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Type-II Matrices Attached to Conference Graphs

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Abstract. We determine the Nomura algebras of the type-II matrices belonging to the Bose-Mesner algebra of a conference graph.

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1. Type-II matrices and nomura algebras

We say that an $n \times n$ matrix W with complex entries is *type II* if

$$W(j,i)(W^{-1})(i,j) = \frac{1}{n}$$

for $i, j = 1, \ldots, n$. So a type-II matrix is invertible and has no zero entry. We use I and J to denote the identity matrix and the matrix of all ones respectively. For each integer $n \geq 2$ and a complex number t satisfying $nt^2 + nt + 1 = 0$, the matrix $I + tJ$ is a type-II matrix. These matrices are known as the Potts models, which are examples of spin models. The matrices of spin models and four-weight spin models are also type II, see [1] and [8].

Type-II matrices arise from combinatorial objects. For instance, any Hadamard matrix is type II. Chris Godsil and the first author have shown that the Bose-Mesner algebra of any strongly regular graphs contains a type-II matrix that is not type-II equivalent to the Potts model, see [6]. Two type-II matrices W and W' are *type-II equivalent* if $W' = M_1 W M_2$ for some monomial matrices M_1 and M_2.

In [9], Jaeger, Matsumoto and Nomura have given the construction of a Bose-Mesner algebra from a type-II matrix. Let W be an $n \times n$ type-II matrix. For $i, j = 1, \ldots, n$, we define a vector $\mathbf{Y}_{ij}$ in $\mathbb{C}^n$ with its k-th entry being

$$\mathbf{Y}_{ij}(k) = \frac{W(k,i)}{W(k,j)}. \tag{1.1}$$

The Nomura algebra of W, denoted by $\mathcal{N}_W$, is defined as the set of $n \times n$ matrices for which $\mathbf{Y}_{ij}$ is an eigenvector, for $i, j = 1, \ldots, n$. It is easy to see that $\mathcal{N}_W = \mathcal{N}_{cW}$ for any non-zero scalar c.

For $M \in \mathcal{N}_W$, we use $\Theta_W(M)$ to denote the matrix whose (i, j)-th entry equals the eigenvalue of M corresponding to the eigenvector $\mathbf{Y}_{ij}$. Note that Θ_W is a linear map, and we use $\mathcal{N}'_W$ to denote its image of $\mathcal{N}_W$. It follows from the definition that W is a type-II matrix if and only if W^T is type II. The following significant result due to Jaeger, Matsumoto and Nomura, Theorem 4 of [9], links a type-II matrix to a pair of Bose-Mesner algebras.

Theorem 1.1 *Let W be an $n \times n$ type-II matrix. Then $\mathcal{N}'_W = \mathcal{N}_{W^T}$. Moreover, $\mathcal{N}_W$ and $\mathcal{N}'_W$ form a formally-dual pair of Bose-Mesner algebras with Θ_W being a formal duality.*

A Bose-Mesner algebra $\mathcal{B}$ is a commutative algebra that contains I and J and is closed under the transpose and the Schur product (which is entrywise multiplication of two matrices). It is well known that $\mathcal{B}$ is a Bose-Mesner algebra if and only if it is equal to the span of the matrices of some association scheme. If $\mathcal{B} = \text{span}(I, J)$ then it has dimension two and we call it the trivial Bose-Mesner algebra. If $\mathcal{B}$ has dimension n then it is the Bose-Mesner of a group scheme. When $\mathcal{N}_W$ has dimension strictly between 2 and n, we may get a pair of association schemes that consist of interesting combinatorial objects.

It is known that the Nomura algebra of a Potts model of order $n \geq 5$ is the trivial Bose-Mesner algebra. It follows from a simple counting argument that the Nomura algebra of a Hadamard matrix is trivial if its order is congruent to 4 mod 8, see Section 5.2 of [9].

Chris Godsil has proved in Section 4 of [5] that a type-II matrix W has two distinct entries if and only if $W = aJ + (b - a)N$ for some $a \neq b$ and N is the incidence matrix of a symmetric design. He has also shown that if $b \neq -a$ and $n > 3$ then the Nomura algebra of W is trivial. In the same paper, he has determined that a symmetric type-II matrix with constant diagonal and quadratic minimal polynomial has the form $aI + bC$ where C is a regular two-graph. Using the method in [5], it can be shown that the Nomura algebra of this type-II matrix is also trivial for $n \geq 5$.

