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A Decomposition of the Descent Algebra of a Finite Coxeter Group

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Abstract. The purpose of this paper is twofold. First we aim to unify previous work by the first two authors, A. Garsia, and C. Reutenauer (see [2], [3], [4], [5] and [10]) on the structure of the *descent algebras* of the Coxeter groups of type A_n and B_n. But we shall also extend these results to the descent algebra of an arbitrary finite Coxeter group W. The descent algebra, introduced by Solomon in [14], is a subalgebra of the group algebra of W. It is closely related to the subring of the Burnside ring $B(W)$ spanned by the permutation representations W/W_J, where the W_J are the parabolic subgroups of W. Specifically, our purpose is to lift a basis of primitive idempotents of the *parabolic Burnside algebra* to a basis of idempotents of the descent algebra.

Keywords: Coxeter groups, idempotents, descent algebra.

1. Introduction

Let (W, S) be a finite Coxeter system. That is, W is a finite group generated by a set S subject to the defining relations

$$(sr)^{m_{sr}} = 1 \quad \text{for all} \quad s, r \in S$$

where m_{sr} are positive integers and $m_{ss} = 1$ for all $s \in S$.

As is well known, W is faithfully represented in the orthogonal group of an inner product space V, which has a basis $\Pi = \{\alpha_s | s \in S\}$ in bijective correspondence with S. The inner product is given by

$$(\alpha_s, \alpha_r) = -\cos(\pi/m_{sr})$$

and the action of W by

$$s(v) = v - 2(\alpha_s, v)\alpha_s$$

for all $r, s \in S$ and $v \in V$. Thus s acts as the reflection in the hyperplane orthogonal to α_s, and, consequently V is called the *reflection representation* of W.

One easily checks that for all $s, r \in S$ we have $\alpha_r = \pm w(\alpha_s)$ in V if and only if $r = wsw^{-1}$ in W.

We call the set $\Phi = \{w(\alpha) | w \in W, \alpha \in \Pi\}$ the *root system* of W, and Π the set of *fundamental* roots. It is well known (see [7]) that Φ can be decomposed as $\Phi = \Phi^+ \uplus \Phi^-$, where every element of Φ^+ (resp. Φ^-) is a linear combination of fundamental roots with coefficients all nonnegative (resp. all nonpositive). Moreover, if $w \in W$ and $\ell(w)$ denotes the length of a minimal expression for w in terms of elements of S, then $\ell(w)$ equals the cardinality of the set $N(w)$, where

$$N(w) = \{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-\}$$

Note that $\ell(vw) = \ell(v) + \ell(w)$ if and only if $N(vw) = w^{-1}(N(v)) \uplus N(w)$.

For each $J \subseteq \Pi$, the *standard parabolic subgroup* W_J is the subgroup of W generated by

$$S_J = \{s \in S | \alpha_s \in J\}$$

Then (W_J, S_J) is also a Coxeter system. If V_J is the subspace of V spanned by J, then the W-action on V yields a W_J-action on V_J, which can be identified with the reflection representation of W_J. The root system of W_J is $\Phi_J = \Phi \cap V_J$; and we write Φ_J^+ for $\Phi^+ \cap V_J$ for Φ_J^- and $\Phi^- \cap V_J$. It is easily shown that $N(w) \subseteq \Phi_J^+$ if and only if $w \in W_J$.

In this paper we study the *descent algebra* (or *Solomon algebra*) $\sum(W)$ of a Coxeter group W. If $w \in W$, then the *descent set* of w is defined to be

$$D(w) = N(w) \cap \Pi = \{\alpha \in \Pi | w(\alpha) \in \Phi^-\}$$

In terms of the generating set S this corresponds to $\{s \in S | \ell(ws) < \ell(w)\}$. If $J \subseteq \Pi$, let

$$X_J = \{w \in W | D(w) \cap J = \emptyset\} = \{w \in W | w(J) \subseteq \Phi^+\}$$

and let

$$x_J = \sum_{w \in X_J} w$$

Define $\sum(W)$ to be the subspace of $\mathbf{Q}(W)$ spanned by all such elements x_J (which are clearly linearly independent).

It has been shown by Solomon [14] that $\sum(W)$ is a subalgebra of $\mathbf{Q}(W)$. More precisely, Solomon has shown that

$$x_J x_K = \sum_{L \subseteq K} a_{JKL} x_L, \tag{1}$$

where

$$a_{JKL} = |\{w \in X_J^{-1} \cap X_K | w^{-1}(J) \cap K = L\}|.$$

In Section 2 we shall prove these facts using techniques that will be developed further in later sections. It is easily shown (Solomon [14]) that the x_K's are linearly independent; thus they form a basis of $\sum(W)$.

