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Matroid Shellability, β-Systems, and Affine Hyperplane Arrangements

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Abstract. The *broken-circuit complex* is fundamental to the shellability and homology of matroids, geometric lattices, and linear hyperplane arrangements. This paper introduces and studies the *β-system* of a matroid, $\beta\mathbf{nbc}(M)$, whose cardinality is Crapo's β-invariant. In studying the shellability and homology of base-pointed matroids, geometric semilattices, and affine hyperplane arrangements, it is found that the β-system acts as the affine counterpart to the broken-circuit complex. In particular, it is shown that the β-system indexes the homology facets for the lexicographic shelling of the reduced broken-circuit complex $\overline{\text{BC}}(M)$, and the basic cycles are explicitly constructed. Similarly, an EL-shelling for the geometric semilattice associated with M is produced, and it is shown that the β-system labels its decreasing chains. Basic cycles can be carried over from $\overline{\text{BC}}(M)$. The intersection poset of any (real or complex) affine hyperplane arrangement $\mathcal{A}$ is a geometric semilattice. Thus the construction yields a set of basic cycles, indexed by $\beta\mathbf{nbc}(M)$, for the union $\bigcup\mathcal{A}$ of such an arrangement.

Keywords: matroid, β-invariant, broken-circuit complex, shellability, affine hyperplane arrangement

0. Introduction

If $\mathcal{A}$ is a finite arrangement of *linear* hyperplanes in $\mathbb{R}^d$ or $\mathbb{C}^d$, then the basic combinatorial structure is the geometric lattice L of intersections, corresponding to a *matroid* M. In this situation the *broken-circuit complex* $\text{BC}(M)$ indexes bases for the homology and the homotopy type of the link, i.e., the intersection of $\bigcup\mathcal{A}$ with the unit sphere in $\mathbb{R}^d$, respectively $\mathbb{C}^d$; see Björner and Ziegler [9].

If $\mathcal{A}$ is an *affine* arrangement of hyperplanes in $\mathbb{R}^d$ or $\mathbb{C}^d$, then the intersection poset is a geometric semilattice L^o. Such lattices were studied by Wachs and Walker [17], who also showed that L^o uniquely determines the intersection lattice L of the linearization of $\mathcal{A}$ and thus the *affine matroid*; that is, L^o determines the pair (M, g), where g is the distinguished element corresponding to the hyperplane at infinity. Here L^o is the poset of all flats of M that do not contain g.

The purpose of this paper is to introduce and study the *β-system* $\beta\mathbf{nbc}(M)$, which is the affine counterpart to the broken-circuit complex. In particular, we show that $\beta\mathbf{nbc}(M)$ is the natural indexing set for the homology of the reduced broken-circuit complex $\overline{\text{BC}}(M)$ of the geometric semilattice L^o and thus of the affine arrangement $\bigcup\mathcal{A}$. (The existence of such indexing systems was previously established by Dayton and Weibel [13] for sufficiently generic arrangements; see Section 4.)

The key technical steps in our work are the construction of the basic spherical cycles in the reduced broken-circuit complex (Theorem 1.7) and of an explicit EL-shelling for the geometric semilattice (Theorem 2.2).

This paper is in many aspects a continuation (and affine counterpart) of Björner's work [3]. Therefore it contains only very brief sketches of the basic facts about broken-circuit complexes and shellability, which can all be found in [3]. For background material on shellability see also [2], [7], and the references therein; for broken circuit complexes see also [8] and [11]; for the geometry of affine arrangements see [18].

For history we refer to [3, Section 7.11]. The broken-circuit construction was pioneered by Whitney and Rota, and the broken-circuit complex was introduced by Wilf and was further studied by Brylawski; see [3] for references. The shellability of the broken-circuit complex was first proved by Billera and Provan and in the lexicographic version by Björner, who also identified the close connection between lexicographic shellability and basis activities. The β-invariant was introduced by Crapo. The relevance of geometric lattices and semilattices and the β-invariant to the study of arrangements was discovered by Zaslavsky. Finally, the theory of geometric semilattices and their shellability is due to Wachs and Walker.

1. Broken-Circuit Complexes and β-Systems

Let M be a (finite) matroid. The construction of $\mathrm{BC}(M)$ and $\beta\mathbf{nbc}(M)$ relies on a linear ordering on the ground set E. In the following the elements of E are identified with the natural numbers in $[n] := \{1, 2, \ldots, n\}$, which specifies a linear order "$<$" on $E = [n]$. As explained in [3] the broken-circuit construction depends on this linear ordering but its main properties do not. It turns out that for the affine situation the "correct" choice is to assume that $g = 1$, i.e., the first element of the matroid corresponds to the hyperplane at infinity.

The key notion is that of a *broken circuit*: a set of the form $C\backslash\min(C)$ obtained by deleting the smallest element of a circuit. The *broken-circuit complex* $\mathrm{BC}(M)$ is the simplicial complex of all subsets of $[n]$ that do *not* contain a broken circuit.

It is easy to see that $\mathrm{BC}(M)$ is a pure, $(r-1)$-dimensional simplicial complex by using the fact that the lexicographically first basis of any flat cannot contain a broken circuit. The facets (maximal faces) of $\mathrm{BC}(M)$ are bases of M; we will refer to them as the set $\mathbf{nbc}(M)$ of *no-broken-circuit bases*, or **nbc**-*bases*, of M.

For any basis B of M and $b \in B$, let $c^*(B, b)$ denote the *basic cocircuit*: the complement of the hyperplane spanned by $B\backslash b$. Similarly, for $p \notin B$ let $c(B, p)$ denote the *basic circuit:* the unique circuit in $B \cup p$. Clearly, $b \notin c^*(B, b)$ and $p \in c(B, p)$.

LEMMA 1.1. [3, Lemma 7.3.1]. *If B is a basis, $b \in B$, $p \notin B$, then*

$$b \in c(B, p) \iff (B \backslash b) \cup p \text{ is a basis} \iff p \in c^*(B, b).$$

In the following B will always denote a basis. An element $b \in B$ is *internally active* if it is the smallest element of $c^*(B, b)$. The set of internally active elements with respect to B is denoted by $\mathrm{IA}(B)$. Similarly, $p \notin B$ is *externally active* if it is the smallest element of $c(B, p)$. The set of externally active elements with respect to B is denoted by $\mathrm{EA}(B)$.

Note that $B \in \mathbf{nbc}(M)$ holds if and only if $\mathrm{EA}(B) = \emptyset$, by definition. Also, 1 is always active, either internally (if $1 \in B$) or externally. Thus $1 \in \mathrm{EA}(B) \cup \mathrm{IA}(B)$ for all bases B. In particular, every facet of $\mathrm{BC}(M)$ contains 1, so the broken circuit complex is a cone with apex 1 over the *reduced broken-circuit complex* $\overline{\mathrm{BC}}(M) := \{A \backslash 1 : A \in \mathrm{BC}(M)\} = \{A' \subseteq [n] \backslash 1 : A' \cup 1 \in \mathrm{BC}(M)\}$. This $\overline{\mathrm{BC}}(M)$ is a pure $(r-2)$-dimensional complex.

