Journal of Algebraic Combinatorics 1 (1992), 235–255

A Rational-Function Identity Related to the Murnaghan–Nakayama Formula for the Characters of S_n

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Received January 10, 1992; Revised July 23, 1992.

Abstract. The Murnaghan–Nakayama formula for the characters of S_n is derived from Young's seminormal representation, by a direct combinatorial argument. The main idea is a rational function identity which when stated in a more general form involves Möbius functions of posets whose Hasse diagrams have a planar embedding. These ideas are also used to give an elementary exposition of the main properties of Young's seminormal representations.

Keywords: symmetric group, representation, character, Young tableau, Möbius function

1. Introduction

Despite their formidable appearance in most treatments of representations of symmetric groups (e.g., [14], [11], [7]), Young's seminormal representations reveal a rich combinatorial structure when probed beneath the surface. Some hints of this may be found, for example, in [3], [6], and [7, chap. 7]. The purpose of this paper is to show that the well known Murnaghan–Nakayama formula for irreducible characters of S_n can be derived from the seminormal representations by a direct combinatorial calculation. In the process we obtain a rational function identity which appears to be new and which involves Möbius functions of posets in a rather unexpected way. This offers a particularly direct approach to defining the seminormal representations and verifying their properties. It is a routine matter to write down the matrices representing adjacent transpositions $\tau_k = (k, k+1)$ and to check that the Coxeter relations

$$\begin{aligned} \tau_k^2 &= \epsilon \\ \tau_k\tau_j &= \tau_j\tau_k, \quad |j-k| \geq 2 \\ \tau_k\tau_{k+1}\tau_k &= \tau_{k+1}\tau_k\tau_{k+1} \end{aligned} \tag{1}$$

are satisfied. This proves that one has constructed a family of representations of S_n. Our results show that the characters, and hence the representations, agree with ones given by other constructions (e.g., [8] or [9]).

As another byproduct, we show that the so-called skew representations of S_n occur naturally as constituents when the seminormal representations are restricted to subgroups S_J, where $J = [j, j+1, \ldots, k]$ is an interval. Again, this follows readily by comparison of characters since the skew characters are known to satisfy a Murnaghan–Nakayama-type recursion. An explicit combinatorial description of the skew representations based on Young's natural representations has been given in [4], but the arguments needed to justify the construction are quite intricate. The seminormal representations evidently offer a simpler and more direct approach to this problem.

Section 2 gives a brief description of the seminormal construction and states the main results. Section 3 contains a proof of the rational-function identity on which the main arguments rest. Section 4 discusses the application to skew representations. Section 5 provides, for completeness, the details to support the claim that verifying the Coxeter relations for the representing matrices is a routine calculation. The ideas in Section 5 must certainly have been known to Young, Rutherford, and others, but I have not seen them written down in coherent form. In verifying the Coxeter relations one can see clearly how the entries of the representing matrices are essentially forced by a few natural combinatorial assumptions.

The idea of deriving formulas for the irreducible characters directly from the seminormal matrices is not new. Indeed, Rutherford takes this approach [11, pp. 71–77] and obtains a result essentially equivalent to Theorem 2.8 in this paper. The methods presented in Section 3 are more general than Rutherford's in several respects, and I hope that some of the key ideas in Sections 2 and 5 are revealed in a way which may not be apparent to the casual reader of [11].

2. Young's Seminormal Representations

From a combinatorial point of view it could be argued that Young's seminormal representations are more natural than the more familiar "natural" representations, which Young developed first. For each $\sigma \in S_n$ one defines an action

$$\rho_\lambda(\sigma) : V \longrightarrow V$$

where $V = \langle T_1, T_2, \ldots, T_{f(\lambda)} \rangle$ is the vector space consisting of all $\mathbb{R}$-linear combinations of the $f(\lambda)$ standard tableaux of shape λ. If T is a tableau, let σT denote the tableau obtained from T by replacing each entry by its image under σ. Since σT is not always standard, one cannot define a "purely" combinatorial action on the basis vectors of V. However, a close approximation is possible in the following sense: We seek a system of parameters $a(T, k)$, $b(T, k)$, $c(T, k)$, where $k = 1, 2, \ldots, n-1$ and T ranges over all standard tableaux of shape λ, yielding an action of the form

$$\rho_\lambda\Big((k, k+1)\Big)T = \begin{cases} a(T, k)T + b(T, k)T' & \text{if } T' = (k, k+1)T \text{ is standard} \\ c(T, k)T & \text{otherwise} \end{cases} \tag{2}$$

We note that T' fails to be standard if and only if k and $k+1$ appear together in a row or column of T. It turns out to be possible to choose $a(T, k)$, $b(T, k)$, $c(T, k)$ in such a way that a representation is obtained. To explain how, we need a few more definitions.

Definition 2.1. If λ is any shape and x denotes a cell in λ, the *content* of x (denoted $c(x)$) is equal to the column index of x minus the row index of x. If x and y are cells of λ, the *(signed) distance* $d(x, y)$ from x to y is defined by

$$d(x, y) = c(y) - c(x)$$

In other words, $d(x, y)$ is the number of steps in a northeast path from x to y, or minus that value if y lies to the southwest of x.

If T is a tableau and a and b are entries in T, we define $d_T(a, b) = d(x, y)$, where x and y are the cells containing a and b, respectively. We also need to introduce a linear order on $\mathrm{SYT}(\lambda)$, the set of standard tableaux of shape λ, as follows: if T_i and T_j are such that the largest disagreeing number occurs in a lower row in T_j, then T_i precedes T_j in the ordering. This is known as the *last-letter (LL) ordering* of tableaux. For example, if $\lambda = \{3, 2\}$, the LL ordering on $\mathrm{SYT}(\lambda)$ is

$$\underset{T_1}{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}} < \underset{T_4}{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}} < \underset{T_5}{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}}$$

THEOREM 2.2 (Young [14]). *Representations of S_n are obtained if one chooses either (A):*

$$a(T, k) = c(T, k) = 1/d_T(k, k+1)$$

$$b(T, k) = \sqrt{1 - a(T, k)^2}$$

or (B):

$$a(T, k) = c(T, k) = 1/d_T(k, k+1)$$

$$b(T, k) = \begin{cases} 1 - a(T, k)^2 & \textit{if } T \textit{ precedes } T' \textit{ in } LL \textit{ order} \\ 1 & \textit{otherwise} \end{cases}$$

Representations (A) and (B) are known as Young's *orthogonal* and *seminormal* representations, respectively. They can be shown to be equivalent by a diagonal transformation. In Section 5 we prove Theorem 2.2 by verifying the relations (1) explicity.