In [10] A. Garsia and C. Reutenauer have given a decomposition of the multiplicative structure of the descent algebra of the symmetric group (the Coxeter group of type A_n). This decomposition exploits the action of the symmetric group on the free Lie algebra in a manner reminiscent of the Poincaré–Birkhoff–Witt theorem. In [2] and [5] we showed that a similar decomposition, as well as related results, also holds for the hyperoctahedral group (type B_n). The object of this paper, and ongoing work, is to extend these results to the descent algebra of any finite Coxeter group.

For a general descent algebra $\sum(W)$ we shall exhibit a new basis consisting of elements e_k, $K \subseteq \Pi$, which are scalar multiples of idempotents, and satisfy $\sum_{K\subseteq\Pi} \mu_K^{\Pi} e_K = 1$ for certain positive constants μ_K^{Π}. Furthermore, for all $J, M \subseteq \Pi$, when $e_J e_M$ is expressed as a linear combination of the e_K's, the only nonzero coefficients correspond to subsets K of M that are equivalent to J, in the sense that $J = w(K)$ for some $w \in W$. As a consequence we obtain a set of idempotents $E_\lambda = \sum_{k\in\lambda} \mu_K^{\pi} e_K$ indexed by equivalence classes λ of subsets of Π, such that

$$E_\lambda E_\mu = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ E_\lambda & \text{if } \lambda = \mu \end{cases} \tag{2}$$

and $\sum_\lambda E_\lambda = 1$. In fact, the E_λ's form a decomposition of the identity into primitive idempotents, and hence the right ideals of $\sum(W)$ which they generate are a full set of indecomposable projective right modules for $\sigma(W)$. Furthermore, the E_λ's induce a decomposition of the action of $\sum(W)$ on $\mathbf{Q}(W)$ by left multiplication:

$$\mathbf{Q}(W) = \bigoplus_\lambda H_\lambda$$

where $H_\lambda = E_\lambda \cdot \mathbf{Q}(W)$. We shall compute the dimension of H_λ in Section 7. Our calculations in Section 7 also show that the unique maximal submodule of $E_\lambda \sum(W)$ is spanned by the differences $e_J - e_K$ for $J, K \in \lambda$. This gives an alternative proof of Solomon's result that the radical of $\sum(W)$ is the subspace spanned all elements $x_J - x_K$ for J and K equivalent subsets of Π. Thus

$$\dim\left(\sqrt{\sum(W)}\right) = 2^{|\pi|} - |\Lambda|$$

where Λ is the set of equivalence classes of subsets of Π.

These constructions have already been carried through for all indecomposable finite Coxeter groups of type A_n (see [10]), and of type B_n (see [2] and [5]). Part of the study of the descent algebra has been carried through with extensive use of the computer algebra system Maple [3].

2. The Solomon Algebra

We start by proving some basic facts concerning the elements x_J defined in Section 1. Proofs of results we assume can be found in Section 2.7 of Carter [7].

If $J \subseteq \Pi$, then each element of W is uniquely expressible in the form du with $d \in X_J$ and $u \in W_J$, and here we have $\ell(du) = \ell(d) + \ell(u)$. Thus X_J is a set of representatives of the cosets wW_J in W. Likewise, if $K \subseteq J \subseteq \Pi$, then $X_J \cap W_J$ is a set of representatives of the cosets wW_K in W_J. In this situation we define

$$x_K^J = \sum_{w \in W_J \cap X_K} w$$

and note that $x_K^\Pi = x_K$. The next two lemmas provide analogues of induction and restriction for Solomon algebras. The connection with induction and restriction of permutation characters will be given in detail in Section 4.

LEMMA 2.1. *If* $K \subseteq J \subseteq \Pi$ *then* $X_K = X_J(W_J \cap X_K)$ *and thus* $x_K = x_J x_K^J$.