The following lemma is the key to the shellability of broken-circuit complexes. (Our formulation is a slight improvement on [3, Lemma 7.3.2].)

LEMMA 1.2. *If B is a basis and $b \in B$, then $B' := (B \backslash b) \cup b'$ is a basis as well, for $b' := \min c^*(B, b)$. If B is an* **nbc**-*basis, then so is B'.*

Proof. The case $b = b'$ is trivial. For $b' \neq b$ the first claim follows from Lemma 1.1. Assume that B' is not an **nbc**-basis; then there is an element $a \notin B'$ with $a = \min c(B', a)$. If B is an **nbc**-basis, then we cannot have $c(B', a) \subseteq B \cup a$, so we know $b' \in c(B', a)$. But this implies $a < b'$ by definition of a, and it implies $a \in c^*(B', b') = c^*(B, b)$ by Lemma 1.1, thus $a \geq b'$. □

Let Δ be a pure simplicial complex of dimension d, that is, such that all maximal faces have dimension d. We will make the usual identification of a simplex in Δ with its set of vertices, so a *face* of the simplex corresponds to a subset of its vertex set. A *facet* is a maximal face.

A *shelling* of Δ is a linear ordering of the set $\mathcal{F}$ of facets in such a way that the intersection of any facet with the previous ones is a nonempty union of $(d-1)$-dimensional faces. In other words, a shelling is a linear ordering of the facets $F_1, F_2, \ldots, F_N$ such that for all $i > 1$ the intersection $F_i \cap \left(\bigcup_{j<i} F_j\right)$ is a nonempty union of facets of the boundary ∂F_i.

In such a shelling F_i is a *homology facet* if the intersection with the previous facets is the whole boundary, i.e., if $F_i \cap \left(\bigcup_{j<i} F_j\right) = \partial F_i$. Write $\mathcal{F}_1 = \{F_{i_1}, \ldots, F_{i_K}\}$ for the set of homology facets, and write $\mathcal{F}_0 = \mathcal{F} \backslash \mathcal{F}_1$ for the nonhomology facets. It is now easy to see that the restriction of the linear order to $\mathcal{F}_0$ is a shelling order of $\bigcup \mathcal{F}_0$ and that $\Delta_0 := \bigcup \mathcal{F}_0$ is a contractible subcomplex of Δ [3, Lemma 7.7.1]. Since contraction of a contractible subcomplex

is a homotopy equivalence (contractible subcomplex lemma, [4, (10.2)]), every shellable simplicial complex has the homotopy type of a wedge of spheres, where the spheres are in bijection with the homology facets: $\Delta \simeq \Delta/\Delta_0 \cong \bigvee_K S^d$. Furthermore, there is a canonical set of basic cycles $\sigma_j (1 \le j \le K)$ for the reduced homology group $\tilde{H}^d(\Delta; \mathbb{Z}) \cong \mathbb{Z}^K$, which is uniquely determined (up to sign) by the condition that the support of σ_j is contained in $F_{i_j} \cup \bigcup \mathcal{F}_0$, with a ± 1 coefficient on F_{i_j} [3, Thm. 7.7.2].

Definition 1.3. Let M be a matroid on the ground set $[n]$. The *β-system* of M is the collection of bases

$$\beta\mathbf{nbc}(M) := \{B : \mathrm{EA}(B) = \emptyset; \mathrm{IA}(B) = \{1\}\}.$$

THEOREM 1.4 (see [3]). *Let M be a matroid of rank r on the set $[n]$.*

(i) *The lexicographic ordering of the facets of $\mathrm{BC}(M)$ is a shelling order for the broken-circuit complex $\mathrm{BC}(M)$. The complex is a cone and hence contractible.*
(ii) *The lexicographic ordering of the facets of $\overline{\mathrm{BC}}(M)$ is a shelling order for the reduced broken-circuit complex $\overline{\mathrm{BC}}(M)$. The set of homology facets for this shelling is $\mathcal{F}_1 = \{B \backslash 1 : B \in \beta\mathbf{nbc}(M)\}$.*

Proof. It follows immediately from Lemma 1.2 that the lexicographic ordering induces shellings [3, Thm. 7.4.3]. Furthermore, $B\backslash 1$ is a homology facet for $\overline{\mathrm{BC}}(M)$ if and only if for every $b \in B\backslash 1$ there is an element b' such that $B' := (B\backslash b) \cup b'$ is an **nbc**-basis that is lexicographically smaller than B. But B' is lexicographically smaller if and only if $b' < b$. If b is not internally active, then b' can be found by Lemma 1.2, and if b is internally active, then b' cannot be found because it must lie in $c^*(B, b)$ by Lemma 1.1. (See Figure 1 for an explicit example.) □

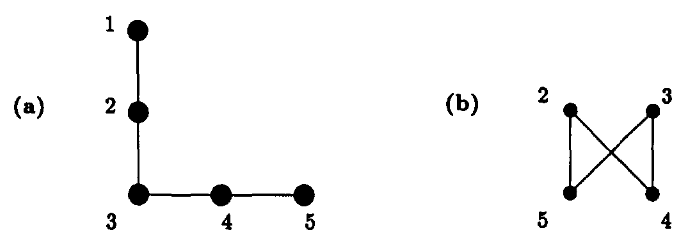

Figure 1. (a) Matroid $(M, 1)$ on 5 points, rank 3 of [3, Example 7.3.5] and (b) reduced broken-circuit complex $\overline{\mathrm{BC}}(M)$. We get $\beta\mathbf{nbc}(M) = \{135\}$. The basic cycle $\overline{\sigma}_{135}$ corresponding to $B = 135$ covers the whole complex.

There are various ways to see that the cardinality of $\beta\mathbf{nbc}(M)$ is Crapo's β-invariant $\beta(M)$ [12]. In fact, $\beta(M)$ is easily seen to be the coefficient t_{01} of the Tutte polynomial $t(M; x, y) = \Sigma t_{ij}x^iy^j$ [12, Thm. V]. A very elementary derivation uses the fact that $|\beta\mathbf{nbc}(M)|$ satisfies the same recursion as $\beta(M)$, namely, $\beta(M) = \beta(M\backslash n) + \beta(M/n)$ if n is neither a loop nor a coloop. Such a recursion for the β-system is given by Theorem 1.5 below. In this connection, recall that there is a similar recursion [11, Prop. 3.2]

$$\mathrm{BC}(M) = \mathrm{BC}(M\backslash n) \uplus \mathrm{BC}(M/n) * n,$$

where $\mathrm{BC}(M/n) * n = \{A \cup n : A \in \mathrm{BC}(M/n)\}$: this recursion holds unless n is a loop of M, in which case $\mathrm{BC}(M) = \emptyset$. It is a basic tool for the homology computations of [9].

