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The Hodge Structure on a Filtered Boolean Algebra

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Abstract. Let $\Delta(B_n)$ be the order complex of the Boolean algebra and let $B(n,k)$ be the part of $\Delta(B_n)$ where all chains have a gap at most k between each set. We give an action of the symmetric group S_l on the l-chains that gives $B(n,k)$ a Hodge structure and decomposes the homology under the action of the Eulerian idempontents. The S_n action on the chains induces an action on the Hodge pieces and we derive a generating function for the cycle indicator of the Hodge pieces. The Euler characteristic is given as a corollary.

We then exploit the connection between chains and tabloids to give various special cases of the homology. Also an upper bound is obtained using spectral sequence methods.

Finally we present some data on the homology of $B(n,k)$.

Keywords: Hodge structure, Boolean algebra, Euler characteristics

1. Introduction

Our main object of study is an algebraic complex $B(n,k)$ that is a filtration of the order complex of the Boolean algebra. More precisely, fix positive integers k and n. Then define $\Delta_l(n,k) = \{\emptyset \subset C_0 \subset \cdots \subset C_l \subset \{1,\ldots,n\}: 0 < |C_{i+1} - C_i| \le k \text{ for all } -1 \le i \le l\}$. Here $C_{-1} = \emptyset$ and $C_{l+1} = \{1,\ldots,n\}$. Let $B_l(n,k)$ be the vector space with basis $\Delta_l(n,k)$ over $\mathbb{C}$. We can now define our main object of study:

Definition 1.1 Fix positive integers n and k. Then define

$$B(n,k) = \bigoplus_{l=0}^{n-2} B_l(n,k). \tag{1}$$

Example 1.2 When $k > n-2$ we get the order complex of the Boolean algebra. When $k = 1$ there are only maximal chains.

We must make this into a complex by defining a boundary operator. Suppose $C = \emptyset \subset C_0 \subset \cdots \subset C_l \subset \{1,\ldots,n\}$ is a chain in $\Delta_l(n,k)$. Then define

$$\partial_l(C) = \begin{cases} \sum_{i=0}^{l} (-1)^i \emptyset \subset \cdots \subset \hat{C}_i \subset \cdots \subset \{1,\ldots,n\}, & \text{if} |C_{i+1} - C_{i-1}| \le k; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

where $\hat{C}_i$ denotes that this set is omitted from the chain. Extend ∂_l linearly to a function on $B_l(n,k)$. Then it is easy to verify that $(B(n,k),\partial)$ is an algebraic complex.

In the case of the Boolean algebra recall that the natural action of the symmetric group S_n on the poset B_n defines an action of S_n on the homology. Since this action doesn't affect step size of chains we also have an action of S_n on $B(n,k)$, and hence on $H_*(B(n,k))$ (we will always consider complex coefficients when we take homology, so we omit the coefficients from the notation). We can now describe the main motivation for considering this complex. If we just consider the trivial representation inside of $B(n,k)$ then we note that since all numbers are equivalent we only need to concern ourselves with the sizes of the various sets. Introducing a formal variable t as a place marker we see that the complex is the vector space over $\mathbb{C}$ generated by $t^{i_1}\otimes\cdots\otimes t^{i_l}$, where $0<i_j\leq k$ and the exponents sum to n. Here the exponents keep track of how the chains grow. Examining the action of the boundary map we find that:

$$H_d(B(n,k))^{S_n}=HH_{d,n}(t\mathbb{C}[t]/(t^{k+1})) \tag{3}$$

where V^G denotes the trivial isotypic component of a representation V of G and HH denotes the Hochschild homology (see Weibel [8] or Loday [5] for information on Hochschild homology). Here the second subscript of HH refers to the grading by degree. In fact the right-hand side of 3 is known, see Hanlon [3]:

$$\dim(HH_{d,n}(t\mathbb{C}[t]/(t^{k+1})))=\begin{cases}1, \text{ if } d \text{ is odd and } \frac{(d+1)(k+1)}{2}=n;\\ 1, \text{ if } d \text{ is even and } \frac{d(k+1)}{2}=n;\\ 0, \text{ otherwise.}\end{cases} \tag{4}$$

