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On Subgraphs in Distance-Regular Graphs

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Abstract. Terwilliger [15] has given the diameter bound $d \leq (s-1)(k-1)+1$ for distance-regular graphs with girth $2s$ and valency k. We show that the only distance-regular graphs with even girth which reach this bound are the hypercubes and the doubled Odd graphs. Also we improve this bound for bipartite distance-regular graphs. Weichsel [17] conjectures that the only distance-regular subgraphs of a hypercube are the even polygons, the hypercubes and the doubled Odd graphs and proves this in the case of girth 4. We show that the only distance-regular subgraphs of a hypercube with girth 6 are the doubled Odd graphs. If the girth is equal to 8, then its valency is at most 12.

Keywords: distance-regular graph, hypercubes, doubled odd graph, subgraph, uniformly geodetic graph.

In this paper we assume that a graph is undirected, without loops or multiple edges and with a finite vertex set. Let Γ be a connected graph. For x, y two vertices of Γ, we denote with $d(x, y)$ the distance between x and y in Γ. If x is a vertex of Γ, we write $\Gamma_i(x)$ for the set of vertices y with $d(x, y) = i$. Instead of $\Gamma_1(x)$ we write $\Gamma(x)$. The valency k_x of a vertex x is the cardinality of $\Gamma(x)$. A graph is regular (with valency k) if each vertex has the same valency k. For x, y two vertices of Γ at distance j we write $c_j(x, y) := |\Gamma_{j-1}(x) \cap \Gamma(y)|, a_j(x, y) := |\Gamma_j(x) \cap \Gamma(y)|$ and $b_j(x, y) := |\Gamma_{j+1}(x) \cap \Gamma(y)|$. We say that the number a_j (resp. b_j, c_j) exists if $a_j(x, y)$ (resp. $b_j(x, y), c_j(x, y)$) does not depend on x, y. We put $\lambda = a_1, \mu = c_2$, when they exist.

The diameter of a connected graph Γ is the maximal distance between two vertices occurring in Γ. The girth of Γ, denoted by g, is the length of a shortest circuit (induced subgraph of valency 2) occurring in Γ.

A connected graph is called uniformly geodetic when for all j the numbers c_j exist. When for all j the numbers a_j, b_j and c_j exist, it is called distance-regular. The intersection array of a distance-regular graph Γ is the array $\{b_0, b_1, \cdots, b_{d-1}; c_1, c_2, \cdots c_d\}$ where d is the diameter of Γ. For a description of the graphs not defined here, see [4].

A graph Γ is bipartite if its vertex set can be partitioned into two classes M and N such that there are no edges between vertices of the same class.

1. Introduction

In this paper we study distance-regular subgraphs of distance-regular graphs. In the second section we give some sufficient conditions to assure that the graph induced by the geodesics between two vertices is distance-regular.

Terwilliger [15] has given the diameter bound $d \leq (s-1)(k-1)+1$ for distance-regular graphs with girth $2s$ and valency $k \geq 3$. In the third section we show that the only distance-regular graphs with even girth which reach this bound are the hypercubes and the doubled Odd graphs (Theorem 6) and give a somewhat improved diameter bound for bipartite distance-regular graphs.

In the fourth section we study distance-regular subgraphs in a hypercube. In this section the subgraphs are not necessarily induced subgraphs. Weichsel [17] has studied them and conjectured that the only distance-regular subgraphs of a hypercube are the even polygons, the hypercubes and the doubled Odd graphs and proved this in the case of girth 4. We show that if the girth is 6, then it must be a doubled Odd graph (Theorem 13). If the girth is equal to 8 then the valency is at most 12 (Theorem 16).

2. Substructures

Let Γ be a graph. For two vertices x, y of Γ, put $C(x,y) := \{z \mid d(z,x)+d(z,y) = d(x,y)\}$. Let $\Delta(x,y)$ denote the graph with vertex set $C(x,y)$ and two vertices $u, v \in C(x,y)$ are adjacent iff uv is an edge in Γ and $d(u,x) \neq d(v,x)$. In this section we investigate when $\Delta(x,y)$ is distance-regular.