THEOREM 1.5. *Let M be a matroid on $[n]$. Then*

$$\beta\mathbf{nbc}(M) = \begin{cases} \beta\mathbf{nbc}(M\backslash n) \uplus \beta\mathbf{nbc}(M/n) * n & \\ & \textit{if } n>1 \textit{ is neither a loop nor a coloop}, \\ \emptyset & \textit{if } n \textit{ is a loop, or if } n>1 \textit{ is a coloop}, \\ \{\{1\}\} & \textit{if } n = 1 \textit{ is a coloop}. \end{cases}$$

Proof. We may assume $n > 1$. If $B \subseteq [n]$ does not contain n, then it is immediate from the definitions that $B \in \beta\mathbf{nbc}(M) \Longleftrightarrow B \in \beta\mathbf{nbc}(M\backslash n)$. If $B \subseteq [n]$ and $n \in B$, then again it is immediate that $B \in \beta\mathbf{nbc}(M) \Rightarrow B\backslash n \in \beta\mathbf{nbc}(M/n)$. For the converse, note that if n is the smallest element in $c^*(B, n)$, then n is a coloop. □

Another basic property of the β-invariant is that it is invariant under duality: $\beta(M) = \beta(M^*)$ if $n > 1$ [12, Thm. IV]. The following gives a "bijective proof" for this by describing a bijection $\beta\mathbf{nbc}(M) \longleftrightarrow \beta\mathbf{nbc}(M^*)$. It was discovered by Biggs [1, Prop. 14.2] for graphs. The straightforward generalization to matroids was first given by Björner in the preprint version of [3].

THEOREM 1.6 (Biggs, Björner). *Let M be a matroid on $[n]$, with $n \geq 2$. Then*

$$\beta\mathbf{nbc}(M^*) = \{[n]\backslash\breve{B} : \breve{B} := (B\backslash 1) \cup 2,\ B \in \beta\mathbf{nbc}(M)\}.$$

Proof. Let $B \in \beta\mathbf{nbc}(M)$. Then clearly $1 \in B$ and $2 \notin B$. From $2 \notin \mathrm{EA}(B)$ we get $1 \in c(B, 2)$, and by Lemma 1.1 $\breve{B} = (B\backslash 1) \cup 2$ is a basis. Given that $\mathrm{EA}_M(B) = \mathrm{IA}_{M^*}([n]\backslash B)$ and $\mathrm{IA}_M(B) = \mathrm{EA}_{M^*}([n]\backslash B)$, it suffices to show that $\mathrm{EA}(\breve{B}) = \{1\}$ and $\mathrm{IA}(\breve{B}) = \emptyset$.

Assume $a \in \mathrm{EA}(\check{B})$; then $a = \min A$ for $A := c(\check{B}, a)$. Now $\mathrm{EA}(B) = \emptyset$, so $2 \in A$, and we conclude that $a = 1$. Hence $\mathrm{EA}(\check{B}) = \{1\}$.

Now assume that $a \in \mathrm{IA}(\check{B})$. If $a = 2$, then $1 \notin c^*(\check{B}, a)$, and thus $1 \in \overline{B\backslash 1}$, so B is not a basis. Thus we assume $a > 2$. From $1 \notin c^*(\check{B}, a)$ we get that $a \notin c(\check{B}, 1) =: A$. We get $2 \in A$, since otherwise B would contain the circuit A. Thus A contains 1 and 2 but misses a. We conclude that $c^*(\check{B}, a) = c^*(B, a)$ and thus that $a \in \mathrm{IA}(B)$. □

Consider a homology facet $B \in \beta\mathbf{nbc}(M)$, and define the map $\varphi : B \longrightarrow [n]$, $b \longmapsto \min c^*(B, b)$. We write the image of this set as $\varphi(B) = \{p_1, \dots, p_k\}_<$ in increasing order, where $B \cap \varphi(B) = \mathrm{IA}(B) = \{1\}$ and thus $\varphi(1) = 1 = p_1$, while $\varphi(b) > 1$ for $b \neq 1$.

Now set $A_i := \{p_i\} \cup \varphi^{-1}(p_i)$ for $1 \leq i \leq k$, where $A_1 = \{1\}$. The sets A_i form a partition of $B \cup \varphi(B)$. With this we associate to $B \in \beta\mathbf{nbc}(M)$ the simplicial complex

$$\overline{\Sigma}_B := \{F \subseteq B \cup \varphi(B) : A_i \nsubseteq F \text{ for } 1 \leq i \leq k\}.$$

The following explicit construction of the basic cycles in $\overline{\mathrm{BC}}(M)$ is a counterpart to Björner's treatment of the independence complex given in [3, Thms. 7.8.3 and 7.8.4].

THEOREM 1.7. *Let M be a matroid on $[n]$ of rank r, and let $\overline{\Sigma}_B$ be the simplicial complex associated to some $B \in \beta\mathbf{nbc}(M)$.*

(i) $B\backslash 1 \in \overline{\Sigma}_B \subseteq \overline{\mathrm{BC}}(M)$: *the complex $\overline{\Sigma}_B$ is an $(r-2)$-dimensional subcomplex of the reduced broken-circuit complex of M.*
(ii) $\overline{\Sigma}_B \cong S^{r-2}$: *the complex $\overline{\Sigma}_B$ is homeomorphic to the $(r-2)$-dimensional sphere.*
(iii) *The simplicial cycles $\overline{\sigma}_B$ associated with the spheres $\overline{\Sigma}_B$, for $B \in \beta\mathbf{nbc}(M)$, form a basis for the integral homology group $\tilde{H}_{r-2}(\overline{\mathrm{BC}}(M); \mathbb{Z})$.*

Proof. For part (ii), consider $D(A_i) := \{F : F \subset A_i\}$. This is the boundary of a simplex of dimension $|A_i| - 1$ and is thus homeomorphic to the $(|A_i| - 2)$-sphere. But $\overline{\Sigma}_B$ is the join of these spheres, so

$$\begin{aligned}\overline{\Sigma}_B = D(A_1) * \cdots * D(A_k) &\cong S^{|A_1|-2} * \cdots * S^{|A_k|-2} \\ &\cong S^{\Sigma_i(|A_i|-1)-1} = S^{|B|-2}.\end{aligned}$$

The proof for (i) relies on the following technical fact:

$(*)$ If $1 < i < j \leq k$ and $b_j \in A_j\backslash p_j$, then $A_i \cap c^*(B, b_j) = \emptyset$.

To see $(*)$, note that $c^*(B, b_j) \cap B = \{b_j\}$, while $A_i \subseteq B \cup p_i$, so the intersection is contained in $\{b_j, p_i\}$. However, $i \neq j$ implies $b_j \notin A_i$, while $i < j$ implies $p_i < p_j = \min c^*(B, b_j)$ and thus $p_i \notin c^*(B, b_j)$.

We will now prove by induction on $|F \cap \varphi(B)|$ that for every facet F of the sphere $\overline{\Sigma}_B$ the set $F \cup 1$ is an **nbc**-basis. This is by assumption true if $|F \cap \varphi(B)| = 0$, that is, if $F \cup 1 = B$. Now assume $|F \cap \varphi(B)| > 0$, and let $p_j = \max F \cap \varphi(B)$. Then there is a unique $b_j \in A_j \backslash F$. We set $F' := (F\backslash p_j) \cup b_j$. Then F' is a facet of $\overline{\Sigma}_B$ which by induction satisfies $F' \cup 1 \in \mathbf{nbc}(M)$.

