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On Schur's Q-functions and the Primitive Idempotents of a Commutative Hecke Algebra*

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Abstract. Let B_n denote the centralizer of a fixed-point free involution in the symmetric group S_{2n}. Each of the four one-dimensional representations of B_n induces a multiplicity-free representation of S_{2n}, and thus the corresponding Hecke algebra is commutative in each case. We prove that in two of the cases, the primitive idempotents can be obtained from the power-sum expansion of Schur's Q-functions, from which follows the surprising corollary that the character tables of these two Hecke algebras are, aside from scalar multiples, the same as the nontrivial part of the character table of the spin representations of S_n.

Keywords: Gelfand pairs, Hecke algebras, symmetric functions, zonal polynomials

0. Introduction

Schur's Q-functions are a family of symmetric polynomials $Q_\lambda(x_1, x_2, \ldots)$ indexed by partitions λ with distinct parts. They were originally defined in Schur's 1911 paper [16] as the Pfaffians of certain skew-symmetric matrices. The main point of Schur's paper was to prove that the Q-functions "encode" the characters of the irreducible projective representations of symmetric groups, in the same sense that Schur's S-functions encode the ordinary irreducible characters of symmetric groups.

In the past 10 years, there have been a number of developments showing that Schur's Q-functions arise naturally in several seemingly unrelated areas, just as Schur's S-functions arise as the answer to a number of natural algebraic and geometric questions. In particular, (1) Sergeev [17] has proved that the Q-functions $Q_\lambda(x_1, \ldots, x_m)$ are (aside from scalar factors) the characters of the irreducible tensor representations of a certain Lie superalgebra $Q(m)$; (2) Pragacz [15] has proved that the cohomology ring of the isotropic Grassmanian Sp_{2n}/U_n is a homomorphic image of the ring generated by Q-functions, and furthermore, this homomorphism maps Q-functions to Schubert cycles; and (3) You [22] has proved that the Q-functions are the polynomial solutions of the BKP hierarchy

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of partial differential equations.

Thus, including Schur's original analysis of the characters of projective representations of symmetric groups, there are at least four "natural" settings where the Q-functions arise. The purpose of this paper is to introduce a fifth setting for Q-functions involving the primitive idempotents of a certain commutative Hecke algebra.

To be more explicit, consider the hyperoctahedral group B_n (the Weyl group of the root system of the same name), embedded in the symmetric group S_{2n} as the centralizer of a fixed-point free involution. Let ρ be one of the four one-dimensional representations of B_n; i.e., 1, δ, ε, or $\delta\varepsilon$, where 1 denotes the trivial representation, δ the restriction to B_n of the sign character of S_{2n}, and ε the composition of the sign character of S_n with the homomorphism $B_n \to S_n$. In each of these four cases, the induction of ρ to S_{2n} is multiplicity free, and so the centralizer of this induced representation in the group algebra of S_{2n} is a commutative Hecke algebra $\mathcal{H}_n^\rho$. Among these four Hecke algebras there are two isomorphisms: $\mathcal{H}_n^1 \cong \mathcal{H}_n^\delta$ and $\mathcal{H}_n^\varepsilon \cong \mathcal{H}_n^{\delta\varepsilon}$; this is a consequence of the fact that δ is the restriction of a linear character of S_{2n} (see Remark 1.1).

For the case $\rho = 1$, the commutativity of $\mathcal{H}_n = \mathcal{H}_n^1$ is equivalent to the well-known fact that (S_{2n}, B_n) is a Gelfand pair. Furthermore, the primitive idempotents of this algebra (or equivalently, the spherical functions of (S_{2n}, B_n)) are known by a theorem of James [6] to be "encoded" by the power-sum expansion of the zonal polynomials of the real symmetric matrices (see Section 7). Since $\mathcal{H}_n \cong \mathcal{H}_n^\delta$, it follows that the primitive idempotents for the case $\rho = \delta$ are essentially the same as for the case $\rho = 1$; however, we should note that Macdonald has shown that the idempotents for this case are closely related to the Jack symmetric functions with parameter $\alpha = 1/2$ [12, §5].

The remaining pair of Hecke algebras, $\mathcal{H}_n^\varepsilon$ and $\mathcal{H}_n^{\delta\varepsilon}$, are the subject of this paper.

Since the two Hecke algebras are isomorphic, it suffices to restrict our attention to the case $\rho = \varepsilon$. We prove (Corollary 3.2) that the dimension of $\mathcal{H}_n^\varepsilon$ is the number of partitions of n into odd parts, and that the primitive idempotents of $\mathcal{H}_n^\varepsilon$, say E_λ, are naturally indexed by partitions λ of n into distinct parts (see Section 4). The main result (Theorem 5.2) shows that the expansion of $Q_\lambda(x_1, x_2, \ldots)$ into power-sum symmetric functions is essentially the same as the expansion $E_\lambda = \sum_{w \in S_{2n}} E_\lambda(w) w$ of E_λ as a member of the group algebra of S_{2n}. Since Schur proved that the irreducible projective characters of S_n also occur as coefficients in the power-sum expansion of the Q-functions, we thus obtain the surprising conclusion (Corollary 6.2) that aside from scalar factors, the character table of projective representations of S_n is essentially the same as the character table of $\mathcal{H}_n^\varepsilon$.

The remainder of the paper is organized as follows. In Section 1, we give a brief survey of the general theory of Hecke algebras, with special emphasis on commutative Hecke algebras induced by one-dimensional representations of the base group. We refer to these as "twisted Gelfand pairs" because they enjoy a theory quite similar to the theory of Gelfand pairs (cf. [3]). In Section 2, we analyze the combinatorial structure of double cosets $B_n\backslash S_{2n}/B_n$; much of the material in this section can be found in equivalent forms elsewhere (e.g., [1], [12,§5]). In the seventh and final section, we rederive the connection between zonal polynomials and the spherical functions of the Gelfand pair (S_{2n}, B_n), in order to contrast this with the twisted case we analyze in Sections 3 to 6.

1. Hecke Algebras

Let G be a finite group, H a subgroup of G, and e an idempotent of the complex group algebra $\mathbf{C}H$. The Hecke algebra of the triple (G, H, e) is the $\mathbf{C}G$-subalgebra

$$\mathcal{H} = \mathcal{H}(G, H, e) = e\mathbf{C}Ge.$$

The Hecke algebra $\mathcal{H}$ is also isomorphic to the (opposite) algebra of endomorphisms of $\mathbf{C}Ge$ that commute with the action of G [2,§11D].

In the following, let ε denote the character of $\mathbf{C}He$ as a representation of H, and let ε^G denote the induced G-character; i.e., the character of $\mathbf{C}Ge$.

For each irreducible character χ of G, let e_χ denote the primitive central idempotent of $\mathbf{C}G$ indexed by χ, so that

$$\mathbf{C}G = \bigoplus_{\chi \in \mathrm{Irr}(G)} \mathbf{C}Ge_\chi$$

is the Wedderburn decomposition of $\mathbf{C}G$ as a direct sum of simple algebras. By Schur's lemma, the centralizer of $\mathbf{C}Ge$ is the direct sum of its projections onto the Wedderburn components of $\mathbf{C}G$, and these projections are matrix algebras of degrees equal to the multiplicities in ε^G of each irreducible character χ. It follows that the Wedderburn decomposition of $\mathcal{H}$ is given by

$$\mathcal{H} = \bigoplus_{\chi \in I_\varepsilon(G)} e\mathbf{C}Ge_\chi e, \tag{1.1}$$

where $I_\varepsilon(G) = \{\chi \in \mathrm{Irr}(G) : \langle \varepsilon^G, \chi \rangle \neq 0\}$. In particular, the primitive central idempotents of $\mathcal{H}$ are of the form

$$E_\chi := ee_\chi e = e_\chi e$$

for $\chi \in I_\varepsilon(G)$. If χ is not a constituent of ε^G(i.e., $\chi \notin I_\varepsilon(G)$), then the projection of $\mathcal{H}$ onto $\mathbf{C}Ge_\chi$ will be zero, and thus

$$e_\chi e = 0 \quad \text{unless} \quad \langle \varepsilon^G, \chi \rangle \neq 0. \tag{1.2}$$

A further consequence of (1.1) is the fact that the irreducible characters of $\mathcal{H}$ are restrictions of those of $\mathbf{C}G$ (cf. Theorem 11.25 of [2]); thus for $w \in G$,

$$\theta_\chi(w) := \chi(ew) = \chi(ewe)$$

is the trace of ewe in the representation of $\mathcal{H}$ afforded by $e\mathbf{C}Ge_\chi e$. Note that since

$$e_\chi = \frac{\deg(\chi)}{|G|} \sum_{w\in G} \chi(w^{-1})w,$$

it follows that

$$E_\chi = \frac{\deg(\chi)}{|G|} \sum_{w\in G} \theta_\chi(w^{-1})w,$$

so one may determine θ_χ from E_χ, and vice-versa.

Remark 1.1. If δ is a linear character of G, then there is an automorphism $a \mapsto a'$ of $\mathbf{C}G$ in which $w \mapsto \delta(w)w$. By restricting this automorphism to $\mathcal{H} = \mathcal{H}(e)$, we obtain an isomorphism $\mathcal{H}(e) \to \mathcal{H}(e')$ of Hecke algebras. Thus, it follows that if the $\mathbf{C}H$-modules generated by two idempotents e and e' differ only by the action of a linear G-character δ, then the idempotents, characters, and representations of either Hecke algebra can be easily obtained from those of the other.