We now fix our attention on the seminormal representations and denote by $\rho_\lambda(\theta)$ the matrix representing a permutation $\theta \in S_n$. Further, we let $\rho_\lambda(\theta)|_{i,j}$ denote the ij entry of that matrix. To compute the character of ρ_λ it suffices to consider permutations θ which are in the *standard form*

$$\theta = (1, 2, \ldots, a_1)(a_1 + 1, \ldots, a_1 + a_2) \cdots (b_{K-1} + 1, \ldots, b_K)$$

where $\{a_1, \ldots, a_K\}$ denotes the cycle types of θ and for notational convenience we have written $b_i = a_1 + \cdots + a_i$ for $i = 1, 2, \ldots, K$. We will say that such permutations are in *standard form*.

Definition 2.3. If T is a standard tableau with n cells and if $\theta \in S_n$ is in standard form, the *θ-weight* of T (denoted $\Delta_\theta(T)$) is defined as follows:

$$\Delta_\theta(T) = \prod_{\substack{1 \le j \le n \\ j \ne b_1, b_2, \ldots, b_k}} \frac{1}{d_T(j, j+1)}$$

When $\theta = (1, 2, \ldots, n)$, we write $\Delta_\theta(T) = \Delta(T)$ and refer to $\Delta(T)$ simply as the *weight* of T.

The following key lemma can be found in Rutherford [11, p. 43].

LEMMA 2.4. *If θ is a standard permutation, then*

$$\rho_\lambda(\theta)|_{i,i} = \Delta_\theta(T_i)$$

Proof. One can write

$$\theta = (1, 2) \cdots (a_1 - 1, a_1)(a_1 + 1, a_1 + 2) \cdots (b_{K-1} + 1, b_{K-1} + 2) \cdots (b_K - 1, b_K)$$

Computing $\rho_\lambda(\theta)T_i$ by using (2), one gets a linear combination of tableaux T, each obtained from T_i by applying a subsequence of transpositions in the above expression for θ. Only one such permutation gives T_i itself (namely, the identity), and it yields a product of a-terms having value equal to $\Delta_\theta(T_i)$, as asserted. $\square$

COROLLARY 2.5. *If θ is a standard permutation, then*

$$\text{Trace } \rho_\lambda(\theta) = \sum_{T \in \mathrm{SYT}(\lambda)} \Delta_\theta(T)$$

This last result may be expressed more conveniently in the following notation:

Definition 2.6. If λ is any shape, let

$$\Delta(\lambda) = \sum_{T \in \mathrm{SYT}(\lambda)} \Delta(T)$$

If λ/μ is a skew shape, then $\Delta(\lambda/\mu)$ is defined in a similar fashion.

Since only the positions and the linear order of symbols are relevant, this definition makes sense if the tableaux (or skew tableaux) have values in any linearly ordered set (for example, a segment in $\{1, 2, \ldots, n\}$). Note also that in Definition 2.6 of $\Delta(\lambda)$ we are taking $\theta = (1, 2, \ldots, n)$.

If in the sum appearing in Corollary 2.5 one collects terms according to the positions occupied by the various segments of numbers $\{1, 2, \ldots, a_1\}$, $\{a_1 + 1, \ldots, a_1 + a_2\}, \ldots, \{b_{K-1} + 1, \ldots, b_K\}$, one obtains the following:

COROLLARY 2.7.

$$\sum_{T \in \mathrm{SYT}(\lambda)} \Delta_\theta(T) = \sum_{\emptyset \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_K = \lambda} \Delta(\lambda_1)\Delta(\lambda_2/\lambda_1)\cdots\Delta(\lambda_K/\lambda_{K-1}) \tag{3}$$

The value of the right-hand side of (3) can be determined by

THEOREM 2.8. *For any partitions $\mu \subseteq \lambda$ we have*

$$\Delta(\lambda/\mu) = \begin{cases} 0 & \textit{unless } \lambda/\mu \textit{ is a skew hook} \\ (-1)^{H-1} & \textit{if } \lambda/\mu \textit{ is a skew hook with } H \textit{ rows} \end{cases}$$

Combining Corollary 2.7 with Theorem 2.8, one gets exactly the Murnaghan–Nakayama formula for characters of S_n (see, for example, [7, p. 60]). It is Theorem 2.8 which we intend to prove in Section 3.

3. A Rational-Function Identity

Let $x_1, x_2, \ldots, x_n$ be indeterminates. We wish to assign to each tableau T of shape λ a weight $w(T)$, as follows. Fix a standard labeling of the cells of λ, for example, the one which labels cells in order from left to right in each row, beginning with the first. (The exact labeling chosen is not important.) We can now regard each tableau of shape λ as a map $T : [n] \longrightarrow [n]$, where $T(i)$ is the entry in the cell labeled i.

Definition 3.1. If T is a tableau of shape λ, then

$$w(T) = \prod_{\alpha=1}^{n-1} (x_{T^{-1}(\alpha+1)} - x_{T^{-1}(\alpha)})^{-1} \tag{4}$$

To understand the definition of $w(T)$ it is helpful to introduce a tableau X_λ of shape λ, obtained by inserting the indeterminates x_i according to the standard labeling of λ. For example, if $\lambda = \{3, 3, 2\}$, then

$$X_\lambda = \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline x_4 & x_5 & x_6 \\ \hline x_7 \\ \cline{1-1} \end{array}$$

Then the denominator of $w(T)$ is determined by taking a "walk" through X_λ according to the route defined by T and taking products of differences. Thus if

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 7 \\ \hline 6 \\ \cline{1-1} \end{array}$$

then

$$w(T) = \frac{1}{(x_4 - x_1)(x_2 - x_4)(x_3 - x_2)(x_5 - x_3)(x_7 - x_5)(x_6 - x_7)}$$

If we set each x_i equal to the content $c(i)$ of the cell labeled i, then $(x_j - x_i) = d(i, j)$ and we obtain the following:

LEMMA 3.2. *Let T be any tableau (or skew tableau). Then*

$$\Delta(T) = w(T)$$

where for $i = 1, \ldots, n$ the value of x_i in $w(T)$ has been replaced by $c(i)$.