It follows from our choice of p_j that the symmetric difference $F' \Delta B$ is contained in $\bigcup_{i<j} A_i$, which means $(F'\Delta B) \cap c^*(B, b_j) \subseteq \bigcup_{i<j} A_i \cap c^*(B, b_j) = \emptyset$ by $(*)$, and thus $c^*(B, b_j) = c^*(F', b_j)$. Therefore $p_j = \min c^*(B, b_j) = \min c^*(F', b_j)$, which yields $F \cup 1 \in \mathbf{nbc}(M)$ by Lemma 1.2.

Furthermore, this shows that $p_j \in \mathtt{IA}(F)$. Thus F is not a homology facet for the lexicographic shelling of $\overline{\mathrm{BC}}(M)$ when $F \neq B\backslash 1$, and $B\backslash 1$ is the only homology facet covered by the sphere $\overline{\Sigma}$, which proves (iii).

In general, the reduced broken-circuit complex is not the union of the spheres $\overline{\Sigma}_B$, in contrast to the situation for the independence complex [3, Cor. 7.8.5]: this can be seen, e.g., in [3, Example 7.4.4(b)].

Analogously to [3, (7.42)], we can also write explicit expressions for the cycles $\overline{\sigma}_B$:

$$\overline{\sigma}_B = \sum_{i_2=0}^{e_2} \cdots \sum_{i_k=0}^{e_k} [a_0^2, \ldots, \widehat{a_{i_2}^2}, \ldots, a_{e_2}^2, \ldots\ldots, a_0^k, \ldots, \widehat{a_{i_k}^k}, \ldots, a_{e_k}^k],$$

where $A_j = \{a_0^j, \ldots, a_{e_j}^j\}_<$ for $2 \leq j \leq k$, and thus $a_0^j = p_j$.

Definition 1.8. A *homotopy basis* for a space T is a map from a wedge of spheres into T that induces a homotopy equivalence.

In this sense the spheres $\overline{\Sigma}_B$ in fact form a homotopy basis for $T := \overline{\mathrm{BC}}(M)$: there is an obvious way to map the wedge of spheres $\bigvee_{B \in \beta\mathbf{nbc}(M)} \overline{\Sigma}_B$ into Δ (since vertex 2 lies in each of the spheres $\overline{\Sigma}_B$), and this map is a homotopy equivalence (again by the contractible subcomplex lemma).

2. β-Systems and Geometric Semilattices

In the following we consider the geometric lattice of flats L associated with M. We use $\hat{0}, \hat{1}$ to denote the minimal and maximal elements of L, respectively. With the additional assumption that M is simple (without loops or parallel elements) we get that the atoms (elements covering $\hat{0}$) are in bijection to the ground set $[n]$. For any flat $y \in L$ we denote by $\Box y \subseteq [n]$ the set of elements of y, with $\Box\hat{0} = \{\text{loops of } M\} = \emptyset$ and $\Box\hat{1} = [n]$.

The affine analogue of the geometric lattice is the poset

$(*)\quad L^o := L\backslash L_{\geq\{1\}} = \{y \in L : 1 \notin \Box y\},$

which is a *geometric semilattice* in the sense of Wachs and Walker. We refer to [17] for a comprehensive study of such posets. By [17, Thm. 3.2] we can use (*) as a definition for geometric semilattices.

An important observation from [17] is that L^o, in fact, determines L uniquely. The poset $L^o \cup \hat{1}$ is a graded lattice of length r. A further result is that the poset L^o is shellable (that is, its order complex $\Delta(L^o)$ is a shellable simplicial complex). This was shown in [17] by proving that $L^o \cup \hat{1}$ has a recursive atom ordering in the sense of [7]; that is, it is *CL-shellable.*

In the following we want to strengthen this result. $L^o \cup \hat{1}$ is in fact *EL-shellable*: the cover relations of $L^o \cup \hat{1}$ can be labeled in such a way that (if we always read labels from bottom to top) in every interval $[x, y] \subseteq L^o \cup \hat{1}$ the lexicographically smallest maximal chain is the unique increasing one. This condition ensures (see [3, Section 7.6], [2]) that the lexicographic order on the maximal chains yields a shelling for the order complex $\Delta(L^o \backslash \hat{0})$ and furthermore that the homology facets for that shelling correspond exactly to the maximal chains with decreasing labels. Note for this that the topologically interesting part of L^o is the *proper part* $\overline{L^o} := L^o \backslash \hat{0}$, whereas $\Delta(L^o)$ and $\Delta(L^o \cup \hat{1})$ are cones and are thus contractible.

The EL-shelling we describe amounts to a special choice in the class of CL-shellings described by Wachs and Walker. In Theorem 2.4 we will see that the decreasing chains of $\Delta(\overline{L^o})$ are labeled by the β**nbc**-bases. This shows in particular that the complexes $\overline{\mathrm{BC}}(M)$ and $\Delta(\overline{L^o})$ are homotopy equivalent; we also use Theorem 2.4 to show that the natural map $\mu : \mathrm{sd}(\overline{\mathrm{BC}}(M)) \longrightarrow \Delta(\overline{L^o})$ induces a homotopy equivalence between the reduced broken-circuit complex $\overline{\mathrm{BC}}(M)$ and the proper part $\overline{L^o} = L^o \backslash \hat{0}$. In particular, μ transports the basic cycles $\mathrm{sd}_{\#}\overline{\sigma}_B$ from $\mathrm{sd}(\overline{\mathrm{BC}}(M))$ to $\Delta(\overline{L^o})$.

The reader may want to compare our approach with that of [8, Thm. 3.12], in which a map in the opposite direction $\pi : \overline{L^o} \longrightarrow \overline{\mathrm{BC}}(M)$ is shown to be a homotopy equivalence with different tools.

Let's get going. For $x \in L^o \cup \hat{1}$ let $A(x)$ be the lexicographically first basis of $M/(1 \cup \Box x)$. Inductively, this can be described as $A(x) = a_1 \cup A(\overline{\Box x \cup a_1})$, where a_1 is the smallest nonloop of $M/(1 \cup \Box x)$.

Now, to label the edges of $L^o \cup \hat{1}$ we define the following for every cover relation $x \lessdot y$ in $L^o \cup \hat{1}$:

$$\overline{\lambda}(x, y) = (\chi(x, y), \lambda(x, y)) := \begin{cases} (0, \min(\Box y \cap A(x))) & \text{if } \Box y \cap A(x) \neq \emptyset, \\ (1, \min(\Box y \backslash \Box x)) & \text{otherwise.} \end{cases}$$

This defines an edge labeling on the poset $L^o \cup \hat{1}$; see Figure 2(b) for a small example. For all coatoms $x \in L^o \cup \hat{1}$ we have $A(x) = A(\hat{1}) = \emptyset$, and thus $(\chi(x, \hat{1}), \lambda(x, \hat{1})) = (1, 1)$. We order the labels lexicographically, with $(\chi, \lambda) < (\chi', \lambda')$ if and only if either $0 = \chi < \chi' = 1$ or $\chi = \chi'$ and $\lambda < \lambda'$.