Furthermore, H. Suzuki and the second author have proved in [7] that the Nomura algebra of a type-II matrix W is imprimitive if and only if W is type-II equivalent to the twisted tensor product of type-II matrices.

An interesting problem is to find type-II matrices that give formally dual pairs of Bose-Mesner algebras of dimension strictly between 2 and n. Motivated by this problem, we consider the type-II matrices in the Bose-Mesner algebra of conference graphs. We report in this paper that if $n > 9$ then the Nomura algebras of these type-II matrices are trivial.

2. Conference graphs

A conference graph G is a strongly regular graph with parameter

$$(4\mu + 1, 2\mu, \mu - 1, \mu)$$

for some positive integer μ. The eigenvalues of G are 2μ and the roots r and s of the equation $x^2 + x - \mu = 0$. So we have

$$\{r, s\} = \left\{ \frac{-1 + \sqrt{4\mu + 1}}{2}, \frac{-1 - \sqrt{4\mu + 1}}{2} \right\}.$$

For basic facts on strongly regular graphs, see [3]. It is easy to verify that the Bose-Mesner algebra of a conference graph is formally self-dual, i.e., the matrix of eigenvalues coincides with the matrix of dual eigenvalues, see [4]. Let A be the adjacency matrix of a conference graph G. It follows from Eq. (32) of [8] that $W_\epsilon = t_0 I + t_1 A + t_2(J - I - A)$ is a type-II matrix if and only if

$$t_1 = \epsilon t, \quad t_2 = t^{-1}, \quad t_0 = -\epsilon st - st^{-1} \quad \text{and} \quad t_0{}^{-1} = -\epsilon st^{-1} - st \tag{2.1}$$

for $\epsilon = \pm 1$. It follows from Remark (ii) on page 41 of [8] that $W_\epsilon = -i W_{-\epsilon}$. Since $\mathcal{N}_W = \mathcal{N}_{cW}$ for any non-zero scalar c, we may assume that $\epsilon = 1$. So Eq. (2.1) becomes $t_0 = t_0{}^{-1} = -s(t + t^{-1})$ which has solutions

$$t_0 = \pm 1, \quad \{t, t^{-1}\} = \left\{ t_0\left(\frac{-s^{-1} + \sqrt{s^{-2} - 4}}{2}\right), t_0\left(\frac{-s^{-1} - \sqrt{s^{-2} - 4}}{2}\right) \right\}.$$

Again since $\mathcal{N}_W = \mathcal{N}_{t_0 W}$, we may assume that $t_0 = 1$ and

$$W = I + tA + t^{-1}(J - I - A). \tag{2.2}$$

Note that t and t^{-1} are the solutions to the quadratic $t^2 + s^{-1}t + 1 = 0$, so is $\bar{t}$. Since for all $\mu \geq 1$

$$s^{-2} - 4 = 4\left(\frac{1}{(1 \pm \sqrt{4\mu + 1})^2} - 1\right) < 0,$$

t is not a real number and we see that $\bar{t} = t^{-1}$. If $\mathbf{Y}_{ij}$ is the eigenvector of W defined in Eq. (1.1), then

$$\overline{\mathbf{Y}_{ij}} = \mathbf{Y}_{ji}. \tag{2.3}$$

3. Nomura algebra of W

A Bose-Mesner algebra $\mathcal{B}$ is a commutative matrix algebra with identity that is closed under conjugate transpose. It is semisimple and it contains a basis of pairwise orthogonal idempotents (with respect to matrix multiplication), called the principle idempotents. In addition, each principle idempotent represents the orthogonal projection on an eigenspace of the matrices in $\mathcal{B}$. See [2] and [4] for more about Bose-Mesner algebras. Let n be the size of matrices in $\mathcal{B}$. When $\mathcal{B} = \text{span}(I, J)$ is the trivial Bose-Mesner algebra, then $\frac{1}{n}J$ and $I - \frac{1}{n}J$ are the principle idempotents of $\mathcal{B}$ corresponding to the eigenspaces $\mathbf{1}$, i.e., the space spanned by the vector of all ones, and $\mathbf{1}^\perp$, i.e., the space orthogonal to $\mathbf{1}$, respectively.