Our first main result is the following:

THEOREM 3.3. *Let λ be any shape (or skew shape). Then*

$$\sum_{T \in \mathrm{SYT}(\lambda)} w(T) = \begin{cases} 0 & \text{if } \lambda \text{ is not connected} \\ \dfrac{\prod_D (x_j - x_i)}{\prod_R (x_j - x_i) \prod_C (x_j - x_i)} & \text{if } \lambda \text{ is connected} \end{cases} \tag{5}$$

where D denotes the set of pairs x_i, x_j in X_λ with $i < j$ which are adjacent in some diagonal, R denotes the set of pairs adjacent in some row, and C denotes the set of pairs adjacent in some column.

For example, if $\lambda = \{3, 3, 2\}$, Theorem 3.3 asserts that

$$\sum_T w(T) = \frac{(x_5 - x_1)(x_6 - x_2)}{(x_2 - x_1)(x_3 - x_2)(x_5 - x_4)(x_6 - x_5)(x_4 - x_1)(x_7 - x_4)(x_5 - x_2)(x_6 - x_3)}$$

It is clear that Theorem 2.8 follows as an immediate consequence of Theorem 3.3. If λ/μ is a skew shape containing two cells in a common diagonal, the numerator in (5) vanishes when x_i's are replaced by $c(i)$'s. On the other hand, if λ/μ is a skew hook, the denominator contributes (-1) for each pair of adjacent vertical cells, thus contributing $(-1)^{H-1}$ altogether.

There is a natural and well known extension of the theory of tableaux to partially ordered sets, in which the role of standard tableaux is played by *linear extensions* (= natural labelings) (see, for example, Stanley [12]). By definition, a linear extension of a poset P with p elements is a bijective map $\sigma : P \longrightarrow [p]$ such that $x <_P y$ implies $\sigma(x) < \sigma(y)$ for all $x, y \in P$. If λ and ν are partitions with $\nu \subseteq \lambda$, we may construct a poset $P = P_{\lambda/\nu} = \{(i, j) | \nu_i \leq j \leq \lambda_i\} \subseteq \mathbb{R} \times \mathbb{R}$. Here $\mathbb{R} \times \mathbb{R}$ is endowed with the usual componentwise ordering. It is clear that every linear extension of $P_{\lambda/\nu}$ determines a standard tableau of shape λ/ν and conversely.

It turns out that an analog of Theorem 3.3 holds for linear extensions, provided that we restrict our attention to posets which are *planar* in the (strong) sense that their Hasse diagrams may be order-embedded in $\mathbb{R} \times \mathbb{R}$, without edge crossings, even when extra bottom and top elements 0 and 1 are added. Thus, for example, all of the posets $P_{\lambda/\nu}$ defined above are planar, but a simple example of a nonplanar poset is shown in Figure 1.

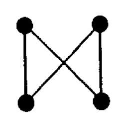


Figure 1.

The more general version of Theorem 3.3 is as follows:

THEOREM 3.4. *Let P be a planar poset, and let $\mathcal{L}(P)$ denote the collection of linear extensions of P. Then*

$$\sum_{\sigma \in \mathcal{L}(P)} w(\sigma) = \begin{cases} 0 & \text{if } P \text{ is not connected} \\ \prod_{a<b} (x_b - x_a)^{\mu(a,b)} & \text{if } P \text{ is connected} \end{cases}$$

where μ denotes the Möbius function of P and the product is over all pairs $a < b$ in P. The weight $w(\sigma)$ of a linear extension is defined as in Definition 3.1.

We will assume the reader is familiar with some basic facts about Möbius functions. For more background refer to [10] or [12]. It is well known that for intervals in $\mathbb{N} \times \mathbb{N}$ of the form of Figure 2 we have $\mu(w, x) = \mu(w, y) = \mu(x, z) = \mu(y, z) = -1$, and $\mu(w, z) = 1$, while $\mu(a, b) = 0$ for all other $a < b$ in

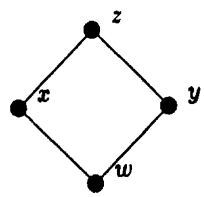


Figure 2.

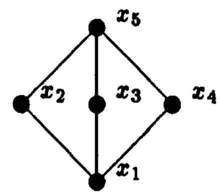


Figure 3.

$\mathbb{N} \times \mathbb{N}$. Accordingly, Theorem 3.3 follows as a special case of Theorem 3.4 since the posets $P_{\lambda/\nu}$ are embedded in $\mathbb{N} \times \mathbb{N}$ in such a way that intervals correspond to intervals.

A more general example is obtained by considering the poset shown in Figure 3, which has six linear extensions (corresponding to the six possible orderings of the middle level). We note that $\mu(1, 5) = 2$ and obtain

$$\sum_{\sigma \in \mathcal{L}(P)} w(\sigma) = \frac{(x_5 - x_1)^2}{(x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}$$

The hypothesis of planarity cannot be omitted from the statement of Theorem 3.4: for example, the result fails to hold for the poset illustrated in Figure 1.

Proof of Theorem 3.4. The proof is by induction. Let $|P| = n$, and assume that the theorem holds for all posets Q with $|Q| < n$ and for all posets Q with $|Q| = n$ that have *more* order relations than does P. The result clearly holds for chains, which have the maximum possible number of order relations among posets with n elements. We consider two cases, according to whether P has a unique maximal element or more than one.