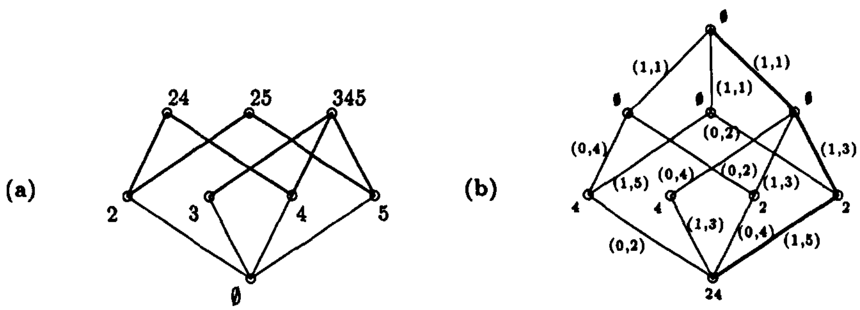

Figure 2. Geometric semilattice L^o corresponding to the matroid $(M, 1)$ of Figure 1. (a) Note that if the barycentric subdivision of the cycle of Figure 1(b) is mapped to L^o, then it covers the whole proper part; at the same time part of the cycle (at the vertex 3) is collapsed. (b) Description of the EL-labeling of the edges of $L^o \cup \hat{1}$. The elements $x \in L^o \cup \hat{1}$ are indexed by $A(x)$. The decreasing chain—corresponding to a homology facet—is drawn with thicker lines.

LEMMA 2.1. *Let* $x \lessdot y < \hat{1}$ in L^o, and let $\lambda := \lambda(x, y)$. Then we have $A(x) = A(y) \uplus \{\lambda'\}$, with $\lambda' = \lambda$ if $\chi(x, y) = 0$ and $\lambda' < \lambda$ if $\chi(x, y) = 1$.

Proof. Let $M' := M/(\overline{1 \cup \Box x})$; then $A(x)$ is the lexicographically smallest basis of M', and $A(y)$ is the lexicographically smallest basis of M'/λ. Constructing $A(y)$ greedily, we see that it coincides with $A(x) = \{b_1, \ldots, b_k\}_<$ in all entries except for the point where for the first time $\lambda \in \overline{\{b_1, \ldots, b_i\}}$, at which point b_i is not taken into $A(y)$, and $b_i \leq \lambda$. Here $A(y) = A(x) \backslash b_i$ for $b_i = \lambda' \leq \lambda$, with equality if and only if $\chi = 0$. □

We are now ready to show the main result of this section.

THEOREM 2.2. *Let* $L^o = L_{\not\geq \{1\}}$ *be a geometric semilattice. Then the labeling* $(x \lessdot y) \mapsto (\chi(x, y), \lambda(x, y))$ *defines an* EL*-shelling of* $L^o \cup \hat{1}$; *that is, for every* $x < y$ *the maximal chain*

$$\mathbf{c}_{x,y} : x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y$$

with the lexicographically smallest sequence of labels is the unique maximal chain in $[x, y]$ *that has increasing labels.*

Proof. By induction on $r(y) - r(x)$ we will show that the lexicographically first chain from x to y has labels $(0, a_1), \ldots, (0, a_i), (1, a_{i+1}), \ldots, (1, a_k)$, where $\{a_1, \ldots, a_i\} = (\Box y \cap A(x))_<$, and that $\{a_{i+1}, \ldots, a_k\}$ is the lexicographically first basis of $M(\Box y)/\Box x_i$, in increasing order. In particular, the lexicographically first chain is increasing.

For this consider $x \lessdot z \leq y$. First assume that $\Box y \cap A(x) \neq \emptyset$. By Lemma 2.1

we get that $\Box y \cap A(x) \supseteq \Box y \cap A(z)$ and thus $\min(\Box y \cap A(x)) \leq \min(\Box y \cap A(z))$, with "$<$" if $z = x_1$. Similarly, if $\Box y \cap A(x) = \emptyset$, then we get $\Box y \backslash \Box x \supseteq \Box y \backslash \Box z$ and thus $\min(\Box y \backslash \Box x) \leq \min(\Box y \backslash \Box z)$, with "$<$" if $z = x_1$.

Now assume that there is a different chain $\mathbf{c}'_{x,y} : x = x'_0 \lessdot x'_1 \lessdot \cdots \lessdot x'_k = y$ with increasing labels. By induction on length we may assume $x_1 \neq x'_1$. We write $\overline{\lambda}_i = (\chi_i, \lambda_i) := \overline{\lambda}(x_{i-1}, x_i)$ and, similarly, $\overline{\lambda}'_i = (\chi'_i, \lambda'_i) := \overline{\lambda}(x'_{i-1}, x'_i)$. By construction we know that $(\chi_1, \lambda_1) < (\chi'_1, \lambda'_1)$.

Case 1: If $\chi_1 = \chi'_1 = 1$, then we get a contradiction from the linear case, as in [2] and [3, Lemma 7.6.2]: we know $\lambda_1 = \min(\Box y \backslash \Box x)$, and from monotonicity we get $\chi_i = \chi'_i = 1$ for all $i \geq 1$. This implies $\lambda_1 = \lambda'_i$ for some $i > 1$, and thus $\overline{\lambda}'_i = (1, \lambda'_i) = (1, \lambda_1) < (1, \lambda'_1) = \overline{\lambda}'_1$, so $\mathbf{c}'_{x,y}$ is not increasing.

Case 2: If $\chi_1 = 0$ and $\chi'_1 = 1$, then we get from the definitions $\Box y \cap A(x) \neq \emptyset$ and $\lambda_1 = \min(\Box y \cap A(x))$. Since $\mathbf{c}'_{x,y}$ is increasing, we have $\chi'_i = 1$ for all i. Thus the labels of $\mathbf{c}'_{x,y}$ are given by $\lambda'_i = \min(\Box x'_{i-1} \backslash \Box x'_i)$. From the linear case (as above) we know that such a chain can only be increasing if $\lambda'_1 = \min(\Box y \backslash \Box x)$, which implies $\lambda'_1 < \lambda_1$.

From Lemma 2.1 we conclude that $A(x'_1) = A(x) \backslash \lambda'$ for some $\lambda' < \lambda'_1 < \lambda_1$. Hence $\lambda_1 \in \Box y \cap A(x'_1)$, and by induction on length we know that the only increasing chain from x'_1 to y has the first label $(0, \lambda_1)$. But we know $\chi'_i = 1$ for all i, a contradiction.

Case 3: If $\chi_1 = \chi'_1 = 0$, then consider the smallest $i \geq 1$ with $\chi'_{i+1} = \chi(x'_i, x'_{i+1}) = 1$. Since $\lambda_1 < \lambda'_1$ with $\lambda_1, \lambda'_1 \in A(x)$, we see from Lemma 2.1 that all elements of $A(x'_i) \backslash A(x'_1)$ are greater than λ'_1. Thus we have $\lambda_1 \in A(x'_i)$, and we get by case 2 that the labels on the chain $x'_i \lessdot \cdots \lessdot x'_k = y$ are not increasing. $\square$

Given any EL-shelling of a bounded poset P, one also has a shelling of the order complex $\Delta(\overline{P})$ of the proper part $\overline{P} = P \backslash \{\hat{0}, \hat{1}\}$. Furthermore, the homology facets of this shelling of $\Delta(\overline{P})$ correspond to the chains of P with decreasing labels [3, Prop. 7.6.4]. To identify them for the above EL-labeling of $P = L^o \cup \hat{1}$ we need another lemma.

