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Modular Adjacency Algebras of Hamming Schemes

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Abstract. To each association scheme G and to each field R, there is associated naturally an associative algebra, the so-called adjacency algebra RG of G over R. It is well-known that RG is semisimple if R has characteristic 0. However, little is known if R has positive characteristic. In the present paper, we focus on this case. We describe the algebra RG if G is a Hamming scheme (and R a field of positive characteristic). In particular, we show that, in this case, RG is a factor algebra of a polynomial ring by a monomial ideal.

Keywords: association scheme, Hamming scheme, modular adjacency algebra

1. Introduction

Let p be a prime number, and let $\mathbb{F}_p$ denote a field with p elements. Let n and q be positive integers, and let $H(n, q)$ denote the Hamming scheme the point set of which consists of all n-tuples of elements of $\{0, 1, \ldots, q-1\}$. It follows from [2, III.Theorem 2.3] that the Frame number of $H(n, q)$ is $q^{n(n+1)}$. Therefore from [1, Theorem 1.1] or [5, Theorem 4.2], we know that $\mathbb{F}_p H(n, q)$ is semisimple iff p does not divide q. Moreover, in Section 2.3 of the present paper, we shall show that, if p divides q, $\mathbb{F}_p H(n, p) \cong \mathbb{F}_p H(n, q)$. Therefore, we shall focus our attention to the investigation of $\mathbb{F}_p H(n, p)$.

From [4, Theorem 3.4, Corollary 3.5] we know that $\mathbb{F}_p$ is a splitting field for $\mathbb{F}_p H(n, p)$. Therefore, if we determine the structure of $\mathbb{F}_p H(n, p)$, we know the structure over any field of characteristic p.

We will describe $\mathbb{F}_p H(n, p)$ as a factor algebra of a polynomial ring by a monomial ideal for the clarity of the structure. A monomial ideal is the ideal that is generated by only monomials.

2. Preparation

For the definitions in this section, refer to [2].

2.1. Association schemes

Let X be a finite set of cardinality n. We define $R_0 := \{(x, x) \mid x \in X\}$. Let $R_i \subseteq X \times X$ be given. We set $R_i^* := \{(z, y) \mid (y, z) \in R_i\}$. Let G be a partition of $X \times X$ such that $R_0 \in G$ and the empty set $\emptyset \notin G$, and assume that, $R_i^* \in G$ for each $R_i \in G$. Then, the pair (X, G)

will be called an *association scheme* if, for all $R_i, R_j, R_k \in G$, there exists an integer p_{ijk} such that, for all $y, z \in X$

$$(y, z) \in R_k \Rightarrow \sharp\{x \in X \mid (y, x) \in R_i, (x, z) \in R_j\} = p_{ijk}.$$

The elements of $\{p_{ijk}\}$ will be called the *intersection numbers* of (X, G).

For each $R_i \in G$, we define the $n \times n$ matrix A_i indexed by the elements of X,

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise,} \end{cases}$$

and this matrix A_i will be called the *adjacency matrix* of R_i.

Let the cardinal number of G be $d + 1$ and let J be the $n \times n$ all 1 matrix. Then, by the definition, it follows that $\sum_{i=0}^{d} A_i = J$. It follows that for all A_i, A_j,

$$A_i A_j = \sum_{k=0}^{d} p_{ijk} A_k.$$

From this fact, we can define an algebra naturally. For the commutative ring R with 1, we put $R(X, G) = \bigoplus_{i=0}^{d} RA_i$ as a matrix ring over R, and it will be called the *adjacency algebra* of (X, G) over R.

For all $i, j, k \in \{0, 1, \ldots, d\}$, we define the matrix B_i by $(B_i)_{jk} = p_{ijk}$. This matrix B_i will be called the *i-th intersection matrix*. It follows that for all B_i, B_j, $B_i B_j = \sum_{k=0}^{d} p_{ijk} B_k$. Therefore we can define an algebra $RB = \bigoplus_{i=0}^{d} RB_i$ for a commutative ring R with 1, and it will be called the *intersection algebra* of (X, G) over R. Then the mapping from the adjacency algebra to the intersection algebra of (X, G) over R, $A_i \mapsto B_i$, is an algebra isomorphism.