For the remainder of this section, we will assume that ε is a linear character of H, and that e is the corresponding primitive idempotent; i.e., $e = |H|^{-1} \sum_{x\in H} \varepsilon(x^{-1})x$. Note that we have

$$ex_1wx_2e = \varepsilon(x_1x_2)ewe \tag{1.3}$$

for all x_1, $x_2 \in H$ and $w \in G$, so it follows that if $w_1,\dots,w_l$ are a set of representatives of the double cosets $H\backslash G/H$, then $\mathcal{H}$ is spanned by $\{ew_1e,\dots,ew_le\}$. Furthermore, since ew_ie and ew_je are supported on disjoint subsets of G (for $i \neq j$), it follows that the nonzero members of $\{ew_1e,\dots,ew_le\}$ form a basis for $\mathcal{H}$.

Let $L(G)$ denote the algebra of functions $f: G \to \mathbf{C}$ under convolution. The mapping $f \mapsto \sum_{w\in G} f(w^{-1})w$ defines an antiisomorphism $L(G) \to \mathbf{C}G$. By (1.3), it follows that $\mathcal{H}$ is antiisomorphic to the subalgebra of functions $f \in L(G)$ satisfying

$$f(x_1wx_2) = \varepsilon(x_1x_2)f(w)$$

for all $x_i \in H$, $w \in G$. The characters θ_χ form a basis for the center of this subalgebra.

If ε is the trivial character of H, then $\mathcal{H}$ is isomorphic to the subalgebra of H-biinvariant functions in $L(G)$. If in addition, $\mathcal{H}$ is commutative (or equivalently, ε^G is multiciplity free), then (G,H) is known as a *Gelfand pair* and the characters θ_χ are known as *spherical functions* [3]. If $\mathcal{H}$ is commutative, but ε is merely a linear (not necessarily trivial) character of H, then we refer to (G,H,ε) as a *twisted Gelfand pair*. (Perhaps it should be called a Gelfand triple.) The characters θ_χ will be referred to as *twisted spherical functions*.

The following result will be used in Sections 6 and 7 to prove the nonnegativity of certain structure constants.

LEMMA 1.2. *Let $\chi \in \mathrm{Irr}(G)$. If e_1 and e_2 are central idempotents for two (possibly distinct) subgroups of G, then $\chi(e_1e_2) \geq 0$.*

Proof. Let $\rho: G \to U_n$ be a unitary representation of G with character χ. If K is some subgroup of G and e_φ is the primitive central idempotent for some $\varphi \in \mathrm{Irr}(K)$, then

$$\rho(e_\varphi) = \frac{\deg(\varphi)}{|K|} \sum_{x \in K} \varphi(x^{-1})\rho(x).$$

Since $\rho(x^{-1}) = \rho(x)^*$ (where * = conjugate transpose), it follows that $\rho(e_\varphi) = \rho(e_\varphi)^*$. Thus, the primitive central idempotents of every subgroup of G are represented as Hermitian matrices by ρ. In particular, $\rho(e_1)$ and $\rho(e_2)$ are Hermitian and idempotent, so

$$\chi(e_1e_2) = \mathrm{tr}\rho(e_1e_2) = \mathrm{tr}\rho(e_1^2e_2^2) = \mathrm{tr}\rho(e_1e_2^2e_1) = \mathrm{tr}\rho(e_1e_2)\rho(e_1e_2)^*.$$

Thus, $\chi(e_1e_2)$ is the trace of a positive semidefinite matrix. □

The analogous result for three idempotents is false. A counterexample can be obtained by taking idempotents from the three 2-element subgroups of S_3.

COROLLARY 1.3. *Let (G, H, ε) be a twisted Gelfand pair. If f is a central idempotent for some subgroup K of G, then there exist scalars $c_\chi \geq 0$ such that*

$$efe = \sum_{\chi \in I_\varepsilon(G)} c_\chi E_\chi.$$

Proof. The idempotents $\{E_\chi : \chi \in I_\varepsilon(G)\}$ form a basis for $\mathcal{H}$, so efe is certainly in their linear span. Since ε^G must be multiplicity free, it follows that E_χ acts as a rank-one idempotent in the χth Wedderburn component of $\mathbf{C}G$, and as zero on the other components. Thus, $c_\chi = \chi(efe) = \chi(ef)$. Apply Lemma 1.2. □

2. On the Gelfand Pair (S_{2n}, B_n)

Let B_n denote the hyperoctahedral group, embedded in S_{2n} as the centralizer of the involution $(1,2)(3,4)\cdots(2n-1,2n)$. Let T_n denote the subgroup (isomorphic to $\mathbf{Z}_2^n$) generated by $(1,2),\ldots,(2n-1,2n)$, and let Σ_n denote the subgroup (isomorphic to S_n) generated by the "double transpositions" $(2i-1,2j-1)(2i,2j)$ for $1 \leq i < j \leq n$. Note that B_n is the semidirect product of T_n and Σ_n.

There is a simple way to describe the double cosets of B_n in S_{2n} (cf. [12,§5]). Given $w \in S_{2n}$, construct a bipartite graph on $4n$ vertices $x_1, y_1, \ldots, x_{2n}, y_{2n}$ by declaring x_i adjacent to y_j if and only if $j = w(i)$. Now perform the series of vertex identifications $x_1 = x_2$, $y_1 = y_2$, $x_3 = x_4$, $y_3 = y_4, \ldots$, thereby obtaining a 2-regular bipartite graph $\Gamma(w)$ on $2n$ vertices.

PROPOSITION 2.1. *Let* $w_1, w_2 \in S_{2n}$.

(a) $\Gamma(w_1) = \Gamma(w_2)$ *if and only if* $T_n w_1 T_n = T_n w_2 T_n$.
(b) $\Gamma(w_1) \cong \Gamma(w_2)$ *if and only if* $B_n w_1 B_n = B_n w_2 B_n$.

Proof.

(a) For $t \in T_n$, the operations $w \mapsto tw$ and $w \mapsto wt$ correspond to the interchanging of identified vertices, and thus have no effect on $\Gamma(w)$. Conversely, if $\Gamma(w_1) = \Gamma(w_2)$, it is easy to see that one can find $t_1, t_2 \in T_n$ such that $t_1 w_1 t_2 = w_2$.
(b) For any double transposition $x \in \Sigma_n$, the operations $w \mapsto xw$ and $w \mapsto wx$ correspond to interchanging two vertices in the same half of the bipartition of $\Gamma(w)$, and thus do not affect the isomorphism class of $\Gamma(w)$. Conversely, since every permutation of the vertices (within a given half of the bipartition) can be obtained by the interchanging of pairs of vertices, it follows that if $\Gamma(w_1) \cong \Gamma(w_2)$, then we can find $x_1, x_2 \in \Sigma_n$ such that $\Gamma(x_1 w_1 x_2) = \Gamma(w_2)$. The result now follows from a. □

From this result it follows that the double cosets $B_n \backslash S_{2n} / B_n$ are in one-to-one correspondence with the isomorphism classes of 2-regular bipartite graphs on $2n$ vertices. Such graphs are disjoint unions of even-length cycles, and are thus indexed by partitions of n. More precisely, we will say that w has *coset-type* $\nu = (\nu_1, \nu_2, \ldots)$ if the cycles of $\Gamma(w)$ are of length $2\nu = (2\nu_1, 2\nu_2, \ldots)$.

A further consequence of Proposition 2.1 is the fact that the double cosets $B_n \backslash S_{2n} / B_n$ are invariant under the map $w \mapsto w^{-1}$, and so by Gelfand's lemma (e.g., [3]), we have

COROLLARY 2.2. (S_{2n}, B_n) *is a Gelfand pair.*

Proof. Let $e_0 = |B_n|^{-1} \sum_{x \in B_n} x$ denote the idempotent associated with the trivial character of B_n. Since $\Gamma(w) \cong \Gamma(w^{-1})$, it follows that $e_0 w e_0 = e_0 w^{-1} e_0$ for all $w \in S_{2n}$. Therefore, since the set of inverses of $w_1 B_n w_2$ is $w_2^{-1} B_n w_1^{-1}$, we have

$$e_0 w_1 e_0 \cdot e_0 w_2 e_0 = e_0 (w_1 e_0 w_2) e_0 = e_0 (w_2^{-1} e_0 w_1^{-1}) e_0 = e_0 w_2 e_0 \cdot e_0 w_1 e_0$$

for all $w_1, w_2 \in S_{2n}$. Thus, the Hecke algebra $e_0 \mathbf{C} S_{2n} e_0$ is commutative. □

For each partition ν of n, let z_ν denote the size of the S_n-centralizer of any permutation having cycles of length ν. Thus $z_\nu = \prod_i m_i! i^{m_i}$, if the multiplicity of i in ν is m_i.

PROPOSITION 2.3 *If* $w \in S_{2n}$ *has coset-type* ν, *then* $|B_n w B_n| = |B_n|^2/z_{2\nu}$.

Proof. First consider the case $w = (1, 2, \dots, 2n)$. Note that w has coset-type (n). It is easy to see that $T_n \cap w T_n w^{-1} = \{\mathrm{id}\}$ (provided that $n > 1$), so $|T_n w T_n| = 2^{2n}$. By Proposition 2.1a, it follows that for any bipartite $2n$-cycle Γ, there are 2^{2n} permutations in S_{2n} whose graph is Γ. Since there are a total of $n!(n-1)!/2$ bipartite $2n$-cycles, we may conclude that there are $2^{2n-1} n!(n-1)! = |B_n|^2/2n$ permutations with coset-type (n). This argument breaks down when $n = 1$, but the formula $|B_n|^2/2n$ remains correct.