LEMMA 2.3. *Let $B = \{b_1, \ldots, b_{r-1}, 1\}_>$ be a basis of M, listed decreasingly. Then for $1 \leq i \leq r-1$*

$$b_i \notin \mathrm{IA}(B) \iff b_i \notin A(\overline{\{b_1, \ldots, b_{i-1}\}}).$$

Proof. For $i = 1$, $b_1 \notin \mathrm{IA}(B)$ implies the existence of $b'_1 := \min c^*(B, b_1) < b_1$ such that $B' := (B \backslash b_1) \cup b'_1$ is a basis. Now $b_1 > \max(B') \geq \max(A(\hat{0}))$ for the lexicographically first basis $A(\hat{0})$ implies that $b_1 \notin A(\hat{0})$.

Conversely, $A(\hat{0})$ contains an element from $c^*(B, b_1)$. Now from $b_1 \geq \max A(\hat{0})$ (which always holds) and $b_1 \notin A(\hat{0})$ we get that $c^*(B, b_1) \cap A(\hat{0})$ contains an element that is smaller than b_1; thus $b_1 \notin \mathrm{IA}(B)$.

For $i > 1$ consider $M' := M/\overline{\{b_1, \ldots, b_{i-1}\}}$ and its basis $B' := B\backslash\{b_1, \ldots, b_{i-1}\}$. Then we get $c^*_{M'}(B, b_i) = c^*_M(B', b_i)$ and $A_M(\overline{\{b_1, \ldots, b_{i-1}\}}) = A_{M'}(\hat{0})$ by definition, which reduces the situation to the basis B' of M' and thus to the case $i = 1$. $\square$

THEOREM 2.4. *The maximal chains of $L^o \cup \hat{1}$ with decreasing labels are exactly the chains*

$$\mathbf{c}_B : \hat{0} < \overline{\{b_1\}} < \overline{\{b_1, b_2\}} < \cdots < \overline{\{b_1, \ldots, b_{r-1}\}} < \hat{1}$$

*for the β**nbc**-bases $B = \{b_1, \ldots, b_{r-1}, 1\}_> \in \beta\mathbf{nbc}(M)$. Their labels are given by*

$$\overline{\lambda}(\mathbf{c}_B) : (1, b_1), (1, b_2), \ldots\ldots, (1, b_{r-1}), (1, 1).$$

Proof. Given B as stated, define $x_i := \overline{\{b_1, \ldots, b_i\}}$, which yields a maximal chain. We want to see that it has decreasing labels as claimed. From part "$\Longrightarrow$" of Lemma 2.3 we get $b_i \notin A(x_{i-1})$. Now consider any $b'_i \in \Box x_i \backslash \Box x_{i-1}$. If $b'_i < b_i$, then $A := c(B, b'_i) \subseteq \{b_1, \ldots, b_{i-1}, b_i, b'_i\}_>$ and $b'_i = \min(A)$, so B contains a broken circuit, contrary to assumption. If $b'_i > b_i$, then $B' := (B\backslash b_i) \cup b'_i$ is a basis with $\{b_i, b'_i\} \subseteq c^*(B', b'_i) = c^*(B, b_i)$ by Lemma 1.1; hence $b'_i \notin A(x_{i-1})$ by part "$\Longrightarrow$" of Lemma 2.3. Thus we have $\Box x_i \cap A(x_{i-1}) = \emptyset$, and B defines a chain with the correct labels.

For the converse, since the last edge of every chain has label $\overline{\lambda}(x_{r-1}, \hat{1}) = (1, 1)$, we get for every decreasing chain that $\chi(x_{i-1}, x_i) = 1$ for all i; that is, $\Box x_i \cap A(x_{i-1}) = \emptyset$ and $b_i := \lambda(x_{i-1}, x_i) = \min(\Box x_i \backslash \Box x_{i-1})$. This yields that $B = \{b_1, \ldots, b_{r-1}, 1\}$ is an **nbc**-basis in decreasing order. Setting $x_i := \overline{\{b_1, \ldots, b_i\}}$, we know $b_i \in \Box x_i$ and $\Box x_i \cap A(x_{i-1}) = \emptyset$; thus $b_i \notin A(x_{i-1})$, from which part "$\Longleftarrow$" of Lemma 2.3 yields $b_i \notin \mathtt{IA}(B)$ for $1 \leq i \leq r-1$. Thus $\mathtt{IA}(B) = \{1\}$ and $B \in \beta\mathbf{nbc}(M)$. $\square$

The final goal of this section is to construct a map $\mu : \mathrm{sd}(\overline{\mathtt{BC}}(M)) \longrightarrow \Delta(L^o\backslash\hat{0})$ and to show that it is a homotopy equivalence. (We know that the complexes are homotopy equivalent, but we need an explicit map in this direction in order to transport the cycles $\overline{\sigma}_B$ to $\Delta(L^o)$.) For this, we use the following lemma.

LEMMA 2.5. *Let $\tau : \Delta \longrightarrow \Delta'$ be a simplicial map of finite simplicial complexes of the same dimension. Assume that Δ and Δ' have shellings such that*

(i) *τ yields a bijection $\tau : \mathcal{F}_1 \longrightarrow \mathcal{F}'_1$ between the homology facets that maps every homology facet of Δ onto a homology facet of Δ', and*
(ii) *τ maps the nonhomology facets of Δ into those of Δ', i.e., $\tau(\bigcup \mathcal{F}_0) \subseteq \bigcup \mathcal{F}'_0$.*

Then τ is a homotopy equivalence.

Proof. We use that $\Delta_0 := \bigcup \mathcal{F}_0$ and $\Delta_0' := \bigcup \mathcal{F}_0'$ are contractible subcomplexes of Δ, respectively Δ'. Thus the vertical maps in the diagram

$$\begin{tikzcd} \Delta \arrow[r, "\tau"] \arrow[d, "\pi"] & \Delta' \arrow[d, "\pi'"] \\ \Delta/\Delta_0 \arrow[r, "\overline{\tau}"] & \Delta'/\Delta_0' \end{tikzcd}$$

are homotopy equivalences by the contractible subcomplex lemma [4, (10.2)]. Furthermore, $\overline{\tau}$ is a map between wedges of spheres: since $\tau : \biguplus \mathcal{F}_1 \longrightarrow \biguplus \mathcal{F}_1'$ is a homeomorphism, we see that $\overline{\tau}$ is a homeomorphism, and the diagram commutes. From this we get that $\tau \sim (\pi')^{-1} \circ \overline{\tau} \circ \pi$, using some homotopy inverse to π', which proves the claim. □

In the following let $P_{\overline{\mathrm{BC}}(M)}$ be the face poset of $\overline{\mathrm{BC}}$, that is, the set of all faces, ordered by inclusion. This includes a minimal element $\hat{0} \in P_{\overline{\mathrm{BC}}(M)}$, corresponding to $\emptyset \in \overline{\mathrm{BC}}(M)$. Note that the order complex $\Delta(P_{\overline{\mathrm{BC}}(M)} \backslash \hat{0})$ is the barycentric subdivision $\mathrm{sd}(\overline{\mathrm{BC}}(M))$.