2.2. *P-polynomial schemes*

A symmetric association scheme $(X, \{R_i\}_{0 \le i \le d})$ is called a *P-polynomial scheme* with respect to the ordering $R_0, R_1, \ldots, R_d$, if there exist some complex coefficient polynomials v_i of degree i $(0 \le i \le d)$ such that $A_i = v_i(A_1)$, where A_i is the adjacency matrix of R_i.

We use the following notation: a tridiagonal matrix

$$B = \begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & a_1 & \ddots & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & \ddots & c_d \\ 0 & & & b_{d-1} & a_d \end{pmatrix}$$

is denoted by

$$\begin{Bmatrix} * & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & * \end{Bmatrix}.$$

Then the following (i) and (ii) are equivalent to each other (see [2, Proposition 1.1]).
(i) B_1 is a tridiagonal matrix with non-zero off-diagonal entries:

$$\begin{Bmatrix} * & 1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_{d-1} & * \end{Bmatrix} (b_i \neq 0, c_i \neq 0).$$

(ii) $(X, \{R_i\}_{0 \leq i \leq d})$ is a P-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$, i.e.,

$$A_i = v_i(A_1) \quad (i = 0, 1, \ldots, d)$$

for some polynomials v_i of degree i.

2.3. *Hamming schemes*

Let Σ be an alphabet of q symbols $\{0, 1, \ldots, q-1\}$. We define Ω to be the set Σ^n of all n-tuples of elements of Σ, and let $\rho(x, y)$ be the number of coordinate places in which the n-tuples x and y differ. Thus $\rho(x, y)$ is the Hamming distance between x and y. we set

$$R_i = \{(x, y) \in \Omega \times \Omega \mid \rho(x, y) = i\},$$

and then $(\Omega, \{R_i\}_{0 \leq i \leq n})$ is an association scheme. This will be called the *Hamming scheme*, and denoted by $H(n, q)$.

We consider the intersection numbers $p_{ijk}^{(n,q)}$ of $H(n, q)$. For the convenience of the argument, we extend the binomial coefficient as follows.

$$\binom{0}{x} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for each integer x and each negative integer y,

$$\binom{x}{y} = 0, \quad \binom{y}{x} = 0.$$

We consider the following three elements in Ω,

$$(\overbrace{0, 0, \ldots, 0}^{k}, 0, \ldots, 0),$$
$$(\alpha_0, \alpha_1, \alpha_*, \beta),$$
$$(\underbrace{1, 1, \ldots, 1}_{k}, 0, \ldots, 0),$$

where $\alpha_0, \alpha_1, \alpha_*$, and β means that there are α_0 0's, α_1 1's, and α_* other symbols among first k figures, and β non-zero symbols among remaining $(n-k)$ figures.

Then if we assume that

$$((0, \ldots, 0), (\alpha_0, \alpha_1, \alpha_*, \beta)) \in R_i,$$
$$((\alpha_0, \alpha_1, \alpha_*, \beta), (1, \ldots, 1, 0, \ldots, 0)) \in R_j,$$
$$((0, \ldots, 0), (1, \ldots, 1, 0, \ldots, 0)) \in R_k,$$

the system of equations must hold that

$$\begin{cases} \alpha_1 + \alpha_* + \beta = i \\ \alpha_0 + \alpha_* + \beta = j \\ \alpha_0 + \alpha_1 + \alpha_* = k. \end{cases}$$

From the definition, since $p_{ijk}^{(n,q)}$ is the total of n-tuples that satisfy the above the system of equations,

$$p_{ijk}^{(n,q)} = \sum_{\beta=0}^{n-k} \binom{k}{k-i+\beta} \binom{i-\beta}{k-j+\beta} \binom{n-k}{\beta} (q-1)^{\beta} (q-2)^{i+j-k-2\beta}.$$

Therefore if $p \mid q$ for some prime number p, $p_{ijk}^{(n,q)} \equiv p_{ijk}^{(n,p)} \pmod{p}$. Since the intersection numbers are the structure constants of the adjacency algebra, $\mathbb{F}_p H(n, q) \cong \mathbb{F}_p H(n, p)$.