Case (i): P has more than one maximal element. Let a and b be two such elements which are chosen as follows: if P is disconnected, then a and b are in different components; if P is connected, then a and b are "adjacent" in the sense that they are not separated by other maximal elements as the boundary of P is traversed. Define new posets P_A and P_B on the same underlying set of points, as follows:

$$P_A = P \cup \{(b, a)\} \cup \{(t, a)\}_{t \le b}$$

$$P_B = P \cup \{(a, b)\} \cup \{(t, b)\}_{t \le q}$$

It is clear that both P_A and P_B are planar, are connected, and have more order relations than does P; hence Theorem 3.4 applies to each. Further,

$$\mathcal{L}(P) = \mathcal{L}(P_A) \cup \mathcal{L}(P_B)$$

since in every linear extension of P exactly one of $\sigma(a) < \sigma(b)$ or $\sigma(b) < \sigma(q)$ holds. Let μ_A and μ_B denote the Möbius functions of P_A and P_B, respectively. By the inductive hypothesis,

$$\begin{aligned}\sum_{\sigma \in \mathcal{L}(P)} w(\sigma) &= \prod_{p \neq q} (x_q - x_p)^{\mu_A(p,q)} + \prod_{p \neq q} (x_q - x_p)^{\mu_B(p,q)} \\ &= \prod_{p \neq q} (x_q - x_p)^{\mu(p,q)} \left\{ \prod_{p \neq q} (x_q - x_p)^{\mu_A(p,q) - \mu(p,q)} + \prod_{p \neq q} (x_q - x_p)^{\mu_B(p,q) - \mu(p,q)} \right\}\end{aligned}$$

We will show that the term in braces is equal to 0 or 1 according to whether P is disconnected or connected. First note that $\mu_A(p, q) = \mu(p, q)$ unless $q = a$ and $p \le b$. This is obvious since all other intervals are identical in P_A. Similarly, $\mu_B(p, q) = \mu(p, q)$ unless $q = b$ and $p \le a$. If P is disconnected, one readily sees that

$$\mu_A(b, a) - \mu(b, a) = \mu_B(a, b) - \mu(a, b) = -1$$

and that all of the other values of μ remain unchanged. Hence the above expression in braces reduces to

$$(x_a - x_b)^{-1} + (x_b - x_a)^{-1} = 0$$

as claimed.

If P is connected, the situation requires more careful analysis. Let $c = a \wedge b$, the greatest lower bound of a and b, which must exist by the planarity and connectivity of P. We claim

$$\begin{aligned} \mu_A(b, a) - \mu(b, a) &= -1 \\ \mu_A(c, a) - \mu(c, a) &= +1 \\ \mu_A(p, a) - \mu(p, a) &= 0 \quad \text{for all other } p \end{aligned} \tag{6}$$

and similar relations hold for μ_B.

To verify this claim it will be convenient to use some of the elementary formalism of Möbius algebras (see [5]). If k is a field, define a vector space kP consisting of all formal k-linear combinations of elements of P. Then the elements

$$\delta_p = \sum_{t \leq p} \mu(t, p)t \tag{7}$$

also form a basis of kP, and for all $p \in P$ we have

$$p = \sum_{t \leq p} \delta_t \tag{8}$$

Indeed, the latter relation can be viewed as *defining* μ. We will refer to the elements δ_p as *primitive idempotents* for P. When P is a $\wedge$-semilattice, the elements δ_p are primitive orthogonal idempotents for kP viewed as a k-algebra with $\wedge$ as multiplication; however, we will not use this fact in the proof.

It is easy to verify that the elements $\bar{\delta}_p$ defined by

$$\begin{aligned} \bar{\delta}_p &= \delta_p \quad \text{for } p \neq a \\ \bar{\delta}_a &= \delta_a - \Sigma_{\substack{t \leq b \\ t \not\leq c}} \delta_t = \delta_a - (b - c) \end{aligned} \tag{9}$$

form a set of primitive idempotents for P_A; that is, the relations (8) hold for all $p \in P_A$. Examination of coefficients in (9) yields (6) immediately.

Returning to the proof of Theorem 3.4, we now see that

$$\begin{aligned} \prod_{p \neq q}(x_q - x_p)^{\mu_A(p,q)-\mu(p,q)} &+ \prod_{p \neq q}(x_q - x_p)^{\mu_B(p,q)-\mu(p,q)} \\ &= (x_a - x_b)^{-1}(x_a - x_c) + (x_b - x_a)^{-1}(x_b - x_c) \\ &= \frac{x_a - x_b}{x_a - x_b} = 1 \end{aligned}$$

and the proof of Case (i) is complete.

Case (ii): Now assume that P has a unique maximal element u. If u itself covers a unique element v, then every linear extension maps u and v to p and $p - 1$, respectively, and hence the factor $(x_u - x_v)^{-1}$ occurs in $w(\sigma)$ for all $\sigma \in \mathcal{L}(P)$. By the inductive hypothesis we have

$$\sum_{\sigma \in \mathcal{L}(P-u)} w(\sigma) = \prod_{p<q\leq v} (x_q - x_p)^{\mu(p,q)}$$

and the result follows from multiplying both sides by $(x_u - x_v)^{-1}$.

Assume, finally, that P has a unique maximal element u which covers at least two distinct elements a and b, which we assume to be "adjacent" in the sense used earlier. We may further assume that $c = a \wedge b$ exists since otherwise the argument of Case (i) can be applied to the dual of P.

Define posets P_A and P_B as in Case (i) by adding the relations (b, a) and (a, b) (respectively) and all other relations implied by transitivity. Again P_A and P_B are connected and planar, and

$$\mathcal{L}(P) = \mathcal{L}(P_A) \cup \mathcal{L}(P_B)$$

Hence to prove Theorem 3.4 we must show that

$$\prod_{p\neq q}(x_q - x_p)^{\mu_A(p,q)-\mu(p,q)} + \prod_{p\neq q}(x_q - x_p)^{\mu_B(p,q)-\mu(p,q)} = 1 \tag{10}$$

One readily verifies that if $\langle \delta_x \rangle_{x\in P}$ is a system of primitive idempotents for P, then $\langle \bar{\delta}_x \rangle_{x\in P_A}$ defined by

$$\begin{aligned} \bar{\delta}_p &= \delta_p && \text{for } p \neq a,\, u \\ \bar{\delta}_a &= \delta_a - (b-c) && \\ \bar{\delta}_u &= \delta_u + (b-c) && \end{aligned}$$

are primitive idempotents for P_A. Accordingly, we obtain

$$\begin{aligned} \mu_A(t, p) &= \mu(t, p) \quad \text{if } \ p \neq a,\, u \\ \mu_A(b, a) - \mu(b, a) &= -1 \\ \mu_A(c, a) - \mu(c, a) &= +1 \\ \mu_A(b, u) - \mu(b, u) &= +1 \\ \mu_A(c, u) - \mu(c, u) &= -1 \end{aligned}$$

with similar relations holding for μ_B. Hence the left-hand side of (10) reduces to

$$\frac{(x_a - x_c)(x_u - x_b)}{(x_a - x_b)(x_u - x_c)} + \frac{(x_b - x_c)(x_u - x_a)}{(x_b - x_a)(x_u - x_c)} = 1$$

as desired, and the proof of Theorem 3.4 is complete. □

We note the following corollary, which follows when P is an antichain. Note that in this case every permutation of $[n]$ is a linear extension.

