Journal of Algebraic Combinatorics, 20, 359–370, 2004

Partial Flocks of Non-Singular Quadrics in $PG(2r+1, q)$

MATTHEW R. BROWN* mbrown@maths.adelaide.edu.au
CHRISTINE M. O'KEEFE†† cokeefe@maths.adelaide.edu.au
School of Pure Mathematics, The University of Adelaide, Adelaide, 5005, Australia

CRISTINA TONESI† tonesi@cage.rug.ac.be
Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, B-9000 Gent, Belgium

Received March 7, 2003; Revised October 2, 2003; Accepted October 27, 2003

Abstract. We generalise the definition and many properties of partial flocks of non-singular quadrics in $PG(3, q)$ to partial flocks of non-singular quadrics in $PG(2r + 1, q)$.

Keywords: flock, partial flock, quadric, exterior set, Thas flock

1. Introduction and definitions

In [10] O'Keefe and Thas investigated the generalisation of a partial flock of a quadratic cone in $PG(3, q)$ to a quadratic cone in $PG(2r + 1, q)$ with point vertex. In a similar way, we generalise a partial flock of a non-singular quadric in $PG(3, q)$ to a non-singular quadric in $PG(2r + 1, q)$.

In $PG(2r + 1, q)$ let $\mathcal{Q}_{2r+1}$ be a non-singular quadric of either elliptic character or of hyperbolic character. A *partial flock* of $\mathcal{Q}_{2r+1}$ of cardinality s is a set of hyperplanes $\{\pi_1, \dots, \pi_s\}$ of $PG(2r + 1, q)$, such that each element of the set intersects the quadric in a non-singular parabolic section and for $k \neq l$ the $(2r - 1)$-dimensional space $\pi_k \cap \pi_l$ meets $\mathcal{Q}_{2r+1}$ in an elliptic quadric. In the case $r = 1$, since an elliptic quadric in $PG(1, q)$ has no points, the above definition coincides with the existing definition of a partial flock of a non-singular quadric in $PG(3, q)$.

Let $\mathcal{Q}_3$ be a non-singular quadric in $PG(3, q)$. If $\mathcal{Q}_3$ is a hyperbolic quadric (respectively, elliptic quadric), then a partition of all (respectively, all but two) points of $\mathcal{Q}_3$ into $q + 1$ disjoint irreducible conics (respectively, $q - 1$ irreducible conics) is called a *flock* of $\mathcal{Q}_3$. Clearly, a flock of $\mathcal{Q}_3$ is a partial flock of maximal size and as such partial flocks generalise this important concept of a flock of a quadric in $PG(3, q)$. If L is a line of $PG(3, q)$ external to $\mathcal{Q}_3$, then the set of irreducible conic sections of $\mathcal{Q}_3$, whose planes contain L, forms a

*Supported in this research by a research grant of Ghent University, number GOA 12050300.
†Supported in this research by a scholarship of the Flemish Community.
††Present address: CSIRO Mathematical and Information Sciences, GPO Box 664, Canberra ACT 2601, Australia. E-mail: Christine.OKeefe@csiro.au

flock of $\mathcal{Q}_3$ called a *linear flock*. Since every non-singular quadric in $PG(3, q)$ admits a linear flock the question of the maximal size of a partial flock is solved. In the more general case of a partial flock of a non-singular quadric in $PG(2r + 1, q)$ the question is open. In Section 5 we discuss bounds on the size of a partial flock.

The concept of a linear flock is generalised to a *linear partial flock* of a non-singular quadric $\mathcal{Q}_{2r+1}$ in $PG(2r + 1, q)$ by taking the hyperplanes intersecting $\mathcal{Q}_{2r+1}$ in a non-singular parabolic quadric and containing a fixed $(2r-1)$-dimensional space meeting $\mathcal{Q}_{2r+1}$ in a non-singular elliptic quadric. We characterise the linear partial flocks in Section 4.

In [12] Thas constructed examples of non-linear flocks of the hyperbolic quadric in $PG(3, q)$, q odd. In Section 6 we generalise this construction method to the case of a non-singular quadric in $PG(2r + 1, q)$, q odd, using interior and exterior sets of quadrics.

For more information on flocks and partial flocks see the survey article of Thas [16].

2. The algebraic condition

In this section we determinate the algebraic conditions for a set of hyperplanes to form a partial flock.

For $q = 2^h$, the map trace : $GF(q) \to GF(2)$, is given by $x \mapsto \sum_{i=0}^{h-1} x^{2^i}$.

Theorem 1 *In $PG(2r + 1, q)$ let $\mathcal{Q}_{2r+1}$ be the non-singular hyperbolic quadric with equation $Q(x_0, x_1, \ldots, x_{2r+1}) = x_0x_1 + x_2x_3 + \cdots + x_{2r}x_{2r+1} = 0$. Let $\mathcal{F} = \{\pi_1, \ldots, \pi_s\}$ be a set of hyperplanes each intersecting $\mathcal{Q}_{2r+1}$ in a non-singular parabolic section with $\pi_k : a_0^{(k)}x_0 + \cdots + a_{2r}^{(k)}x_{2r} + a_{2r+1}^{(k)}x_{2r+1} = 0$ where $a_i^{(k)} \in GF(q)$ and*

$$a_0^{(k)}a_1^{(k)} + a_2^{(k)}a_3^{(k)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(k)} \neq 0. \tag{1}$$

If q is odd, then $\mathcal{F}$ is a partial flock of $\mathcal{Q}_{2r+1}$ if and only if

$$\begin{aligned}&\left(a_0^{(k)}a_1^{(l)} + a_1^{(k)}a_0^{(l)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(l)} + a_{2r+1}^{(k)}a_{2r}^{(l)}\right)^2\\ &\quad - 4\left(a_0^{(k)}a_1^{(k)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(k)}\right)\left(a_0^{(l)}a_1^{(l)} + \cdots + a_{2r}^{(l)}a_{2r+1}^{(l)}\right)\end{aligned} \tag{2}$$

is a non-square in $GF(q)$ for all $k, l \in \{1, \ldots, s\}$ and $k \neq l$.
If q is even, then $\mathcal{F}$ is a partial flock of $\mathcal{Q}_{2r+1}$ if and only if

$$\operatorname{trace}\left(\frac{\left(a_0^{(k)}a_1^{(k)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(k)}\right)\left(a_0^{(l)}a_1^{(l)} + \cdots + a_{2r}^{(l)}a_{2r+1}^{(l)}\right)}{\left(a_0^{(k)}a_1^{(l)} + a_1^{(k)}a_0^{(l)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(l)} + a_{2r+1}^{(k)}a_{2r}^{(l)}\right)^2}\right) = 1 \tag{3}$$

for all $k, l \in \{1, \ldots, s\}$ and $k \neq l$.