The Hamming scheme $H(n, q)$ is a P-polynomial scheme (see [2]), and

$$B_1 = \left\{ \begin{matrix} * & 1 & \cdots & i & \cdots & n \\ 0 & q-2 & \cdots & i(q-2) & \cdots & n(q-2) \\ n(q-1) & (n-1)(q-1) & \cdots & (n-i)(q-1) & \cdots & * \end{matrix} \right\}.$$

For the remainder of this paper, let p be a fixed prime number. Therefore we set $H(n) := H(n, p)$. And we denote the intersection numbers, the adjacency matrices, and the intersection matrices of $H(n)$ respectively by $p_{ijk}^{(n)}$, $A_i^{(n)}$, $B_i^{(n)}$ and so on.

If we index the adjacency matrices by a suitable order, for example, the lexicographic order on Σ^n, then it follows that

$$A_i^{(n+1)} = I \otimes A_i^{(n)} + K \otimes A_{i-1}^{(n)} \quad \text{for } \forall i \in \{0, 1, \ldots, n+1\},$$

where I is the $p \times p$ identity matrix, K is the $p \times p$ matrix such that the diagonal entries are 0 and the others 1, $A_{-1}^{(n)} = A_{n+1}^{(n)} = O$ (the $p^n \times p^n$ zero matrix), and $\otimes$ is the Kronecker product. The Kronecker product $A \otimes B$ of matrices A and B is defined as follows. Suppose $A = (a_{ij})$. Then $A \otimes B$ is obtained by replacing the entry a_{ij} of A by the matrix $a_{ij}B$, for all i and j. The most important property of this product is that, provided the required products exist,

$$(A \otimes B)(X \otimes Y) = AX \otimes BY.$$

3. $H(p^r - 1)$

The intersection numbers are the structure constants of the adjacency algebra. Therefore, if we consider the adjacency algebra over a field of characteristic p, we may consider the intersection numbers modulo p.

The size of the adjacency matrix of $H(n)$ is p^n. Therefore, the adjacency algebra of $H(n)$ over a field of characteristic p is local. Moreover the unique irreducible representation is $A_i \mapsto p_{i\ i^*\ 0}$ (see [4, Theorem 3.4, Corollary 3.5]). Therefore $\mathbb{F}_p$ is a splitting field for $\mathbb{F}_p H(n)$. Thus, if we determine the structure of $\mathbb{F}_p H(n)$, we know the structure over any field of characteristic p.

For the remainder of this paper, since we consider the adjacency algebras only over $\mathbb{F}_p$, we set $\mathfrak{A}_n := \mathbb{F}_p H(n)$.

By the definition,

$$B_1^{(p^r-1)} = \begin{pmatrix} B_1^{(p-1)} & & & \\ & B_1^{(p-1)} & & \\ & & \ddots & \\ & & & B_1^{(p-1)} \end{pmatrix},$$

therefore if we set $A_i^{(p-1)} = v_i(A_1^{(p-1)})$, it follows that for $0 \le \alpha \le p-1$,

$$A_{pi+\alpha}^{(p^r-1)} = v_\alpha\big(A_1^{(p^r-1)}\big)A_{pi}^{(p^r-1)}.$$

Then since any $c_i^{(p-1)} \not\equiv 0 \pmod p$, we can define v_α over $\mathbb{F}_p$ for $0 \le \alpha \le p-1$. For calculating $B_{pi+\alpha}^{(p^r-1)}$, we prepare the following theorem and corollary.

Theorem 1 (*Lucas' theorem* [3, Theorem 3.4.1]) *Let p be prime, and let*

$$m = a_0 + a_1 p + \cdots + a_k p^k,$$
$$n = b_0 + b_1 p + \cdots + b_k p^k,$$

where $0 \leq a_i, b_i < p$ *for* $i = 0, 1, \ldots, k-1$. *Then*

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} (\text{mod p}).$$

Corollary 2 *Let* $p, m,$ *and* n *be as in Theorem 1. Then, for any two elements* α *and* β *in* $\{0,1,\ldots,p-1\}$, *we have*

$$\binom{pm+\alpha}{pn+\beta} \equiv \binom{m}{n}\binom{\alpha}{\beta} (\text{mod p}).$$