COROLLARY 3.5. *For any integer $n > 1$ we have*

$$\sum_{\sigma\in S_n} \prod_{i=1}^{n-1} \frac{1}{(x_{\sigma_{i+1}} - x_{\sigma_i})} = 0$$

Although it is reminiscent of other formulae in the theory of Lagrange interpolation (and can be proved easily by using those methods), we have not found it stated explicitly in the literature. Guo-Niu Han has shown us an easy direct proof of Corollary 3.5: simply argue that the weights sum to zero over each cyclic class of permutations, i.e., over cosets of the subgroup $\langle(1, 2, \ldots, n)\rangle$.

4. Application: Skew Representations

The skew representations of S_n are defined for any skew shape λ/μ having n

cells. They are most easily defined to be those representations corresponding to the skew Schur functions

$$s_{\lambda/\mu} = \sum_{\eta} c^{\lambda}_{\mu\eta} s_{\eta}$$

where s_η denotes an ordinary Schur function and $c^{\lambda}_{\mu\eta}$ is a Littlewood–Richardson coefficient (see [9], for example, for details). Equivalently, they are representations formed by taking direct sums of irreducible representations with multiplicities determined by the $c^{\lambda}_{\mu\eta}$. It is known [7, p. 64] that the characters of skew representations can be computed by an analog of the Murnaghan–Nakayama formula: if $\theta = \theta_1\theta_2\cdots\theta_K$ is a permutation whose cycles θ_i have length m_i, then

$$\chi^{\lambda/\mu}(\theta) = \sum_{\widehat{\lambda/\mu}} (-1)^{H-1} \chi^{\widehat{\lambda/\mu}}(\hat{\theta}) \tag{11}$$

where the sum is over all skew shapes $\widehat{\lambda/\mu}$ obtained by removing a skew hook of length m_K from the border of λ/μ, H denotes the number of rows of the skew hook, and $\hat{\theta}$ denotes the permutation obtained by removing the last cycle from θ. The formula can also be given in nonrecursive form:

$$\chi^{\lambda/\mu}(\theta) = \sum_{\mu\subseteq\lambda_1\subseteq\cdots\subseteq\lambda_K=\lambda} (-1)^{\sum(H_i-1)} \tag{12}$$

where the sum is over all sequences such that λ_i/λ_{i-1} is a skew hook of length m_i, and H_i denotes its length.

We observe that representations with these characters occur naturally as constituents of the seminormal representation ρ_λ. If μ has m cells and λ has $m+n$ cells, then S_n is naturally embedded in S_{m+n} as the set of all permutations fixing $1, 2, \ldots, m$. We denote this subgroup explicitly by S_N. Now consider how the seminormal representation ρ_λ (acting on the vector space V spanned by all tableaux of shape λ) restricts to S_N. Clearly, the tableaux of shape λ containing $\{1, 2, \ldots, m\}$ as a fixed subtableau T span a subspace V_T invariant under the action of $\rho_\lambda|_{S_N}$, for each T. Furthermore, V is the direct sum of all such V_T. Let $\rho_{\lambda/T}$ denote this action, for each T. Then we may write

$$\rho_\lambda|_{S_N} = \bigoplus_{\mu} \bigoplus_{T} \rho_{\lambda/T}$$

where the first sum is over all partitions μ of m contained in λ, and the second sum is over all standard tableaux T of shape μ. From (2) it is clear that the action of S_N on V_T depends only on μ and not on T, so we may define

$$\rho_{\lambda/\mu} = \rho_{\lambda/T}$$

Finally, it is clear that a formula for the character of $\rho_{\lambda/\mu}$ can be computed as in (3), and Theorem 2.8 yields (12) immediately. This shows that $\rho_{\lambda/\nu}$ has character $\chi^{\lambda/\mu}$ and hence must be the skew representation corresponding to λ/μ.

5. The Coxeter Relations

The goal of this section is to prove Theorem 2.2 by writing down the Coxeter relations (1) explicitly in matrix form and determining what is required to satisfy them. In other words, we find exactly what properties $a(T, k)$, $b(T, k)$, and $c(T, k)$ must satisfy in order for (2) to define a representation of S_n.

THEOREM 5.1. *Suppose that parameters $a(T, k)$, $b(T, k)$, and $c(T, k)$ are such that* (2) *defines a representation of S_n. Suppose further that $b(T, k) \neq 0$ whenever $T' = (k, k+1)T$ is a standard tableau. Then there exist a constant $\varepsilon = \pm 1$ and constants ψ_T defined for each $T \in \mathrm{SYT}(\lambda)$ such that for all T, k*

$$a(T, k) = c(T, k) = \varepsilon / d_T(k, k+1) \tag{13}$$

$$b(T, k) b(T', k) = 1 - a(T, k)^2 \tag{14}$$

$$b(T, k) = \frac{\psi_T}{\psi_{T'}} \sqrt{1 - a(T, k)^2} \tag{15}$$

Conversely, any choice of parameters satisfying (13)–(15) *defines a representation of S_n.*

Remark 1. The choice of $\varepsilon = \pm 1$ determines whether the representation obtained corresponds to λ or to λ^* (the conjugate of λ).

Remark 2. Clearly, (14) is implied by (13) and (15).

