{wt}(T)$ is a dominant weight, they are obtained by considering all

the factorizations $w(T) \equiv yw(Y)$ in the plactic monoid with $y \neq 1$ a letter and Y a semi-standard tableau. The use of non initial cocyclages with symplectic tableaux is problematic because it can make appear loops in the cyclage graphs. For example consider $Z = \begin{array}{|c|c|} \hline \bar{3} & 3 \\ \hline \bar{1} & \\ \hline 1 & \\ \hline \end{array}$. Then $w(Z) = (3\bar{3} \bar{1})1 \equiv_3 \bar{2}(2\bar{1}1) \equiv_3 \bar{2} \bar{1}21$. Now if we compute $P(\xi(\bar{2} \bar{1}21))$ we obtain $Z' = P(\bar{1}21\bar{2}) = \begin{array}{|c|c|} \hline \bar{2} & \bar{1} & 2 \\ \hline 1 & & \\ \hline \end{array}$. So we have a loop since $U^{(3)}(Z') = Z$ (see the cyclage graph $\Gamma(\begin{array}{|c|c|c|c|} \hline \bar{1} & \bar{1} & 1 & 1 \\ \hline \end{array})$ above).

(iii) For type A , every semi-standard tableau belongs to the cyclage graph containing a row tableau. By considering $\Gamma(\begin{array}{|c|c|} \hline \bar{2} & 1 \\ \hline \bar{1} & 2 \\ \hline \end{array})$ we see that such a property is false with the symplectic tableaux even if we consider non initial cyclages. This explains why we have to consider the cocyclage operation and not the cyclage one.

4.3. Reduction operations

Consider $T \in \mathbf{ST}(\mu)$ with $\mu \in P_n^+$. If the cocyclage operation is not authorized for T , then T does not contain any letters n . Indeed there exists $p \in \{1, \dots, n\}$ such that $\mu_{\bar{p}}$ is equal to the number of columns of T . Thus $\mu_{\bar{n}} = \mu_{\bar{p}}$ since $\mu_{\bar{n}} \geq \mu_{\bar{p}}$. So each column of T contains a letter \bar{n} and no letter n . Let T_{\S} be the tableau obtained first by erasing the letters \bar{n} in T next by applying t to the letters $x \in T$ such that $\bar{n} < x < n$. It is easy to verify that $T_{\S} \in \mathbf{ST}(\mu')$ with $\mu' = (\mu_{\bar{n}-1}, \dots, \mu_{\bar{1}}, 0) \in P_n^+$. Now if the cocyclage operation is not authorized on T_{\S} , we can compute $(T_{\S})_{\S}$ and so on to obtain a symplectic tableau \hat{T} which is either a column of weight 0 (eventually empty) either a symplectic tableau for which the cocyclage operation is authorized. We will say that \hat{T} is obtained by reduction operations from T . By convention we set $\hat{T} = T$ if the cocyclage operation is already defined for the symplectic tableau T .

Remark When a reduction operation is done in $T \in \mathbf{ST}(\mu)$ with $\mu \in P_n^+$, $\mu_{\bar{n}}$ is equal to the first part of the shape of T . To define a charge statistic on symplectic tableaux related to Kostka-Foulkes polynomials, it seems natural by Lemma 2.1.3 to impose that the charges associated to T and \hat{T} should be equal as we will do in Section 5.

From $T \in \mathbf{ST}(\mu)$ we can compute a sequence of symplectic tableaux by setting $T_0 = T$ and

$$T_{k+1} = \begin{cases} U(\hat{T}_k) & \text{if } U(\hat{T}_k) \text{ is not a column} \\ (\widehat{U(\hat{T}_k)}) & \text{otherwise} \end{cases} .$$

while \hat{T}_k is not a column.