Now we want to calculate $B_{pi+\alpha}^{(p^r-1)}$, that is the coefficients of $A_{pi+\alpha}^{(p^r-1)} A_{pj+\beta}^{(p^r-1)}$. But it is enough to investigate $A_{pi}^{(p^r-1)} A_{pj}^{(p^r-1)}$, i.e. $p_{pi\ pj\ k}^{(p^r-1)}$ because we know $v_\alpha(A_1^{(p^r-1)}) v_\beta(A_1^{(p^r-1)})$.
Here we recall from Section 2.3 that

$$p_{pi\ pj\ k}^{(p^r-1)} = p_{pi\ pj\ k}^{(p^r-1,p)} = \sum_{s=0}^{p^r-1-k} \binom{k}{k-pi+s}\binom{pi-s}{k-pj+s}\binom{p^r-1-k}{s} \times (p-1)^s (p-2)^{pi+pj-k-2s}.$$

We assume that $k = k' + pk''$ and $s = s' + ps''$ where $0 \leq k', s' < p$. Then by Corollary 2, it follows that

$$0 < s' < p-k' \Rightarrow \binom{k}{k-pi+s} \equiv 0 \pmod{p},$$
$$p-1-k' < s' < p \Rightarrow \binom{p^r-1-k}{s} \equiv 0 \pmod{p},$$

and if $s' = 0$,

$$k' \neq 0 \Rightarrow \binom{pi-s}{k-pj+s} \equiv 0 \pmod{p}.$$

Therefore it follows that if $k = pk''$,

$$\begin{aligned}
p_{pi\,pj\,k}^{(p^r-1)} &= \sum_{s=0}^{p^r-1-k} \binom{k}{k-pi+s}\binom{pi-s}{k-pj+s}\binom{p^r-1-k}{s} \\
&\quad \times (p-1)^s (p-2)^{pi+pj-k-2s} \\
&\equiv \sum_{s''=0}^{p^{r-1}-1-k''} \binom{pk''}{pk''-pi+ps''}\binom{pi-ps''}{pk''-pj+ps''}\binom{p^r-1-pk''}{ps''} \\
&\quad \times (p-1)^{ps''}(p-2)^{pi+pj-pk''-2ps''} \\
&\equiv \sum_{s''=0}^{p^{r-1}-1-k''} \binom{k''}{k''-i+s''}\binom{i-s''}{k''-j+s''}\binom{p^{r-1}-1-k''}{s''}\binom{p-1}{0} \\
&\quad \times (p-1)^{s''}(p-2)^{i+j-k''-2s''} \\
&\equiv p_{ijk''}^{(p^{r-1}-1)} \pmod{p},
\end{aligned}$$

and if $p \nmid k$, $p_{pi\ pj\ k}^{(p^r-1)} \equiv 0 \pmod{p}$.

Thus

$$\begin{aligned}
A_{pi+\alpha}^{(p^r-1)} A_{pj+\beta}^{(p^r-1)} &= v_\alpha\big(A_1^{(p^r-1)}\big) v_\beta\big(A_1^{(p^r-1)}\big) A_{pi}^{(p^r-1)} A_{pj}^{(p^r-1)} \\
&\equiv \sum_{k=0}^{p^{r-1}-1} \sum_{\gamma=0}^{p-1} p_{ijk}^{(p^{r-1}-1)} p_{\alpha\beta\gamma}^{(p-1)} A_{pk+\gamma}^{(p^r-1)}.
\end{aligned}$$

By the above argument, it follows that

$$B_{pi+\alpha}^{(p^r-1)} = B_i^{(p^{r-1}-1)} \otimes B_\alpha^{(p-1)}.$$

Repeating the same argument, we know that for each non-negative integer m such that $0 \le m \le p^r - 1$ and $m = m_0 p^0 + m_1 p^1 + \cdots + m_{r-1} p^{r-1}$,

$$B_m^{(p^r-1)} = B_{m_{r-1}}^{(p-1)} \otimes B_{m_{r-2}}^{(p-1)} \otimes \cdots \otimes B_{m_0}^{(p-1)}.$$

From this fact, we obtain that

$$\mathfrak{A}_{p^r-1} \cong \overbrace{\mathfrak{A}_{p-1} \otimes \mathfrak{A}_{p-1} \otimes \cdots \otimes \mathfrak{A}_{p-1}}^{r}.$$

Theorem 3 $\mathfrak{A}_{p-1} \cong \mathbb{F}_p C_p \cong \mathbb{F}_p[X]/\langle X^p \rangle$

Proof: Since $B_1^{(p-1)} - B_0^{(p-1)}$ is nilpotent and its rank is $p-1$, the theorem holds. □

Therefore the following theorem holds.