Now in the general case, let $X \cup Y$ denote the bipartition of the vertices in $\Gamma(w)$. Define a partition π of X by declaring two members of X to be in the same block of π if they belong to the same cycle of $\Gamma(w)$. Similarly define a partition σ of Y. Second, define a bijection between the blocks of π and σ by declaring $A \leftrightarrow B$ if A and B share vertices of the same cycle of $\Gamma(w)$. Note that any bijection between π and σ that preserves the cardinality of blocks could arise in this manner. Furthermore, if A and B are any such pair of corresponding blocks with $|A| = |B| = k$, then the restriction of $\Gamma(w)$ to $A \cup B$ could have arisen from any permutation of coset-type (k); from the above calculation, we know that there are $2^{2k}(k!)^2/2k$ such permutations.

Hence, one may obtain all $w \in S_{2n}$ for which $\Gamma(w)$ consists of cycles of length 2ν by first choosing π and σ in $(n!/\prod_i \nu_i! m_i(\nu)!)^2$ ways, then choosing the bijection $\pi \leftrightarrow \sigma$ in $\prod_i m_i(\nu)!$ ways, and then for each pair of corresponding blocks of sizes $\nu_1, \nu_2, \dots$, choosing permutations of coset-types $(\nu_1), (\nu_2), \dots$ in $\prod_i 2^{2\nu_i}(\nu_i!)^2/2\nu_i$ ways. Thus, we conclude that there are

$$\left[n! / \prod_i \nu_i! m_i(\nu)! \right]^2 \cdot \prod_i m_i(\nu)! \cdot 2^{2n} \prod_i \frac{(\nu_i!)^2}{2\nu_i} = \frac{|B_n|^2}{z_{2\nu}}$$

permutations of coset-type ν. □

3. A twisted Gelfand pair

Let ε denote the linear character of B_n whose restriction to Σ_n is the sign character, and whose restriction to T_n is trivial. Let $e = |B_n|^{-1} \sum_{x \in B_n} \varepsilon(x) x \in \mathbf{C}B_n$ denote the corresponding primitive idempotent.

In the following, we will need to specify representatives w_ν for each of the double cosets of B_n in S_{2n}. In order to avoid awkward developments later on, these choices cannot be entirely arbitrary. First, for the case $\nu = (n)$, we define

$$w_{(n)} := (1, 2, \dots, 2n),$$

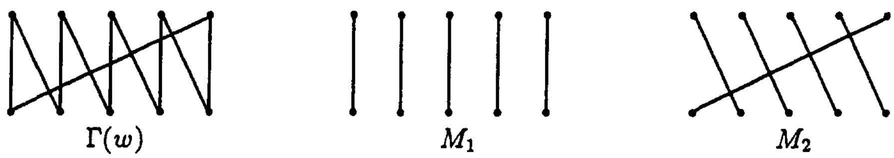

Figure 1.

and then for the general case $\nu = (\nu_1, \ldots, \nu_l)$, we define

$$w_\nu = w_{(\nu_1)} \circ \cdots \circ w_{(\nu_l)},$$

where the operation $x \circ y$ (for $x \in S_{2i}$, $y \in S_{2j}$) denotes the embedding of $S_{2i} \times S_{2j}$ in S_{2i+2j} with S_{2i} acting on $\{1, \ldots, 2i\}$ and S_{2j} acting on $\{2i+1, \ldots, 2i+2j\}$. Note that since $\Gamma(w_1 \circ w_2) = \Gamma(w_1) \cup \Gamma(w_2)$ (disjoint union), it is clear that w_ν does indeed have coset-type ν.

Following the notation of Section 2, let $X \cup Y$ denote the bipartition of $\Gamma(w)$. By a well-known fact from graph theory, every regular bipartite graph on $X \cup Y$ can be partitioned into disjoint perfect matchings of X and Y (i.e., 1-regular graphs). Furthermore, since $|X| = |Y| = n$, any perfect matching of X and Y can be regarded as a permutation of n objects. In particular, any perfect matching M has a well-defined sign $\mathrm{sgn}(M)$, relative to the "identity matching" that arises from $\Gamma(\mathrm{id})$.

In the following, OP_n denotes the set of partitions of n into odd parts.

LEMMA 3.1. *Let $w \in S_{2n}$ be of coset type ν.*

(a) If $\nu \in OP_n$ and $w = x_1 w_\nu x_2$ (where $x_1, x_2 \in B_n$) then every factorization of $\Gamma(w)$ into perfect matchings yields two matchings whose signs both equal $\varepsilon(x_1 x_2)$.
(b) If $\nu \notin OP_n$, then there exist $x_1, x_2 \in B_n$ such that $x_1 w_\nu x_2 = w_\nu$ and $\varepsilon(x_1 x_2) = -1$.

Proof.

(a) In the special case $w = w_{(n)}$, there is only one way to partition $\Gamma(w)$ (a $2n$-cycle) into two perfect matchings; these two matchings are displayed in Figure 1. Note that the permutations defined by these matchings are the identity permutation and an n-cycle. Assuming that n is odd, then both of these are even permutations. Therefore, if $\nu \in OP_n$, then every partition of $\Gamma(w_\nu)$ into two perfect matchings M_1 and M_2 yields two permutations that are products of odd-length cycles and thus $\mathrm{sgn}(M_1) = \mathrm{sgn}(M_2) = 1$.

Now suppose that $w = x_1 w_\nu x_2$ for some $x_1, x_2 \in B_n$. Note that replacing w with $t_1 w t_2$, x_1 with $t_1 x_1$, and x_2 with $x_2 t_2$ (with $t_1, t_2 \in T_n$) has no effect on $\Gamma(w)$ (Proposition 2.1.a) or $\varepsilon(x_1 x_2)$, so it suffices to assume that $x_1, x_2 \in \Sigma_n$.

However, as discussed in the proof of Proposition 2.16, the effect on $\Gamma(w_\nu)$ of left and right multiplication by Σ_N is to permute the vertices in either half of the bipartition of $\Gamma(w_\nu)$. This in turn induces an action $M \mapsto x_1 M x_2$ on perfect matchings M with the property that $\mathrm{sgn}(x_1 M x_2) = \varepsilon(x_1 x_2)\mathrm{sgn}(M)$. Therefore, if $M_1 \cup M_2$ is a partition of $\Gamma(w)$ into disjoint perfect matchings, then $M_1' = x_1^{-1} M_1 x_2^{-1}$ and $M_2' = x_1^{-1} M_2 x_2^{-1}$ define a partition of $\Gamma(w_\nu)$ into perfect matchings with the property that $\mathrm{sgn}(M_1) = \varepsilon(x_1 x_2)\mathrm{sgn}(M_1')$ and $\mathrm{sgn}(M_2) = \varepsilon(x_1 x_2)\mathrm{sgn}(M_2')$. However, we have already noted that all such partitions of $\Gamma(w_\nu)$ must have $\mathrm{sgn}(M_1') = \mathrm{sgn}(M_2') = 1$.

(b) Let Γ be a bipartite $2k$-cycle, and let v be any vertex of Γ. There is a bipartition-preserving automorphism of Γ that interchanges each pair of vertices at distance i from v $(1 \leq i < k)$, and fixes the unique vertex at distance k from v. This automorphism is a product of $k-1$ transpositions, and is therefore odd if k is even. Hence, if some part of ν is even, one can find $x_1, x_2 \in \Sigma_n$ with $\varepsilon(x_1 x_2) = -1$ and $\Gamma(x_1 w_\nu x_2) = \Gamma(w_\nu)$. By Proposition 2.1.a, it follows that $w_\nu = t_1 x_1 w_\nu x_2 t_2$ for some $t_1, t_2 \in T_n$. Since $\varepsilon(t_1 t_2) = 1$, the result follows. □

Let $\mathcal{H}_n^\varepsilon$ denote the Hecke algebra of the triple (S_{2n}, B_n, e). For each partition ν of n, let us define

$$K_\nu := e w_\nu e = \frac{1}{|B_n|^2} \sum_{x_1, x_2 \in B_n} \varepsilon(x_1 x_2) x_1 w_\nu x_2. \tag{3.1}$$

Clearly, the K_ν's span $\mathcal{H}_n^\varepsilon$.

COROLLARY 3.2.

(a) If $\nu \notin OP_n$, then $K_\nu = 0$.
(b) If $\nu \in OP_n$, then the coefficient of w_ν in K_ν is $z_{2\nu}/|B_n|^2$.
(c) $\{K_\nu : \nu \in OP_n\}$ is a basis of $\mathcal{H}_n^\varepsilon$.
(d) $(S_{2n}, B_n, \varepsilon)$ is a twisted Gelfand pair.

Proof.

(a) If $x_1, x_2 \in B_n$ satisfy $w_\nu = x_1 w_\nu x_2$, then one has $K_\nu = \varepsilon(x_1 x_2) K_\nu$, by (1.3). However by Lemma 3.1.b, if $\nu \notin OP_n$, then there exist choices for x_1 and x_2 with $\varepsilon(x_1 x_2) = -1$. Thus $K_\nu = 0$ in such cases.
(b) In view of (3.1), the coefficient of w_ν in K_ν is (aside from a factor of $|B_n|^{-2}$) the sum of $\varepsilon(x_1 x_2)$, where $x_1, x_2 \in B_n$ range over all solutions to $w_\nu = x_1 w_\nu x_2$. By Lemma 3.1.a, all such solutions have the property that $\varepsilon(x_1 x_2) = 1$, so this number is in fact equal to $|B_n|^2/|B_n w_\nu B_n|$. Apply Proposition 2.3.
(c) Parts (a) and (b) imply $K_\nu = 0$ if and only if $\nu \notin OP_n$. Since the nonzero K_ν's must form a basis of $\mathcal{H}_n^\varepsilon$ (cf. the discussion in Section 1), the result

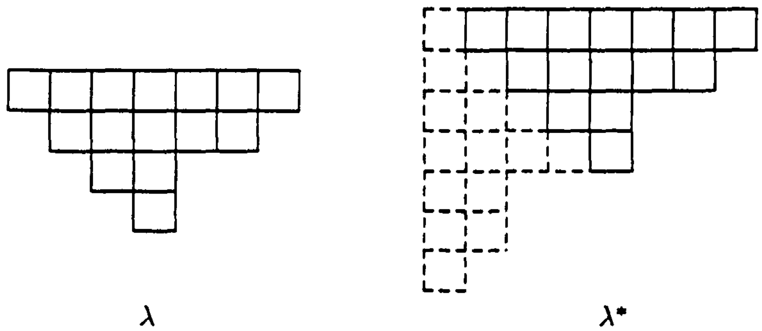

Figure 2.

follows.