Proof: Let β be the bilinear form of $\mathcal{Q}_{2r+1}$ and $\perp_{2r+1}$ the polarity of $\mathcal{Q}_{2r+1}$. The hyperplane π_k has a non-singular parabolic intersection with $\mathcal{Q}_{2r+1}$ if $Q(\pi_k^{\perp_{2r+1}}) = a_0^{(k)}a_1^{(k)} + a_2^{(k)}a_3^{(k)} +$

$\cdots + a_{2r}^{(k)} a_{2r+1}^{(k)} \neq 0$. Now $\pi_k \cap \pi_l$ meets $\mathcal{Q}_{2r+1}$ in a $(2r-1)$-dimensional elliptic quadric if and only if $\langle \pi_k^{\perp_{2r+1}}, \pi_l^{\perp_{2r+1}} \rangle$ is an exterior line to $\mathcal{Q}_{2r+1}$. That is

$$Q\big(\big(a_1^{(k)}, a_0^{(k)}, \ldots, a_{2r+1}^{(k)}, a_{2r}^{(k)}\big) + \lambda\big(a_1^{(l)}, a_0^{(l)}, \ldots, a_{2r+1}^{(l)}, a_{2r}^{(l)}\big)\big) = 0 \tag{4}$$

has no solution for $\lambda \in GF(q)$, and so

$$\begin{aligned} &\lambda\beta\big(\big(a_1^{(k)}, a_0^{(k)}, \ldots, a_{2r+1}^{(k)}, a_{2r}^{(k)}\big), \big(a_1^{(l)}, a_0^{(l)}, \ldots, a_{2r+1}^{(l)}, a_{2r}^{(l)}\big)\big) \\ &\quad + Q\big(a_1^{(k)}, a_0^{(k)}, \ldots, a_{2r+1}^{(k)}, a_{2r}^{(k)}\big) + \lambda^2 Q\big(a_1^{(l)}, a_0^{(l)}, \ldots, a_{2r+1}^{(l)}, a_{2r}^{(l)}\big) = 0 \end{aligned}$$

has no solution. Using, the discriminant when q is odd and the trace map when q is even, on the quadratic above gives the algebraic conditions. □

Theorem 2 *In $PG(2r+1, q)$ let $\mathcal{Q}_{2r+1}$ be an elliptic quadric with equation $Q(x_0, x_1, \ldots, x_{2r+1}) = f(x_0, x_1) + x_2x_3 + \cdots + x_{2r}x_{2r+1} = 0$, where f is an irreducible quadratic form of a suitable type. If q is odd, then $f(x_0, x_1) = x_0^2 - \eta x_1^2$ where η is a fixed non-square of $GF(q)$; and if q is even $f(x_0, x_1) = x_0^2 + x_0x_1 + \rho x_1^2$ where* $\mathrm{trace}(\rho) = 1$. *Let $\mathcal{F} = \{\pi_1, \ldots, \pi_s\}$ be a set of hyperplanes intersecting $\mathcal{Q}_{2r+1}$ in a non-singular parabolic section, with $\pi_k : a_0^{(k)}x_0 + \cdots + a_{2r}^{(k)}x_{2r} + a_{2r+1}^{(k)}x_{2r+1} = 0$ where $a_i^{(k)} \in GF(q)$. If q is odd, then*

$$\frac{\big(a_0^{(k)}\big)^2}{4} - \frac{\big(a_1^{(k)}\big)^2}{4\eta} + a_2^{(k)}a_3^{(k)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(k)} \neq 0 \tag{5}$$

for $k = 1, \ldots, s$ and $\mathcal{F}$ is a partial flock if and only if

$$\begin{aligned} &\left(\frac{a_0^{(k)}a_0^{(l)}}{2} - \frac{a_1^{(k)}a_1^{(l)}}{2\eta} + a_2^{(k)}a_3^{(l)} + \cdots + a_{2r+1}^{(k)}a_{2r}^{(l)}\right)^2 \\ &\quad -4\left(\frac{\big(a_0^{(k)}\big)^2}{4} - \frac{\big(a_1^{(k)}\big)^2}{4\eta} + a_2^{(k)}a_3^{(k)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(k)}\right) \\ &\quad \times \left(\frac{\big(a_0^{(l)}\big)^2}{4} - \frac{\big(a_1^{(l)}\big)^2}{4\eta} + a_2^{(l)}a_3^{(l)} + \cdots + a_{2r}^{(l)}a_{2r+1}^{(l)}\right) \end{aligned}$$

is a square in $GF(q)$. If q is even and if we write

$$\theta_k = \rho\big(a_0^{(k)}\big)^2 + a_0^{(k)}a_1^{(k)} + \big(a_1^{(k)}\big)^2 + a_2^{(k)}a_3^{(k)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(k)}, \tag{6}$$

then $\theta_k \neq 0$ for $k = 1, \ldots, s$ and $\mathcal{F}$ is a partial flock if and only if

$$\mathrm{trace}\left(\frac{\theta_k\theta_\ell}{\big(a_0^{(k)}a_1^{(l)} + a_1^{(k)}a_0^{(l)} + \cdots + a_{2r}^{(k)}a_{2r+1}^{(l)} + a_{2r+1}^{(k)}a_{2r}^{(l)}\big)^2}\right) = 0$$

for all $k, l \in \{1, \ldots, s\}$ and $k \neq l$.

Proof: As in the proof of hyperbolic case in Theorem 1, we use the quadratic form Q and the bilinear form β associated with $\mathcal{Q}$. We obtain a quadratic equation in λ which must have two solutions. □

3. Degenerate partial flocks

Definition 3 Let $\mathcal{F} = \{\pi_1, \ldots, \pi_s\}$ be a partial flock of a non-singular quadric $\mathcal{Q}_{2r+1}$ in $PG(2r+1, q)$ and let $\Sigma(\mathcal{F}) = \cap_{k=1}^{s} \pi_k$. If $\Sigma(\mathcal{F})$ contains a non-singular hyperbolic section of $\mathcal{Q}_{2r+1}$, then we say that $\mathcal{F}$ is *degenerate*; otherwise we say that $\mathcal{F}$ is *non-degenerate*.