Theorem 4 *For each positive integer r, $\mathfrak{A}_{p^r-1}$ is isomorphic to the group algebra of the elementary abelian group of order p^r over $\mathbb{F}_p$.*

4. The structure of $\mathfrak{A}_n$

In the previous section, we considered the structure of $\mathfrak{A}_{p^r-1}$. To determine the structure of $\mathfrak{A}_n$, in general, we construct an algebra homomorphism $\mathfrak{A}_{n+1} \to \mathfrak{A}_n$.

From Section 2.3, $A_i^{(n+1)} = I \otimes A_i^{(n)} + K \otimes A_{i-1}^{(n)}$. This means that $\mathfrak{A}_{n+1}$ is a subalgebra of $\mathfrak{A}_1 \otimes \mathfrak{A}_n$. The unique irreducible representation of $\mathfrak{A}_1$ is $A_0^{(1)} \mapsto 1$, $A_1^{(1)} \mapsto -1$.

Therefore we can define naturally the mapping f_{n+1} for each positive integer n by

$$\begin{aligned} &f_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n \\ &A_i^{(n+1)} = I \otimes A_i^{(n)} + K \otimes A_{i-1}^{(n)} \mapsto A_i^{(n)} - A_{i-1}^{(n)}. \end{aligned}$$

Proposition 5 *For each positive integer n, $f_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$ above is an algebra epimorphism.*

By Theorem 4, $\mathfrak{A}_{p^r-1}$ is isomorphic to $\mathbb{F}_p(\underbrace{C_p \times C_p \times \cdots \times C_p}_{r})$ for each positive integer r. Let $x_1, x_2, \ldots, x_r$ be the generators of each C_p starting from the right. Then the element of $\mathfrak{A}_{p^r-1}$ corresponding to x_i by the algebra homomorphism above, is $A_{p^{i-1}}^{(p^r-1)}$.

From the representation theory of the finite group, there exists the algebra isomorphism g from the quotient ring $\mathfrak{P}_r = F_p[X_1, X_2, \ldots, X_r]/\langle X_1^p, \cdots, X_r^p\rangle$ of the polynomial ring of r variables over $\mathbb{F}_p$ to $\mathbb{F}_p(\underbrace{C_p \times C_p \times \cdots \times C_p}_{r})$ by $g(X_i) = 1 - x_i$. Therefore we can define an algebra isomorphism $s_r : \mathfrak{P}_r \to \mathfrak{A}_{p^r-1}$ by

$$s_r(X_i) = A_0^{(p^r-1)} - A_{p^{i-1}}^{(p^r-1)}.$$

We define a weight function wt on the set of the monomials of $\mathfrak{P}_r$ by

$$wt(X_i) = p^{i-1}, \quad wt\left(\prod_j X_j^{k_j}\right) = \sum_j k_j p^{j-1}.$$

Proposition 6 *For each positive integer m such that $1 \le m \le p-1$,*

$$\left(A_0^{(p^r-1)} - A_{p^i}^{(p^r-1)}\right)^m = m! \sum_{n=0}^{m} \binom{m}{n} (-1)^n A_{np^i}^{(p^r-1)}.$$

And if $i \neq j, 0 \leq \alpha, \beta \leq p-1$,

$$A_{\alpha p^i}^{(p^r-1)} A_{\beta p^j}^{(p^r-1)} = A_{\alpha p^i+\beta p^j}^{(p^r-1)}.$$

Proof: We obtain the first equation by the induction and the second equation by considering tensor expression of $B_{\alpha p^i}^{(p^r-1)}$ (see Section 3). □

Let $Y_i = X_{i_0}^{k_0} X_{i_1}^{k_1} \dots X_{i_s}^{k_s}$ be the monomial of $\mathfrak{P}_r$ such that $wt(Y_i) = i$. Then by the above two equations, the following Proposition holds.