Remark 3. It will be convenient to have an alternative version of conditions (14)–(15), which serve to define $b(T, k)$. First, note that it suffices to define $b(T, k)$ whenever k lies below $k+1$ in T since then $b(T', k)$ is determined by (14). It is known that the standard Young tableaux of shape λ form an interval in the *weak order* of S_n (see [1] for definitions), which we denote by $\mathrm{SYT}_W(\lambda)$. Here one associates each tableau with its "column word," reading entries from top to bottom in each column, starting with the first. In this order T_1 is the smallest tableau and T_i is covered by T_j in $\mathrm{SYT}_W(\lambda)$ if $T_j = (k, k+1)T_i$ with k appearing below $k+1$ in T_i. We can thus interpret $b(T, k)$ as a function on covering pairs in $\mathrm{SYT}_W(\lambda)$. A function $f(T_i, T_j)$ on covering pairs in $\mathrm{SYT}_W(\lambda)$ will be called *path independent* if the product of f over covering pairs in a saturated chain depends only on the endpoints or, equivalently, if there exists a "potential function" ϕ defined on $\mathrm{SYT}_W(\lambda)$ such that $f(T_i, T_j) = \phi(T_j)/\phi(T_i)$.

PROPOSITION 5.2. *If $b(T, k)$ is any path-independent function on covering pairs*

$T \prec T'$ in $\mathrm{SYT}_W(\lambda)$, there exist parameters ψ_T defined for each $T \in \mathrm{SYT}(\lambda)$ such that (14)–(15) *hold, and conversely.*

Proof. This is a simple consequence of the fact that $\sqrt{1 - a(T, k)^2}$ (indeed any function dependent only on the location of k and $k+1$ in T) is path independent on $\mathrm{SYT}_W(\lambda)$. □

Remark 4. Choosing $\psi_T = 1$ for all T gives Young's orthogonal representation. By Proposition 5.2 and the observation made in its proof, there exists a unique function $\phi(T)$ on $\mathrm{SYT}(\lambda)$ such that

$$\frac{\phi(T')}{\phi(T)} = 1 - \frac{1}{d_T(k, k+1)^2} \tag{16}$$

where k appears below $k+1$ in T and $T' = (k, k+1)T$, that is, T' covers T in $\mathrm{SYT}_W(\lambda)$. Taking $\psi_T = \sqrt{\phi(T)}$ gives Young's seminormal form. The function $\phi(T)$ defined by (16) is the *tableau function* constructed explicitly by Rutherford in [11, p. 47].

Proof of Theorem 5.1. Suppose that ρ is a representation defined by (2) and satisfies the conditions of Theorem 5.1 (that is, $b(T, k) \neq 0$ whenever T' is standard). We will show that (13)–(15) are satisfied. For notational convenience let

$$M_k = \rho\Big((k, k+1)\Big)$$

Each M_k is composed of diagonal blocks of the form

$$\begin{pmatrix} a & b \\ b' & a' \end{pmatrix} \quad \text{or} \quad (c)$$

with b and b' not equal to zero. It is easy to check that $M_k^2 = I$ implies that for each block

$$\begin{aligned} a + a' &= 0 \\ bb' &= 1 - a^2 \\ c &= \pm 1 \end{aligned} \tag{17}$$

First, let $|j - k| > 2$, with $j < k$, and let T be a tableau such that both pairs $\{j, j+1\}$ and $\{k, k+1\}$ lie in distinct rows and columns of T. Denote by

$$\{T_{1234}, T_{2134}, T_{1243}, T_{2143}\} \tag{18}$$

the orbit of T under the action of $S_{\{j,j+1\}} \times S_{\{k,k+1\}}$. Here T_{pqrs} denotes the tableau obtained from T by arranging j, $j+1$, k, $k+1$ in an order consistent with p, q, r, s. The blocks of M_j and M_k corresponding to $T_{1234}, T_{2134}, T_{1243}, T_{2143}$ have the form

$$A = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1' & -a_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & b_2' & -a_2 \end{pmatrix}, \quad B = \begin{pmatrix} a_3 & 0 & b_3 & 0 \\ 0 & a_4 & 0 & b_4 \\ b_3' & 0 & -a_3 & 0 \\ 0 & b_4' & 0 & -a_4 \end{pmatrix}$$

from which we compute

$$AB-BA = \begin{pmatrix} 0 & b_1(a_4-a_3) & b_3(a_1-a_2) & (b_1b_4-b_3b_2) \\ b_1'(a_3-a_4) & 0 & (b_1'b_3-b_4b_2') & b_4(a_2-a_1) \\ b_3'(a_2-a_1) & (b_2b_4'-b_3'b_1) & 0 & b_2(a_3-a_4) \\ (b_2'b_3'-b_1'b_4') & b_4'(a_1-a_2) & b_2'(a_4-a_3) & 0 \end{pmatrix} \tag{19}$$

Setting each entry equal to zero, we find

$$a_1 = a_2, \quad a_3 = a_4 \tag{20}$$

$$b_1b_4 = b_3b_2 \tag{21}$$

That the remaining diagonal conditions are equivalent to (21) follows from the relation $b_ib_i' = 1 - a_i^2$. From (20) we draw the following important conclusion:

PROPOSITION 5.3. *If T^* is obtained from T by permuting integers $i < k$ or integers $j > k+1$, then $a(T, k) = a(T^*, k)$ and $c(T, k) = c(T^*, k)$.*

Proof. For $a(T, k)$ this follows from (20) since such permutations can always be achieved by a sequence of transpositions of adjacent elements (see, for example, [1]). The proof for $c(T, k)$ is similar. □

Next, suppose that T is a tableau containing k, $k+1$, $k+2$ in distinct rows and columns, with k below $k+1$ below $k+2$. The orbit of T under the action of $S_{\{k,k+1,k+2\}}$ contains six tableaux, represented (in the notation of (18)) by

$$T_{123} \quad T_{213} \quad T_{132} \quad T_{231} \quad T_{312} \quad T_{321}$$

when written in LL order. If A and B denote the corresponding blocks of M_k and M_{k+1} (respectively), then A and B have the form

$$A = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 \\ b_1' & -a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & 0 & 0 \\ 0 & 0 & b_2' & -a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & b_3 \\ 0 & 0 & 0 & 0 & b_3' & -a_3 \end{pmatrix}$$

$$B = \begin{pmatrix} A_3 & 0 & B_3 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & B_2 & 0 \\ B_3' & 0 & -A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & 0 & B_1 \\ 0 & B_2' & 0 & 0 & -A_2 & 0 \\ 0 & 0 & 0 & B_1' & 0 & -A_1 \end{pmatrix}$$