Lemma 4 *Let $\mathcal{Q}_{2r+1}$ be a non-singular quadric in $PG(2r+1, q)$ with polarity $\perp_{2r+1}$ and let $\mathcal{F} = \{\pi_1, \ldots, \pi_s\}$ be a degenerate partial flock of $\mathcal{Q}_{2r+1}$. Let $\mathcal{H}_m$ be an m-dimensional, non-singular hyperbolic section of $\mathcal{Q}_{2r+1}$ such that $\mathcal{H}_m \subset \Sigma(\mathcal{F})$ and let $\mathcal{Q}_{2r-m} = \langle \mathcal{H}_m \rangle^{\perp_{2r+1}} \cap \mathcal{Q}_{2r+1}$ with polarity $\perp_{2r-m}$. Then $\mathcal{F}' = \{(\pi_1^{\perp_{2r+1}})^{\perp_{2r-m}}, \ldots, (\pi_s^{\perp_{2r+1}})^{\perp_{2r-m}}\}$ is a partial flock of $\mathcal{Q}_{2r-m}$ of size s.*

Conversely if $\mathcal{Q}_{2r-m}$ is a non-singular sub-quadric of $\mathcal{Q}_{2r+1}$ of dimension $2r-m$ with polarity $\perp_{2r-m}$ such that $\langle \mathcal{Q}_{2r-m} \rangle^{\perp_{2r+1}} \cap \mathcal{Q}_{2r+1} = \mathcal{Q}_m$ is hyperbolic, and if $\mathcal{F}' = \{\pi_1, \ldots, \pi_s\}$ is a partial flock of $\mathcal{Q}_{2r-m}$, then $\mathcal{F} = \{(\pi_1^{\perp_{2r-m}})^{\perp_{2r+1}}, \ldots, (\pi_s^{\perp_{2r-m}})^{\perp_{2r+1}}\}$ is a degenerate partial flock of $\mathcal{Q}_{2r+1}$ with $\langle \mathcal{Q}_m \rangle \subset \Sigma(\mathcal{F})$.

Proof: Suppose $\mathcal{F} = \{\pi_1, \ldots, \pi_s\}$ is the degenerate partial flock of $\mathcal{Q}_{2r+1}$. The point $\pi_k^{\perp_{2r+1}} \in \langle \mathcal{Q}_{2r-m} \rangle$ for $k = 1, \ldots, s$ and the line $\langle \pi_k^{\perp_{2r+1}}, \pi_l^{\perp_{2r+1}} \rangle$ is an external line or a secant line to $\mathcal{Q}_{2r-m}$ for all $k, l \in \{1, \ldots, s\}$, $k \neq l$, according to whether the character of $\mathcal{Q}_{2r+1}$ is hyperbolic or elliptic. Hence $(\pi_k^{\perp_{2r+1}})^{\perp_{2r-m}} \cap (\pi_l^{\perp_{2r+1}})^{\perp_{2r-m}}$ is a $(2r-m-2)$ -dimensional non-singular elliptic section of $\mathcal{Q}_{2r-m}$ for all $k, l \in \{1, \ldots, s\}$, $k \neq l$. The result follows. □

Remark 5 Lemma 4 says that we can generalise a partial flock of a quadric to a degenerate partial flock in higher dimensions. In particular we can generalise the flocks of $Q^+(3, q)$ to degenerate partial flocks of $Q^+(2r+1, q)$.

Remark 6 In Lemma 4 since $\mathcal{Q}_{2r-m}$ is a sub-quadric of $\mathcal{Q}_{2r+1}$ it follows that for any $\pi \in \mathcal{F}$ we have $(\pi^{\perp_{2r+1}})^{\perp_{2r-m}} \subset \pi$. Hence $\mathcal{F}'$ is obtained by intersecting the elements of $\mathcal{F}$ with $PG(2r-m, q) = \langle \mathcal{Q}_{2r-m} \rangle$.

4. The linear partial flocks

Let $\mathcal{Q}_{2r+1}$ be a non-singular quadric in $PG(2r+1, q)$ and let $PG(2r-1, q)$ be a $(2r-1)$-dimensional subspace of $PG(2r+1, q)$ such that $PG(2r-1, q) \cap \mathcal{Q}_{2r+1}$ is a non-singular elliptic quadric. Then the set $\{\pi_1, \ldots, \pi_s\}$ of hyperplanes containing $PG(2r-1, q)$ and meeting $\mathcal{Q}_{2r+1}$ in a non-singular parabolic quadric is called a *linear partial flock* of $\mathcal{Q}_{2r+1}$. In the case where $\mathcal{Q}_{2r+1}$ is hyperbolic $s = q+1$ and when $\mathcal{Q}_{2r+1}$ is elliptic $s = q-1$.

We now characterise the linear partial flocks.

Theorem 7 *Let $\mathcal{F} = \{\pi_1, \dots, \pi_s\}$ be a partial flock of size s of the non-singular hyperbolic quadric $\mathcal{H}_{2r+1}$ in $PG(2r+1, q)$. For distinct $k, l \in \{1, \dots, s\}$ let $\mathcal{E}_{kl} = \pi_k \cap \pi_l \cap \mathcal{H}_{2r+1}$. If for any fixed such k, l the elements of $\mathcal{F}$ cover the points of $\mathcal{H}_{2r+1} \setminus \mathcal{E}_{kl}$, then $s \geq q+1$ and if $s = q+1$, then $\mathcal{F}$ is linear.*

Proof: For $\pi_m \in \mathcal{F}$, let $\mathcal{P}_m = \pi_m \cap \mathcal{H}_{2r+1}$. Let $\mathcal{S} = \mathcal{H}_{2r+1} \setminus \{\mathcal{P}_k \cup \mathcal{P}_l\}$ and suppose that the elements of $\mathcal{F} \setminus \{\pi_k, \pi_l\}$ cover the points of $\mathcal{S}$. For $p \in \mathcal{S}$, let N_p denote the number of elements of $\mathcal{F} \setminus \{\pi_k, \pi_l\}$ on p. By hypothesis, $N_p \geq 1$ for $p \in \mathcal{S}$. Now count the ordered pairs (p, π_m), where $p \in \mathcal{S}$, $\pi_m \in \mathcal{F} \setminus \{\pi_k, \pi_l\}$ and $p \in \pi_m$; it follows that

$$\begin{aligned}
&|\mathcal{H}_{2r+1}| - |\mathcal{P}_k| - |\mathcal{P}_l| + |\mathcal{E}_{kl}| = |\mathcal{S}| \leq \sum_{p \in \mathcal{S}} N_p \\
&= \sum_{\pi_m \in \mathcal{F}\setminus\{\pi_k,\pi_l\}} (|\mathcal{P}_m| - |\pi_m \cap (\pi_k \cup \pi_l) \cap \mathcal{H}_{2r+1}|) \\
&= \sum_{\pi_m \in \mathcal{F}\setminus\{\pi_k,\pi_l\}} (|\mathcal{P}_m| - |\pi_m \cap \pi_k \cap \mathcal{H}_{2r+1}| - |\pi_m \cap \pi_l \cap \mathcal{H}_{2r+1}| \\
&\quad + |\pi_m \cap \pi_k \cap \pi_l \cap \mathcal{H}_{2r+1}|) \\
&= \sum_{\pi_m \in \mathcal{F}\setminus\{\pi_k,\pi_l\}} (|\mathcal{P}_m| - 2|\mathcal{E}_{kl}| + |\pi_m \cap \mathcal{E}_{kl}|) \leq (s-2)(|\mathcal{P}_m| - |\mathcal{E}_{kl}|).
\end{aligned}$$