Proposition 7

$$\begin{aligned} s_r(Y_i) &= \prod_{j=0}^{s} \left(A_0^{(p^r-1)} - A_{p^{i_j-1}}^{(p^r-0)}\right)^{k_j} \\ &= \left(\prod_{j=0}^{s} k_j!\right) \sum_{n=0}^{p^r-1} \binom{i}{n} (-1)^n A_n^{(p^r-1)}. \end{aligned}$$

Proof: The first equation means that the expansion of $(A_0^{(p^r-1)} - A_{p^i}^{(p^r-1)})^m$ is the formula that one expands $(X^0 - X^{p^i})^m$ and replaces X^n with $A_n^{(p^r-1)}$ and multiplies it by $m!$. The second equation means that we can apply the same way to $\prod_{j=0}^{s}(A_0^{(p^r-1)} - A_{p^{i_j}-1}^{(p^r-1)})^{k_j}$. Namely, $\prod_{j=0}^{s}(A_0^{(p^r-1)} - A_{p^{i_j}-1}^{(p^r-1)})^{k_j}$ is the formula that one expands $\prod_{j=0}^{s}(X^0 - X^{p^{i_j-1}})^{k_j} = (X^0 - X^1)^i$ and replaces X^n with $A_n^{(p^r-1)}$ and multiplies it by $\prod_{j=0}^{s} k_j!$. □

Then the following theorem, that is the main theorem in this paper, holds.

Theorem 8 *We set* $\mathfrak{P} = \mathbb{F}_p[X_1, X_2, \dots]/\langle X_1^p, X_2^p \dots\rangle$, *and for each positive integer* n, *we set*

$$W_n = \langle x \mid x \textit{ is the monomial of } \mathfrak{P} \textit{ such that } wt(x) > n\rangle.$$

Then it holds that $\mathfrak{P}/W_n \cong \mathfrak{A}_n$ *as algebras.*

Proof: It is enough that we show that,

$$\mathfrak{P}_r/W_n \cong \mathfrak{A}_n \quad \text{for } n < p^r.$$

Furthermore it is enough that we show that for each positive integer n such that $n \leq p^r - 1$, $Y_n \in \mathrm{Ker} f_n f_{n+1} \dots f_{p^r-1} s_r$. Since

$$
\begin{aligned}
& f_n f_{n+1} \dots f_{p^r-1} s_r(Y_n) \\
&= \left(\prod_{j=0}^{s} k_j!\right) f_n f_{n+1} \dots f_{p^r-1} \left(\sum_{i=0}^{p^r-1} \binom{n}{i} (-1)^i A_i^{(p^r-1)} \right) \\
&= \left(\prod_{j=0}^{s} k_j!\right) f_n f_{n+1} \dots f_{p^r-2} \left(\sum_{i=0}^{p^r-2} \left(\binom{n}{i} (-1)^i - \binom{n}{i+1} (-1)^{i+1} \right) A_i^{(p^r-2)} \right) \\
&= \left(\prod_{j=0}^{s} k_j!\right) (-1) f_n f_{n+1} \dots f_{p^r-2} \left(\sum_{i=0}^{p^r-2} \binom{n+1}{i+1} (-1)^{i+1} A_i^{(p^r-2)} \right) \\
&= \left(\prod_{j=0}^{s} k_j!\right) (-1)^{p^r-n} \sum_{i=0}^{n-1} \binom{p^r}{i+p^r-n} (-1)^{i+p^r-n} A_i^{(n-1)} \\
&= 0,
\end{aligned}
$$

the theorem holds. □

Remark 1 We set $G_{n,q} = S_q\ wr\ S_n$, $H_{n,q} = S_{q-1}\ wr\ S_n$ for positive integers n, q. Let K be a field. Then $KH(n, q)$ and the Hecke algebra $\mathrm{End}_{KG_{n,q}}(1_{H_{n,q}}^{G_{n,q}})$ are isomorphic as algebras (see [2, III.2]). Therefore we also could determine the structure of $End_{KG_{n,q}}(1_{H_{n,q}}^{G_{n,q}})$. In particular, Theorem 4 means that for each positive integer r, if $n = p^r - 1$, the Hecke algebra $\mathrm{End}_{\mathbb{F}_p G_{n,p}}(1_{H_{n,p}}^{G_{n,p}})$ is isomorphic to the group algebra of the elementary abelian group of order p^r.

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