Proposition 4.3.1 *The sequence (T_n) is finite without repetition and the last symplectic tableau obtained is a column of weight 0 (eventually empty).*

Proof: Suppose first that there is a loop in the sequence (T_n) that is, there exists two integers k and s such that $T_k = T_{k+s}$. Then $T_i = \hat{T}_i$ for any $i = k, \dots, k + s - 1$. Choose

$p \in \{k, \dots, k + s - 1\}$ such that the number of columns of T_p is minimal among all the tableaux $T_i, i = k, \dots, k + s - 1$. Denote by r the number of columns of T_p . Then by assertion 1 of Lemma 4.2.3, T_k, \dots, T_{k+s-1} contain r or $r + 1$ columns since every $T_i, i = k, \dots, k + s - 1$ can be obtained by cocyclage operations from T_p . We must have $N_r(T_k) \leq \dots \leq N_r(T_{k+s})$. This implies that $N_r(T_k) = \dots = N_r(T_{k+s-1})$ for $T_{k+s} = T_k$. Then by assertion 2 of Lemma 4.2.3 we can write $w(T_i) = x_{i,*}w(T_{i,*}), i = k, \dots, k + s$ with

$$w(T_{k+s,*}) < \dots < w(T_{s,*}).$$

We obtain a contradiction since $w(T_{k+s,*}) = w(T_{s,*})$. It means that there is no loop in the sequence (T_n) . Hence this sequence is without repetition.

Now suppose that this sequence is infinite. Then there exists an integer a such that the sequence $(T_{n+a})_{n \geq 0}$ is infinite without reduction operation. In the proof of Lemma 4.2.4 the hypothesis $\mu = 0$ is only used to assure that the sequence of the cocycled tableaux is defined. It means that this lemma is still true for the sequence $(T_{n+a})_{n \geq 0}$. Thus we can define by induction as in proof of Proposition 4.2.2 an infinite sequence of tableaux $(T_{v_j})_{j \geq 0}$ such that $T_{j_0} = T_{n+a}$ and for any $j, T_{v_{j+1}}$ has one column less than T_{v_j} . We derive a contradiction since the number of columns of T_a is finite. It means that the sequence (T_n) is finite.

Finally T_u the last tableau of this sequence is necessarily a column such that $\hat{T}_u = T_u$ that is, T_u is a column of weight 0. □

Example 4.3.2 The cocyclage operation is not authorized for $T = \begin{array}{|c|c|} \hline \bar{3} & \bar{3} \\ \hline \bar{2} & \bar{1} \\ \hline 1 & \\ \hline \end{array}$. We have $\hat{T} =$

$$\begin{array}{|c|c|} \hline \bar{3} & \bar{2} \\ \hline 2 & \\ \hline \end{array}. \text{ Then } T_1 = \begin{array}{|c|c|} \hline \bar{3} & 2 \\ \hline \bar{2} & \\ \hline \end{array} \text{ and } T_2 = \widehat{\begin{array}{|c|} \hline \bar{3} \\ \hline \bar{2} \\ \hline 2 \\ \hline \end{array}} = \begin{array}{|c|} \hline \bar{3} \\ \hline 3 \\ \hline \end{array}.$$

4.4. Embedding of cyclage graphs

Each connected component $\Gamma(T)$ contains a tableau Y which admits no cocyclage. This tableau is necessarily unique. Suppose that $Y \in \mathbf{ST}(m)$. Then $\Gamma(T) \subset \mathbf{ST}(m)$ thus is finite. Moreover for any $Z \in \Gamma(T)$ there exists an integer k such that $U^{(k)}(Z) = T$. This means that $\Gamma(T)$ has a tree structure.

Proposition 4.4.1 *Let $\mu \in P_n$ and consider $T_\mu \in \mathbf{ST}(\mu)$. Suppose that there exists $j \leq i \leq n$ such that $\mu_{\bar{i}} > \mu_{\bar{j}} \geq 0$. Set $\nu \in P_n$ defined by $\nu_{\bar{k}} = \mu_{\bar{k}}$ for $k \neq i, j$, $\nu_{\bar{i}} = \mu_{\bar{i}} - 1$ and $\nu_{\bar{j}} = \mu_{\bar{j}} + 1$. Then there exists a tableau $T_\nu \in \mathbf{ST}(\nu)$ and a unique embedding from $\Gamma(T_\mu)$ to $\Gamma(T_\nu)$ which commutes with U and preserves the shape of the tableaux.*