So $|\mathcal{H}_{2r+1}| - |\mathcal{E}_{kl}| \leq s\,(|\mathcal{P}_m| - |\mathcal{E}_{kl}|)$ for any $m \in \{1, \dots, s\}$ and on substitution we obtain $q+1 \leq s$. If $s = q+1$, then equality must hold throughout the expression and so $N_p = 1$ for all $p \in \mathcal{S}$. Thus $\mathcal{F}$ partitions $\mathcal{H}_{2r+1} \setminus \mathcal{E}_{kl}$ and each element of $\mathcal{F}$ contains $\mathcal{E}_{kl}$; so the flock is linear. □

Theorem 8 *Let $\mathcal{F} = \{\pi_1, \dots, \pi_s\}$ be a partial flock of size s of the elliptic quadric $\mathcal{E}_{2r+1}$ in $PG(2r+1, q)$ with polarity $\perp_{2r+1}$. For distinct fixed $k, l \in \{1, \dots, s\}$ let $\mathcal{E}_{kl} = \pi_k \cap \pi_l \cap \mathcal{E}_{2r+1}$ and $\mathcal{E}_{kl}^{\perp_{2r+1}} \cap \mathcal{E}_{2r+1} = \{x, y\}$. If the elements of $\mathcal{F}$ cover the points of $\mathcal{E}_{2r+1} \setminus \{x\mathcal{E}_{kl} \cap y\mathcal{E}_{kl}\}$, then $s \geq q-1$ and if $s = q-1$, then $\mathcal{F}$ is linear.*

Proof: For $\pi_m \in \mathcal{F}$, let $\mathcal{P}_m = \pi_m \cap \mathcal{E}_{2r+1}$. Let $\mathcal{S} = \mathcal{E}_{2r+1} \setminus \{x\mathcal{E}_{kl} \cup y\mathcal{E}_{kl}\}$ and suppose that the elements of $\mathcal{F}$ cover the points of $\mathcal{S}$. For $z \in \mathcal{S}$, let N_z denote the number of elements of $\mathcal{F}$ on z. By hypothesis, $N_z \geq 1$ for $z \in \mathcal{S}$. Now count the ordered pairs (z, π_m), where $z \in \mathcal{S}$, $\pi_m \in \mathcal{F}$ and $z \in \pi_m$; it follows that

$$\begin{aligned}
&|\mathcal{E}_{2r+1}| - |x\mathcal{E}_{kl}| - |y\mathcal{E}_{kl}| + |\mathcal{E}_{kl}| = |\mathcal{S}| \leq \sum_{z \in \mathcal{S}} N_z \\
&= \sum_{\pi_m \in \mathcal{F}} (|\mathcal{P}_m| - |\pi_m \cap (x\mathcal{E}_{kl} \cup y\mathcal{E}_{kl})|) \\
&= \sum_{\pi_m \in \mathcal{F}} (|\mathcal{P}_m| - |\pi_m \cap x\mathcal{E}_{kl}| - |\pi_m \cap y\mathcal{E}_{kl}| + |\pi_m \cap x\mathcal{E}_{kl} \cap y\mathcal{E}_{kl}|).
\end{aligned}$$

There are three different possibilities for $\pi_m \cap \mathcal{E}_{kl}$: $\mathcal{E}_{kl}$, $\mathcal{P}_{2r-2}$ a $(2r-2)$-dimensional, non-singular parabolic section of $\mathcal{E}_{2r+1}$ or $v\mathcal{E}_{2r-3}$ a quadratic cone with vertex a point v and base a $(2r-3)$-dimensional, non-singular elliptic quadric of $\mathcal{E}_{2r+1}$. If $\pi_m \cap \mathcal{E}_{kl} = \mathcal{E}_{kl}$, then $\pi_m \cap x\mathcal{E}_{kl} = \pi_m \cap y\mathcal{E}_{kl} = \mathcal{E}_{kl}$, otherwise $\pi_m \cap x\mathcal{E}_{kl}$ may either be a cone with vertex x and base $\pi_m \cap \mathcal{E}_{kl}$ or a $(2r-1)$-dimensional, non-singular elliptic section of $\mathcal{E}_{2r+1}$, and similarly for $\pi_m \cap y\mathcal{E}_{kl}$. By calculating the value of $|\pi_m \cap x\mathcal{E}_{kl}| + |\pi_m \cap y\mathcal{E}_{kl}| - |\pi_m \cap x\mathcal{E}_{kl} \cap y\mathcal{E}_{kl}|$ for all of these possibilities, we have that $|\pi_m \cap x\mathcal{E}_{kl}| + |\pi_m \cap y\mathcal{E}_{kl}| - |\pi_m \cap x\mathcal{E}_{kl} \cap y\mathcal{E}_{kl}| \geq |\mathcal{E}_{kl}|$. Thus $|\mathcal{S}| \leq s(|\mathcal{P}_m| - |\mathcal{E}_{kl}|)$. On substitution we find $s \geq q-1$. If $s = q-1$, then equality must hold throughout the expression and $N_z = 1$ for all $z \in \mathcal{S}$. Thus $\mathcal{F}$ partitions $\mathcal{E}_{2r+1} \setminus \{x\mathcal{E}_{kl} \cup y\mathcal{E}_{kl}\}$ and each of the elements of $\mathcal{F}$ contains $\mathcal{E}_{kl}$; so the flock is linear. □

5. Upper bounds on the size of a partial flock

In this section we look at the known bounds on the largest possible size of a partial flock.