PROPOSITION 1. *Let e be an integer; $e \geq 2$. Let Γ be a uniformly geodetic graph (or, more generally, a graph such that c_i exists for $2 \leq i \leq e$) such that $c_{e-1} < c_e$. Then we have the following:*

(*i*) *For all vertices u, u' at distance e there exists a bijective map $\phi : \Gamma(u) \cap \Gamma_{e-1}(u') \to \Gamma_{e-1}(u) \cap \Gamma(u')$ such that $d(v, \phi(v)) > e-2$,*
(*ii*) *$c_i + c_{e-i} \leq c_e$ for all i, $1 \leq i \leq e-1$.*

Proof. Let u, u' be two vertices at distance e. Put $S := \Gamma(u) \cap \Gamma_{e-1}(u')$ and $S' := \Gamma_{e-1}(u) \cap \Gamma(u')$.

(i) Define the set P_s by $P_s := \{s' \in S' \mid d(s,s') \geq e-1\}$ for $s \in S$. Analogously we define $P'_{s'}$ for $s' \in S'$. Note that $|P_s| = |P'_{s'}| = c_e - c_{e-1}$. Let Δ be a graph with vertex set $S \cup S'$ such that $\Delta(s) = P_s$ and $\Delta(s') = P'_{s'}$ for $s \in S, s' \in S'$. Then Δ is a regular bipartite graph and thus has a complete matching (cf. [13], Theorem 7.5.2). So we have *(i)*.
(ii) Let v be a vertex such that $d(u,v) = i$ and $d(u',v) = e-i$ for an integer $1 \leq i \leq e-1$. Denote $A = \{s \in S \mid d(s,v) = i-1\}$ and $B = \{s' \in$

$S' \mid d(s',v) = e-i-1\}$. It follows that $\phi(A) \cap B = \emptyset$. Now we get $c_i + c_{e-i} = |A| + |B| = |\phi(A) \cup B| \le c_e$. So we are done. □

THEOREM 2. *Let e be an integer, $e \ge 2$. Let Γ be a uniformly geodetic graph such that*

(*i*) $c_i + c_{e-i} = c_e$ *for all i, $0 < i < e$, and*
(*ii*) *Γ does not contain two edges xy and zw such that $d(x,z) = e$ and $d(x,w) = d(y,z) = d(y,w) = e-1$.*

Then for any two vertices u, u' of Γ at distance e, the subgraph $\Delta(u,u')$ is a bipartite distance-regular graph with intersection array $\{c_e, c_{e-1}, \cdots, 1; 1, c_2, \cdots, c_e\}$.

Proof. Let u, u' be two vertices of Γ at distance e. Let S, S' be defined as in the proof of the previous lemma. Let $s \in S$, then $d(u',s) = e-1$ and so there is a unique vertex in S', say s', such that $d(s,s') \ge e-1$, because $c_{e-1} = c_e - 1$. We have $d(u,u') = e, d(u,s') = d(u',s) = e-1$ and $d(u,s) = d(u',s') = 1$ and hence, by (*ii*), we get $d(s,s') \ne e-1$. So $d(s,s') = e$.

Now we will show: $C(u,u') = C(s,s')$. Let $v \in C(u,u') \backslash \{u,u'\}$. Then $d(v,u) = i$ and $d(v,u') = e-i$ for an integer i, $0 < i < e$. Let $A := \{t \in S \mid d(v,t) = i-1\}$, $B := \{t' \in S' \mid d(v,t') = e-i-1\}$ and $A' := \{t' \in S' \mid d(a,t') = e$ for an $a \in A\}$. Now we get $A' \cap B = \emptyset$ and thus

$$c_e = c_i + c_{e-i} = |A| + |B| = |A'| + |B| = |A' \cup B| \le c_e.$$

So we have shown that $s \notin A$ implies $s' \in B$ and therefore $v \in C(s,s')$ and thus we get $d(s,v) = i-1$ or $d(s,v) = i+1$. We conclude that $C(u,u') \subseteq C(s,s')$, but these two sets have the same cardinality and thus they are equal.

