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A Higman-Haemers Inequality for Thick Regular Near Polygons

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Abstract. In this note we will generalize the Higman-Haemers inequalities for generalized polygons to thick regular near polygons.

Keywords: regular near polygon, distance-regular graph

1. Introduction

The reader is referred to the next section for the definitions.

Generalized n-gons of order (s, t) were introduced by Tits in [12]. Although formally n is unbounded, a famous theorem of Feit-G. Higman asserts that, apart from the ordinary polygons, finite examples can exist only for $n = 3, 4, 6, 8$ or 12. (See [5] and [3, Theorem 6.5.1].)

If $s > 1$ and $t > 1$, then $n = 12$ is not possible. Moreover in the case of $n = 4, 6, 8$, D.G. Higman [8, 9] and Haemers [7] showed that s and t are bounded from above by functions in t and s, respectively. To show this they used the Krein condition. (See also [3, Theorem 6.5.1].)

Let Γ be a thick regular near $2d$-gon of order (s, t) and let $t_i := c_i - 1$ for all $1 \le i \le d$. Brouwer and Wilbrink [4] showed

$$\sum_{i=0}^{d-1} \left(\frac{-1}{s^2}\right)^i \prod_{j=1}^{i} \left(\frac{t - t_j}{1 + t_j}\right) \ge 0.$$

This was proved from the Krein condition $q_{dd}^d \ge 0$. If d is even, then $1 + t \le (s^2 + 1)(1 + t_{d-1})$.

A similar result was shown by Mathon for regular near hexagons.

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In this note we are going to show that for thick regular near $2d$-gons of order (s, t), t is bounded from above by a function of s and the diameter d.

In particular, we show the following results. We will only use the multiplicity of the smallest eigenvalue to show those results.

Theorem 1 *Let Γ be a distance-regular graph of order (s, t) with $s > 1$. Let d be the diameter of Γ, $r := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$ and $\rho := \frac{d}{r}$. Suppose $-t-1$ is an eigenvalue of Γ. Then $t < s^{4\rho-1}$.*

Corollary 2 *Let Γ be a thick regular near $2d$-gon of order (s, t). Let $r := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$ and $\rho := \frac{d}{r}$. Then the following hold.*
(1) $t < s^{4\rho-1}$.
(2) *If $r \notin \{1, 2, 3, 5\}$, then $t < s^7$.*

A generalized $2d$-gon of order (s, t) is a regular near $2d$-gon of order (s, t) with $d = r+1$. It is known that if a generalized $2d$-gon of order (s, t) exists, then there exists a generalized $2d$-gon of order (t, s) which is known as *dual.* So as a consequence of this corollary we will show that for generalized $2d$-gons we can bound s and t by functions in t and s, respectively.

Let Γ be a generalized $2d$-gon of order (s, t). Then the following hold.

(1) If $s > 1$, then $t < s^{\frac{3d+1}{d-1}}$.
(2) If $t > 1$, then $s < t^{\frac{3d+1}{d-1}}$.

The bound given by Higman [8, 9] and Haemers [7] can be proved without using the Krein condition although the bound proved here is a bit weaker.

Let us consider another consequence of Corollary 2. Suppose it is true that for given s and t there are only finitely many regular near $2d$-gons of order (s, t). Then for given $s' > 1$ there are only finitely many regular near $2d$-gons of order (s', t') with $r = \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\} \geq 6$. Furthermore, for a regular near $2d$-gons of order (s', t') the diameter d is bounded by a function in s'.

2. Definitions

Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph without loops or multiple edges. For vertices x and y in Γ we denote by $\partial_\Gamma(x, y)$ the distance between x and y in Γ. For a vertex x in Γ and a set L of vertices we define $\partial_\Gamma(x, L) := \min\{\partial_\Gamma(x, z) \mid z \in L\}$.

The *diameter* of Γ, denoted by d, is the maximal distance of two vertices in Γ. We denote by $\Gamma_i(x)$ the set of vertices which are at distance i from x.