Definition 9 An ovoid of a non-singular quadric $\mathcal{Q}_{2r+1}$ in $PG(2r+1, q)$ is a set of points on $\mathcal{Q}_{2r+1}$ which has exactly one point in common with every maximal singular space on $\mathcal{Q}_{2r+1}$. A *partial ovoid* of $\mathcal{Q}_{2r+1}$ is a set of points on $\mathcal{Q}_{2r+1}$ which has at most one point in common with any maximal singular space on $\mathcal{Q}_{2r+1}$.

An ovoid of $Q^-(2r+1, q)$ has size $q^{r+1}+1$ and an ovoid of $Q^+(2r+1, q)$ has size q^r+1 (see [7, Theorem AVI.2.1]).

Adapting [3] we have the following theorems relating partial flocks of the non-singular quadrics $Q^+(2r+1, q)$ and $Q^-(2r+1, q)$ and partial ovoids of $Q^+(2r+3, q)$.

Theorem 10 *Let $\mathcal{F} = \{\pi_1, \ldots, \pi_s\}$ be a partial flock of a non-singular quadric $\mathcal{Q}_{2r+1}$ in $PG(2r+1, q)$. Then there exists a partial ovoid $\mathcal{O}$ of $Q^+(2r+3, q)$ with cardinality $s(q+1)$ if $\mathcal{Q}_{2r+1}$ is elliptic and with cardinality $s(q-1)+2$ if $\mathcal{Q}_{2r+1}$ is hyperbolic.*

Proof: Embed $\mathcal{Q}_{2r+1}$ into $Q^+(2r+3, q)$ as the intersection of $Q^+(2r+3, q)$ with a $(2r+1)$-dimensional subspace Σ. Let $\perp_{2r+3}$ be the polarity of $Q^+(2r+3, q)$. For $k, \ell \in \{1, \ldots, s\}$, $k \neq \ell$, we have that $\pi_k^{\perp_{2r+3}}$ and $\pi_\ell^{\perp_{2r+3}}$ are conic planes, with conics $C_k = \pi_k^{\perp_{2r+3}} \cap Q^+(2r+3, q)$ and $C_\ell = \pi_\ell^{\perp_{2r+3}} \cap Q^+(2r+3, q)$. Now $\langle \pi_k^{\perp_{2r+3}}, \pi_\ell^{\perp_{2r+3}} \rangle$ intersects $Q^+(2r+3, q)$ in a three-dimensional non-singular elliptic quadric and hence no two points of $C_k \cup C_\ell$ are collinear in $Q^+(2r+3, q)$. Thus $\mathcal{O} = C_1 \cup C_2 \cup \cdots \cup C_s$ is a partial ovoid of $Q^+(2r+3, q)$ of size $s(q+1)$ if $\mathcal{Q}_{2r+1}$ is elliptic and $s(q-1)+2$ if $\mathcal{Q}_{2r+1}$ is hyperbolic. □

Comparing the size of the partial ovoid of $Q^+(2r+3, q)$ in Theorem 10 with the size of an ovoid of $Q^+(2r+3, q)$, gives an upper bound on the size of a partial flock of $\mathcal{Q}_{2r+1}$.

Theorem 11 *Let $\mathcal{Q}_{2r+1}$ be a non-singular quadric of $PG(2r+1, q)$ and let $\mathcal{F}$ be a partial flock of $\mathcal{Q}_{2r+1}$. Then*

$$|\mathcal{F}| \leq \begin{cases} \dfrac{q^{r+1}+1}{q+1} & \text{if } \mathcal{Q}_{2r+1} \text{ is elliptic,} \\ \dfrac{q^{r+1}-1}{q-1} & \text{if } \mathcal{Q}_{2r+1} \text{ is hyperbolic.} \end{cases}$$

Remark 12 For some cases the upper bound is not integral. Thus for these cases a partial flock cannot give rise to an ovoid. In particular a partial flock of $Q^-(2r+1, q)$ may not give rise to an ovoid of $Q^+(2r+3, q)$, as above, if r is odd.

Definition 13 Let $\mathcal{Q}_{2r+1}$ be a non-singular quadric of $PG(2r+1, q)$ and X a set of points of $PG(2r+1, q)$ not on $\mathcal{Q}_{2r+1}$. The set X is called an *exterior set* with respect to $\mathcal{Q}_{2r+1}$ if the span of any two points in X is a line exterior to $\mathcal{Q}_{2r+1}$. The set X is called an *interior set* with respect to $\mathcal{Q}_{2r+1}$ if the span of any two points is a line interior to $\mathcal{Q}_{2r+1}$.

Lemma 14 *Let $\mathcal{F} = \{\pi_1, \pi_2, \ldots, \pi_s\}$ be a partial flock of $Q^+(2r+1, q)$ and let $\perp$ be the polarity of $Q^+(2r+1, q)$. Then the set $\{\pi_1^\perp, \pi_2^\perp, \ldots, \pi_s^\perp\}$ is an exterior set of $Q^+(2r+1, q)$.*

In [3] De Clerck and Thas proved that the size of an exterior set X of $Q^+(2r+1, q)$ is at most $\frac{q^{r+1}-1}{q-1}$; if X has exactly $\frac{q^{r+1}-1}{q-1}$ points then it is called a *maximal exterior set*, abbreviated to MES. The maximal exterior sets have been completely classified by De Clerck and Thas (see [3]). In the case where there is no MES, the bound is decreased by Klein [9]. Klein gave a recursive bound for the size of an exterior set, that is

$$M(2r+1, q) \leq \frac{q^{r+1}-1}{q^r-1} M(2r-1, q)$$

where $M(2k+1, q) = \max\{|X|,\ X \text{ is an exterior set of } Q^+(2k+1, q)\}$. Klein [9] observed that by setting $M(3, q) = q+1$ (the known maximal size of an exterior set) the recursive formula gives the bound of De Clerck and Thas [3]. For many cases Klein improved the bound for $Q^+(5, q)$ and hence, by the recursion formula, the general bound. These results on an exterior set give corresponding results on a partial flock of $Q^+(2r+1, q)$.