(d) We claim that $ew_\nu e = ew_\nu^{-1}e$ for all partitions ν. If $\nu \notin OP_n$ then both are zero and there is nothing to prove. Otherwise, since a partition of $\Gamma(w_\nu)$ into perfect matchings can be obtained by inverting a similar partition of $\Gamma(w_\nu^{-1})$, Lemma 3.1.a implies that $w_\nu^{-1} = x_1 w_\nu x_2$ for some $x_1, x_2 \in B_n$ with $\varepsilon(x_1 x_2) = 1$. Thus $ew_\nu e = ew_\nu^{-1}e$, by (1.3). The commutativity of $\mathcal{H}_n^\varepsilon$ now follows by reasoning analogous to the proof of Corollary 2.2. □

4. The twisted spherical functions of $(S_{2n}, B_n, \varepsilon)$

Let DP (resp., DP_n) denote the set of partitions (resp., partitions of n) with distinct parts. For each $\lambda \in DP_n$, there is a corresponding partition λ^* of $2n$ whose definition is best explained in terms of Young diagrams. One starts with the shifted diagram of λ, which consists of the cells $D = \{(i,j) \in \mathbf{Z}^2 : 1 \le i \le j \le \lambda_i + i - 1\}$, and then one adjoins a shifted "transpose" of this diagram, embedded as $\{(i, j-1) : (j,i) \in D\}$. The union of these two sets defines the (unshifted) diagram of the partition λ^*. For example, if $\lambda = (7,5,2,1)$, then $\lambda^* = (8,7,5,5,2,2,1)$; see Figure 2.

The partitions λ^* occur in the following symmetric function identity due to Littlewood [10, p. 238] (cf. also Example I.5.9 of [11])

$$\prod_{i \le j}(1 + x_i x_j) = \sum_{\lambda \in DP} s_{\lambda^*}(x_1, x_2, \ldots),$$

where $s_\mu(x_1, x_2, \ldots)$ denotes the Schur function indexed by μ. Since the induction of characters from a wreath product $S_m \wr S_n$ to S_{mn} corresponds to plethysm of symmetric functions [11, p. 66], one may easily deduce that the induction of ε

from $B_n = S_2 \wr S_n$ to S_{2n} has the decomposition

$$\varepsilon^{S_{2n}} = \sum_{\lambda \in DP_n} \chi^{\lambda^*}, \tag{4.1}$$

where χ^μ denotes the irreducible character of the symmetric group indexed by μ.

Let e_μ denote the primitive central idempotent of $\mathbf{C}S_n$ corresponding to χ^μ; i.e.,

$$e_\mu = \frac{1}{H_\mu} \sum_{w \in S_n} \chi^\mu(w) w,$$

where $H_\mu = n!/\deg(\chi^\mu)$ denotes the product of the hook lengths of the Young diagram of μ [7, p. 56]. Since $\varepsilon^{S_{2n}}$ is evidently multiplicity free, (4.1) provides another proof of the fact that $(S_{2n}, B_n, \varepsilon)$ is a twisted Gelfand pair. It also provides a basis of orthogonal idempotents for $\mathcal{H}_n^\varepsilon$; namely,

$$E_\lambda := e_{\lambda^*} e = e e_{\lambda^*} e,$$

where λ ranges over DP_n.

Let ξ^λ denote the twisted spherical function associated with E_λ; i.e, the function on S_{2n} defined by $\xi^\lambda(w) = \chi^{\lambda^*}(ewe) = \chi^{\lambda^*}(ew)$ (cf. Section 1). Since

$$\xi^\lambda = (x_1 w x_2) = \varepsilon(x_1 x_2) \xi^\lambda(w)$$

for all $x_1, x_2 \in B_n$, $w \in S_{2n}$, it follows that the ξ^λ's are determined by their values on a set of representatives for $B_n \backslash S_{2n} / B_n$, and thus by the values $\xi^\lambda(w_\nu) = \chi^{\lambda^*}(K_\nu)$ for $\nu \in OP_n$.

PROPOSITION 4.1. *If* $\lambda \in DP_n$, *then*

$$E_\lambda = \frac{|B_n|^2}{H_{\lambda^*}} \sum_{\nu \in OP_n} \frac{1}{z_{2\nu}} \chi^{\lambda^*}(K_\nu) K_\nu = \frac{1}{H_{\lambda^*}} \sum_{w \in S_{2n}} \xi^\lambda(w) w.$$

Proof. We know that $\{K_\nu : \nu \in OP_n\}$ is a basis of $\mathcal{H}_n^\varepsilon$, so $E_\lambda = \sum_{\nu \in OP_n} a_{\lambda,\nu} K_\nu$ for suitable scalars $a_{\lambda,\nu}$. Since the coefficient of w_ν in K_μ is $z_{2\nu}\delta_{\mu,\nu}/|B_n|^2$ (Corollary 3.2.b), it follows that $z_{2\nu} a_{\lambda,\nu}/|B_n|^2$ is the coefficient of w_ν in E_λ. However,

$$E_\lambda = e e_{\lambda^*} e = \frac{1}{|B_n|^2} \cdot \frac{1}{H_{\lambda^*}} \sum_{x_1, x_2 \in B_n} \sum_{w \in S_{2n}} \varepsilon(x_1 x_2) \chi^{\lambda^*}(w) x_1 w x_2,$$

so the coefficient of w_ν in E_λ is also equal to

$$\frac{1}{|B_n|^2} \cdot \frac{1}{H_{\lambda^*}} \sum_{x_1, x_2 \in B_n} \varepsilon(x_1 x_2) \chi^{\lambda^*}(x_1 w_\nu x_2) = \frac{1}{H_{\lambda^*}} \chi^{\lambda^*}(K_\nu) \qquad \square$$

For any $w \in S_{2n}$ and $a = \sum a_w w \in \mathbf{C}S_{2n}$, let $[w]a = a_w$ denote the coefficient operator, and let $\overline{a} = \sum \overline{a}_w w$ denote complex conjugation. Since $ewe = ew^{-1}e$ for all $w \in S_{2n}$ (cf. the proof of Corollary 3.2.d), it follows that $\mathcal{H}_n^\epsilon$ is invariant under the linear transformation of $\mathbf{C}S_{2n}$ induced by $w \mapsto w^{-1}$, and hence we may create an inner product on $\mathcal{H}_n^\epsilon$ by defining

$$\langle a, b\rangle := [\mathrm{id}]a\overline{b}.$$

Both the K_ν's and the E_λ's are orthogonal with respect to this inner product. To make this precise, consider the following.

PROPOSITION 4.2.

(a) $\langle K_\alpha, K_\beta\rangle = \frac{z_{2\alpha}}{|B_n|^2}\delta_{\alpha,\beta}$ *for* $\alpha, \beta \in OP_n$.
(b) $\langle E_\lambda, E_\mu\rangle = \frac{1}{H_{\lambda^*}}\delta_{\lambda,\mu}$ *for* $\lambda, \mu \in DP_n$.
(c) $\langle E_\lambda, K_\nu\rangle = \frac{1}{H_{\lambda^*}}\xi^\lambda(w_\nu)$ *for* $\lambda \in DP_n, \nu \in OP_n$.

Proof.

(a) Since w and w^{-1} belong to the same double coset of $B_n\backslash S_{2n}/B_n$, it follows that [id] $K_\alpha K_\beta = 0$ unless $\alpha = \beta$. In that case, [id] K_α^2 is the sum of the squares of the coefficients in K_α. By Corollary 3.2.b, these coefficients are all equal to $\pm z_{2\alpha}/|B_n|^2$. Hence, by Proposition 2.3,

$$[\mathrm{id}]K_\alpha^2 = \frac{z_{2\alpha}^2}{|B_n|^4}|B_n w_\alpha B_n| = \frac{z_{2\alpha}}{|B_n|^2}.$$

(b) Note that $\chi^{\lambda^*}(e) = \chi^{\lambda^*}(K_{(1^n)}) = 1$, since e acts as a rank-one idempotent on the Wedderburn component of $\mathbf{C}S_{2n}$ indexed by λ^*. We therefore have $[\mathrm{id}]E_\lambda = H_{\lambda^*}^{-1}$, by Proposition 4.1. However, the E_λ's are orthogonal idempotents, so $E_\lambda E_\mu = \delta_{\lambda,\mu}E_\lambda$, and hence, $[\mathrm{id}]E_\lambda E_\mu = H_{\lambda^*}^{-1}\delta_{\lambda,\mu}$.
(c) Apply Proposition 4.1 and part (a). □

In terms of the twisted spherical functions ξ^λ, the orthogonality of the E_λ's can be equivalently expressed as

$$\frac{1}{H_{\lambda^*}}\sum_{w\in S_{2n}} \xi^\lambda(w)\xi^\mu(w) = \delta_{\lambda,\mu}.$$