A computation now gives $ABA - BAB$

$$= \begin{pmatrix} * & * & * & (a_1 - A_1)b_2B_3 & (a_3 - A_3)b_1B_2 & * \\ * & * & (a_2 - A_2)b_1'B_3 & * & * & (A_1 - a_1)B_2b_3 \\ * & (a_2 - A_2)b_1B_3' & * & * & * & (A_3 - a_3)B_1b_2 \\ (a_1 - A_1)b_2'B_3' & * & * & * & (A_2 - a_2)B_1b_3' & * \\ (a_3 - A_3)b_1'B_2' & * & * & (A_2 - a_2)B_1'b_3 & * & * \\ * & (A_1 - a_1)B_2'b_3' & (A_3 - a_3)B_1'b_2' & * & * & * \end{pmatrix} \tag{22}$$

where we have temporarily suppressed some of the entries. Setting the remaining entries equal to zero, we obtain $A_i = a_i$ for $i = 1, 2, 3$, and it follows that

$$b_ib_i' = B_iB_i' = 1 - a_i^2 \tag{23}$$

Now, computing $ABA - BAB$ again, we obtain

$$\begin{pmatrix} (a_1 - a_3)D & b_1D & -B_3D & 0 & 0 & C \\ b_1'D & -(a_1 + a_2)D & 0 & -\frac{b_1'}{B_1}C & -B_2D & 0 \\ -B_3'D & 0 & (a_2 + a_3)D & b_2D & -\frac{b_3'}{B_3}C & 0 \\ 0 & \frac{b_2'}{B_2}\frac{b_3'}{B_3}C & b_2'D & -(a_1 + a_2)D & 0 & -B_1D \\ 0 & -B_2'D & \frac{b_1'}{B_1}\frac{b_2'}{B_2}C & 0 & (a_2 + a_3)D & b_3D \\ -\frac{b_1'}{B_1}\frac{b_2'}{B_2}\frac{b_3'}{B_3}C & 0 & 0 & -B_1'D & b_3'D & (a_1 - a_3)D \end{pmatrix} \tag{24}$$

where we have written

$$D = a_1a_3 - a_1a_2 - a_2a_3 \tag{25}$$

$$C = b_1B_2b_3 - B_1b_2B_3 \tag{26}$$

Setting $D = C = 0$, we obtain

$$\frac{1}{a_2} = \frac{1}{a_3} - \frac{1}{a_1} \tag{27}$$

$$b_1B_2b_3 = B_1b_2B_3 \tag{28}$$

Combining (21) and (28), we obtain the following:

PROPOSITION 5.4. *The function $b(T, k)$ is path independent on* $\mathrm{SYT}_W(\lambda)$. *Hence, the conditions of Proposition* 5.2 *are satisfied and $b(T, k)$ is determined by* (15).

Proof. It is well known [13] that given any two saturated chains in the weak order of S_n (and hence in $\mathrm{SYT}_W(\lambda)$), one can deform one into another by a sequence of transformations corresponding to applications of the elementary Coxeter relations (1). Path independence for these transformations follows immediately from (21) and (28). □

Only a few details now need to be resolved to complete the proof. The first step will be to compute

$$M_1 = \rho\Big((1, 2)\Big)$$

explicitly. Let T be a tableau containing

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

as a subtableau, and let $T' = (2, 3)T$. Consider the 2×2 subblocks of

$$M_1 = \rho\Big((1, 2)\Big) \quad \text{and} \quad M_2 = \rho\Big((2, 3)\Big)$$

determined by T and T'. These have the form

$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & b \\ b' & -a \end{pmatrix} \tag{29}$$

respectively, with $x^2 = y^2 = 1$ and $bb' = 1 - a^2$, as noted earlier. A computation gives

$$ABA - BAB = \begin{pmatrix} a - a^2(x-y) - y & b(xy - a(x-y)) \\ b'(xy - a(x-y)) & -a + a^2(x-y) - x \end{pmatrix} \tag{30}$$

Setting (30) equal to zero, we deduce that $x = -y$ and hence $xy = -1$. Further, $b \neq 0$ implies $x - y = -2y$ and $a = 1/2y$. We let $\varepsilon = y$. Since the value of $c(T, 1)$ depends only on the location of 1 and 2, it follows that

$$c(T, 1) = \begin{cases} \varepsilon & \text{if 1 and 2 are in the same row of } T \\ -\varepsilon & \text{if 1 and 2 are in the same column of } T \end{cases}$$

Thus if $\varepsilon = 1$, for example, we have

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1/2 & b \\ b' & -1/2 \end{pmatrix}$$

Next we show that for all k

$$c(T, k) = \begin{cases} \varepsilon & \text{if } k \text{ and } k+1 \text{ are in the same row of } T \\ -\varepsilon & \text{if } k \text{ and } k+1 \text{ are in the same column of T} \end{cases}$$

that is, the pattern determined by M_1 holds for all k. The proof is by induction on k. Let T be a tabelau in which k and $k+1$ are in the same row. (A similar argument applies if they are in the same column.) If k is not in the first column, we can find a tableau T' containing $k-1, k, k+1$ in the same row. By Proposition 5.3, we have $c(T, k) = c(T', k)$. By the inductive hypothesis $c(T', k-1) = \varepsilon$, and by (1)

$$c(T', k-1)c(T', k)c(T', k-1) = c(T', k)c(T', k-1)c(T', k)$$

which implies $c(T', k) = \varepsilon$, as claimed. On the other hand, if k is in the first column (and assuming $k > 1$), we can find a tableau T' containing

$$\begin{array}{|c|c|}\hline k-2 & k-1 \\ \hline k & k+1 \\ \hline\end{array}$$

as a subtableau, and again $c(T, k) = c(T', k)$. We let $T'' = (k-1, k)T'$ and note that T'' precedes T' in LL order. Consider the 2×2 submatrices of M_{k-2}, M_{k-1}, and M_k determined by T' and T'', which have the form

$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ b' & -a \end{pmatrix}, \quad C = \begin{pmatrix} x' & 0 \\ 0 & y' \end{pmatrix}$$

respectively. From the Coxeter relations we deduce $(AB)^3 = (CB)^3 = I$, then $A = \pm C$, and, finally, $A = C$. However, $y = c(T', k-2) = \varepsilon$ by the inductive hypothesis. Hence $y' = c(T', k) = \varepsilon$, as asserted.