Proof. If $d \in X_J$ and $w \in W_J \cap X_K$, then $w(K) \subseteq \Phi_J^+$, whence $dw(K) \subseteq d(\Phi_J^+) \subseteq \Phi^+$. It follows that $dw \in X_K$ and this shows that

$$\{dw | d \in X_J, w \in W_J \cap X_K\} \subseteq X_K$$

Since the number of products dw is $|W : W_J||W_J : W_K| = |X_K|$ we see that equality holds; and, on taking sums, we have $x_K = x_J x_K^J$. □

LEMMA 2.2. *For all* $J, K \subseteq S$

$$X_K = \biguplus_{d \in X_{JK}} (W_J \cap X_{J \cap d(K)})d,$$

where $X_{JK} = X_J^{-1} \cap X_K$*; and thus*

$$x_K = \sum_{d \in X_{JK}} x_{J \cap d(K)}^J d.$$

Proof. First note that if $d \in X_{JK}$ and $u \in W_J \cap X_{J \cap d(K)}$, then $d \in X_J^{-1}$ and $u \in W_J$; so an element of W can arise as a product ud in at most one way.

Let $w \in X_K$ and write $w = ud$ with $d \in X_J^{-1}$ and $u \in W_J$. Since $\ell(ud) = \ell(u) + \ell(d)$ we have $N(d) \subseteq N(ud) = N(w)$, and so $d \in X_K$. Thus $d \in X_{JK}$, and futhermore

$$u(J \cap d(K)) \subseteq ud(K) = w(K) \subseteq \Phi^+$$

so that $u \in W_{J \cap d(K)}$.

It remains to prove that $ud \in X_K$ whenever $d \in X_{JK}$ and $u \in W_J \cap X_{J \cap d(K)}$. Since a fundamental root cannot be nontrivially expressed as a positive linear combination of positive roots, we see that $K \cap d^{-1}(\Phi_J^+) = K \cap d^{-1}(J)$. But $d(K) \subseteq \Phi^+$ (since $d \in X_K$) and so $d(K) \subseteq (\Phi^+ \backslash \Phi_J^+) \cup (J \cap d(K))$. It follows that $ud(K) \subseteq u(\Phi^+ \backslash \Phi_J^+) \cup u(J \cap d(K) \subseteq \Phi^+$, since $u(\Phi^+ \backslash \Phi_J^+) \subseteq \Phi^+$ for $u \in W_J$, and therefore $ud \in X_K$, as required. □

Lemma 2.2 shows that each element of W is uniquely expressible in the form udw with $w \in W_K, d \in X_{JK}$ and $u \in W_J \cap X_{J\cap d(K)}$. Moreover, in this situation $\ell(udw) = \ell(u) + \ell(d) + \ell(w)$. It follows readily that each double coset $W_J w W_K$ contains a unique $d \in X_{JK}$, and that $W_J \cap dW_K d^{-1} = W_{J\cap d(K)}$.

For $J, K \subseteq \Pi$ we write $J \sim K$ whenever $w(J) = K$ for some $w \in W$ (that is, J and K are equivalent) and $J \preceq K$ whenever J is equivalent to a subset of K. The next lemma shows that this equivalence relation is the one used by Solomon in [14].

LEMMA 2.3. *If $J, K \subseteq \Pi$, then $J \sim K$ if and only if W_J and W_K are conjugate, and $J \preceq K$ if and only if W_J is conjugate to a subgroup of W_K.*

Proof. Suppose that $w \in W$ satisfies $w^{-1}W_J w \subseteq W_K$. If d is the shortest element in $W_J w W_K$, then $d^{-1}W_J d \subseteq W_K$, and therefore

$$W_{J\cap d(K)} = W_J \cap dW_K d^{-1} = W_J.$$

Thus $J \cap d(K) = J$ and therefore $d^{-1}(J) \subseteq K$. All assertions of the lemma now follow. □

LEMMA 2.4. *If $J \subseteq \Pi$ and $d \in W$ with $d^{-1}(J) \subseteq \Pi$, then $X_J d = X_{d^{-1}(J)}$.*

Proof. For $w \in X_{d^{-1}(J)}$, it is clear that $wd^{-1} \in X_J$, and conversely for $w \in X_J$, that $wd \in X_{d^{-1}(J)}$. □

The following result due to Solomon is easily derived from these lemmas.