Proof: We have seen that $\Gamma(T_\mu)$ has a finite number of vertices. Write $m \geq n$ for the lowest integer such that $\Gamma(T_\mu)$ is contained in $\mathbf{ST}(m)$. By abuse of notation we also denote μ and v the weights of P_m defined by $\mu = (0, \dots, 0, \mu_{\bar{n}}, \dots, \mu_{\bar{1}})$ and $v = (0, \dots, 0, v_{\bar{n}}, \dots, v_{\bar{1}})$. Let $\sigma \in W_m$ such that $\sigma(\bar{i}) = \bar{m}$, $\sigma(\bar{j}) = \overline{m-1}$ and $\sigma(\bar{k}) = \bar{k}$ for $k \neq i, j$. Set $\sigma(\mu) = \mu'$ and $\sigma(v) = v'$. Write $T_{\mu'} = \sigma(T_\mu)$. We have $\Gamma(T_{\mu'}) \subset \mathbf{ST}(m)$. Then for any $T \in \Gamma(T_{\mu'})$, $\tilde{f}_{m-1}(w(T)) \neq 0$. Indeed the crystal graph $G_m^{(m-1)}$ obtained by erasing in G_m all the arrows of color $i \neq m-1$ and all the letters $x \notin \{\bar{m}, \overline{m-1}, m-1, m\}$ is a $U_q(\mathfrak{sl}_2)_{m-1}$ -crystal where $U_q(\mathfrak{sl}_2)_{m-1}$ is the sub-algebra of $U_q(\mathfrak{sp}_{2m})$ isomorphic to $U_q(\mathfrak{sl}_2)$ generated by e_{m-1} , f_{m-1} and t_{m-1} . The vertex $w(T)_{m-1}$ of $G_m^{(m-1)}$ obtained from $w(T)$ is of weight $(\mu'_{\bar{m}}, \mu'_{\overline{m-1}}) \neq 0$. Since $\mu_{\bar{i}} > \mu_{\bar{j}} \geq 0$ we have $\mu'_{\bar{m}} > \mu'_{\overline{m-1}} \geq 0$. Thus $w(T)_{m-1}$ is a highest weight vertex and there is an arrow of color $m-1$ and length $\mu'_{\bar{m}} - \mu'_{\overline{m-1}}$ which starts from $w(T)$.

Now consider $T \in \Gamma(T_{\mu'})$ such that $U(T) = T'$ is defined. Write $w(T) = x_* w(T_*)$. We must have $x_* \neq \bar{m}$ since the cocyclage operation is authorized for T . Moreover $x_* \neq m$. Otherwise the first column of T' would contain the letters m and \bar{m} (because $\mu'_{\bar{m}} > 0$) and $T' \notin \mathbf{ST}(m)$. We are going to prove that

$$\tilde{f}_{m-1}(x_* w(T_*)) = x_* \tilde{f}_{m-1}(w(T_*)). \tag{19}$$

It suffices to establish (19) for $x_* = m-1$. When $x_* = m-1$ there is no letter $\overline{m-1}$ in the second row of T since $T' \in \mathbf{ST}(m)$. Thus all the letters $\overline{m-1}$ of T belong to its first row. Denote by T_1 the sub-tableau of T containing all the columns whose the lowest letter is \bar{m} . The tableau T can be regarded as the juxtaposition $T_1 T_2$ of the tableaux T_1 and T_2 where T_2 is the sub-tableau obtained by considering the columns of T which do not occur in T_1 . Then T_1 do not contain any letter m or $\overline{m-1}$ and T_2 do not contain any letter \bar{m} . Suppose that $\tilde{f}_{m-1}(x_* w(T_*)) = \tilde{f}_{m-1}(x_*) w(T_*)$. Then with the notation of Note 2.2.2 we can write $\rho(w(T_2)) = (+)^s$ since $x_* = m-1 = +$ is not ignored during the encoding procedure. The pairs (\pm) ignored are pairs $(m-1, \overline{m-1})$ or $(m-1, m)$ for \bar{m} do not belong to $w(T_2)$. Since $\rho(w(T_2))$ contains only symbols $+$, all the letters $\overline{m-1}$ can be paired with letters $m-1$. Thus the number of letters $m-1$ in $w(T_2)$ is strictly greater than that of letters $\overline{m-1}$. It is also true for $w(T)$ because $w(T_1)$ does not contain any letter $\overline{m-1}$. This contradicts the inequality $\mu'_{m-1} \geq 0$. Thus (19) is true.