We remark that the product-of-hook-lengths H_{λ^*} can also be expressed in terms of *shifted* hook lengths. To be more precise, choose some $\lambda \in DP_n$ with l parts, let D denote the shifted diagram of λ (as defined at the beginning of

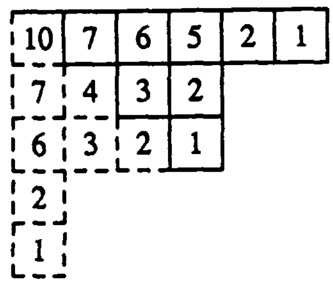

Figure 3.

this section), and let D^* denote the (unshifted) diagram of λ^*. The shifted hook lengths of λ can be defined as the set of ordinary hook lengths of D^* belonging to cells of D (cf. Example III.7.8 of [11]). For example, the set of shifted hook lengths for $\lambda = (5,2,1)$ are $7,6,5,3,2,2,1,1$, as illustrated in Figure 3. It is not difficult to verify that the hook lengths of λ^* belonging to cells of $D^* - D$ are nearly identical to those of D; the only difference is that the hook lengths $\lambda_1,\ldots,\lambda_l$ in column l of D are replaced by the hook lengths $2\lambda_1,\ldots,\ 2\lambda_l$ on the main diagonal of $D^* - D$ (cf. Figure 3). Thus we have

$$H_{\lambda^*} = 2^l (H'_\lambda)^2, \tag{4.2}$$

where H'_λ denotes the product of the shifted hook lengths of λ. Like the classical hook length formula, the quantity $n!/H'_\lambda$ counts the number of standard *shifted* Young tableaux of shape λ [11, p. 135].

5. The main result

Let $\Lambda = \oplus_{n\geq 0}\Lambda^n$ denote the graded **C**-algebra of symmetric functions in the variables $x_1, x_2 \ldots$ [11], and let $p_r = p_r(x_1, x_2, \ldots) = x_1^r + x_2^r + \cdots$ denote the rth power-sum symmetric function. For each partition $\nu = (\nu_1, \ldots, \nu_l)$, define $p_\nu = p_{\nu_1} \ldots p_{\nu_l}$. By the fundamental theorem on symmetric functions one knows that the p_r's are algebraically independent generators of Λ.

Following [19], let $\Omega = \oplus_{n\geq 0}\Omega^n$ denote the graded subalgebra of Λ generated by 1 and the odd power sums p_{2r+1}. Note that $\{p_\nu : \nu \in OP_n\}$ is a basis of Ω^n. There is a convenient inner product $[\cdot,\cdot]$ on Ω defined by

$$[p_\mu, p_\nu] := 2^{-\ell(\mu)} z_\mu \delta_{\mu,\nu}$$

for all $\mu, \nu \in OP$, where $\ell(\mu)$ denotes the number of parts in μ.

There is another useful set of generators $q_1, q_2, q_3, \ldots$ for Ω; these can be defined by means of the generating function

$$Q(t) = \sum_{n\geq 0} q_n(x_1, x_2, \ldots)t^n = \prod_{i\geq 1} \frac{1 + x_i t}{1 - x_i t}.$$

It is easy to show that

$$\log Q(t) = 2\sum_{r\geq 0} \frac{1}{2r+1} p_{2r+1} t^{2r+1}, \tag{5.1}$$

so the q_n's do generate Ω. If we define $q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots$ for all partitions λ, then the q_λ's will span Ω; however, they are not linearly independent. In fact, it is not hard to show that both $\{q_\lambda : \lambda \in DP\}$ and $\{q_\lambda : \lambda \in OP\}$ are bases of Ω [19,§5].

Schur's Q-functions are a family of symmetric functions Q_λ ($\lambda \in DP$) that form an orthogonal basis of Ω. They have several equivalent definitions: (1) as generating functions for a certain type of shifted tableaux [19], (2) as Hall–Littlewood symmetric functions $Q_\lambda(x; t)$ with parameter $t = -1$ [11], (3) as Pfaffians of certain skew-symmetric matrices defined over Ω [16], [20], and (4) as ratios of certain Pfaffians defined over $\mathbf{Z}[x_1, \ldots, x_m]$ [14].

For our purposes, we prefer to adopt the following definition of the Q-functions; it is analogous to the definition of the Jack symmetric functions [12] [18]. It is also similar to the definition of Q-functions used by Hoffman and Humphreys [4].

THEOREM 5.1. *The symmetric functions Q_λ are the unique homogeneous basis of Ω satisfying:*

(a) $[Q_\lambda, Q_\mu] = 2^{\ell(\lambda)} \delta_{\lambda,\mu}$ *for* $\lambda, \mu \in DP$.
(b) For any partition μ, $[q_\mu, Q_\lambda] = 0$ *unless* $\lambda \geq \mu$ *in the "dominance" partial order (i.e.,* $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ *for all* $i \geq 1$*).*
(c) $[Q_\lambda, p_1^n] > 0$ *for* $\lambda \in DP_n$.

It is clear that this result determines the Q-functions. Indeed, if there were another basis Q'_λ with these properties, then the transition matrix between the two bases would have to be unitary (by (a)) and triangular (by (b)), and thus $Q'_\lambda = c_\lambda Q_\lambda$ for some $c_\lambda \in \mathbf{C}$ with $|c_\lambda| = 1$. Part (c) then forces $c_\lambda = 1$. It is also clear that this result provides a simple algorithm for constructing the Q-functions: one starts with the Ω-basis $\{q_\mu : \mu \in DP\}$, linearly ordered in a fashion compatible with the dominance order, and then one applies the Gram–Schmidt algorithm to create an orthogonal basis. What is not clear *a priori* is that the resulting orthogonal basis has a transition matrix with respect to $\{q_\mu : \mu \in DP\}$ that is not merely triangular, but in fact satisfies the much stronger hypotheses of (b).

For a proof that the tableaux definition of the Q-functions satisfies Theorem 5.1, see Section 6 of [19]; for a proof starting from the Hall–Littlewood definition, see Chapter III of [11].

We now define the *characteristic map* ch: $\mathcal{H}_n^\varepsilon \to \Omega^n$ to be the unique linear isomorphism satisfying

$$ch(K_\nu) = 2^{\ell(\nu)}p_\nu$$

for all $\nu \in OP_n$. This map is analogous to the Frobenius characteristic map between the class functions on S_n and symmetric functions of degree n [11,§7]. Since $z_{2\alpha} = 2^{\ell(\alpha)}z_\alpha$, Proposition 4.2.a implies that this map is essentially an isometry; i.e.,

$$\langle a,b\rangle = \frac{1}{|B_n|^2}[ch(a), ch(b)] \tag{5.2}$$

for all $a, b \in \mathcal{H}_n^\varepsilon$.

It is also possible to impose a graded algebra structure on the space

$$\mathcal{H}^\varepsilon := \bigoplus\nolimits_{n\geq 0}\mathcal{H}_n^\varepsilon$$

so that the characteristic map is an algebra isomorphism. For this, it is convenient to first extend the operation $x \circ y$ of Section 3 bilinearly so as to define an embedding of $\mathbf{C}S_{2i} \otimes \mathbf{C}S_{2j}$ in $\mathbf{C}S_{2i+2j}$. Also, to avoid ambiguity in what follows, we write e_n for e, to emphasize the dependence on n. In these terms, one may define the algebra structure of $\mathcal{H}^\varepsilon$ by setting

$$a * b = e_{i+j}(a \circ b)e_{i+j}$$

for all $a \in \mathcal{H}_i^\varepsilon$, $b \in \mathcal{H}_j^\varepsilon$. Since $e_ie_{i+j} = e_{i+j}e_i = e_{i+j}$, it follows that $e_iw_1e_i * e_jw_2e_j = e_{i+j}(w_1 \circ w_2)e_{i+j}$ for all $w_1 \in S_{2i}$, $w_2 \in S_{2j}$. In particular, $K_\mu * K_\nu = K_{\mu\cup\nu}$, where $\mu\cup\nu$ denotes multiset union of partitions, since $w_\mu \circ w_\nu$ and $w_{\mu\cup\nu}$ are B_n-conjugates. Hence,

$$ch(K_\alpha)ch(K_\beta) = 2^{\ell(\alpha)+\ell(\beta)}p_{\alpha\cup\beta} = ch(K_\alpha * K_\beta),$$

so the characteristic map is indeed an isomorphism of graded algebras.

We are now ready to state the main result.

THEOREM 5.2. *For any* $\lambda \in DP_n$, *we have*

$$ch(E_\lambda) = 2^{n-\ell(\lambda)}g^\lambda Q_\lambda,$$

where $g^\lambda = n!/H'_\lambda$ *denotes the number of shifted standard tableaux of shape* λ.

As the first step towards the proof, we need to explicitly evaluate the twisted spherical function ξ^λ in the case $\lambda = (n)$.

LEMMA 5.3. *For* $\nu \in OP_n$, *we have* $\xi^{(n)}(w_\nu) = 2^{-(n-\ell(\nu))}$.

Proof. If $\lambda = (n)$, the $\lambda^* = (n+1, 1^{n-1})$. Furthermore, the partition $(n, 1^n)$ is not of the form λ^* for any $\lambda \in DP_n$, so the restriction of $\chi^{(n,1^n)}$ to $\mathcal{H}_n^\varepsilon$ is zero. Thus we have $\xi^{(n)}(w) = \chi(ewe) = \chi(ew)$, where

$$\chi = \chi^{(n+1,1^{n-1})} + \chi^{(n,1^n)}.$$

By the Littlewood–Richardson rule [7] [11], one knows that χ is the induction of the outer tensor product of the trivial and sign characters from $S_n \times S_n$ to S_{2n}, or equivalently, the character of $\bigwedge^n(C^{2n})$, where S_{2n} acts on $\mathbf{C}^{2n}$ by permuting some basis $v_1, \ldots, v_{2n}$.