Theorem 15 *If $\mathcal{F}$ is a partial flock of $Q^+(2r+1, q)$, then $|\mathcal{F}| \leq \frac{q^{r+1}-1}{q^r-1} M(2r-1, q)$.*

6. Generalized Thas partial flocks of non-singular quadrics in $PG(2r+1, q)$, q odd

We recall from [4] that if $\mathcal{Q}_3$ is a non-singular hyperbolic quadric in $PG(3, q)$, with q odd, then on the set of all irreducible conics sections of $\mathcal{Q}_3$ it is possible to define the following equivalence relation: two conics C_1 and C_2 are equivalent if and only if there is an irreducible conic C on $\mathcal{Q}_3$ which is tangent to both C_1 and C_2. There are two equivalence classes under the equivalence relation and the two classes are said to be *opposite*. We can extend this equivalence relation to apply to the planes of $PG(3, q)$ meeting $\mathcal{Q}_3$ in a conic. Suppose L

is a line not meeting $\mathcal{Q}_3$ and let $L^{\perp_3}$ be the polar line of L with respect to $\mathcal{Q}_3$. Of the $q+1$ conic planes on L there are $\frac{q+1}{2}$ in each class. Let V be the set of $\frac{q+1}{2}$ conic planes on L of one class. Clearly V is a partial flock of $\mathcal{Q}_3$. If we define W to be the set of $\frac{q+1}{2}$ conic planes containing $L^{\perp_3}$ with the opposite class (respectively, same class) as those of V when $q \equiv 1 \pmod 4$, (respectively, $q \equiv -1 \pmod 4$), then $V \cup W$ is a non-linear flock of $\mathcal{Q}_3$. These are the *Thas flocks*, constructed by Thas in [12]. For an elliptic quadric it is possible to introduce the same equivalence relation on conic sections of the quadric and the same construction of a flock. In this case the construction yields linear flocks (see [4]).

If we employ the polarity of the hyperbolic quadric $\mathcal{Q}_3$, then the equivalence relation on conic planes becomes an equivalence relation on points not on $\mathcal{Q}_3$, and a flock becomes an exterior set. Two points x and y are equivalent if there is a third point z such that $\langle x, z\rangle$ and $\langle y, z\rangle$ are both tangents to $\mathcal{Q}_3$. Viewed in this way the Thas construction gives an exterior set from the union of two exterior sets, both of which have all their elements in the same class. Similarly, if $\mathcal{Q}_3$ is elliptic, then the polarity of $\mathcal{Q}_3$ gives rise to an equivalence relation on points not on $\mathcal{Q}_3$, and a flock becomes an interior set.

Extending these ideas to general dimension $2r+1$ we will give constructions for interior and exterior sets of non-singular quadrics, and hence of partial flocks of non-singular quadrics.

6.1. An equivalence relation on points not on a quadric

Let $\mathcal{Q}_{2r+1}$ be a non-singular quadric in $PG(2r+1, q)$, q odd, with polarity $\perp_{2r+1}$. Let $\mathcal{Q}_{2r+1}$ have quadratic form $Q(x)$ and associated bilinear form $\beta(x, y)$. Given this and following Fisher and Thas [4], we now define the following operations: $y \cdot z = \beta(y, z)$, $\|y\| = y \cdot y$, $y \times z = (y \cdot z)^2 - \|y\|\|z\|$. It follows that $y \times z$ is the discriminant of the equation $Q(y + \lambda z) = 0$ for $\lambda \in GF(q) \setminus \{0\}$. The number of the solutions of this equation determines whether $\langle y, z\rangle$ is an exterior line to the quadric, a tangent line to the quadric or a secant line to the quadric respectively. In particular we have the following:

$$|\langle y, z\rangle \cap \mathcal{Q}_{2r+1}| = 2 \iff y \times z \text{ is a non-zero square,}$$
$$|\langle y, z\rangle \cap \mathcal{Q}_{2r+1}| = 1 \iff y \times z = 0,$$
$$|\langle y, z\rangle \cap \mathcal{Q}_{2r+1}| = 0 \iff y \times z \text{ is a non-square.}$$

We say that $y \sim z$ if there exists a point v such that $\langle y, v\rangle$ and $\langle z, v\rangle$ are both tangent lines to $\mathcal{Q}_{2r+1}$. Otherwise we write $y \not\sim z$. The relation $\sim$ is an equivalence relation on the set of non-singular points of $PG(2r+1, q)$ and also on the set of hyperplane sections which are non-singular parabolic quadrics, mentioned in the introduction to Section 6.

Theorem 16 *Let y and z be two points of $PG(2r+1, q) \setminus \mathcal{Q}_{2r+1}$, then $y \sim z$ if and only if $\|y\|\|z\|$ is a square in $GF(q)$.*

Proof: Suppose that $y \sim z$. Then there exists a point v such that $\langle y, v\rangle$ and $\langle z, v\rangle$ are both tangent lines to $\mathcal{Q}_{2r+1}$ and hence $y \times v = z \times v = 0$. Thus $(y \cdot v)^2 (z \cdot v)^2 = \|v\|^2\|y\|\|z\|$ and $\|y\|\|z\|$ is a square.

Conversely, suppose that $\|y\|\|z\|$ is a square. If $y = z$, then there is a point v such that $y \sim v$ and $z \sim v$. So now suppose that $y \neq z$. Let v be a fixed point of $\mathcal{Q}_{2r+1}$ such that we have $v \cdot y = 0$ and $v \cdot z \neq 0$. Let $t = y + hv$ where $h \in GF(q) \setminus \{0\}$ and so $\|t\| = \|y\| \neq 0$ and $y \times t = 0$. The equation $z \times t = 0$ is a quadratic equation in h with discriminant $4\,(z \cdot v)^2\,\|z\|\|y\|$. Since this is a non-zero square, there is at least one value of h such that $z \times t = 0$ and so $y \sim z$. □

Theorem 17 *If $y \sim z$ and v is a non-singular point such that $v \cdot y = v \cdot z = 0$, then $\langle y, v\rangle$ and $\langle z, v\rangle$ are either both exterior lines or both secant lines to $\mathcal{Q}_{2r+1}$. If*

$$\begin{aligned} V_{ext} &= \{v : v \cdot y = v \cdot z = 0,\ \langle y, v\rangle \text{ and } \langle z, v\rangle \text{ are exterior lines}\} \quad \text{and} \\ V_{sec} &= \{v : v \cdot y = v \cdot z = 0,\ \langle y, v\rangle \text{ and } \langle z, v\rangle \text{ are secant lines}\}, \end{aligned}$$

then V_{ext} consists exactly of the set of non-singular points of $y^{\perp_{2r+1}} \cap z^{\perp_{2r+1}}$ of one class and V_{sec} exactly of the non-singular points on $y^{\perp_{2r+1}} \cap z^{\perp_{2r+1}}$ of the other class. Further, V_{ext} has the same class as y,z if and only if $q \equiv -1 \pmod 4$ and V_{sec} has the same class as y,z if and only if $q \equiv 1 \pmod 4$.