THEOREM 2.5. (Solomon) *For all $J, K \subseteq \Pi$*

$$x_J x_K = \sum_{L \subseteq K} a_{JKL} x_L$$

Proof.

$$\begin{aligned} x_J x_K &= x_J \sum_{d \in X_{JK}} x^J_{J\cap d(K)} d && \text{by Lemma 2.2} \\ &= \sum_{d \in X_{JK}} x_{J\cap d(K)} d && \text{by Lemma 2.1} \\ &= \sum_{d \in X_{JK}} x_{d^{-1}(J)\cap K} && \text{by Lemma 2.4} \\ &= \sum_{L} a_{JKL} x_L \end{aligned}$$

Obviously $a_{JKL} = 0$ when $L \not\subseteq K$. Thus the theorem is proved. □

PROPOSITION 2.6. *Let a^M_{LJK} denote the structure constants of the descent algebra $\sum(W_M)$ corresponding to the x^M_N basis. If $J, N \subseteq \Pi$, then*

$$x_N x_J = \sum_{K \subseteq J} \left(\sum_{L \subseteq M} a_{NML} a^M_{LJK} \right) x_K$$

for all $M \subseteq \Pi$ such that $J \subseteq M$. Thus the structure constants satisfy the identities

$$a_{NJK} = \sum_{L \subseteq M} a_{NML} a^M_{LJK}$$

for all M containing J.

Proof. We have

$$\begin{aligned} x_N x_J &= x_N x_M x^M_J \\ &= \left(\sum_{L \subseteq M} a_{NML} x_L \right) x^M_J \\ &= \sum_{L \subseteq M} a_{NML} x_M x^M_L x^M_J \\ &= \sum_{L \subseteq M} a_{NML} x_M \left(\sum_{K \subseteq J} a^M_{LJK} x^M_K \right) \\ &= \sum_{\substack{L,K \\ L \subseteq M, K \subseteq J}} a_{NML} a^M_{LJK} x_K. \end{aligned}$$

This proves the first assertion of the theorem, and comparison with

$$x_N x_J = \sum_{K \subseteq J} a_{NJK} x_K$$

completes the proof. □

3. Reduction to indecomposable finite Coxeter groups

Throughout this section we suppose that J and K are mutually orthogonal subsets of Π such that $\Pi = J \cup K$. In this case $W = W_J \times W_K$ and $\mathbf{Q}W \simeq \mathbf{Q}W_J \otimes \mathbf{Q}W_K$. We shall show that a similar decomposition holds for the Solomon algebra of W.

LEMMA 3.1. *For $L \subseteq J$ and $M \subseteq K$ we have*

$$x^J_L x^K_M = x_{L \cup M}.$$

Proof. Given $d \in X_{L\cup M}$ there is a unique decomposition $d = d_J d_K$ with $d_J \in W_J$ and $d_K \in W_K$. As d_K fixes every element of J, it follows that

$$d_J(L) = d_J d_K(L) = d(L) \subseteq \Phi^+$$

and hence $d_J \in X_L^J$. Similarly, $d_K \in X_M^K$.

Conversely, if $d_J \in X_L^J$ and $d_K \in X_M^K$, then

$$d_J d_K(L \cup M) \subseteq d_J(L \cup \Phi_K^+) \subseteq \Phi^+$$

whence $d_J d_K \in X_{L\cup M}$. It follows that $X_{L\cup M} = X_L^J \times X_M^K$ and that $x_{L\cup M} = x_L^J x_M^K$. □

As an immediate consequence of this lemma, we have

PROPOSITION 3.2. *The function* $\varphi : \sum(W_J) \otimes \sum(W_K) \to \sum(W)$ *defined by* $\varphi(u \otimes v) = uv$, *is an isomorphism of algebras.*

This shows that we may reduce our discussion to the indecomposable finite Coxeter groups.

4. The parabolic Burnside ring

For each $J \subseteq \Pi$ we have a permutation representation of W on the set W/W_J of cosets $W_J w$. The orbits of W on $W/W_J \times W/W_K$ have representatives of the form $(W_J d, W_K)$, where $d \in X_{JK}$; and the stablizer of $(W_J d, W_K)$ in W is $d^{-1} W_J d \cap W_K = W_{d^{-1}(J)\cap K}$. Thus

$$W/W_J \times W/W_K = \sum_{L \subseteq K} a_{JKL} W/W_L \tag{3}$$

where the a_{JKL}'s are defined as in Section 1. This proves that the representations W/W_J span a subring $\mathcal{PB}(W)$ of the Burnside ring of W. We call this the *parabolic Burnside ring* of W. On comparing (3) and (1) we see that there is a homomorphism $\theta : \sum(W) \to \mathcal{PB}(W)$ which takes x_J to the element of $\mathcal{PB}(W)$ represented by W/W_J. Note that θ is not, in general an isomorphism because W/W_J and W/W_K represent the same element of $\mathcal{PB}(W)$ whenever $J \sim K$.