Denote by V the symplectic tableau of reading $\tilde{f}_{m-1}(w(T_{\mu'}))$. We are going to prove that $\Psi : \Gamma(T_{\mu'}) \rightarrow \Gamma(V)$ defined by $\Psi(T) = S$ if and only if $w(S) = \tilde{f}_{m-1}(w(T))$ is an embedding which commutes with U and preserves the shape of the tableaux. We have $\Psi(U(T)) = P(\tilde{f}_{m-1}(w(T_*)x_*))$. Suppose that there exists $p \in \{1, \dots, m\}$ such that v'_p is equal to r the number of columns of S . Then since $U(T)$ is defined and $v'_k = \mu'_k$ for any $k \neq m, m-1$ we must have $p \in \{m, m-1\}$. If $v'_m = r$ then we have $\mu'_m = r+1$ which is impossible for T contains only r columns. If $v'_{m-1} = r$ then we obtain $\mu'_m \geq r$ since $\mu'_{m-1} = r-1$ and $\mu'_m > \mu'_{m-1}$. This contradicts the fact that the cocyclage operation is authorized for T . Hence the cocyclage operation is authorized for S and we

can write

$$\begin{aligned} U(S) &= U(\Psi(T)) = P(\xi(\tilde{f}_{m-1}(x_*w(T_*)))) = P(\xi(x_*(\tilde{f}_{m-1}(w(T_*)))))) \\ &= P(\tilde{f}_{m-1}(w(T_*))x_*). \end{aligned}$$

Thus we have to show that

$$\tilde{f}_{m-1}(w(T_*)x_*) = \tilde{f}_{m-1}(w(T_*))x_*.$$

By (4) it is equivalent to

$$\varphi_{m-1}(w(T_*)) > \varepsilon_{m-1}(x_*). \tag{20}$$

We have seen that $x_* \neq m$ and for $x_* \neq \overline{m-1}$ (20) is true since $\varepsilon_{m-1}(x_*) = 0$ and $\varphi_{m-1}(w(T_*)) \geq 1$. Suppose that $x_* = \overline{m-1}$. Then the vertex of $G_m^{(m-1)}$ obtained from $w(T_*)$ as above is of weight $(\mu_{\overline{m}}, \mu_{\overline{m-1}} - 1)$. Thus $\varphi_{m-1}(w(T_*)) \geq \mu_{\overline{m}} - \mu_{\overline{m-1}} + 1 \geq 2$ for $\mu_{\overline{m}} > \mu_{\overline{m-1}}$. So (20) is satisfied. It is clear that T and $\Psi(T)$ have the same shape. Moreover by (iii) of Lemma 4.2.1 Ψ is the unique map from $\Psi \Gamma(T_{\mu'})$ to $\Gamma(V)$ which commutes with U and preserves the shape of the tableaux. Finally, using σ^{-1} we obtain from Ψ a unique embedding Ψ_{σ} satisfying $\Psi_{\sigma}(T) = \sigma^{-1}\Psi\sigma(T)$ from $\Gamma(T_{\mu})$ to $\Gamma(T_{\nu})$ with $T_{\nu} = \sigma^{-1}\Psi\sigma(T_{\mu})$. \square

Note that T_{ν} is not unique in general since $\Gamma(\nu) = \{\Gamma(T), T \in \mathbf{ST}(\mu)\}$ may contain fewer connected components isomorphic to $\Gamma(T_{\nu})$.