The approach we take here is to show that the Nomura algebra of W has $\mathbf{1}^\perp$ as an eigenspace. Hence $\mathcal{N}_W$ has only two principle idempotents, and is therefore equal to the span of I and J.

Let $\langle \mathbf{u}, \mathbf{v}\rangle$ denote the Hermitian product of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{C}^n$, i.e., $\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^n \mathbf{u}(i)\overline{\mathbf{v}(i)}$.

Lemma 3.1 *Let W be the type-II matrix defined in Eq.* (2.2). *If the Hermitian product of $\mathbf{Y}_{\alpha\beta}$ and $\mathbf{Y}_{x\alpha}$ is non-zero for all adjacent vertices α and β and for all $x \neq \alpha, \beta$, then $\mathcal{N}_W$ is trivial.*

Proof: Now $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{x\alpha}\rangle \neq 0$ implies that $\mathbf{Y}_{\alpha\beta}$ and $\mathbf{Y}_{x\alpha}$ belong to the same eigenspace. This eigenspace, denoted by U, contains $\mathbf{Y}_{x\alpha}$ for all $x \neq \alpha, \beta$.

Since each vertex in G has degree $2\mu \geq 2$, the vertex α has a neighbour β' distinct from β. Applying the hypothesis to α and β', we conclude that $\mathbf{Y}_{\alpha\beta'}$ and $\mathbf{Y}_{y\alpha}$, for all $y \neq \alpha, \beta'$, belong to the same eigenspace denoted by U'.

Since $n = 4\mu+1 \geq 5$, there exists a vertex z distinct from α, β and β'. Now $\mathbf{Y}_{z\alpha}$ belongs to both U and U'. Hence $U = U'$ and it contains

$$\mathbf{Y}_{\beta'\alpha}, \quad \mathbf{Y}_{\beta\alpha}, \quad \text{and} \quad \mathbf{Y}_{x\alpha} \quad \text{for all } x \neq \alpha.$$

Now

$$S = \{\mathbf{Y}_{x\alpha} \mid x \neq \alpha\} \subseteq U$$

is a set of $n-1$ columns of $\Delta_\alpha W$, where Δ_α is the diagonal matrix with its (i,i)-th entry being $W(i,\alpha)^{-1}$. It follows from the definition of type-II matrix that W has no zero entry and is invertible. Hence both Δ_α and $\Delta_\alpha W$ are invertible. Note that $\mathbf{Y}_{\alpha\alpha} = \mathbf{1} \notin S$ is the α-th column of $\Delta_\alpha W$. We see that $S \subseteq U$ is the set of $n-1$ linearly independent vectors in $\mathbf{1}^\perp$. Now $\mathcal{N}_W$ contains J which has $\mathbf{1}$ and $\mathbf{1}^\perp$ as its eigenspaces. So $\mathbf{1}^\perp$ is an eigenspace of $\mathcal{N}_W$ and $I - \frac{1}{n}J$ is a principle idemptent of $\mathcal{N}_W$. Therefore $\mathcal{N}_W = \text{span}(I, J)$ is the trivial Bose-Mesner algebra. $\square$

Let α, β and γ be vertices of a conference graph G with parameters $(4\mu+1, 2\mu, \mu-1, \mu)$. For $x_\alpha \in \{\alpha, \bar{\alpha}\}$, $x_\beta \in \{\beta, \bar{\beta}\}$ and $x_\gamma \in \{\gamma, \bar{\gamma}\}$, we define $\Gamma_{x_\alpha x_\beta x_\gamma}$ to be the set of vertices that are adjacent (not adjacent) to x_v if $x_v = v$ ($x_v = \bar{v}$, respectively), for $v = \alpha, \beta, \gamma$. For instance $\Gamma_{\alpha\beta\gamma}$ is the set of common neighbours of α, β and γ while $\Gamma_{\alpha\beta\bar{\gamma}}$ is the set of common neighbours of α and β that are not adjacent to γ. Now the vertex set of G is partitioned into

$$\{\alpha, \beta, \gamma\} \cup \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\alpha\bar{\beta}\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}.$$

By Eq. (2.3), we have

$$\mathbf{Y}_{\alpha\beta}(x)\overline{\mathbf{Y}_{\gamma\alpha}(x)} = \mathbf{Y}_{\alpha\beta}(x)\mathbf{Y}_{\alpha\gamma}(x) = \frac{W(x,\alpha)^2}{W(x,\beta)W(x,\gamma)}.$$