A subgroup of W is said to be *parabolic* if it is conjugate to a standard parabolic subgroup W_J for some $J \subseteq \Pi$. For each $v \in V$, the stabilizer in W of v,

$$\mathrm{Stab}_W(v) = \{w \in W | w(v) = v\}$$

is a parabolic subgroup. Indeed, the set

$$C = \{u \in V | (\alpha, u) \geq 0 \quad \text{for all} \quad \alpha \in \Pi\}$$

is a fundamental domain for the action of W, and we may choose $t \in W$ such that $t(v) \in C$. Then (see Steinberg [15])

$$t\,\mathrm{Stab}_w(v)t^{-1} = \mathrm{Stab}_w(t(v)) = W_J,$$

where $J = \{\alpha \in \Pi | (\alpha, t(v)) = 0\}$.

Since W_J stabilizes $J^\perp$ it follows that $w \in W_J$ stabilizes $v \in V$ if and only if it stabilizes the orthogonal projection of v in V_J. Hence $\mathrm{Stab}_{W_J}(v)$ is a parabolic subgroup of W_J. It follows by induction that the pointwise stabilizer, $\mathrm{Stab}_W(P)$, of an arbitrary subset P of V, is a parabolic subgroup of W. Since $\mathrm{Stab}_W(P \cup Q) = \mathrm{Stab}_W(P) \cap \mathrm{Stab}_W(Q)$ we see that the intersection of two parabolic subgroups is again parabolic; this also follows from the fact, mentioned in Section 2, that $W_J \cap dW_K d^{-1} = W_{J \cap d(K)}$ whenever $d \in X_{JK}$.

If g is an arbitrary orthogonal transformation on V, define

$$[V, g] = \{(1 - g)(v) | v \in V\}$$

and

$$C_v(g) = \{v \in V | g(v) = v\}$$

and let $\tau(g) = \dim[V, g]$. It is easily checked that $[V, g]$ is the orthogonal complement of $C_v(g)$ in V. Furthermore, if $0 \neq v \in V$ and r is the reflection in the hyperplane orthogonal to v, then

$$\tau(rg) = \begin{cases} \tau(g) + 1 & \text{if } v \notin [V, g] \\ \tau(g) - 1 & \text{if } v \in [V, g] \end{cases} \tag{4}$$

Thus $\tau(g)$ is the length of a minimal expression for g as a product of reflections. In [7] Carter proves that every element $w \in W$ can be written as a product of $\tau(w)$ reflections in W. (We include a proof in Lemma 4.3.)

Following Solomon [14], for $w \in W$, we define

$$A(w) = \{y \in W | [V, y] \subseteq [V, w]\} = \{y \in W | C_V(w) \subseteq C_V(y)\}.$$

Equivalently, $A(w) = \mathrm{Stab}_W(C_V(w))$. In particular, $A(w)$ is a parabolic subgroup of W. We say that w is of *type* J if $A(w)$ is conjugate to W_J. We shall sometimes say that w is of type λ, where λ is the equivalence class of J, since (by Lemma 2.3) J is determined by w only to within equivalence. It is clear that $A(twt^{-1}) = tA(w)t^{-1}$, and hence conjugate elements have the same type.

Observe that the maps $P \mapsto \mathrm{Stab}_W(P)$ and $H \mapsto C_V(H)$, where H is a subgroup of W, form a Galois connection between the partially ordered set of subspaces of V and the partially ordered set of subgroups of W, in the sense that $P \subseteq C_V(H)$ if and only if $H \subseteq \mathrm{Stab}_W(P)$. The parabolic subgroups are the closed subgroups of W for this Galois connection; that is, H is parabolic if and only if $H = \mathrm{Stab}_W(C_V(H))$. Thus if H is any subgroup of W, then $\mathrm{Stab}_W(C_V(H))$

is the smallest parabolic subgroup of W containing H. In particular, if $w \in W$, then $A(w)$ is the smallest parabolic subgroup containing w, and so w is of type J if and only if $J \subseteq \Pi$ is minimal subject to W_J containing a conjugate of w.

LEMMA 4.3. *Let $J \subseteq \Pi$ and suppose that $w \in W$ is of type J. Then*

(1) if $K \subseteq \Pi$ and W_K contains a conjugate of w, then $J \preceq K$,
(2) $\tau(w) = |J|$,
(3) w can be written as a product of $|J|$ reflections in W.