Corollary 4.4.2 *Let $\mu \in P_n^+$ and $T_{\mu} \in \mathbf{ST}(\mu)$. Write respectively m and p for the sum of the non zero parts and the number of zero parts in μ . Define $\kappa \in P_{m+p}^+$ by $\kappa_{\bar{i}} = 1$ for $p+1 \leq i \leq m+p$ and $\kappa_{\bar{i}} = 0$ otherwise. Then there exists a tableau $T_{\kappa} \in \mathbf{ST}(\kappa)$ and a unique embedding of $\Gamma(T_{\mu})$ into $\Gamma(T_{\kappa})$ which commutes with U and preserves the shape of the tableaux.*

Proof: The corollary directly follows by composing embeddings obtained in the previous Proposition. \square

Example 4.4.3 If $n = 3$ and $\mu = (2, 1, 0)$ then $\kappa = (1, 1, 1, 0)$. The cyclage graph $\Gamma(\overline{3} \overline{3} \overline{2} \overline{1} \overline{1})$ of (18) may be uniquely embedded in $\Gamma(\overline{4} \overline{3} \overline{2} \overline{1} \overline{1})$.

5. A charge for symplectic tableaux

5.1. Definition of ch_n

Definition 5.1.1 Let C be a column of weight 0. Write $E_C = \{i \geq 1, i \in C, i + 1 \notin C\}$. The charge $ch_n(C)$ of the column C is

$$ch_n(C) = 2 \sum_{i \in E_C} (n - i).$$

Note that for any column T of weight 0,

$$ch_{n+1}(C) = ch_n(C) + 2\text{card}(E_C) \quad \text{and} \quad ch_{n+1}(t(C)) = ch_n(C).$$

Moreover for any $i \geq 1$, we have $\varepsilon_i(w((C))) = \begin{cases} 1 & \text{if } i \in E_C \\ 0 & \text{otherwise} \end{cases}$ (see (5)). Thus for any n -admissible column C of weight 0

$$ch_n(C) = 2 \sum_{i=1}^{n-1} (n - i) \varepsilon_i(w(C)). \quad (21)$$

Now consider $T \in \mathbf{ST}(\mu)$ with $\mu \in P_n^+$. Let $\{T_0, \dots, T_p\}$ with $T_p = C_T$ a column of weight 0 be the sequence obtained from T as in Proposition 4.3.1).

Definition 5.1.2 The charge $ch_n(T)$ is

$$ch_n(T) = ch_n(C_T) + p.$$

Remark

- (i) For any tableau T we have by Lemma 4.2.1 (ii)

$$ch_{n+1}(t(T)) = ch_n(T).$$

- (ii) If the cocyclage operation is authorized in T , then $ch_n(U(T)) = ch_n(T) - 1$. Otherwise $ch_n(\hat{T}) = ch_n(T)$.

5.2. Conjectures

Conjecture 5.2.1 Let Λ_k^A and Λ_{n-k}^A be respectively the k -th and $(n - k)$ -th fundamental weights of $U_q(\mathfrak{sl}_n)$. Set $\lambda_k = \Lambda_k^A + \Lambda_{n-k}^A$. Then we have the equality:

$$K_{\Lambda_{2k}, 0}(q) = K_{\lambda_k, 0}^A(q^2)$$

where $K_{\lambda_k, 0}^A(q^2)$ is the Kostka-Foulkes polynomial for the root system A_{n-1} corresponding to $\mu = 0$ evaluated at q^2 .