It is easy to verify that

$$\mathbf{Y}_{\alpha\beta}(x)\overline{\mathbf{Y}_{\gamma\alpha}(x)} = \begin{cases} 1 & \text{if } x \in \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \\ t^2 & \text{if } x \in \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\alpha\bar{\beta}\gamma} \\ t^{-2} & \text{if } x \in \Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma} \\ t^4 & \text{if } x \in \Gamma_{\alpha\bar{\beta}\bar{\gamma}} \\ t^{-4} & \text{if } x \in \Gamma_{\bar{\alpha}\beta\gamma}. \end{cases}$$

Hence the Hermitian product

$$\begin{aligned} \langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle = {} & \frac{W(\alpha,\alpha)^2}{W(\alpha,\beta)W(\alpha,\gamma)} + \frac{W(\beta,\alpha)^2}{W(\beta,\beta)W(\beta,\gamma)} \\ & + \frac{W(\gamma,\alpha)^2}{W(\gamma,\beta)W(\gamma,\gamma)} + |\Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| + |\Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}}|t^2 \\ & + |\Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma}|t^{-2} + |\Gamma_{\bar{\alpha}\beta\gamma}|t^{-4} + |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}|t^4. \end{aligned} \tag{3.1}$$

In the following computation, we let α and β be adjacent vertices in a conference graph G. We now check that for $\mu > 2$ and for all $\gamma \neq \alpha, \beta$, the Hermitian product $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle$ is non-zero.

We first consider the case where γ is adjacent to both α and β. We have $W(\alpha,\beta) = W(\alpha,\gamma) = W(\beta,\gamma) = t$. We use Γ_v to denote the set of neighbours of v in G. Then we get

$$\begin{aligned} \Gamma_\alpha &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\alpha\bar{\beta}\bar{\gamma}} \cup \{\beta,\gamma\} \\ \Gamma_\beta &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \{\alpha,\gamma\} \\ \Gamma_\gamma &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma} \cup \{\alpha,\beta\} \\ \Gamma_\alpha \cap \Gamma_\beta &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \{\gamma\} \\ \Gamma_\alpha \cap \Gamma_\gamma &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \{\beta\} \\ \Gamma_\beta \cap \Gamma_\gamma &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \{\alpha\} \\ V(G) &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\alpha\bar{\beta}\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \cup \{\alpha,\beta,\gamma\}. \end{aligned}$$

Now we translate the above to the following system of equations.

$$\begin{aligned} 2\mu &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| + 2 \\ 2\mu &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| + 2 \\ 2\mu &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| + 2 \\ \mu - 1 &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + 1 \\ \mu - 1 &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + 1 \\ \mu - 1 &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + 1 \\ 4\mu + 1 &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| + |\Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| \\ &\quad +3. \end{aligned} \tag{3.2}$$

Solving this system of equations, we get

$$|\Gamma_{\alpha\beta\gamma}| = m, \quad |\Gamma_{\bar{\alpha}\beta\gamma}| = |\Gamma_{\alpha\bar{\beta}\gamma}| = |\Gamma_{\alpha\beta\bar{\gamma}}| = |\Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| = \mu - 2 - m,$$
$$\text{and} \quad |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| = |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| = |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| = 2 + m,$$

for some non-negative integer m. Using Eq. (3.1), we have

$$\begin{aligned}\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle &= t^{-2} + 2t + (\mu - 2) + 2(\mu - 2 - m)t^2 + 2(m+2)t^{-2} \\ &\quad + (\mu - 2 - m)t^{-4} + (m+2)t^4.\end{aligned}$$

It follows from $\bar{t} = t^{-1}$ that the real part of this product is

$$\begin{aligned}\frac{1}{2}(\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle + \overline{\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle}) &= \frac{1}{2}((t^2 + t^{-2}) + 2(t + t^{-1}) + 2(\mu - 2) \\ &\quad + 2\mu(t^2 + t^{-2}) + \mu(t^4 + t^{-4})).\end{aligned}$$

Note that $-s(t + t^{-1}) = 1$ which leads to

$$\begin{aligned}&t^2 + t^{-2} = (t + t^{-1})^2 - 2 = s^{-2} - 2 \quad \text{and} \\ &t^4 + t^{-4} = (t + t^{-1})^4 - 4(t^2 + t^{-2}) - 6 = s^{-4} - 4s^{-2} + 2.\end{aligned}$$