Let us take the vectors $v_{i_1} \wedge \cdots \wedge v_{i_n}$ with $1 \leq i_1 < \cdots < i_n \leq 2n$ as a basis of $\bigwedge^n(\mathbf{C}^{2n})$, and define

$$u = (v_1 + v_2) \wedge \cdots \wedge (v_{2n-1} + v_{2n}) \in \bigwedge^n(C^{2n}).$$

If the indices $2j-1$ and $2j$ both occur among $i_1,\ldots,i_n$, then the action of the transposition $t = (2j-1, 2j) \in T_n$ on $v_{i_1} \wedge \cdots \wedge v_{i_n}$ amounts to negation. Since $et = e$, it follows that $e(v_{i_1} \wedge \cdots \wedge v_{i_n}) = 0$ unless

$$i_1 \in \{1,2\}, i_2 \in \{3,4\}, \ldots, i_n \in \{2n-1, 2n\}. \tag{5.3}$$

In that case, it is easy to show that $\Sigma_{t \in T_n} t(v_{i_1} \wedge \cdots \wedge v_{i_n}) = u$. Since $xu = \varepsilon(x)u$ for $x \in \Sigma_n$, we therefore have

$$e(v_{i_1} \wedge \cdots \wedge v_{i_n}) = 2^{-n}u. \tag{5.4}$$

whenever (5.3) is satisfied.

To compute the trace of $w_\nu e$ acting on $\bigwedge^n(\mathbf{C}^{2n})$, let $(i_1,\ldots,i_n)$ be an n-tuple satisfying (5.3), and consider the special case $\nu = (n)$. Recall from Section 3 that $w_{(n)} = (1, 2, \ldots, 2n)$, so we have

$$w_{(n)}u = (v_2 + v_3) \wedge (v_3 + v_4) \wedge \cdots \wedge (v_{2n} + v_1).$$

The only basis vectors satisfying (5.3) that occur in this expression are $v_2 \wedge v_4 \wedge \cdots \wedge v_{2n}$ and

$$v_3 \wedge \cdots \wedge v_{2n-1} \wedge v_1 = (-1)^{n-1} v_1 \wedge v_3 \wedge \cdots \wedge v_{2n-1}.$$

Assuming n is odd, these vectors will both provide positive contributions to the trace of $w_{(n)}e$. In view of (5.4), this trace must be $2^{-(n-1)}$.

For arbitrary $\nu \in OP_n$, we have $w_\nu = w_{(\nu_1)} \circ \cdots \circ w_{(\nu_l)}$, so the action of w_ν on u will produce 2^l-basis vectors satisfying (5.3), each occurring with coefficient 1. Thus by (5.4), we conclude that

$$\xi^{(n)}(w_\nu) = \chi(w_\nu e) = 2^{-n+\ell(\nu)}. \qquad \square$$

COROLLARY 5.4. $ch(E_{(n)}) = 2^{n-1} q_n$.

Proof. Since $H_{(n+1,1^{n-1})} = 2(n!)^2$, Proposition 4.1 and Lemma 5.3 imply

$$E_{(n)} = 2^{n-1} \sum_{\nu \in OP_n} \frac{1}{z_\nu} K_\nu.$$

On the other hand, exponentiating (5.1) yields

$$q_n = \sum_{\nu \in OP_n} \frac{2^{\ell(\nu)}}{z_\nu} p_\nu. \qquad \square$$

Proof of Theorem 5.2. For $\lambda \in DP_n$, define $Y_\lambda \in \Omega^n$ by setting $\mathrm{ch}(E_\lambda) = 2^{n-\ell(\lambda)} g^\lambda Y_\lambda$. By Proposition 4.2.b and (5.2), we have

$$[Y_\lambda, Y_\mu] = 2^{-2(n-\ell(\lambda))} (g^\lambda)^{-2} \frac{|B_n|^2}{H_{\lambda^*}} \delta_{\lambda,\mu}.$$

However,

$$\frac{|B_n|^2}{H_{\lambda^*}} = 2^{2n-\ell(\lambda)} \left(\frac{n!}{H'_\lambda} \right)^2 = 2^{2n-\ell(\lambda)} (g^\lambda)^2$$

by (4.2), so $[Y_\lambda, Y_\mu] = 2^{\ell(\lambda)} \delta_{\lambda,\mu}$. Thus the Y_λ's satisfy part (a) of Theorem 5.1.

Now since the characteristic map is an algebra isomorphism, Corollary 5.4 implies

$$q_\mu = 2^{n-1} ch(E_{(\mu_1)} * \cdots * E_{(\mu_l)})$$

for any partition $\mu = (\mu_1, \ldots, \mu_l)$ of n. Therefore, to prove that Y_λ satisfies part (b) of Theorem 5.1, it suffices to show that

$$(E_{(\mu_1)} * \cdots * E_{(\mu_l)}) E_\lambda = 0 \qquad (5.5)$$

unless $\lambda \geq \mu$.

If H is any subgroup of S_{2n}, and e_0 is an idempotent in $\mathbf{C}H$ that generates a $\mathbf{C}H$-module with character θ, then by (1.2) we have $e_\alpha e_0 = 0$ unless χ^α occurs with nonzero multiplicity in $\theta^{S_{2n}}$. Since

$$(E_{(\mu_1)} * \cdots * E_{(\mu_l)}) E_\lambda = e(e_{(\mu_1)^*} \circ \cdots \circ e_{(\mu_l)^*}) e_{\lambda^*} e,$$

we may therefore establish (5.5) by proving that

$$\langle (\chi^{(\mu_1)^*} \times \cdots \times \chi^{(\mu_l)^*})^{S_{2n}}, \chi^{\lambda^*} \rangle = 0 \qquad (5.6)$$

unless $\lambda \geq \mu$. (Here we are using $\times$ to denote the outer tensor product of characters.)

To prove this, let $D(\nu)$ denote the Young diagram of a partition ν. The Littlewood–Richardson rule implies that if α and β are partitions of $n-k$ and n, and if $(r, 1^{k-r})$ is any hook-shaped partition of k, then

$$\langle (\chi^{(r,1^{k-r})} \times \chi^\alpha)^{S_n}, \chi^\beta \rangle = 0$$

unless $D(\alpha) \subset D(\beta)$ and $D(\beta) - D(\alpha)$ is a disjoint union of border strips; i.e., a subset of $\mathbf{Z}^2$ containing no 2×2 square. By the transitivity of induction, we may iterate this result and conclude that (5.6) holds unless there exists a sequence $D_0,\dots,D_l$ of Young diagrams satisfying

(1) $\emptyset = D_0 \subset \cdots \subset D_l = D(\lambda^*)$,
(2) $|D_i| - |D_{i-1}| = 2\mu_i$,
(3) $D_i - D_{i-1}$ contains no 2×2 squares.

However, properties (1) and (3) imply that every cell of D_i must belong to the union of the first i rows and columns of $D(\lambda^*)$; there are a total of $2\lambda_1 + \cdots + 2\lambda_i$ such cells in $D(\lambda^*)$. On the other hand, property (2) implies that D_i contains $2\mu_1 + \cdots + 2\mu_i$ cells, so if χ^{λ^*} is a constituent of $(\chi^{(\mu_1)^*} \times \cdots \times \chi^{(\mu_l)^*})^{S_{2n}}$, we must have

$$2\lambda_1 + \cdots + 2\lambda_i \geq 2\mu_1 + \cdots + 2\mu_i$$

for all $i \geq 1$; i.e., $\lambda \geq \mu$. Thus Y_λ satisfies part (b) of Theorem 5.1.

To complete the proof, we need only to establish that Y_λ satisfies part (c) of Theorem 5.1. For this we claim

$$\begin{aligned}[Y_\lambda, p_1^n] &= \frac{1}{2^{2n-l(\lambda)}g^\lambda}[ch(E_\lambda), ch(K_{(1^n)})] = \frac{|B_n|^2}{2^{2n-l(\lambda)}g^\lambda}\langle E_\lambda, e\rangle \\ &= \frac{|B_n|^2}{2^{2n-l(\lambda)}g^\lambda}\langle E_\lambda, E_\lambda\rangle = \frac{|B_n|^2}{2^{2n-l(\lambda)}g^\lambda H_{\lambda*}} = \frac{(n!)^2}{g^\lambda (H'_\lambda)^2} = g^\lambda > 0, \qquad (5.7)\end{aligned}$$

by successive applications of (5.2), Proposition 4.2.b, and (4.2). □

6. Ramifications

Following [19], let $\tilde{S}_n$ denote the double cover of S_n generated by elements $\sigma_1,\dots,\sigma_{n-1}$ and a central involution -1, subject to the relations

$$\sigma_i^2 = -1, \qquad (\sigma_i\sigma_j)^2 = -1 \quad (|i-j| \geq 2), \qquad (\sigma_i\sigma_{i+1})^3 = -1.$$

Let $\tilde{A}_n$ denote the subgroup of $\tilde{S}_n$ that doubly covers the alternating group.

The irreducible representations of $\tilde{S}_n$ can be divided into two families; one consisting of representations in which -1 acts trivially (these are essentially equivalent to representations of S_n), and the other consisting of representations in which -1 acts as scalar multiplication by -1. The latter are known as the *spin representations* of S_n, and were first considered by Schur in 1911 [16]. We will briefly summarize here a few aspects of spin representations; for details and proofs, see [19]. (Other sources include [4], [8].)