THEOREM 2.6. *The matroid closure operator defines an order- and rank-preserving map*

$$P_{\overline{\mathrm{BC}}(M)} \longrightarrow L^o.$$

The induced simplicial map of order complexes

$$\mu : \mathrm{sd}(\overline{\mathrm{BC}}(M)) \longrightarrow \Delta(L^o \backslash \hat{0})$$

is a homotopy equivalence.

Proof. For any $A \in \overline{\mathrm{BC}}(M)$ we have that $A \uplus 1 \in \mathrm{BC}(M)$ is an independent set; thus $1 \notin \overline{A}$, and $A \in L^o$ has rank $|A|$. The closure map is clearly order preserving.

By Theorem 2.4 the maximal chain in $P_{\overline{\mathrm{BC}}(M)}$

$$\mathbf{d}_B : \emptyset \subset \{b_1\} \subset \{b_1, b_2\} \subset \cdots \subset \{b_1, \ldots, b_{r-1}\}$$

for $B = \{b_1, \ldots, b_{r-1}, 1\}_>$ is mapped to the maximal chain in L^o

$$\mathbf{c}_B : \hat{0} < \overline{\{b_1\}} < \overline{\{b_1, b_2\}} < \cdots < \overline{\{b_1, \ldots, b_{r-1}\}}.$$

Now we use the (simple) fact that if Δ is any shellable simplicial complex, then its barycentric subdivision $\mathrm{sd}(\Delta)$ is shellable as well. Moreover, the construction of [2, Thm. 5.1] shows that we can prescribe a homology facet of $\mathrm{sd}(\Delta)$ within every homology facet of Δ. Applied to $\Delta = \overline{\mathrm{BC}}(M)$, with the use of Theorem

1.4(ii), this means that $P_{\overline{\mathrm{BC}}(M)}$ has a shelling in which the homology facets are exactly given by $\{\mathbf{d}_B : B \in \beta\mathbf{nbc}(M)\}$.

Thus we can apply Lemma 2.5, once we have shown that for $B \in \beta\mathbf{nbc}(M)$ no other maximal chain

$$\mathbf{d}_{B'} : \emptyset \subset \{b'_1\} \subset \{b'_1, b'_2\} \subset \ldots \subset \{b'_1, \ldots, b'_{r-1}\}$$

is mapped to $\mathbf{c}_B$. Thus assume that $\mathbf{d}_{B'}$ exists, and set

$$x_i = \overline{\{b_1, \ldots, b_{r-1}\}} = \overline{\{b'_1, \ldots, b'_{r-1}\}}.$$

We have to show that $b'_i = b_i$ for all i.

For this we have $b_i, b'_i \in \Box x_i \backslash \Box x_{i-1}$, with $b_i = \min(\Box x_i \backslash \Box x_{i-1})$ from the proof of Theorem 2.4, and thus $b_i \leq b'_i$ for all i. Let i be minimal such that $b_i < b'_i$. Then we get $A := c(B', b_i) \subseteq \{b'_1, \ldots, b'_i, b_i\}$, with $b_i < b'_i$ and with $b_i < b_j = b'_j$ for $j < i$. Hence $b_i = \min A$, and B' contains a broken circuit. This contradiction shows that $\mathbf{d}_B = \mathbf{d}_{B'}$, and Lemma 2.5 finishes the proof. □

COROLLARY 2.7. *If $L^o = \{x \in L : 1 \notin x\}$ is the geometric semilattice associated to a base-pointed matroid (M, 1) of rank r, then the cycles*

$$\mu_\# \circ \mathrm{sd}_\#(\overline{\sigma}_B) : B \in \beta\mathbf{nbc}(M)$$

form a basis for $\tilde{H}_{r-2}(\Delta(L^o\backslash\hat{0}); \mathbb{Z})$. In fact, they are the basic cycles associated to the shelling of $\Delta(L^o\backslash\hat{0})$ given by Theorem 2.2.

We do not know whether the cycles $\mu_\# \circ \mathrm{sd}_\#(\overline{\sigma}_B)$ are spherical in general. It is not clear that they have ± 1 coefficients on all of their simplices.

3. Geometric Semilattices and Affine Hyperplane Arrangements

The following well-known proposition identifies the combinatorial structure of an affine hyperplane arrangement $\mathcal{A}$ over any field.

PROPOSITION 3.1. *Let $\mathcal{A}$ be an affine hyperplane arrangement over a field k. Then the set $L^o := \{\bigcap \mathcal{A}_0 \neq \emptyset : \mathcal{A}_0 \subseteq \mathcal{A}\}$ of nonempty intersections of subsets of $\mathcal{A}$, ordered by reverse inclusion (including a minimal element corresponding to the empty intersection), is a geometric semillatice.*

Proof. Consider a linearization or projectivization $\hat{\mathcal{A}}$ of $\mathcal{A}$ that includes the hyperplane H_∞ at infinity. The intersection lattice L of $\mathcal{A}$ is a geometric lattice in which a distinguished atom a_∞ corresponds to H_∞.

Under the canonical embedding $\mathcal{A} \hookrightarrow \hat{\mathcal{A}}$ the nonempty flats of $\mathcal{A}$ correspond exactly to those flats of $\hat{\mathcal{A}}$ that are not contained in H_∞; that is, $L^o = L\backslash[a_\infty, \hat{1}] = L_{\not\geq a_\infty}$. □

We are considering only the cases of real or complex arrangements here. To unify their treatment one might at this point be tempted to generalize to arbitrary affine *c-arrangements* in the sense of Goresky and MacPherson [14, p. 239]: arrangements of codimension c in some $\mathbb{R}^N$ such that every nonempty intersection has a codimension that is a multiple of c. However, affine 2-arrangements like the one given by the subspaces $V_1 = \{x = y = 0\}$, $V_2 = \{z = w = 0\}$, and $V_3 = \{x = z = 1\}$ in $\mathbb{R}^4$ show that this does not work in general: for this arrangement $L^o \cup \hat{1}$ is not even graded. (It does not help to require that $N = cd - 1$.)

For general c-arrangements it is easy to see that all intervals of the intersection semilattice L^o are geometric lattices [14, III.4.1]; thus L^o is a bouquet of geometric lattices in the sense of [15]. However, this is not sufficient to make L^o into a geometric semilattice (see [17, Thm. 2.1]), which would guarantee reasonable topological properties for $\Delta(L^o \backslash \hat{0})$.

The connection between the topology of an affine hyperplane arrangement and its geometric semilattice is a very special case of the following fact.

LEMMA 3.2. *Let $\mathcal{A}$ be a finite set of affine subspaces in a real vector space, and let P be a poset that is isomorphic to all the nonempty intersections of subspaces in $\mathcal{A}$, ordered by reverse inclusion. Let $\phi : P \longrightarrow \bigcup \mathcal{A}$ be an arbitrary map which to every $p \in P$ assigns a point on the corresponding subspace $V_p \in \mathcal{A}$. Then the map*

$$\Phi : \Delta(P) \longrightarrow \bigcup \mathcal{A}$$

obtained by linear extension of ϕ over the simplices of $\Delta(P)$ is a homotopy equivalence.