Proof: Consider a non-singular point v such that $v \cdot y = v \cdot z = 0$, that is $v \in y^{\perp_{2r+1}} \cap z^{\perp_{2r+1}}$. Thus $v \times y = -\|v\|\|y\|$, $v \times z = -\|v\|\|z\|$ and so $(v \times y)\,(v \times z) = \|v\|^2\|y\|\|z\|$ which is a non-zero square by Theorem 16. It follows that $v \times y$ and $v \times z$ are either both square or both non-square. Thus $\langle y, v\rangle$ and $\langle z, v\rangle$ are either both exterior lines to $\mathcal{Q}_{2r+1}$ or both secant lines to $\mathcal{Q}_{2r+1}$. Thus v is in one of V_{ext}, V_{sec}. If $v \in V_{\text{ext}}$, then $v \times y = -\|v\|\|y\|$ is a non-square and so, by Theorem 16, v is in the same class as y (and z) if and only if -1 is a non-square, that is, if and only if $q \equiv -1 \pmod 4$. By similar arguments, all elements of V_{sec} are in the same class as y, z if and only if $q \equiv 1 \pmod 4$. □

6.2. *Construction method for exterior and interior sets of non-singular quadrics*

We now give the generalized Thas construction method for the exterior and interior sets using exterior sets and interior sets of quadrics of lower dimensions.

Definition 18 Let $\mathcal{F} = \{y_1, \ldots, y_s\}$ be an exterior (respectively, interior) set of a non-singular quadric $\mathcal{Q}_{2r+1}$ of $PG(2r+1, q)$, q odd, such that $y_k \sim y_l$ for all $k, l \in \{1, \ldots, s\}$, $k \neq l$. We call such a set *homogeneous* in $\mathcal{Q}_{2r+1}$. Otherwise $\mathcal{F}$ is said to be *inhomogeneous*. We say that a homogeneous exterior (respectively, interior) set has the same class as its elements.

Lemma 19 *Let $\mathcal{F} = \{y_1, \ldots, y_s\}$ and $\mathcal{F}' = \{v_1, \ldots, v_t\}$ be two homogeneous exterior (respectively, interior) sets with respect to a non-singular quadric $\mathcal{Q}_{2r+1}$ of $PG(2r+1, q)$, q odd, such that*

$$y_k \cdot v_l = 0 \quad \text{for all } k = 1, \cdots, s \quad \text{and} \quad l = 1, \ldots, t.$$

(i) *If $\mathcal{F}$ and $\mathcal{F}'$ are in the same class and* $q \equiv -1 \pmod 4$ (*respectively,* $q \equiv 1 \pmod 4$), *then* $\mathcal{F} \cup \mathcal{F}'$ *is a homogeneous exterior* (*respectively*, *interior*) *set of* $\mathcal{Q}_{2r+1}$.

(ii) *If $\mathcal{F}$ and $\mathcal{F}'$ are in opposite classes and* $q \equiv 1 \pmod 4$ (*respectively,* $q \equiv -1 \pmod 4$), *then* $\mathcal{F} \cup \mathcal{F}'$ *is an inhomogeneous exterior* (*respectively*, *interior*) *set of* $\mathcal{Q}_{2r+1}$.

Proof: Follows from Theorem 17. □

By this method we can "patch" together homogeneous exterior (respectively, interior) sets of non-singular quadrics to form an exterior (respectively, interior) set in a higher dimensional non-singular quadric.

Theorem 20 *Let $\mathcal{Q}_{2r+1}$ be a non-singular quadric in $PG(2r+1, q)$, q odd. Let $\mathcal{Q}_m$ be an m-dimensional*, *non-singular section of $\mathcal{Q}_{2r+1}$ and let $\mathcal{Q}_{2r-m} = \langle \mathcal{Q}_m \rangle^{\perp_{2r+1}} \cap \mathcal{Q}_{2r+1}$. Let $\mathcal{F} = \{y_1, \dots, y_s\}$ and $\mathcal{F}' = \{v_1, \dots, v_t\}$ be homogeneous exterior* (*respectively*, *interior*) *sets of $\mathcal{Q}_m$ and $\mathcal{Q}_{2r-m}$ respectively.*

(i) *If $\mathcal{F}$ and $\mathcal{F}'$ are in the same class with respect to $\mathcal{Q}_{2r+1}$ and* $q \equiv -1 \pmod 4$ (*respectively*, $q \equiv 1 \pmod 4$), *then $\mathcal{F} \cup \mathcal{F}'$ is a homogeneous exterior* (*respectively*, *interior*) *set of $\mathcal{Q}_{2r+1}$.*

(ii) *If $\mathcal{F}$ and $\mathcal{F}'$ are in opposite classes with respect to $\mathcal{Q}_{2r+1}$ and* $q \equiv 1 \pmod 4$ (*respectively*, $q \equiv -1 \pmod 4$), *then $\mathcal{F} \cup \mathcal{F}'$ is an inhomogeneous exterior* (*respectively*, *interior*) *set of $\mathcal{Q}_{2r+1}$.*

Proof: Follows from Lemma 19. □

In the following theorems we investigate the largest known constructions of exterior and interior sets given by the generalized Thas construction method. We consider the homogeneous and inhomogeneous cases separately since in the homogeneous case we may use the method repeatedly, while in the inhomogeneous case only once.

Theorem 21 *For* $q \equiv -1 \pmod 4$ *and* $r \geq 0$, *the generalized Thas construction method gives rise to homogeneous exterior sets of the following sizes*:

$$Q^+(2r+1, q) \colon \begin{cases} r(q+1)/2 + 1 & \text{if } r \text{ is even,} \\ (r+1)(q+1)/2 & \text{if } r \text{ is odd; and} \end{cases}$$

$$Q^-(2r+1, q) \colon \begin{cases} (r+1)(q+1)/2 & \text{if } r \text{ is even,} \\ r(q+1)/2 + 1 & \text{if } r \text{ is odd.} \end{cases}$$

Proof: We can use the generalized Thas construction method to construct an exterior set of $Q^+(2r+1, q)$, $q \equiv -1 \pmod 4$ in two ways. Firstly we take non-singular sections $Q^+(2k+1, q)$ and $Q^+(2(r-k-1)+1, q)$ of $Q^+(2r+1, q)$ which are polar with respect to the polarity of $Q^+(2r+1, q)$ and then combine homogeneous exterior sets, of the same class, of these quadrics. The other way is to do the same with a polar $Q^-(2k+1, q)$ and $Q^-(2(r-k-1)+1, q)$ pair. Thus we can prove the theorem by using induction on r.

For the case $r = 0$, we see that $Q^+(1, q)$ has a largest exterior set of size 1 (and so homogeneous) and $Q^-(1, q)$ has a largest homogeneous exterior set of size $(q+1)/2$, and the theorem is satisfied for $r = 0$. Next we consider $r > 0$ and suppose that the theorem is satisfied for all r' with $0 \le r' < r$.