Gerstenhaber and Schack [1] defined a notion of Hodge decomposition for Hochschild homology and Hanlon [4] was able to extend this notion to posets that admit a Hodge structure. A Hodge structure defined in [4] on a poset is an action of S_l on chains of length l which satisfies certain relations. Hanlon showed that the following actions define a Hodge structure on B_n:

Definition 1.3 Let $C=\emptyset\subset C_0\subset\cdots\subset C_l\subset\{1,\ldots,n\}$ be a chain of subsets. Fix $1\leq i\leq l$. Let $A=C_{i+1}-C_i$. Then for the transposition $\tau=(i,i+1)\in S_l$ define

$$\tau C=\emptyset\subset\cdots C_{i-1}\subset C_{i-1}\cup A\subset C_{i+1}\cdots\subset\{1,\ldots,n\}.$$

Note that this action will also work on $B(n,k)$. Now that we have described what the Hodge structure of $B(n,k)$ is (even though this is not poset homology) we need to talk about the Hodge decomposition. Gerstenhaber and Schack defined pairwise orthogonal idempotents summing to the identity in $\mathbb{C}S_n$, $e_n^{(1)},\ldots,e_n^{(n)}$ called the Eulerian idempotents [1]. Unfortunately their definition is not conducive to computation, see Loday [6] for a more concrete definition. Since we have a sequence of S_l actions on $B(n,k)$ we can define:

$$B_l^{(j)}(n,k)=e_l^{(j)}\cdot B_l(n,k).$$

The results of Hanlon (which follow Gerstenhaber and Schack) show that the Hodge structure relations give us:

$$\partial_l\colon \quad B_l^{(j)}(n,k) \to B_{l-1}^{(j)}(n,k).$$

Thus we have that:

$$H_l(B(n,k)) = \bigoplus_j H_l^{(j)}(B(n,k))$$

and this is the **Hodge decomposition of** $H_*(B(n,k))$. Further the action of S_n on $B(n,k)$ clearly commutes with the action of S_l on $(l-1)$-chains. Thus each $H_d^{(j)}(B(n,k))$ is an S_n representation. Hence we have

$$H_d^{(j)}(B(n,k))^{S_n} = HH_{d,n}^{(j)}(t\mathbb{C}[t]/(t^{k+1})). \tag{5}$$

Unfortunately the complex is not a homology sphere. Thus by employing the theory in Hanlon [4] we are only able to get results on the Euler characteristic of each Hodge piece. In particular we have:

Theorem 1.4 *Let* $\chi_j^{n,k} = \sum_l (-1)^l \dim(B_l^{(j)}(n,k))$ *denote the Euler characteristic of the* j*th Hodge piece of* $B(n,k)$*. Then we have*

$$\sum_n (-1)^n \sum_j \lambda^j Z\big(\chi_j^{n,k}\big) = -\prod_l \left(1 + a_l[Z(\epsilon_1) + \cdots + Z(\epsilon_k)]\right)^{-\frac{1}{l}\sum_{d|l}\mu(d)\lambda^{\frac{l}{d}}}.$$

Here if f is a class function of S_n, then $Z(f)$ is the cycle indicator, that is

$$Z(f) = \frac{1}{n!}\sum_{\sigma\in S_n} f(\sigma)Z(\sigma)$$

where $j_i(\sigma)$ denotes the number of i cycles in σ and

$$Z(\sigma) = a_1^{j_1(\sigma)} a_2^{j_2(\sigma)} \cdots a_p^{j_p(\sigma)}.$$

Also the bracket operation is defined as

$$A[B] = A[a_i \leftarrow B[a_j \leftarrow a_{ij}]]$$

where $\leftarrow$ denotes substitution. For example if $A = a_1^2 + a_2$ and $B = a_3 + a_1$ then $A[B] = (a_3 + a_1)^2 + a_6 + a_2$. Also here ϵ_i denotes the trivial representation of S_n and $\mu(d)$ denotes the number-theoretic Möbius function.