Let vw be an edge in $\Delta(s,s')$. Let $d(v,u) = i$. Then $d(v,s) = i-1$ or $d(v,s) = i+1$. Thus if $d(w,u) = i$, then $d(w,s) = d(v,s)$, but this is impossible because vw is an edge in $\Delta(s,s')$. So vw is an edge in $\Delta(u,u')$. With induction on $\min\{d(w,u), d(w,u')\}$ it is easy to prove that for all vertices $w \in C(u,u')$ there is a unique $w' \in C(u,u')$ such that $d(w,w') = e$. Furthermore for such a pair we have $\Delta(u,u') = \Delta(w,w')$.

So we have shown that the subgraph $\Delta(u,u')$ is a bipartite distance-regular graph with intersection array $\{c_e, c_{e-1}, \cdots, 1; 1, c_2, \cdots, c_e\}$. □

Remark 1. For $c_i = i$, Mulder [11, 12] has shown the previous theorem without assumption (*ii*). More examples are given below.

PROPOSITION 3. *If Γ is the collinearity graph of a near polygon then (ii) holds.*

Proof. Let $e \ge 3$. Suppose there are two edges xy and zw in Γ such that $d(x,z) = e$ and $d(x,w) = d(y,z) = d(y,w) = e-1$. Let x and y lie on line l. There is a vertex u on l such that $d(u,w) = e-2$, but then $d(u,z) \leq d(u,w)+d(w,z) = e-1$. So $d(z,x) = e, d(z,y) = e-1$ and $d(z,u) \leq e-1$ and thus $u = y$, contradiction. $\square$

Examples. (*i*) For $e = 2$ we find the not very surprising statement that in graphs with $\mu = 2$ and without induced $K_{2,1,1}$ any two vertices at distance 2 determine a quadrangle. In particular this holds for grids $m \times n$, so that λ need not be small. (*ii*) Graphs with $(c_i)_{i\leq e} = (1,1,2,2,3,\cdots)$ (e odd) contain doubled Odd graphs. For example, this holds for Odd graphs and doubled Odd graphs. Thus, apart from the obvious inclusions $O_m \subseteq O_{m+1}$ and $2O_m \subseteq 2O_{m+1}$ we have $2O_m \subseteq O_{2m}$ ($e = 2m-1, m \geq 1$).

COROLLARY 4. *Let Γ be a distance-regular graph with $c_i = 1$, $c_{i+1} = \ldots = c_{2i} = 2, c_{2i+1} = 3$ and $a_1 = \ldots = a_{2i-1} = 0, a_{2i} \leq 2$. Then $i \leq 2$. Furthermore one of the following holds*

(*i*) *$i = 1$ and any two vertices at distance 3 determines a unique 3-cube,*
(*ii*) *$i = 2$ and Γ is a Odd graph or a doubled Odd graph.*

Proof. By the previous theorem and Proposition 3, for any pair of vertices x, y at distance $2i+1$ the subgraph induced by $C(x,y)$ is a bipartite distance-regular graph with $k = 3$, $c_i = 1, c_{i+1} = \cdots c_{2i} = 2$ and $c_{2i+1} = 3$. By Damerell [7] and also by Bannai and Ito [1] there are no Moore graphs with diameter at least 3 and valency at least 3. Hence we get $i \leq 2$.
By Ray-Chaudhuri and Sprague [14] and Koolen [9] a distance-regular graph with parameters $d \geq 5, c_2 = 1, c_3 = c_4 = 2$ and $a_1 = a_2 = a_3 = 0$ is an Odd graph or a doubled Odd graph. $\square$

Remark 2. The case $i = 1$ of the previous corollary is contained in Brouwer [2].