First we consider constructions for $Q^+(2r+1, q)$ in the case where r is odd. If k is odd, then it follows that $r-k-1$ is also odd and using a polar $Q^+(2k+1, q)$, $Q^+(2(r-k-1)+1, q)$ pair yields a homogeneous exterior set of size $(k+1)(q+1)/2 + (r-k)(q+1)/2 = (q+1)(r+1)/2$. A polar $Q^-(2k+1, q)$, $Q^-(2(r-k-1)+1, q)$ pair gives a set of size $k(q+1)/2 + 1 + (r-k-1)(q+1)/2 + 1 = (q+1)(r-1)/2 + 2$. If k is even, then $r-k-1$ is also even and we obtain exterior sets of size $(r-1)(q+1)/2 + 2$ and $(r+1)(q+1)/2$. If r is even and k is odd, then it follows that $r-k-1$ is even. A polar $Q^+(2k+1, q)$, $Q^+(2(r-k-1)+1, q)$ pair gives a set of size $r(q+1)/2 + 1$ and a polar $Q^-(2k+1, q)$, $Q^-(2(r-k-1)+1, q)$ pair gives a set of size $(r-1)(q+1)/2+2$, smaller than $(r+1)(q+1)/2$. If r and k are even, then $r-k-1$ is odd and this case is equivalent to the one just considered.

Now we consider $Q^-(2r+1, q)$ and working analogously to the $Q^+(2r+1, q)$ case we have proved our result by induction. □

We have a similar result for homogeneous interior sets.

Theorem 22 *For $q \equiv 1 \pmod 4$ and $r \ge 0$ the generalized Thas construction method gives rise to homogeneous interior sets of the following sizes*:

$$Q^+(2r+1, q) : (r+1)(q-1)/2,$$
$$Q^-(2r+1, q) : r(q-1)/2 + 1.$$

Now we consider the construction of inhomogeneous partial flocks using the generalized Thas method.

Theorem 23 *Let $\mathcal{Q}_{2r+1}$ be a non-singular quadric in $PG(2r+1, q)$, q odd. If $q \equiv 1 \pmod 4$, then the generalized Thas construction gives rise to an inhomogeneous exterior set of size $q+1$; and if $q \equiv -1 \pmod 4$ an inhomogeneous interior set of size $q-1$.*

Proof: In this case using the generalized Thas construction we may only combine two homogeneous exterior sets or interior sets, respectively. Using linear examples gives the above results. □

Remark 24 The sizes of the exterior and interior sets constructed above are not necessarily the biggest possible using the generalized Thas construction. Since the construction may be applied for any homogeneous exterior or interior set, discovery of new "big" homogeneous exterior/interior sets could possibly lead to bigger exterior/interior sets using the generalized Thas construction.

Since a partial flock of $Q^-(2r+1,q)$ is equivalent to an interior set of $Q^-(2r+1,q)$ and a partial flock of $Q^+(2r+1,q)$ is equivalent to an exterior set of $Q^+(2r+1,q)$ we have the following result by combining the previous three theorems.

Theorem 25 *For non-singular quadrics in $PG(2r+1,q)$, $r \geq 1$, the generalized Thas construction method gives rise to partial flocks of the following sizes*:

$$Q^+(2r+1,q)\colon \begin{cases} r(q+1)/2+1 & \text{if } r \text{ is even and } q \equiv -1 \pmod 4,\\ (r+1)(q+1)/2 & \text{if } r \text{ is odd and } q \equiv -1 \pmod 4,\\ q+1 & \text{if } q \equiv 1 (\mathrm{mod}\ 4); \end{cases}$$

$$Q^-(2r+1,q)\colon \begin{cases} r(q-1)/2+1 & \text{if } q \equiv 1 \pmod 4,\\ q-1 & \text{if } q \equiv -1 \pmod 4. \end{cases}$$

References

1. L. Bader, G. Lunardon, and J.A. Thas, "Derivation of flocks of quadratic cones," *Forum Math.* **2** (1999), 163–174.
2. W.E. Cherowitzo, T. Penttilla, I. Pinneri, and G.F. Royle, "Flocks and ovals," *Geom. Dedicata* **60** (1996), 17–37.
3. F. De Clerck and J.A. Thas, "Exterior sets with respect to the hyperbolic quadric in $PG(2n-1,q)$," *Finite Geometries, Leture Notes in Pure and Appl. Math.* **103** (1985), 83–91.
4. J.C. Fisher and J.A. Thas, "Flocks in $PG(3,q)$," *Math. Z.* **169** (1979), 1–11.
5. J.W.P. Hirschfeld, *Projective Geometries over Finite Fields*, Oxford University Press, Oxford, 1979.
6. J.W.P. Hirschfeld and J.A. Thas, "Sets of type $(1,n,q+1)$ in $PG(d,q)$," *Proc. London Math. Soc.* **41**(3) (1981), 254–278.
7. J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries*, Oxford University Press, Oxford, 1991.
8. J.W.P. Hirschfeld and J.A. Thas, "The characterization of projections of quadrics over finite fields of even order," *J. London Math. Soc.* **22**(2) (1980), 226–238.
9. A. Klein, "Exterior sets of hyperbolic quadrics," *Bull. Belg. Math. Soc.* **7** (2000), 321–331.
10. C.M. O'Keefe and J.A. Thas, "Partial flocks of quadratic cones with a point vertex in $PG(n,q)$, n odd," *J. of Algebraic Combin.* **6** (1997), 377–392.
11. L. Storme and J.A. Thas, "k-arcs and partial flocks," *Linear Algebra Appl.* **226/228** (1995), 32–45.
12. J.A. Thas, "Flocks of non-singular ruled quadrics in $PG(3,q)$," *Atti Acad. Naz. Lincei Rend.* **59** (1975), 83–85.
13. J.A. Thas, "Generalized quadrangles and flocks of cones," *European J. Combin.* **8** (1987), 441–452.
14. J.A. Thas, "Flocks, maximal exterior sets, and inversive planes. Finite geometries and combinatorial designs (Lincoln, NE, 1987)," *Contemp. Math.* **111**, *Amer. Math. Soc.*, Providence, RI (1990), 187–218.
15. J.A. Thas, "Maximal exterior sets of hyperbolic quadrics; the complete classification," *J. Comb. Theory Ser.* A **56** (1991), 303–308.
16. J.A. Thas, "Flocks and partial flocks of quadrics: A survey," *J. Statist. Plan. Inference* **94** (2001), 335–348.