The rest of the paper is organized as follows: Section 2 proves Theorem 1.4, Section 3 gives various partial results and Section 4 gives some data we have generated.

2. Hodge results

Our goal is to derive an expression for the generating function of the Hodge pieces, that is to prove Theorem 1.4.

We do this by following Section 2 of Hanlon [4]. The idea is to use the following identity (See proof of Theorem 2.1 in Hanlon [4]):

$$\sum_{j\geq 1}\sum_{p\geq 0}\sum_{\tau\in S_p}\left[e_p^{(j)}\right]_\tau Z(\tau)\lambda^j=\prod_l (1+(-1)^l a_l)^{-\frac{1}{l}\sum_{d|l}\mu(d)\lambda^{\frac{l}{d}}}. \tag{6}$$

In addition to this identity we will also need a result concerning the following object:

Definition 2.1 Let $\tau\in S_{u+1}$. Then define $\omega_\tau^{(n)}$ to be the class function whose value on $\sigma\in S_n$ is the number of u-chains fixed in $B(n,k)$ by (τ,σ).

The result we need is:

Lemma 2.2 *Fix $\tau\in S_{u+1}$. Then*

$$\sum_n Z\left(\omega_\tau^{(n)}\right)=Z(\tau)[Z(\epsilon_1)+\cdots+Z(\epsilon_k)] \tag{7}$$

Proof: (See proof of Lemma 2.2 in [4]). Let $T_1,\ldots,T_s$ be the cycles of τ with $|T_i|=t_i$ and $\sigma\in S_n$. Let C be a chain fixed by (τ,σ) and A_i be the subset added in C at step u_i. For C to be fixed we need $\sigma(A_i)=\sigma(A_{i-1})$ (here $A_0=A_l$). Then $|A_i|=c$ for some number c independent of i. Let $A=\cup_{i=1}^l A_i$, $\sigma_i=\sigma|_{A_i}$. Our requirement on σ_i is that it is an injective function from A_i to A_{i-1} with $Z(\sigma|_A)=x_l[Z(\sigma_l\sigma_{l-1}\cdots\sigma_1)]$. In fact we get

$$\sum_{\sigma_1,\ldots,\sigma_l} Z(\sigma|A)=(c!)^{l-1}\sum_{\sigma\in S_c} a_l[Z(\sigma)]=(c)^l a_l[Z(\epsilon_m)]. \tag{8}$$

So to calculate the number of chains fixed by (τ,σ), first pick for each T_i a subset S_i to play the role of A. There are $\binom{n}{m_1t_1,\ldots m_st_s}$ ways to do this. Then we need to divide each S_i into equal pieces to be added at each step of T_i. This can be done in $\binom{m_it_i}{m_i,\ldots,m_i}$ ways. Combining these results with Eq. (8) we get

$$\begin{aligned}\sum_n Z\left(\omega_\tau^{(n)}\right)&=\sum_n\frac{1}{n!}\sum_{\sigma\in S_n}|\{\text{chains fixed by }(\tau,\sigma)\}|Z(\sigma)\\&=\sum_{k\geq m_j\geq 1}\frac{1}{n!}\binom{n}{m_1t_1,\ldots,m_st_s}\prod_i\binom{m_it_i}{m_i,\ldots m_i}(m_i)^{t_i}a_{t_i}\left[Z\left(\epsilon_{m_i}\right)\right]\\&=\sum_{k\geq m_j\geq 1}\prod_{i=1}^s a_{t_i}\left[Z\left(\epsilon_{m_i}\right)\right]=\prod_{i=1}^s a_{t_i}[Z(\epsilon_1)+\cdots+Z(\epsilon_k)].\end{aligned}$$

□

Now that we have Lemma 2.2 we can proceed with proving Theorem 1.4.