This was first proved by Goresky and MacPherson [14, III.2.3]. There are various simple alternative proofs. It is easily derived from contractible carrier or nerve lemmas [4, Section 10], as shown in [6, Prop. 4.1], and it follows as a special case of the diagram technique of Ziegler and Živaljević [20, Thm. 2.1]. Also, it is easy to explicitly construct a homotopy inverse along the lines of [14, III.2.5]; see [19]. For the special case of hyperplane arrangements one can also apply Quillen's fiber lemmas [4, (10.5)], as shown in [13, Thm. 3.12], or Whitehead's theorem, as shown in [9].

Combining Theorem 2.5 with Lemma 3.2, we get the following result, which amounts to an explicit construction of basic cycles for the homology of any affine hyperplane arrangement in terms of its affine matroid.

THEOREM 3.3. *Let $\mathcal{A}$ be a finite real or complex hyperplane arrangement. Let L^o be the associated geometric semilattice. Then the map Φ constructed in Lemma 3.2 induces a homotopy equivalence*

$$\Phi : \Delta(L^o \backslash \hat{0}) \longrightarrow \bigcup \mathcal{A}.$$

In particular, $\bigcup \mathcal{A}$ *is homotopy equivalent to a wedge of* $\beta(M)$ $(r-2)$*-spheres, where* M *is the matroid of rank* r *associated with* L^o*. Furthermore, the cycles*

$$\Phi_\# \circ \mu_\# \circ \mathrm{sd}_\#(\overline{\sigma}_B) : B \in \beta\mathbf{nbc}(M)$$

form a basis for $\tilde{H}_{r-2}(\bigcup \mathcal{A}; \mathbb{Z})$ *and a homotopy basis for* $\bigcup \mathcal{A}$ *in the sense of Definition* 1.8.

In Figure 3, the map Φ is illustrated for an arrangement whose semilattice is given by Figure 2(a).

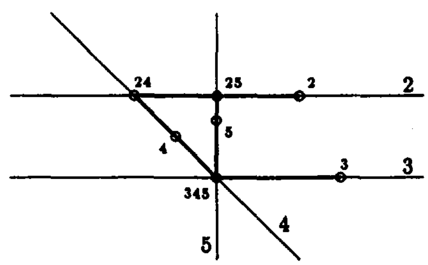

Figure 3. Arrangement with base-pointed matroid $(M, 1)$. Circles indicate possible points $\phi(x)$ for $x \in L^o \backslash \hat{0}$, and thicker lines indicate the corresponding image of $\Phi_\# \circ \mu_\#(\overline{\sigma}_{135})$.

4. Geometry of Affine Hyperplane Arrangements

In [13] Dayton and Weibel study affine hyperplane arrangements $\mathcal{A}$ in $\mathbf{k}^{n+1}$ without reference to matroid-theory language. We will translate their results in square brackets. Dayton and Weibel consider only arrangements $\mathcal{A}$ [with affine matroid (M, g)] that are "admissible" [that is, $\mathcal{A}$ is sufficiently generic, so that $g \in \overline{C}$ for every circuit C of M]. They introduce an invariant $g(\mathcal{A})$ [$= \beta(M)$] for every such arrangement and derive its basic properties. Then they define polysimplicial spheres [the arrangements given by the facets of a product of simplices] and show that every admissible arrangement has a basic set of such spheres [i.e., a set of spheres satisfying a recursion such as that of Theorem 1.5]. This main result [13, Prop. 2.7] can thus be seen as a special case of our Theorem 3.3.

Now specialize to $\mathbf{k} = \mathbb{R}$. To get a set of basic cycles for $\bigcup \mathcal{A}$ one could take the boundaries of the bounded regions. However, this does not generalize to the complex case. (Note also that the regions cannot be characterized by their sets of facet hyperplanes: arbitrarily many regions can be bounded by all hyperplanes [16].)

Instead, one might try to find subarrangements with exactly one bounded region, such as those given by a product of simplices. More generally (consider the facet planes of a square pyramid), the arrangements with $\beta(M) = 1$ correspond to series–parallel graphs [10]. They are basic objects in the category of base-pointed matroids (M, g), as constructed by Brylawski [10]. However, our map $\Phi \circ \mu$, as well as Dayton and Weibel's embeddings of polysimplicial spheres, are inherently *nonlinear*. In fact, in general there need not be any full-dimensional subarrangements with $\beta = 1$ that could support a linear spherical matroid cycle in $\mathcal{A}$. This is demonstrated in our final result. The following construction essentially applies the "Lawrence construction" [5, Section 9.3] to the uniform matroid $U_{2,n+2}$.

PROPOSITION 4.1. *For every $n \geq 1$ there exists an arrangement $\mathcal{A}_n = \{H_1, \ldots, H_{2n+2}\}$ of $2n+2$ real affine hyperplanes in $\mathbb{R}^{2n}$ with exactly n bounded $2n$-dimensional regions such that for every $i \in \{1, \ldots, 2n+2\}$, $\mathcal{A}\backslash H_i$ has no bounded region.*

Proof. Let $U_{2,n+2}$ denote the uniform matroid of rank 2 on the set $[n+2]$. This matroid can be coordinatized by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 2 & \cdots & n \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

corresponding to the affine arrangement of $n+1$ points $0, 1, \ldots, n$ on the affine line $\mathbb{R}$. Clearly, the number of bounded regions of this arrangement is $\beta(U_{2,n+2}) = n$.

Let $\tilde{U}_{2,n+2}$ be obtained by doubling all points of $U_{2,n+2}$ except the first one, yielding a matroid of rank 2 on $[2n+3]$ with the property that $\tilde{U}_{2,n+2}/i$ has a loop for all $i > 1$. This matroid is represented by

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \cdots & n & n \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

It still has $\beta(\tilde{U}_{2,n+2}) = n$, since extension of parallel elements does not change the β-invariant [12].

Now dualization yields the matroid $M_n := (\tilde{U}_{2,n+2})^*$ of rank $2n+1$ on $[2n+3]$ with $\beta(M_n) = n$, which has the property that every deletion of an element other than 1 has a coloop. Furthermore, every cocircuit of $\tilde{U}_{2,n+2}$ has at least $2n+1 > 2$ elements, so M_n is a simple matroid. It can be coordinatized by

$$\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 1 & \\ 1 & 1 & \\ \vdots & \vdots & -\mathbf{I} \\ n & 1 & \\ n & 1 & \end{array}\right).$$

Using row transformations that make the first column into a unit vector, we find that the corresponding affine arrangement $\mathcal{A}_n$ can be represented by

$$\begin{aligned} x_1 + \frac{n-1}{n}(x_2 + x_3) + \cdots + \frac{1}{n}(x_{2n-2} + x_{2n-1}) &= 1, \\ x_i &= 0 \quad \text{for } 1 \leq i \leq 2n, \\ \frac{1}{n}(x_2 + x_3) + \cdots + \frac{n-1}{n}(x_{2n-2} + x_{2n-1}) + x_{2n} &= -1. \qquad \square \end{aligned}$$

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