Remark 3. Related work is done by Brouwer and Wilbrink [5], Chima [6] (cf. [4], Proposition 4.3.14), Ivanov [8] and Brouwer [3] (cf. [4], Proposition 4.3.11) and Koolen [10]. Brouwer and Wilbrink have investigated when there are geodetically closed substructures in near polygons. Chima has shown that in a distance-regular graph with parameters $c_2 = 2, c_3 = 4, a_1 = 0$ and $a_2 \leq 3$ any two vertices at distance 3 determine a unique distance-regular graph, which is isomorphic to the incidence graph of the biplane 2-(7,4,2). Ivanov and Brouwer have given conditions to assure that graphs contain geodetically closed Moore geometries. Koolen has shown that in a distance-regular graph with parameters $c_2 = 1, c_3 = 2, c_4 = 3$ and $a_1 = a_2 = a_3 = 0$ any two vertices at distance 4 determine a unique Pappus graph.

3. On the Terwilliger bound

In this section we give the proofs of the results on the Terwilliger bound we have mentioned in the Introduction.

THEOREM 5. *The only distance-regular graphs with girth* $2s-1$ *or* $2s$, *valency* $k \geq 3$, $c_s \geq 2$, $c_s \geq a_{s-1}$ *and diameter* $d \geq (s-1)(k-1)+1$ *are the hypercubes and the doubled Odd graphs.*

Proof. First let $s = 2$. Then $d \geq k$ and by Terwilliger [16] (cf. [4], Corollary 5.2.4) we must have a hypercube.

Now let $s \geq 3$. Terwilliger [15] has proved that for a distance-regular graph with girth $2s-1$ or $2s$, valency $k \geq 3$, $c_s \geq 2$ and $c_s \geq a_{s-1}$ we have $c_i \geq c_{i-s+1}+1$ for $s \leq i \leq d$, and $b_i \leq b_{i-s+1}-1$ for $s-1 \leq i \leq d$. If the graph is nonbipartite, then it easy follows that $d \leq (s-1)(k-1)$. So the graphs must be bipartite. Now the only way to reach $d = (s-1)(k-1)+1$ is that $c_{2s-2} = 2$ and $c_{2s-1} = 3$. By Corollary 4 the only possible graphs are the doubled Odd graphs. □

THEOREM 6. *For a bipartite distance-regular graph with valency* $k \geq 3$, *and girth* $2s \geq 6$, *not a doubled Odd graph, the diameter* d *is bounded by*

$$d \leq (s-1)(k-1) - \lfloor \frac{k-3}{2} \rfloor.$$

Proof. Recall that $c_{i-s+1}+1 \leq c_i$ for $i = s, \cdots, d$. Suppose that $c_{2(s-1)} = 2$, this means that $d > 2(s-1)$ and $c_{2s-1} \geq c_s + 1 \geq 3$. But if $c_{2s-1} = 3$ then by Corollary 4 we have a doubled Odd graph. Now there are two cases:

CASE 1: $c_{2(s-1)} \geq 3$.

We shall show with induction that

$$c_{2i(s-1)-i+1} \geq 1+2i \text{ or } c_{2i(s-1)-i+2} \geq 2+2i \text{ if } d > 2i(s-1)-i+1.$$

For $i = 1$ it is true.

Let now $d > 2i(s-1)-i+1$, $c_{2i(s-1)-i+1} \leq 2i$ and $c_{2i(s-1)-i+2} \leq 2i+1$ for some $i \geq 2$. If $c_{2(i-1)(s-1)-i+3} \geq 2i$, then $c_{2i(s-1)-i+1} \geq c_{2(i-1)(s-1)-i+3+s-1} \geq 2i+1$, contradiction. So $c_{2(i-1)(s-1)-i+2} \geq 2i-1$ and thus $c_{2i(s-1)-i+1} \geq c_{2(i-1)(s-1)-i+2+s-1} \geq 2i$ and $c_{2i(s-1)-i+2} \geq 2i-1+2 = 2i+1$. We have shown that $c_{2i(s-1)-i+1} = 2i$ and $c_{2i(s-1)-i+2} = 2i+1$. Now $c_{2(s-1)} + c_{2(i-1)(s-1)-i+2} \geq 3+2i-1 = 2i+2$, but this is impossible by Proposition 1.