So we have

$$\begin{aligned}&\frac{1}{2}(\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle + \overline{\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle}) \\ &\quad = \frac{1}{2}(s^{-2} - 2 - 2s^{-1} + 2\mu - 4 + 2\mu(s^{-2} - 2) + \mu(s^{-4} - 4s^{-2} + 2)) \\ &\quad = \frac{-1}{2s^4}((6s^2 - 4s + 8\mu + 3)(s^2 + s - \mu) - (4\mu + 1)(3s - 2\mu)).\end{aligned}$$

Since s is a root of $x^2 + x - \mu = 0$, the above expression equals zero if and only if

$$s = \frac{2\mu}{3}.$$

Substituting the above equation into $s^2 + s - \mu = 0$ yields

$$\frac{\mu(4\mu - 3)}{9} = 0.$$

The only integral solution to this equation is $\mu = 0$. So if $\mu > 0$ then $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle$ is non-zero.

Secondly, we assume that γ is adjacent to α but not to β. So we have $W(\alpha,\beta) = W(\alpha,\gamma) = t$ and $W(\beta,\gamma) = t^{-1}$. Similar to the first case, we can set up a system of equations like that of Eq. (3.2). Solving it, we get

$$\begin{aligned} |\Gamma_{\alpha\beta\gamma}| = |\Gamma_{\alpha\bar\beta\bar\gamma}| = m, \quad & |\Gamma_{\bar\alpha\beta\gamma}| = |\Gamma_{\alpha\bar\beta\gamma}| = |\Gamma_{\alpha\beta\bar\gamma}| = |\Gamma_{\bar\alpha\bar\beta\bar\gamma}| = \mu - 1 - m, \\ \text{and} \quad & |\Gamma_{\bar\alpha\beta\bar\gamma}| = |\Gamma_{\bar\alpha\bar\beta\gamma}| = m + 1, \end{aligned}$$

for some non-negative integer m. By Eq. (3.1), we get

$$\begin{aligned} \langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle = {} & t^{-2} + 2t^3 + (\mu - 1) + 2(\mu - 1 - m)t^2 + 2(m + 1)t^{-2} \\ & + (\mu - 1 - m)t^{-4} + mt^4, \end{aligned}$$

and

$$\begin{aligned} \langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle + \overline{\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle} = {} & (t^2 + t^{-2}) + 2(t^3 + t^{-3}) + 2(\mu - 1) \\ & + 2\mu(t^2 + t^{-2}) + (\mu - 1)(t^4 + t^{-4}). \end{aligned}$$

Since $t^3 + t^{-3} = (t + t^{-1})^3 - 3(t + t^{-1}) = -s^{-3} + 3s^{-1}$, we get

$$\begin{aligned} & \langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle + \overline{\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle} \\ & = (s^{-2} - 2) + 2(3s^{-1} - s^{-3}) + 2(\mu - 1) + 2\mu(s^{-2} - 2) + (\mu - 1)(s^{-4} - 4s^{-2} + 2) \\ & = -s^{-4}((6s^2 - 12s + 8\mu + 7)(s^2 + s - \mu) - (4\mu + 1)(5s - 2\mu - 1)). \end{aligned}$$

Again $s^2 + s - \mu = 0$, the above expression equals zero if and only if

$$s = \frac{2\mu + 1}{5}.$$

Substituting this value into $s^2 + s - \mu$ yields

$$\frac{(4\mu - 3)(\mu - 2)}{25} = 0.$$

The only integral solution to this equation is 2. So if $\mu > 2$ then the Hermitian product of $\mathbf{Y}_{\alpha\beta}$ and $\mathbf{Y}_{\gamma\alpha}$ is non-zero.