Proof: (See proof of Theorem 2.1 in [4]) We wish to get an expression for

$$\sum_n (-1)^n \sum_j \lambda^j Z\big(\chi_j^{n,k}\big).$$

We can rewrite this as:

$$\sum_j \lambda^j \sum_n \frac{1}{n!} \sum_{\sigma \in S_n} \chi_j^{n,k}(\sigma) Z(\sigma)(-1)^n. \tag{9}$$

If we denote the jth Hodge piece of the chain complex by $C_*^{(j)}$ we have that

$$\begin{aligned}\chi_j^{n,k} &= \sum_{i=0}^{n-2} (-1)^i \operatorname{tr}\big(\sigma\big|_{C_{r-i}^{(j)}}\big) = \sum_{i=0}^{r} (-1)^i \operatorname{tr}\big(\sigma e_{r+1-i}^{(j)}\big|_{C_{r-i}}\big) \\ &= \sum_{i=0} (-1)^i \sum_{\tau \in S_{r+1-i}} \big[e^{(j)}\big]_\tau \operatorname{tr}\big(\sigma\tau\big|_{C_{r-i}}\big).\end{aligned}$$

Combining this with Eq. (9) we get

$$\begin{aligned}&\sum_j \lambda^j \sum_n \frac{(-1)^n}{n!} \sum_{\sigma \in S_n} \sum_{i=0}^{n} (-1)^i \sum_{\tau \in S_{n+1-i}} \big[e_{n+1-i}^{(j)}\big]_\tau \omega_\tau^{(n)}(\sigma) Z(\sigma) \\ &\quad = \sum_j \lambda^j \sum_{p=1}^{\infty} (-1)^{p+1} \sum_{\tau \in S_p} \big[e_p^{(j)}\big]_\tau \left(\sum_n \frac{1}{n!} \sum_{\sigma \in S_n} \omega_\tau^{(n)}(\sigma) \right) Z(\sigma).\end{aligned}$$

Then by applying Lemma 2.2 we get

$$= \sum_j \sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \sum_{\tau \in S_p} \big[e_p^{(j)}\big]_\tau \lambda^j Z(\tau)[Z(\epsilon_1) + \cdots + Z(\epsilon_k)],$$

which by Eq. (6) becomes

$$-\prod_l (1 + a_l[Z(\epsilon_1) + \cdots + Z(\epsilon_k)])^{-\frac{1}{l}\sum_{d|l} \mu(d)\lambda^{\frac{l}{d}}}.$$

□

Corollary 2.3 *Let $\chi_{n,k}$ denote the Euler characteristic on $B(n,k)$. Fix k. Then*

$$\sum_n (-1)^n Z(\chi_{n,k}) = -\frac{1}{1 + Z(\epsilon_1) + \cdots + Z(\epsilon_k)}. \tag{10}$$

Proof: We need to evaluate the result from Theorem 1.4 with $\lambda = 1$. Using the well-known identity

$$\sum_{d|l} \mu(d) = \begin{cases} 1, & \text{if } l = 1; \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

we see that we get

$$\sum_n (-1)^n Z(\chi_{n,k}) = -\frac{1}{1 + Z(\epsilon_1) + \cdots + Z(\epsilon_k)}$$

since $a_1[A] = A$ for all A. □

We can derive a further result from Theorem 1.4.

Proposition 2.4 *Fix k and let η_n be the sign representation of S_n. Then we have*:

$$\sum_n (-1)^n \left(\sum_l < \eta_n, \chi_l^{n,k} > \lambda^l \right) x^n = -\frac{1}{1 - x\lambda} \tag{12}$$

Proof:

$$\begin{aligned}
&\sum_n (-1)^n \left(\sum_l < \eta_n, \chi_l^{n,k} > \lambda^l \right) x^n \\
&= \sum_{n,l} (-1)^n \frac{\lambda^l}{n!} \sum_{\sigma \in S_n} \chi_l^{n,k}(\sigma) \eta_n(\sigma) x^n \\
&= \sum_{n,l} (-1)^n \frac{\lambda^l}{n!} \sum_{\sigma \in S_n} \chi_l^{n,k}(\sigma) (Z(\sigma)[a_l \leftarrow (-1)^{l+1} x^l]) \\
&= \sum_n (-1)^n \sum_l \lambda^l \left(\sum_{\sigma \in S_n} \frac{1}{n!} \chi_l^{n,k}(\sigma) (Z(\sigma)[a_l \leftarrow (-1)^{l+1} x^l]) \right) \\
&= \sum_n (-1)^n \sum_l \lambda^l Z\big(\chi_l^{n,k}\big)[a_l \leftarrow (-1)^{l+1} x^l] \\
&= -\prod_l (1 + a_l[Z(\epsilon_1) + \cdots + Z(\epsilon_k)])^{-\frac{1}{l} \sum_{d|l} \mu(d) \lambda^{\frac{l}{d}}} [a_l \leftarrow (-1)^{l+1} x^l].
\end{aligned} \tag{13}$$