It now remains to determine $a(T, k)$ and $b(T, k)$ when k and $k+1$ appear in different rows and columns of T. We proceed by induction on $|d_T(k, k+1)|$, supposing first that $d_T(k, k+1) = 2$. We can find a tableau T' containing

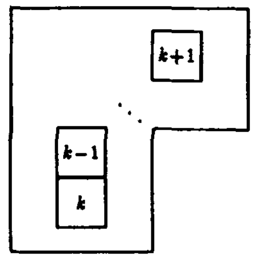
$$\begin{array}{|c|c|}\hline k-1 & k+1 \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

as a subtableau and such that $a(T, k) = a(T', k)$, $b(T, k) = b(T', k)$. Letting $T'' = (k, k+1)T'$ and considering the 2×2 submatrices of M_{k-1} and M_k determined by T' and T'', we obtain

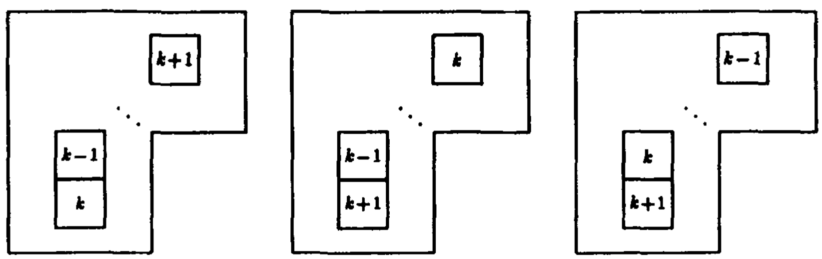
$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & b \\ b' & -a \end{pmatrix}$$

and equations (30) imply (as before) that $a = \varepsilon/2$, which proves that $a(T, k) = \varepsilon/2 = \varepsilon/d(k, k+1)$ when $d(k, k+1) = 2$. A similar argument holds when $d(k, k+1) = -2$.

Next, suppose that $d(k, k+1) = d > 2$. We can find a tableau T' containing



as a subtableau and such that $a(T, k) = a(T', k)$, $b(T, k) = b(T', k)$. The orbit of T' under the action of $S_{\{k-1,k,k+1\}}$ contains three tableaux T', T'', T''', illustrated by



The submatrices of M_{k-1} and M_k determined by T', T'', T''' have (respectively) the form

$$A = \begin{pmatrix} -\varepsilon & 0 & 0 \\ 0 & a_1 & b_1 \\ 0 & b_1' & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & -b_2 & 0 \\ b_2' & -a_2 & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}$$

where $a_1 = \varepsilon / d_{T'''}(k-1, k)$ by the inductive hypothesis and $a_2 = a(T', k)$ is what we seek to determine. Now

$$ABA - BAB = \begin{pmatrix} (a_2+\varepsilon)D & b_2 D & 0 \\ b_2' D & -(a_1+a_2)D & -b_1 D \\ 0 & b_1' D & (a_1-\varepsilon)D \end{pmatrix} \tag{31}$$

where $D = a_1 a_2 - \varepsilon a_1 + \varepsilon a_2$. Setting $ABA - BAB = 0$, we see that $D = a_1 a_2 - \varepsilon a_1 + \varepsilon a_2 = 0$ or, equivalently,

$$\begin{aligned} \varepsilon / a_2 &= \varepsilon / a_1 + 1 \\ &= d_{T'''}(k-1, k) + 1 \\ &= d_{T'}(k, k+1) \end{aligned}$$

which implies $a(T', k) = a_2 = \varepsilon/d_{T'}(k, k+1)$, as claimed. A similar argument works when $d_T(k, k+1) = d < -2$. By induction this establishes (13) for all T and k and completes the proof of necessity.

Much less work is required to show sufficiency, and most of it is already complete. Suppose that the parameters $a(T, k)$, $b(T, k)$, $c(T, k)$ have been chosen to satisfy (13)–(15), and let $M_k = \rho((k, k+1))$ denote the matrix obtained from (2), as before. We must show that the matrices M_k satisfy (1).

It is clear from (13)–(14) and the argument at the beginning of this proof that $M_k^2 = I$ for all k. To verify that $M_k M_j = M_j M_k$ when $|j - k| \geq 2$, we examine the block submatrices determined by the action of $S_{\{j,j+1\}} \times S_{\{k,k+1\}}$ on $\mathrm{SYT}(\lambda)$. The only nontrivial case occurs for tableaux in which the pairs $\{j, j+1\}$ and $\{k, k+1\}$ occur in distinct rows and columns, and $M_j M_k - M_k M_j = 0$ follows immediately from (19).

To verify that $M_{k+1} M_k M_{k+1} = M_k M_{k+1} M_k$ for all k, we need to look at the blocks of M_k and M_{k+1} determined by the orbits of $S_{\{k,k+1,k+2\}}$ on tableaux. There are several cases, depending on which of k, $k+1$, $k+2$ lie in the same row or column. All but one of these cases have been considered already in this proof, and $ABA - BAB = 0$ follows immediately from (30), (31), (24) in each of those cases. The only unexamined case is an easy one, namely, when k, $k+1$, $k+2$ occur T in a subtableau of the form

$$\begin{array}{cc} & \boxed{k+1} \\ \boxed{\;k\;} & \boxed{k+2} \end{array}$$

If $T' = (k, k+1)T$, the corresponding submatrices of M_k and M_{k+1} are

$$A = \begin{pmatrix} \varepsilon/2 & b \\ b' & -\varepsilon/2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$

with $bb' = 3/4$, and the relation $ABA - BAB$ follows immediately. This completes the proof of Theorem 5.1. □

COROLLARY 5.5. *Any two representations of S_n defined by* (2) *and satisfying the conditions of Theorem 5.1 are equivalent by a diagonal transformation.*

Proof. This is an immediate consequence of property (15). □

Acknowledgment

This paper was written while the author was a guest at the University of

Strasbourg. The author thanks D. Foata and R. Seroul for their hospitality and technical support during that visit. This work was supported in part by National Science Foundation grant DMS-9005666.

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