For each partition ν of n, choose an element $\sigma_\nu \in \tilde{S}_n$ whose S_n-image is of cycle-type ν. Every $\sigma \in \tilde{S}_n$ will be conjugate to $\pm\sigma_\nu$ for some ν, so the character φ of any spin representation of S_n is completely determined by the values $\varphi(\sigma_\nu)$.

It turns out that σ_ν is conjugate to $-\sigma_\nu$ (and therefore $\varphi(\sigma_\nu) = -\varphi(\sigma_\nu) = 0$) unless either $\nu \in OP_n$, or else $n - \ell(\nu)$ is odd and $\nu \in DP_n$. Note that, in the former case, one has $\sigma_\nu \in \tilde{A}_n$, whereas in the latter case, one has $\sigma_\nu \in \tilde{S}_n - \tilde{A}_n$.

The irreducible spin characters of S_n can be indexed by partitions $\lambda \in DP_n$, although the indexing is not entirely one to one. When $n - \ell(\lambda)$ is even, λ indexes a single-spin character φ^λ; it is invariant under multiplication by the sign character. In such cases one therefore has $\varphi^\lambda(\sigma) = 0$ for $\sigma \notin \tilde{A}_n$, and $\varphi^\lambda(\sigma_\nu) \neq 0$ only if $\nu \in OP_n$. When $n - \ell(\lambda)$ is odd, there are two characters indexed by λ; they differ only by multiplication by the sign character. Thus for $\nu \in OP_n$, we may unambiguously write $\varphi^\lambda(\sigma_\nu)$ for the common value of these two characters at σ_ν. For $\nu \in DP_n$ with $n - \ell(\nu)$ odd, the two characters are both zero at σ_ν unless $\nu = \lambda$; in that case, the two values are

$$\varphi^\lambda(\sigma_\lambda) = \pm(-1)^{(n-\ell(\nu)+1)/4}\sqrt{z_\lambda/2}.$$

The main result of Schur's paper [16] is the following theorem, which shows that the representatives σ_ν can be chosen so that the nontrivial part of the character table of $\tilde{S}_n$ (i.e., the values $\varphi^\lambda(\sigma_\nu)$ for $\lambda \in DP_n$ and $\nu \in OP_n$) is encoded by the transition matrix between the Q-functions and the power sums.

THEOREM 6.1 (Schur). *If $\lambda \in DP_n$, then*

$$Q_\lambda = c_\lambda 2^{\ell(\lambda)/2} \sum_{\nu \in OP_n} \frac{1}{z_\nu} 2^{\ell(\nu)/2} \varphi^\lambda(\sigma_\nu) p_\nu,$$

where $c_\lambda = \sqrt{2}$ if $n - \ell(\lambda)$ is odd, and $c_\lambda = 1$ if $n - \ell(\lambda)$ is even.

On the other hand, Theorem 5.2 establishes that the transisition matrix between the Q_λ's and p_ν's is essentially $\xi^\lambda(w_\nu)$, aside from scalar factors. Hence, the character tables of $\tilde{S}_n$ and the Hecke algebra $\mathcal{H}_n^\varepsilon$ are closely related. To make this precise, we first note that

$$\begin{aligned}[Q_\lambda, p_\nu] &= 2^{-(n-\ell(\lambda)+\ell(\nu))} \frac{|B_n|^2}{g^\lambda} \langle E_\lambda, K_\nu \rangle \\ &= 2^{-(n-\ell(\lambda)+\ell(\nu))} \frac{|B_n|^2}{g^\lambda H_{\lambda^*}} \xi^\lambda(w_\nu) = 2^{n-\ell(\nu)} g^\lambda \xi^\lambda(w_\nu),\end{aligned}$$

by successive applications of Theorem 5.2, Proposition 4.2.c and (4.2). However, Theorem 6.1 implies that

$$[Q_\lambda, p_\nu] = c_\lambda 2^{(\ell(\lambda)-\ell(\nu))/2} \varphi^\lambda(\sigma_\nu),$$

and, in particular, since $[Q_\lambda, p_1^n] = g^\lambda$ (cf. (5.7)), we have

$$\deg(\varphi^\lambda) = 2^{\lfloor (n-\ell(\lambda))/2 \rfloor} g^\lambda.$$

Comparing the two expressions for $[Q_\lambda, p_\nu]$, we obtain

COROLLARY 6.2. *If $\lambda \in DP_n$ and $\nu \in OP_n$, then*

$$\varphi^\lambda(\sigma_\nu) = 2^{(n-\ell(\nu))/2}\, deg(\varphi^\lambda)\xi^\lambda(w_\nu).$$

There is a combinatorial rule for explicitly evaluating the scalar products $[Q_\lambda, p_\nu]$, and hence, for evaluating both spin characters and twisted spherical functions. The first version of this rule was given by Morris [13], although the formulation we will describe here is taken from [21].

Let $D'(\lambda)$ denote the shifted diagram of any $\lambda \in DP$, as in Section 4. A cell $(i,j) \in D'(\lambda)$ is said to belong to the *kth diagonal* if $j - i = k$. If $D'(\mu) \subseteq D'(\lambda)$, then the difference $D = D'(\lambda) - D'(\mu)$ is said to be a *border strip* if it is rookwise connected and contains at most one cell on each diagonal. The *height h(D)* is the number of nonempty rows. We say that D is a *double strip* if it is rookwise connected and the number of cells on the kth diagonal is a nonincreasing function of k, starting with two cells on the 0th diagonal. We define the height $h(D)$ of a double strip to be $|D - D_0|/2 + h(D_0)$, where D_0 denotes the border strip formed by the one-celled diagonals of D. For simplicity, we write $h(\lambda - \mu)$ for the height of any border strip or double strip of the form $D'(\lambda) - D'(\mu)$.

THEOREM 6.3. *If $\lambda \in DP_n$, $\nu \in OP_{n-r}$ and r is odd, then*

$$[Q_\lambda, p_{\nu\cup(r)}] = -2 \sum_{\alpha \in DP_{n-r}} (-1)^{h(\lambda-\alpha)}[Q_\alpha, p_\nu] - \sum_{\beta \in DP_{n-r}} (-1)^{h(\lambda-\beta)}[Q_\beta, p_\nu],$$

where the first sum is restricted to those α for which $D'(\lambda) - D'(\alpha)$ is a double strip, and the second sum is restricted to those β for which $D'(\lambda) - D'(\beta)$ is a border strip.

A proof can be found in Section 5 of [21].

As a second remark, we mention that there is an analogue of the Littlewood–Richardson rule for the multiplication of Q-functions [19]; i.e., there is a combinatorial rule for evaluating the structure constants $[Q_\mu Q_\nu, Q_\lambda]$, where $\lambda \in DP_n$, $\mu \in DP_k$, and $\nu \in DP_{n-k}$. In the context of spin characters, this amounts to a rule for specifying the irreducible decomposition of the restriction of φ^λ from $\tilde{S}_n$ to a double cover of $S_k \times S_{n-k}$.

By Theorem 5.2, these structure constants also arise in the algebra $\mathcal{H}^\epsilon$. Indeed, we have

$$E_\mu * E_\nu = \sum_{\lambda \in DP_n} b^\lambda_{\mu,\nu} E_\lambda,$$

where

$$b^\lambda_{\mu,\nu} \quad = \quad H_{\lambda^*}\langle E_\mu * E_\nu, E_\lambda\rangle$$

$$= 2^{2n-\ell(\mu)-\ell(\nu)-\ell(\lambda)} g^\mu g^\nu g^\lambda \frac{H_{\lambda^*}}{|B_n|^2}[Q_\mu Q_\nu, Q_\lambda]$$

$$= 2^{-(\ell(\mu)+\ell(\nu))}\frac{g^\mu g^\nu}{g^\lambda}[Q_\mu Q_\nu, Q_\lambda],$$

by successive applications of Proposition 4.2.b, Theorem 5.2, and (4.2).

Since $2^{-(\ell(\mu)+\ell(\nu))}[Q_\mu Q_\nu, Q_\lambda]$ is known to be a nonnegative integer [19, §8], it follows that $g^\lambda b^\lambda_{\mu,\nu}$ is also a nonnegative integer. Although we are unaware of any explanation of this intrinsic to Hecke algebras, we *can* deduce the nonnegativity of $b^\lambda_{\mu,\nu}$ directly. Indeed, since $e_{\mu^*} \circ e_{\nu^*}$ is a central idempotent for the subgroup $S_{2k} \times S_{2n-2k}$ of S_{2n}, Corollary 1.3 implies that the E_λ-expansion of $E_\mu * E_\nu = e(e_{\mu^*} \circ e_{\nu^*})e$ is nonnegative.

7. Remarks on zonal polynomials

The spherical functions for the Gelfand pair (S_{2n}, B_n) have been analyzed via techniques similar to those we developed in Sections 3-5 (see especially [1], [6], and [12, §5]). For the sake of comparison, we describe here the principal features of this analysis; the details are somewhat simpler than the twisted case.

In the following, 1_{B_n} denotes the trivial character of B_n, $e_0 = |B_n|^{-1}\sum_{x\in B_n} x$ denotes the corresponding idempotent of $\mathbf{C}B_n$, and $\mathcal{H}_n = e_0\mathbf{C}S_{2n}e_0$ denotes the Hecke algebra of the triple (S_{2n}, B_n, e_0).