A connected graph Γ with diameter d is called *distance-regular* if there are numbers

$$c_i \ (1 \leq i \leq d), \quad a_i \ (0 \leq i \leq d) \quad \text{and} \quad b_i \ (0 \leq i \leq d-1)$$

such that for any two vertices x and y in Γ at distance i the sets

$$\Gamma_{i-1}(x) \cap \Gamma_1(y), \quad \Gamma_i(x) \cap \Gamma_1(y) \quad \text{and} \quad \Gamma_{i+1}(x) \cap \Gamma_1(y)$$

have cardinalities c_i, a_i and b_i, respectively. Then Γ is regular with valency $k := b_0$.

Let Γ be a distance-regular graph with diameter d. The array

$$\iota(\Gamma) = \left\{ \begin{matrix} * & c_1 & \cdots & c_i & \dots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_i & \dots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_i & \dots & b_{d-1} & * \end{matrix} \right\}$$

is called the *intersection array of* Γ. Define $r = r(\Gamma) := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$. The *numerical girth of* Γ is $2r + 2$ if $c_{r+1} \neq 1$ and $2r + 3$ if $c_{r+1} = 1$.

By an eigenvalue of Γ we will mean an eigenvalue of its adjacency matrix A. Its multiplicity is its multiplicity as eigenvalue of A. Define the polynomials $u_i(x)$ by

$$\begin{aligned} &u_0(x) := 1, \quad u_1(x) := x/k, \quad \text{and} \\ &c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x) = x u_i(x), \quad i = 1, 2, \dots, d-1. \end{aligned}$$

Let $k_i := |\Gamma_i(x)|$ for all $0 \leq i \leq d$ which does not depend on the choice of x.

Let θ be an eigenvalue of Γ with multiplicity m. It is well-known that

$$m = \frac{|V\Gamma|}{\sum_{i=0}^{d} k_i u_i(\theta)^2}.$$

For more information on distance-regular graphs we would like to refer to the books [1–3] and [6].

A graph Γ is said to be *of order* (s, t) if $\Gamma_1(x)$ is a disjoint union of $t + 1$ cliques of size s for every vertex x in Γ. In this case, Γ is a regular graph of valency $k = s(t + 1)$ and every edge lies on a clique of size $s + 1$. A clique of size $s + 1$ is called a *singular line* of Γ.

A graph Γ is called (the collinearity graph of) *a regular near* $2d$*-gon of order* (s, t) if it is a distance-regular graph of order (s, t) with diameter d and $a_i = c_i(s - 1)$ for all $1 \leq i \leq d$.

A regular near $2d$-gon is called *thick* if $s > 1$.

A *generalized* $2d$*-gon of order* (s, t) is a regular near $2d$-gon of order (s, t) with $d = r+1$.

More information on regular near $2d$-gons and generalized $2d$-gons will be found in [3, Sections 6.4–6.6].

3. Proof of the theorem

In this section we prove our theorem. First we recall the following result.

Proposition 3 [11, Proposition 3.3] *Let Γ be a distance-regular graph with valency k, numerical girth g such that each edge lies in an (a_1+2)-clique. Let h be a positive integer. Suppose $\theta = -\frac{k}{a_1+1}$ be an eigenvalue of Γ with multiplicity m. Then the following hold.*
(1) *If $g \geq 4h$, then*

$$m \geq 1 + \frac{ka_1}{a_1+1}\frac{b_1^h - 1}{b_1 - 1}.$$

(2) *If $g \geq 4h+2$, then*

$$m \geq \frac{1}{a_1+1} + \frac{a_1(a_1+2)}{a_1+1}\frac{b_1^{h+1}-1}{b_1-1}.$$

Lemma 4 *Let Γ be a distance-regular graph of order (s,t) with diameter d. Suppose $-t-1$ is an eigenvalue of Γ with multiplicity m. Then for any integer i with $0 \leq i \leq d$, the following hold.*
(1) *Let C be a clique of size $s+1$ and $x \in V\Gamma$ with $\partial_\Gamma(x, C) = i$. Then*

$$\alpha_i := |\{z \in C \mid \partial_\Gamma(x,z) = i\}|$$

does not depend on the choice of C and x. Furthermore, $\partial_\Gamma(x, C) \leq d-1$ for any vertex x in Γ.
(2) *There exists an integer γ_i such that $c_i = \gamma_i\alpha_{i-1}$ and $b_i = (t+1-\gamma_i)(s+1-\alpha_i)$.*
(3) *Let $u_j := u_j(-t-1)$ for all $0 \leq j \leq d$. Then for all $1 \leq j \leq d$ we have*

$$u_j = \left(\frac{-\alpha_{j-1}}{s+1-\alpha_{j-1}}\right)u_{j-1}.$$

In particular,

$$u_i^2 \geq \left(\frac{1}{s}\right)^{2i}.$$

(4) *$m \leq s^{2d}$ with equality if and only if $s = 1$.*

Proof: (1) See [4, Lemma 13.7.2].