Write $B(2k)_0 = \{b \in B(\Lambda_{2k}), \text{wt}(b) = 0\}$. By identifying $U_q(\mathfrak{sl}_n)$ with the subalgebra of $U_q(\mathfrak{sp}_{2n})$ generated by the Chevalley's generators e_i, f_i and $t_i, i = 1, \dots, n - 1$, $B(\Lambda_{2k})$ has a structure of crystal graph for $U_q(\mathfrak{sl}_n)$ obtained by erasing all the arrows of color 0 that we denote $B^A(\Lambda_{2k})$. This graph decomposes into non isomorphic connected components and in this decomposition $B_0(2k)$ is exactly the set of vertices of weight 0 of the connected component isomorphic to $B^A(\lambda_k)$

By Theorem 5.1 of [12] we can write

$$K_{\lambda_k, 0}^A(q^2) = \sum_{w(C) \in B(2k)_0} q^{2d'(w(C))}$$

with $d'(w(C)) = \sum_{i=1}^{n-1} (n - i)\varepsilon_i(w(C))$. Then it follows from (21) that Conjecture 5.2.1 is equivalent to the equality

$$K_{\Lambda_{2k}, 0}(q) = \sum_{w(C) \in B(2k)_0} q^{\text{ch}_n(C)}.$$

So by Corollary 3.2.4, this conjecture is true for $k = 1$.

More generally many computations suggest that ch_n is an analogue for the root system C_n of Lascoux-Schützenberger's charge on semi-standard tableaux.

Conjecture 5.2.2 Consider $\lambda, \mu \in P_n^+$. Then

$$K_{\lambda, \mu}(q) = \sum_{w(T) \in B(\lambda)_\mu} q^{\text{ch}_n(T)}$$

where $B(\lambda)_\mu = \{T \in B(\lambda), \text{wt}(T) = \mu\}$.

Example 5.2.3

1. Suppose $n = 4$, $\lambda^{(1)} = (2, 1, 1, 1)$ and $\mu^{(1)} = (1, 1, 1, 0)$. There are 4 tableaux in $\mathbf{ST}(4)$ of shape $\lambda^{(1)}$ and weight $\mu^{(1)}$. They appear in the cyclage graph of Example 4.4.3. Set

$$C = \left(\begin{array}{c} \boxed{4} \\ \boxed{3} \\ \boxed{2} \\ \boxed{1} \\ \boxed{1} \end{array} \right).$$

We have $\text{ch}_4(C) = 0$ since $\widehat{C} = \begin{bmatrix} \bar{4} \\ 4 \end{bmatrix}$. Hence the charges of these 4 tableaux are 1, 2, 3 and

4. This gives the Kostka-Foulkes polynomial $K_{\lambda^{(1)}, \mu^{(1)}}(q) = q + q^2 + q^3 + q^4$.

2. Suppose $n = 3$, $\lambda^{(2)} = (2, 2, 0)$ and $\mu^{(2)} = (0, 0, 0)$. There are 6 tableaux in $\mathbf{ST}(3)$ of shape $\lambda^{(2)}$ and weight $\mu^{(2)}$:

$$T_1 = \begin{bmatrix} \bar{2} & 1 \\ \bar{1} & 2 \end{bmatrix}, T_2 = \begin{bmatrix} \bar{2} & \bar{1} \\ 1 & 2 \end{bmatrix}, T_3 = \begin{bmatrix} \bar{3} & 1 \\ \bar{1} & 3 \end{bmatrix}, T_4 = \begin{bmatrix} \bar{3} & \bar{1} \\ 1 & 3 \end{bmatrix}, T_5 = \begin{bmatrix} \bar{3} & 2 \\ \bar{2} & 3 \end{bmatrix} \quad \text{and} \quad T_6 = \begin{bmatrix} \bar{3} & \bar{2} \\ 2 & 3 \end{bmatrix}.$$

Note that $T_2, T_3 \in \Gamma(\begin{bmatrix} \bar{1} & \bar{1} & 1 & 1 \end{bmatrix})$ and $\Gamma(T_1)$ is given in (18). We obtain

$$\text{ch}_3(T_1) = 2 + \text{ch}_3 \left(\begin{bmatrix} \bar{2} \\ \bar{1} \\ 1 \\ 2 \end{bmatrix} \right) = 4, \quad \text{ch}_3(T_2) = 4 + \text{ch}_3 \left(\begin{bmatrix} \bar{3} \\ \bar{1} \\ 1 \\ 3 \end{bmatrix} \right) = 8$$

and $\text{ch}_3(T_3) = 6$. Moreover $T_5 = t(T_1)$ and $T_6 = t(T_2)$. Thus $\text{ch}_3(T_5) = 4 - 2 = 2$ and $\text{ch}_3(T_6) = 8 - 2 \times 2 = 4$. By an easy computation we obtain

$$U^{(4)}(T_4) = \left(\begin{bmatrix} \bar{4} \\ \bar{1} \\ 1 \\ 4 \end{bmatrix} \right).$$