Thirdly, suppose γ is adjacent to β but not to α. Then $W(\alpha,\gamma) = t^{-1}$ and $W(\alpha,\beta) = W(\beta,\gamma) = t$. From a system of equations similar to that of Eq. (3.2), we get

$$\begin{aligned} |\Gamma_{\alpha\beta\gamma}| = |\Gamma_{\bar\alpha\beta\bar\gamma}| = m, \quad & |\Gamma_{\bar\alpha\beta\gamma}| = |\Gamma_{\alpha\bar\beta\gamma}| = |\Gamma_{\alpha\beta\bar\gamma}| = |\Gamma_{\bar\alpha\bar\beta\bar\gamma}| = \mu - 1 - m, \\ \text{and} \quad & |\Gamma_{\alpha\bar\beta\bar\gamma}| = |\Gamma_{\bar\alpha\bar\beta\gamma}| = m + 1, \end{aligned}$$

for some non-negative integer m. By Eq. (3.1),

$$\begin{aligned}\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle = {} & 1 + t + t^{-3} + (\mu - 1) + 2(\mu - 1 - m)t^2 + (2m+1)t^{-2} \\ & + (\mu - 1 - m)t^{-4} + (m+1)t^4.\end{aligned}$$

The imaginary part of $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle$ is

$$\begin{aligned}\frac{1}{2}(\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle - \overline{\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle}) = {} & \frac{1}{2}\big((t - t^{-1}) - (t^3 - t^{-3}) + (2\mu - 4m - 3)(t^2 - t^{-2}) \\ & + (2m + 2 - \mu)(t^4 - t^{-4})\big).\end{aligned}$$

Note that $t^2 - t^{-2} = (t - t^{-1})(-s)^{-1}$,

$$\begin{aligned}t^3 - t^{-3} &= (t - t^{-1})(t^2 + 1 + t^{-2}) = (t - t^{-1})(s^{-2} - 1), \quad \text{and} \\ t^4 - t^{-4} &= (t - t^{-1})(t^3 + t + t^{-1} + t^{-3}) = (t - t^{-1})(2s^{-1} - s^{-3}).\end{aligned}$$

So the imaginary part of $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle$ equals

$$\begin{aligned}&= \frac{(t - t^{-1})}{2}(1 - (s^{-2} - 1) - (2\mu - 4m - 3)s^{-1} + (2m + 2 - \mu)(2s^{-1} - s^{-3})) \\ &= \frac{(t - t^{-1})(1 + 2s)}{2s^3}\left(s^2 + 3s + 4sm - 2s\mu - 2m - 2 + \mu\right).\end{aligned}$$

Now we have seen in Section 2 that $s^2 + s - \mu = 0$, so

$$s = -\frac{1}{2} \pm \frac{\sqrt{4\mu + 1}}{2}$$

and $t^{-1} = \bar{t} \neq t$. So the imaginary part of $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha}\rangle$ equals zero if and only if

$$\begin{aligned}m &= \frac{-s^2 - 3s - \mu + 2s\mu + 2}{2(2s - 1)} \\ &= \frac{-(s^2 + s - \mu) + 2(\mu - 1)(s - 1)}{2(2s - 1)}. \\ &= \frac{(\mu - 1)(s - 1)}{2s - 1} \\ &= \frac{(\mu - 1)(-3 \pm \sqrt{4\mu + 1})}{2(-2 \pm \sqrt{4\mu + 1})} \\ &= \frac{(\mu - 1)}{2}\left(1 - \frac{1}{(-2 \pm \sqrt{4\mu + 1})}\right).\end{aligned}$$

For $\mu > 1$, m is an integer only when $4\mu + 1$ is a perfect square. Suppose $\sqrt{4\mu + 1}$ is an integer then we can write $\sqrt{4\mu + 1} = 2b + 1$ for some integer $b \geq 1$. So $\mu = b^2 + b$ and

$$m = \frac{(b^2 + b - 1)(b - 1)}{(2b - 1)}, \quad \text{or} \quad m = \frac{(b^2 + b - 1)(b + 2)}{(2b + 3)}.$$

Observe that

$$8m = (4b^2 + 2b - 7) + \frac{1}{2b - 1}, \quad \text{or} \quad 8m = (4b^2 + 6b - 5) - \frac{1}{2b + 3},$$

respectively. In either case, $8m$ is not an integer for $b > 1$. But $m = |\Gamma_{\alpha\beta\gamma}|$ is an integer. So the imaginary part of $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle$ is not zero for $\mu > 2$.