Let P_n denote the set of partitions of n. The elements $C_\nu := e_0 w_\nu e_0 \in \mathcal{H}_n$ ($\nu \in P_n$) clearly form a basis of $\mathcal{H}_n$. On the other hand, the following well-known Schur function identity due to Littlewood [10, p. 238] (cf. also [11])

$$\prod_{i\le j}(1-x_ix_j)^{-1} = \sum_\lambda s_{2\lambda}(x_1, x_2, \ldots)$$

implies the induction rule

$$1_{B_n}^{S_{2n}} = \sum_{\lambda\in P_n}\chi^{2\lambda},$$

and therefore the elements

$$F_\lambda := e_0 e_{2\lambda} e_0 = e_{2\lambda} e_0 \in \mathcal{H}_n$$

form a basis of orthogonal idempotents for $\mathcal{H}_n$. The spherical function θ^λ corresponding to F_λ is given by

$$\theta^\lambda(w) := \chi^{2\lambda}(e_0 w e_0),$$

and the analogue of Proposition 4.1 is

$$F_\lambda = \frac{|B_n|^2}{H_{2\lambda}}\sum_{\nu\in P_n}\frac{1}{z_{2\nu}}\chi^{2\lambda}(C_\nu)C_\nu = \frac{1}{H_{2\lambda}}\sum_{w\in S_{2n}}\theta^\lambda(w)w. \tag{7.1}$$

As in the twisted case, we may impose a graded algebra structure on $\mathcal{H} := \oplus_{n\geq 0}\mathcal{H}_n$ by defining $a * b = e_0(a \circ b)e_0$ for all $a \in \mathcal{H}_i$ and $b \in \mathcal{H}_j$. In particular, one has

$$C_\mu * C_\nu = e_0(w_\mu \circ w_\nu)e_0 = C_{\mu\cup\nu}, \tag{7.2}$$

so the product is evidently commutative and associative.

Since $\mathcal{H}_n$ is invariant under the linear transformation of $\mathbf{C}S_{2n}$ induced by $w \mapsto w^{-1}$, it follows that

$$\langle a, b\rangle := [\mathrm{id}]a\bar{b}$$

defines an inner product on $\mathcal{H}_n$, just as it does for $\mathcal{H}_n^\varepsilon$. The orthogonality relations analogous to Proposition 4.2 are

$$\langle C_\alpha, C_\beta\rangle = \frac{z_{2\alpha}}{|B_n|^2}\delta_{\alpha,\beta} \tag{7.3a}$$

$$\langle F_\lambda, F_\mu\rangle = \frac{1}{H_{2\lambda}}\delta_{\lambda,\mu} \tag{7.3b}$$

$$\langle F_\lambda, C_\alpha\rangle = \frac{1}{H_{2\lambda}}\theta^\lambda(w_\alpha) \tag{7.3c}$$

for all partitions $\lambda, \mu, \alpha, \beta$ of n.

One may define a characteristic map $ch: \mathcal{H} \to \Lambda$ by setting

$$ch(C_\alpha) := p_\alpha.$$

By (7.2), this map is an isomorphism of graded algebras, and by (7.3a) one has

$$\langle a, b\rangle = \frac{1}{|B_n|^2}\langle ch(a), ch(b)\rangle_2,$$

where $\langle\cdot,\cdot\rangle_2$ denotes the inner product on Λ defined by

$$\langle p_\alpha, p_\beta\rangle_2 = 2^{\ell(\alpha)}z_\alpha\delta_{\alpha,\beta}.$$

The subscript "2" here is used to distinguish this from the usual inner product on Λ in which $\langle p_\alpha, p_\beta\rangle = z_\alpha\delta_{\alpha,\beta}$.

The characteristics of the idempotents F_λ are the symmetric functions known to statisticians as *zonal polynomials*. To be more explicit, let us define

$$Z_\lambda := \frac{H_{2\lambda}}{|B_n|}ch(F_\lambda) = \sum_{\nu\in P_n}\frac{|B_n|}{z_{2\nu}}\chi^{2\lambda}(C_\nu)p_\nu \in \Lambda^n$$

for all partitions λ of n. We may regard Z_λ as a polynomial function of an $m\times m$ symmetric matrix A by treating $Z_\lambda(A)$ as the value of Z_λ at the eigenvalues $x_1,\ldots, x_m$ of A (or equivalently, by identifying the power sum p_r with the matrix function $\mathrm{tr}(A^r)$). On the other hand, the action $A \mapsto XAX^t$ of $GL_m(\mathbf{R})$ on symmetric matrices extends to an action of $GL_m(\mathbf{R})$ on polynomial functions of symmetric matrices. In these terms, the zonal polynomials Z_λ for $\ell(\lambda) \le m$ are the unique polynomials (up to scalar multiplication) that (1) are invariant under the action of the orthogonal group, and (2) generate irreducible $GL_m(\mathbf{R})$-modules. This is essentially the content of Theorem 4 of [6].

There is also a characterization of the zonal polynomials analogous to Theorem 5.1; it is a consequence of the fact that zonal polynomials are the special case $\alpha = 2$ of the Jack polynomials [12] and [18]. Indeed, this was presumably one of the motivations for the original definition of Jack polynomials [5].

To describe this characterization, let us first define

$$\zeta_n = \sum_{\nu \in P_n} \frac{1}{z_{2\nu}} p_\nu,$$

and more generally, $\zeta_\mu := \zeta_{\mu_1}\zeta_{\mu_2}\cdots$ for any partition μ. Alternatively, one may define $\zeta_n(x_1, x_2, \ldots)$ as the coefficient of t^n in $\prod_{i\ge 1}(1 - x_i t)^{-1/2}$. It is not difficult to show that $\{\zeta_n : n \ge 1\}$ is an algebraically independent set of generators for Λ, so that $\{\zeta_\mu : \mu \in P_n\}$ is a basis of Λ^n.

THEOREM 7.1. *The symmetric functions Z_λ are the unique homogeneous basis of Λ satisfying:*

(a) $\langle Z_\lambda, Z_\mu\rangle_2 = H_{2\lambda}\delta_{\lambda,\mu}$ *for* $\lambda, \mu \in P_n$.
(b) $\langle \zeta_\mu, Z_\lambda\rangle_2 = 0$ *unless* $\lambda \ge \mu$.
(c) $\langle Z_\lambda, p_1^n\rangle_2 > 0$ *for* $\lambda \in P_n$.

Proof. Part (a) is a consequence of (7.3b) and the fact that the characteristic map is (essentially) an isometry. For (b), observe that in the special case $\lambda = (n), \chi^{2\lambda}$ is the trivial character of S_{2n}, so $\chi^{2\lambda}(C_\nu) = 1$ for all $\nu \in P_n$, and hence

$$Z_{(n)} = \sum_{\nu \in P_n} \frac{|B_n|}{z_{2\nu}} p_\nu = 2^n n! \zeta_n.$$

Since the characteristic map is an algebra isomorphism, it follows that

$$2^n \mu_1! \cdots \mu_l! \zeta_\mu = Z_{(\mu_1)} \cdots Z_{(\mu_l)} = \frac{(2\mu_1)!}{2^{\mu_1}\mu_1!} \cdots \frac{(2\mu_l)!}{2^{\mu_l}\mu_l!} ch(F_{(\mu_1)} * \cdots * F_{(\mu_l)})$$

for any partition $\mu = (\mu_1, \ldots, \mu_l)$ of n. Therefore, to prove (b) it suffices to show that

$$(F_{(\mu_1)} * \cdots * F_{(\mu_l)})F_\lambda = 0$$

unless $\lambda \ge \mu$. For this, let $K_{\alpha,\beta}$ denote the multiplicity of χ^α in the induction of the trivial character from $S_{\beta_1} \times S_{\beta_2} \times \cdots$ to S_{2n}. It is well-known (e.g., [11, p.

57]) that $K_{\alpha,\beta} = 0$ unless $\alpha \geq \beta$. Thus by (1.2), we have $(e_{(2\mu_1)} \circ \cdots \circ e_{(2\mu_l)})e_{2\lambda} = 0$ unless $2\lambda \geq 2\mu$ (or equivalently, $\lambda \geq \mu$), and therefore,

$$(F_{(\mu_1)} * \cdots * F_{(\mu_l)})F_\lambda = (e_{(2\mu_1)} \circ \cdots \circ e_{(2\mu_l)})e_{2\lambda}e_0 = 0.$$

unless $\lambda \geq \mu$. Finally, to prove (c), note that by (7.3c),

$$\langle Z_\lambda, p_1^n \rangle_2 = \frac{H_{2\lambda}}{|B_n|}\langle ch(F_\lambda), ch(C_{(1^n)})\rangle_2 = |B_n| \cdot H_{2\lambda}\langle F_\lambda, C_{(1^n)}\rangle = |B_n| \geq 0.$$

Since (a) and (b) uniquely determine the zonal polynomials up to a linear transformation that is both triangular and unitary, there can be only one basis that also satisfies (c). □

Unlike the twisted case discussed in Section 6, there is no known combinatorial rule for the explicit evaluation of the spherical functions θ^λ (or equivalently, for evaluating $\langle Z_\lambda, p_\nu\rangle_2$), nor is there any known rule for evaluating $\langle Z_\mu Z_\nu, Z_\lambda\rangle_2$, although Stanley has a conjecture about Jack symmetric functions [18, §8] that would imply that $\langle Z_\mu Z_\nu, Z_\lambda\rangle_2$ is a nonnegative integer.

We remark that the same reasoning used in Section 6 does show that $\langle Z_\mu, Z_\nu, Z_\lambda\rangle_2$ is nonnegative. Indeed, aside from a (positive) scalar multiple, these quantities also arise as the structure constants in the expansion

$$F_\mu * F_\nu = \sum_\lambda a_{\mu,\nu}^\lambda F_\lambda;$$

by Corollary 1.3, these must be nonnegative.

We should note that the nonnegativity of $\langle Z_\mu Z_\nu, Z_\lambda\rangle_2$ is also a direct consequence of the interpretation of Z_λ as a spherical function for the Gelfand pair $(GL_m(\mathbf{R}), O_m(\mathbf{R}))$ [9].

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