So the next step is to calculate $(a_l[Z(\epsilon_i)])[a_l \leftarrow (-1)^{l+1} x^l]$. Notice a term $a = a_1^{p_1} \ldots a_r^{p_r}$ is first sent to $a_l^{p_1} \ldots a_{lr}^{p_r}$ and then finally to

$$(-1^{l+1} x^l)^{p_1} \ldots (-1^{lr+1} x^{rl})^{p_n} = (-1)^{ln + \sum_{i=1}^r p_i} x^{nl}. \tag{14}$$

Recall that the sign of a is $(-1)^{\sum_{i=1}^{r}(i+1)p_i} = (-1)^{n+\sum_{i=1}^{r} p_i}$. Thus using this we get that

$$(a_l[Z(\epsilon_n)])[a_l \leftarrow (-1)^{l+1}x^l] = x^{ln} \sum_{\sigma \in S_n} (-1)^{nl}(-1)^n \operatorname{sgn}(\sigma).$$

This sum is proportional to the intertwining number of ϵ_n and η_n. Thus it is zero unless $n = 1$. Hence returning to Eq. (13) we get

$$-\prod_l (1 + x^l)^{-\frac{1}{l}\sum_{d|l}\mu(d)\lambda^{\frac{l}{d}}}.$$

Applying Eq. (6.2a) in [4] we get the above result. □

Thus the sign representation appears only once and in the top Hodge piece. Later we will give another way to derive this result.

3. Results on $H_*(B(n,k))$

In this Section we mention various results and comments. The first remark is that we can identify a chain with a tabloid as follows: row i of the tabloid is $C_i - C_{i-1}$. Using this we can refine Proposition 2.4.

Proposition 3.1 *The sign representation occurs only in the top homology class and with multiplicity one. Further if $k > 1$ this is the only representation in the top homology class.*

Proof: Write the chain complex $B(n, k) = \bigoplus_u C_u$ where the sum is over all compositions of n with maximum part k and C_u corresponds to chains of type u. A chain in C_u corresponds to a tabloid of shape u. The sign representation of S_n corresponds to the shape (1^n). The multiplicity of the sign representation in C_u corresponds to the number of semistandard Young tableaux of shape (1^n) and type u (see Section 2.9 in [7]). Thus this only occurs when $u = (1^n)$. Thus on $C_{(1^n)}$ the boundary operator is zero, so the sign representation survives in homology.

Note further that if $k > 1$ then all chains in dimension $n-3$ are there. Hence in H_{n-2} there are no cycles that are not in the Boolean algebra. Thus we only get the sign representation. □

We also have a second method of calculating the Euler characteristic:

Proof: Recall (See Section 2.11 in [7]) that the S_n-module of tabloids of shape μ is isomorphic to $\bigoplus_\lambda K_{\lambda,\mu} S^\lambda$ where $K_{\lambda,\mu}$ is the Kostka number and S^λ is the Specht module of shape λ.

$$\sum_n \chi_{n,k} = \sum_n (-1)^n \sum_{\mu|\mu_i \le k} \sum_\lambda K_{\lambda,\mu} S^\lambda = \sum_n (-1)^n \sum_\lambda S^\lambda \sum_{\mu|\mu_i \le k} K_{\lambda,\mu}.$$

So the coefficient of S^λ is the number of semistandard Young tableaux of shape λ and the multiplicity of any number is no bigger than k. Thus if we think about building up such a tableaux we can add no more than k blocks at once. Thus we get the above equation. $\square$

So we can interpret the righthand side of Eq. (10) as being the signed sum of all ways of adding at most k blocks. This idea leads to the calculation of $H_*(B(n, n-2))$.