Lastly, suppose γ is not adjacent to α nor β. So $W(\alpha, \beta) = t$ and $W(\alpha, \gamma) = W(\beta, \gamma) = t^{-1}$. Using the same argument as above, we get

$$|\Gamma_{\alpha\beta\gamma}| = |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| = |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| = |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| = m, \quad |\Gamma_{\bar{\alpha}\beta\gamma}| = |\Gamma_{\alpha\bar{\beta}\gamma}| = \mu - m, \quad \text{and}$$
$$|\Gamma_{\alpha\beta\bar{\gamma}}| = |\Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| = \mu - 1 - m,$$

for some non-negative integer m. By Eq. (3.1), the Hermitian product

$$\begin{aligned}\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle = 1 + t^3 + t^{-1} + (\mu - 1) + (2\mu - 1 - 2m)t^2 + 2mt^{-2} \\ + (\mu - m)t^{-4} + mt^4.\end{aligned}$$

The imaginary part of this product is

$$\begin{aligned}&\frac{1}{2}(\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle - \overline{\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle}) \\ &\quad = \frac{1}{2}((t^3 - t^{-3}) - (t - t^{-1}) + (2\mu - 1 - 4m)(t^2 - t^{-2}) + (2m - \mu)(t^4 - t^{-4})) \\ &\quad = \frac{(t^{-1} - t)(2s + 1)}{2s^3}((s^2 + s - \mu) + (2s\mu - 2s - 4sm + 2m)) \\ &\quad = \frac{\pm(t^{-1} - t)\sqrt{4\mu + 1}}{s^3}\,(s(\mu - 1) - (2s - 1)m)\end{aligned}$$

which equals zero if and only if

$$\begin{aligned}m &= \frac{(\mu - 1)s}{2s - 1} \\ &= \frac{(\mu - 1)}{2}\left(\frac{-1 \pm \sqrt{4\mu + 1}}{-2 \pm \sqrt{4\mu + 1}}\right) \\ &= \frac{(\mu - 1)}{2}\left(1 + \frac{1}{(-2 \pm \sqrt{4\mu + 1})}\right).\end{aligned}$$

Again when $\mu > 1$, m is an integer only if $\sqrt{4\mu + 1}$ is an integer. Now suppose $\sqrt{4\mu + 1} = 2b + 1$ for some integer $b \geq 1$. Then $\mu = b^2 + b$ and

$$m = \frac{(b^2 + b - 1)b}{2b - 1}, \quad \text{or} \quad m = \frac{(b^2 + b - 1)(b + 1)}{2b + 3}.$$

Observe that

$$8m = (4b^2 + 6b - 1) - \frac{1}{2b - 1}, \quad \text{or} \quad 8m = (4b^2 + 2b - 3) + \frac{1}{2b + 3},$$

respectively. In either case, $8m$ is not an integer for $b > 1$. But $m = |\Gamma_{\alpha\beta\gamma}|$ is an integer, hence the imaginary part of $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle$ is not zero for $\mu > 2$.

Now we have shown that $\langle \mathbf{Y}_{\alpha\beta}, \mathbf{Y}_{\gamma\alpha} \rangle$ is non-zero for all $\gamma \neq \alpha, \beta$. Applying Lemma 3.1, we conclude that the Nomura algebra of W is trivial when $\mu > 2$.

Theorem 3.2 *Let W' be a type-II matrix in the Bose-Mesner algebra of the conference graph with parameters* $(4\mu + 1, 2\mu, \mu - 1, \mu)$. *If* $\mu > 2$ *then* $\mathcal{N}_{W'}$ *is trivial.*

Proof: If W' is a non-zero scalar multiple of the type-II matrix W defined in Equation (2.2) then it follows from Lemma 3.1 that $\mathcal{N}_{W'} = \mathcal{N}_W$ is trivial. Otherwise, W' is a non-zero scalar multiple of the Potts model and $\mathcal{N}_{W'}$ is also trivial. $\square$

When $\mu = 1$, the conference graph is the cycle on five vertices, see page 671 of [3]. The type-II matrix W defined by Eq. (2.2) is type-II equivalent to the cyclic type-II matrix of size five [10]. Its Nomura algebra is isomorphic to the Bose-Mesner algebra of the cyclic group of five elements.

When $\mu = 2$, the conference graph is the point graph of the generalized quadrangle of order (2, 1) (or the 3×3 grid), see page 671 of [3]. Solving $s^2 + s - 2 = 0$ we have $s = 1, -2$. If $s = 1$ then W is type-II equivalent to the tensor product of two copies of the Potts model of size three. The Nomura algebra of W is isomorphic to the Bose-Mesner algebra of $C_3 \otimes C_3$. If $s = -2$ then the Nomura algebra of W is trivial.

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