Hence $\text{ch}_3(T_4) = 4 + 2(2 - 1) = 6$. This gives the Kostka-Foulkes polynomial $K_{\lambda^{(2)}, \mu^{(2)}}(q) = q^2 + 2q^4 + 2q^6 + q^8$.

Remark

(i) Once ch_n defined on \mathbf{ST} , it is possible to define ch_n for any words of \mathcal{C}_∞^* by setting

$$\text{ch}_n(w) = \text{ch}_n(P(w)).$$

Then given $w_1, w_2 \in \mathcal{C}_n^*$, the congruence $w_1 \equiv w_2$ implies that $\text{ch}_n(w_1) = \text{ch}_n(w_2)$, that is ch_n is a plactic invariant. We recover a property of the Lascoux-Schützenberger’s charge ch_A for type A [14] [20]. Nevertheless, it seems difficult to define ch_n directly

on words as it is possible for ch_A . In [12], the statistic ch_A is characterized in terms of the combinatorics of crystal graphs. We have not found such a characterization for the symplectic charge ch_n .

- (ii) It seems to be impossible to define a simple charge statistic on $ST(n)$ by using a cocyclage operation taking into account the contraction relation (9) and relevant for computing Kostka-Foulkes polynomials. Consider for example $T = \begin{array}{|c|c|c|} \hline \bar{1} & \bar{1} & 1 \\ \hline 1 & & \end{array}$ for $n = 3$. If we apply cocyclages operations based on the complete insertion scheme (with the contraction relations) we obtain the symplectic tableaux of $ST(3)$,

$$\begin{array}{|c|c|c|}, \begin{array}{|c|c|} \hline \bar{2} & \bar{1} \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \bar{2} & 1 & 2 \\ \hline \bar{1} & & \end{array}, \begin{array}{|c|c|} \hline \bar{2} & 1 \\ \hline \bar{1} & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline \bar{3} & 3 \\ \hline \bar{1} & \\ \hline 1 & \\ \hline \end{array} \text{ and } \begin{array}{|c|} \hline \bar{1} \\ \hline 1 \\ \hline \end{array} \text{ (since } \begin{array}{|c|} \hline \bar{3} \\ \hline \bar{1} \\ \hline 1 \\ \hline 3 \\ \hline \end{array}$$

is not a 3-admissible column). We know by Proposition 3.2.3 that a charge for T must necessarily be odd and by Corollary 3.2.4 a charge for $\begin{array}{|c|} \hline \bar{1} \\ \hline 1 \\ \hline \end{array}$ must be even. So we can not deduce the charge of T from that of $\begin{array}{|c|} \hline \bar{1} \\ \hline 1 \\ \hline \end{array}$ by simply counting the number of cocyclage operations.

- (iii) Lascoux-Schützenberger’s proof of the equality

$$K_{\lambda, \mu}(q) = \sum_{w(T) \in B(\lambda, \mu)} q^{ch_A(T)}$$

for type A is based on the Morris recurrence formula. We have seen that Theorem 3.2.1 can be regarded as an analogue of this formula for type C_n . It permits to decompose a Kostka-Foulkes polynomial for type C_n in terms of Kostka-Foulkes polynomials for type C_{n-1} . Unfortunately a charge statistic must take into account the contraction relations to be compatible with the decomposition obtained in this way since the partitions λ such that $B(\lambda)$ appears in a decomposition of type $B(\gamma) \otimes B((r)_{n-1})$ may be such that $|\lambda| < |\mu|$. This is a reason why we are not able to deduce Conjecture 5.2.2 from Theorem 3.2.1.

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