First let $k = 2l$. If $d > 2(l-1)(s-1)-l+2$, then $c_{2(l-1)(s-1)-l+2} \geq 2l-1$ or $c_{2(l-1)(s-1)-l+3} \geq 2l$. We get $d \leq 2(l-1)(s-1)-l+2+s-1 = (2l-1)(s-1)-l+2$. Now let $k = 2l+1$. If $d \geq 2l(s-1)-l+2$, then $c_{2l(s-1)-l+1} \geq 2l+1$ or $c_{2l(s-1)-l+2} \geq 2l+2$, but both are impossible. So $d \leq 2l(s-1)-l+1$.

CASE 2: $c_{2s-1} \geq 4$.
In the same way as in Case 1 we can show that

$$c_{2i(s-1)-i+3} \geq 2i+2 \text{ or } c_{2i(s-1)-i+4} \geq 2i+3 \text{ if } d > 2i(s-1)-i+3.$$

First let $k = 2l$. If $d \geq 2(l-1)(s-1)-l+4$, then $c_{2(l-1)(s-1)-l+3} \geq 2l$, or $c_{2(l-1)(s-1)-l+4} \geq 2l+1$, but both are impossible. Thus $d \leq 2(l-1)(s-1)-l+3 \leq (2l-1)(s-1)-l+1$.

Now let k $=2l+1$. If $d \geq 2(l-1)(s-1)-l+4$, then $c_{2(l-1)(s-1)-l+3} \geq 2l$, or $c_{2(l-1)(s-1)-l+4} \geq 2l+1$. So $d \leq (2l-1)(s-1)-l+3 \leq 2l(s-1)-l+1$.

The conclusion is that

$$d \leq (k-1)(s-1) - \lfloor \frac{k-3}{2} \rfloor.$$

Remark 4. The above bound is tight. The Foster graph with intersection array $\{3, 2, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 2, 2, 2, 3\}$, has diameter 8, girth 10 and valency 3.

4. On distance-regular subgraphs of a cube

In this section subgraphs are not necessarily induced subgraphs. We show that the only distance-regular subgraphs of a hypercube with girth 6 are the doubled Odd graphs. Also we show that distance-regular subgraphs of a cube with girth 8 have valency at most 12. First we give some notation. From now on we say cube instead of hypercube. Let Γ be a subgraph of a cube. Let x be a vertex of Γ. Without loss of generality we represent x with the vector $(000\cdots0)$. A vertex y lies on level r with weight s if $d_\Gamma(x,y) = r$ and y is represented by a vector of weight s in the cube.

First we give two elementary lemmas.

LEMMA 7. *Let Γ be a uniformly geodetic subgraph of a cube with parameters $(c_i)_i$ and let y be a vertex on level i with weight i. Then for each $j, 0 \leq j \leq i$, we have*

$$\binom{i}{j} \geq \frac{c_i c_{i-1} \cdots c_{i-j+1}}{c_1 c_2 \cdots c_j}.$$

Proof. We calculate the number of vertices z on level j with $d(z,y) = i-j$. On the one hand we have that this number is

$$\frac{c_i c_{i-1} \cdots c_{i-j+1}}{c_1 c_2 \cdots c_j}.$$

On the other hand we can consider such vertices as j-subsets of a i-set. So we are done. ◻

LEMMA 8. *Let Γ be a uniformily geodetic subgraph of a cube with parameters $(c_i)_i$. Let y be a vertex on level i with weight i. If $c_{i-j} = c_i - e$, then*

$$\sum_{s=0}^{e} \binom{i-c_i}{j-s} \binom{c_i}{s} \geq \frac{c_i c_{i-1} \cdots c_{i-j+1}}{c_1 c_2 \cdots c_j}. \tag{1}$$

Proof. We calculate the number of vertices z on level j with $d(z, y) = i - j$. This number equals the right side of (1).