(2) Let x and y be vertices in Γ at distance i. Let γ_i be the number of singular lines through y at distance $i-1$ from x. Each such clique has α_{i-1} vertices which are at distance $i-1$ from x. Hence we have $c_i = \gamma_i\alpha_{i-1}$. There are $t+1-\gamma_i$ singular lines through y at distance i from x. Each such clique has $s+1-\alpha_i$ vertices which are at distance $i+1$ from x. Then we have $b_i = (t+1-\gamma_i)(s+1-\alpha_i)$.

(3) We prove the first assertion by induction on j. The case $j=1$ is true since $u_0 = 1, u_1 = -\frac{1}{s}$ and $\alpha_0 = 1$.

Assume $1 \leq j \leq d-1$ and $\alpha_{j-1}u_{j-1} = -(s+1-\alpha_{j-1})u_j$. Then we have

$$\begin{aligned} b_j u_{j+1} &= (-t-1-a_j)u_j - c_j u_{j-1} \\ &= \{-t-1-(t+1)s + c_j + b_j\}u_j + \gamma_j(s+1-\alpha_{j-1})u_j \\ &= \{-(t+1)(s+1) + \gamma_j\alpha_{j-1} + (t+1-\gamma_j)(s+1-\alpha_j) \\ &\quad + \gamma_j(s+1-\alpha_{j-1})\}u_j \\ &= -(t+1-\gamma_j)\alpha_j u_j \end{aligned}$$

from (2). The first assertion is proved. Since

$$\left(\frac{-\alpha_{j-1}}{s+1-\alpha_{j-1}}\right)^2 \geq \left(\frac{1}{s}\right)^2,$$

the second assertion follows from the first assertion.

(4) We have

$$M := \sum_{i=0}^{d} k_i u_i^2 \geq \sum_{i=0}^{d} k_i \left(\frac{1}{s}\right)^{2i} \geq \left(\frac{1}{s}\right)^{2d} \sum_{i=0}^{d} k_i = \frac{|V\Gamma|}{s^{2d}}.$$

Hence

$$m = \frac{|V\Gamma|}{M} \leq s^{2d}.$$

□

Proof of Theorem 1: We remark that $a_1 = s-1$ and $b_1 = st$. Let g be the numerical girth of Γ.

First we assume r is odd with $r = 2h-1$. Then $g \geq 2r+2 = 4h$ and

$$m > \frac{ka_1}{a_1+1} b_1^{h-1} = (t+1)(s-1)(st)^{h-1} > s^{h-1}t^h$$

from Proposition 3 (1). It follows, by Lemma 4 (4), that

$$s^{(4h-2)\rho} = s^{2d} \geq m > s^{h-1}t^h.$$

The desired result is proved.

Second we assume r is even with $r = 2h$. Then $g \geq 2r+2 = 4h+2$ and $m > b_1^h$ from Proposition 3 (2). Hence we have

$$s^{4h\rho} = s^{2d} \geq m > (st)^h.$$

The desired result is proved. □

In [10], we have shown the following result.

Proposition 5 *Let Γ be a thick regular near $2d$-gon with $r = r(\Gamma)$. If $2r + 1 \leq d$ then for any integer q with $r + 1 \leq q \leq d - r$ there exists a regular near $2q$-gon as a strongly closed subgraph in Γ. In particular, $r \in \{1, 2, 3, 5\}$.*

Proof of Corollary 2: It is known that a regular near $2d$-gon of order (s, t) has an eigenvalue $-t - 1$. Moreover if $r \notin \{1, 2, 3, 5\}$, then $d \leq 2r$ from Proposition 5. Therefore the corollary is a direct consequence of Theorem 1. □

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