Proposition 3.2 *Fix n, then $H_d(B(n, n-2))$ is zero except in dimension 1, where it is $(S^{n-1,1})^2 \bigoplus S^n$.*

Proof: By the next result we know that for $1 \leq i < n-2$ we have $H_i(B(n, n-2)) = 0$ and that $H_{n-2}(B(n, n-2)) = S^{1^n}$. Fix n. If we examine the righthand side of Eq. (10) with $k = n-2$ then the only difference from the Boolean algebra case is we cannot add a block of size $n-1$. In the case $k = n-1$ the righthand side evaluates to $(n) + (-1)^{n-1}(1^n)$. Now if we could add a block of size $n-1$ we could only add it to a block of size 1. Hence we get $(1) * (n-1) + (n-1) * (1) = 2(n) + 2(n-1, n)$. So we subtract these from the $k = n-1$ case to get that the Euler characteristic of the $n-2$ case is $-(n) - 2(n, n-1) + (-1)^{n-1}(1^n)$. Thus $(n) + 2(n, n-1)$ must appear in dimension 1. $\square$

Now we give our last result, using spectral sequences.

Proposition 3.3 (Upper-triangularity) *Fix n and k. Then for $n-k+1 \leq i < n-2$ we have $H_i(B(n,k)) = 0$ and $H_{n-2}(B(n,k)) = S^{1^n}$.*

Proof: Let $F(\emptyset \subset C_1 \subset \cdots \subset C_d \subset \{1, \ldots, n\}) = |C_1|$. The boundary on the E^1 is the normal boundary map except we cannot remove the first set. Thus if we look at the E^2 term of the induced spectral sequence it is easy to see that

$$E_d^2(n,k) \cong \oplus_{A \subset \{1,\ldots,n\}:|A| \leq k} H_{d-1}(n - |A|, k). \tag{15}$$

From this the claim follows easily by induction. $\square$

4. Data

Note that for $k = 1$ we get the regular representation in the top dimension and for $k = n-1$ we get the homology of a sphere. By examining Table 1 it appears that we get not only upper-triangularity but also there appears to be a lower-triangularity. However at $n = 6$ one can verify that for $k = 2$ there are in the Euler characteristic partitions with a negative coefficient. Hence this pattern does not continue.

Table 1. Homologies.

k	1	2	3	4
$H_0(B(3,k))$	0	S^3		
$H_1(B(3,k))$	$\mathbb{C}S_3$	S^{111}		
$H_0(B(4,k))$	0	0	S^4	
$H_1(B(4,k))$	0	$S^{31} \oplus S^{31} \oplus S^4$	0	
$H_2(B(4,k))$	$\mathbb{C}S_4$	S^{1111}	S^{1111}	
$H_0(B(5,k))$	0	0	0	S^5
$H_1(B(5,k))$	0	0	$S^{41} \oplus S^{41} \oplus S^5$	0
$H_2(B(5,k))$	0	$S^{311} \oplus S^{311} \oplus S^{32} \oplus S^{41} \oplus S^{41}$	0	0
$H_3(B(5,k))$	$\mathbb{C}S_5$	S^{11111}	S^{11111}	S^{11111}

Table 2. Character values for $n = 4$.

λ	1111	211	22	31	4
$d = 0, k = 1$	0	0	0	0	0
$d = 1, k = 1$	0	0	0	0	0
$d = 2, k = 1$	24	0	0	0	0
$d = 0, k = 2$	0	0	0	0	0
$d = 1, k = 2$	7	3	−1	1	−1
$d = 2, k = 2$	1	−1	1	1	−1
$d = 0, k = 3$	1	−1	1	1	−1
$d = 1, k = 3$	0	0	0	0	0
$d = 2, k = 3$	1	−1	1	1	−1

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