Suppose that precisely $c_i - s$ neighbours of y on level $i - 1$ (considered as $(i-1)$-sets) contains the vertex z (considered as a j-set). Then $0 \leq s \leq e$ and we find that the number of vertices z is at most the left hand side of (1). □

The proof of Weichsel [17], Theorem 5 shows

LEMMA 9. *Let Γ be a distance-regular subgraph of a cube with valency k and girth $2t \geq 6$.*

(i) If v is a vertex of Γ on level r and of weight r, then $2c_{r-1} + c_r \leq 2r - 1$.
(ii) If $k \geq r$, then $2c_{r-1} + c_r \leq 2r - 1$.

The next lemma is a modification of Theorem 5 of Weichsel [17].

LEMMA 10. *Let Γ be a distance-regular subgraph of a cube with valency k and girth $2t$. (i) If $t \geq 3$ and v is a vertex on level r and of weight r for an $r \geq 3$, then $c_{r-1} < \frac{2r-1}{3}$.*
(ii) If $t \geq 3$ and $k \geq r \geq 3$, then $c_{r-1} < \frac{2r-1}{3}$.

Proof. *(i)* & *(ii)*. Suppose $c_{r-1} = \frac{2r-1}{3}$. Then by Lemma 9, we have $c_r = c_{r-1}$. By Lemma 8 we get $c_r \leq \frac{r}{2}$ and thus $\frac{2r-1}{3} \leq \frac{r}{2}$. This implies $r \leq 2$. □

PROPOSITION 11. *The Pappus graph is not a subgraph of a cube.*

Proof. The Pappus graph has intersection array $\{3,2,2,1;1,1,2,3\}$ and is the unique graph with this intersection array. Let Γ be the Pappus graph. Let $v_i, i = 1,2,3$, be three vertices such that $d(v_i, v_j) = 4, i \neq j$. We have

$$\Gamma_2(v_1) \cap \Gamma_2(v_2) = \Gamma_2(v_1) \cap \Gamma_2(v_2) \cap \Gamma_2(v_3) = \{w_1, w_2, w_3, z_1, z_2, z_3\},$$

such that $d(w_i, w_j) = d(z_i, z_j) = 4$ for $i \neq j$ and $d(w_i, z_j) = 2$. Let v_1 have weight 0. If v_2 has weight 4 then $\{w_1, w_2, w_3, z_1, z_2, z_3\}$ are all 2-subsets of a 4-set, and so there must be an i and j such that $d(w_i, z_j) = 4$, contradiction.

Let now v_2 and v_3 be represented by $(1100\ldots0)$ and $(10100\ldots0)$. But then at least five of $\{w_1, w_2, w_3, z_1, z_2, z_3\}$ are represented by a word of weight 2 with an 1 on the first position and this is impossible. □

THEOREM 12. *Let Γ be a distance-regular graph with valency k and girth 6. If Γ is a subgraph of a cube then Γ is the doubled Odd graph with valency k.*

Proof. Ray-Chaudhuri and Sprague [14] have shown that a bipartite distance-regular graph with $c_2 = 1$, $c_3 = c_4 = 2$ is a doubled Odd graph. If $k \geq 5$, then by Lemma 10 we get $c_4 \leq 2$ and hence $c_3 = c_4 = 2$, and so Γ is a doubled Odd graph.

If $k = 4$, then by Lemma 9 we have $c_3 = 2$. Then we have one of the following possibilities.

(*i*) $c_4 = 2$,

(*ii*) $c_4 = 3, c_5 = 3, c_6 = 4$,

(*iii*) $c_4 = 3, c_5 = 4$,

(*iv*) $c_4 = 4$.

There are no bipartite distance-regular graphs with the parameters of cases (*ii*), (*iii*) and (*iv*).

if $k = 3$, then we have the following possibilities.

(*i*) $c_3 = c_4 = 2$,

(*ii*) $c_3 = 2, c_4 = 3$,

(*iii*) $c_3 = 3$.

The only possible graph in case (*ii*) is the Pappus graph and this graph is ruled out by Proposition 11. Case (*iii*) is not possible by [17], Theorem 7. □

LEMMA 13. *Let Γ be a distance-regular subgraph of a cube with girth 8. If $c_{11} \leq 5$, then there is no vertex on level 11 with weight 11.*

Proof. Suppose $c_{11} \leq 5$ and there is a vertex y on level 11 with weight 11. By Lemma 8 we have $c_9 \leq 4$. Also by this lemma we have $c_{10} \leq 4$ or $c_8 \leq 3$ and thus $c_8 \leq 3$. Then we have $c_6 = 2$ or $c_6 = 3$. If $c_6 = 2$, then $c_3 = 1, c_4 = c_6 = 2$ and $c_7 = 3$. If $c_6 = 3$, then $c_3 = 1, c_4 = c_5 = 2, c_6 = c_8 = 3$ and $c_9 = 4$. There are no bipartite distance-regular graphs with $k = 3, c_3 = 1, c_4 = c_6 = 2$ and $c_7 = 3$, or with $k = 4, c_3 = 1, c_4 = c_5 = 2, c_6 = c_8 = 3$ and $c_9 = 4$. So, by Theorem 2, there are in both cases there are no bipartite distance-regular graph with these c_i's. □

LEMMA 14. *Let Γ be a distance-regular subgraph of a cube with girth 8. Then there is no vertex on level 14 with weight 14.*

Proof. Suppose there is a vertex on level 14 with weight 14.

If $c_{10} \geq 6$ then by Lemma 8 we have $c_{12} \geq 7$, and thus $c_{10}c_{11}c_{12} > \binom{12}{3}$. This is a contradiction with Lemma 7. So $c_{10} \leq 5$. By Lemma 8 we get $c_8 \leq 4, c_6 \leq 3$ and $c_4 \leq 2$.
Suppose $c_{11} \geq 6$. Then $c_{13} \geq 7$ by Lemma 8. If $c_{14} = 7$, then by Lemma 8 we get a contradiction. If $c_{14} \geq 8$, then by Lemma 7 we also get a contradiction. So $c_{11} \leq 5$ and by previous lemma we are done. □

THEOREM 15. *Let Γ be a distance-regular subgraph of a cube with girth 8. Then its valency k is at most 12.*

Proof. If $k \geq 14$, then by [17], Lemma E, we have a vertex on level 14 with weight 14 and so by Lemma 14 we are done.

If $k = 13$, then we have a vertex, say y, on level 13 with weight 13. Suppose there is no vertex z on level 14 with weight 14. We calculate now the number of vertices u on level 10 with $d(u,y) = 3$. On one hand there must be a vertex v on level 12 with $d(v,y) = 1$ and $d(v,u) = 2$. So the number is at most $\binom{13}{3} - \binom{13-c_{13}}{3}$ On the other hand we have that this number is equal to $c_{13}c_{12}c_{11}$. Suppose that $c_{11} \geq 6$. Then $c_{12} \geq 6$ and $c_{13} \geq 7$. We have $\binom{13}{3} < 8.6.6$ and thus $c_{13} = 7$. This is also impossible. So $c_{11} \leq 5$ and we are done by Lemma 13. □

Remark 5. There are more examples of distance-regular subgraphs in distance-regular graphs. Some interesting examples are the Peterson graph in $J(6,3)$, the Shrikhande graph and the 4×4-grid in the halved 6-cube and the point-block incidence graph of the Fano plane in $J(7,3)$. The last example is not an isometric subgraph of $J(7,3)$.

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