Proof. Replacing w by a conjugate of itself, we may assume that $w \in W_J$. Since w has type J it is not contained in any proper parabolic subgroup of W_J.

If $t \in W$ and $t^{-1}wt \in W_K$, then $w \in W_J \cap tW_Kt^{-1}$, a parabolic subgroup of W_J. It follows that $W_J \cap tW_Kt^{-1} = W_J$. Now Lemma 3 gives $J \preceq K$, proving (1).

The generators of W_J all fix $J^{\perp}$ pointwise, and so $J^{\perp} \subseteq C_V(w)$. Taking orthogonal complements gives $[V, w] \subseteq V_J$. If $[V, w] \neq V_J$, we deduce that V_J contains a nonzero $v \in C_V(w)$, and hence that $w \in \mathrm{Stab}_{W_J}(v)$, a proper parabolic subgroup of W_J. This is a contradiction, and therefore $[V, w] = V_J$. Thus

$$\tau(w) = \dim[V, w] = \dim V_J = |J|$$

proving (2).

Since $[V, w] = V_J$ it follows from (4) that $\tau(sw) = \tau(w) - 1$ whenever $s \in S_J$. Hence sw has type K for some $K \subseteq \Pi$ with $|K| = |J| - 1$. Arguing by induction we deduce that sw is a product of $|J| - 1$ reflections in W, and therefore $w = s(sw)$ is a product of $|J|$ reflections. □

For $J \subseteq S$, let c_J be the product of the reflections $s, s \in S_J$, taken in some fixed order. The conjugacy class of c_J in W_J is independent of the order, and the elements of this class are called the *Coxeter elements* of W_J. Since J is a linearly independent set it is clear that $[V, c_J] = V_J$, and so c_J has type J. We note as a consequence that the parabolic subgroups of W are precisely the subgroups $A(w)$.

PROPOSITION 4.4. *If $J, K \subseteq \Pi$, then c_J is conjugate to c_K if and only if $J \sim K$.*

Proof. If c_J and c_K are conjugate, then they are of the same type—that is, $J \sim K$. Conversely, if $J = d(K)$ for some $d \in W$, then $dS_Jd^{-1} = S_K$, and so dc_Jd^{-1}, being a product of the reflections in S_K, is conjugate to c_K. □

Let $\varphi_J = \mathrm{Ind}_{W_J}^{W} 1$, the character of W induced from the trivial character of W_J. In other words, φ_J is the character corresponding to the permutation representation W/W_J.

THEOREM 4.5. *The assignment $W/W_J \mapsto \varphi_J$ defines an isomorphism Θ from $\mathcal{PB}(W)$ to the ring of* **Q**-*linear combinations of the φ_J. Thus we may identify $\mathcal{PB}(W)$ with this ring of class functions.*

Proof. If $J \sim K$, the representations $W/_{W_J}$ and W/W_K are equal in $\mathcal{PB}(W)$ and hence

$$\varphi_J = \Theta(W/W_J) = \Theta(W/W_K) = \varphi_K$$

This makes it legitimate to write φ_λ instead of φ_J where λ is the equivalence class of J. For each equivalence class μ choose an element c_μ of type μ: for example, a Coxeter element. Since W_J contains an element of type K if and only if $K \preceq J$ it is clear that $\varphi_\lambda(c_\mu) \neq 0$ if and only if $\mu \preceq \lambda$. For a suitable ordering of the rows and columns, the matrix $\varphi_\lambda(c_\mu))_{\lambda,\mu}$ is upper triangular with nonzero diagonal entries. Therefore the φ_λ are linearly independent. □

Induction and restriction of characters give rise to maps between $\mathcal{PB}(W_J)$ and $\mathcal{PB}(W)$. For the case of induction, the permutation representation W_J/W_K in $\mathcal{PB}(W_J)$ induced to $\mathcal{PB}(W)$ is simply W/W_K. By Lemma 2.1 the analogue of induction for the Solomon algebras is left multiplication by x_J. More generally, if $J \subseteq M$, define $\mathrm{Ind}_J^M : \sum(W_J) \to \sum(W_M)$ by

$$\mathrm{Ind}_J^M(x) = x_J^M x$$

The restriction of W_M/W_K to $\mathcal{PB}(W_J)$ is obtained by considering the orbits of W_J on the cosets $W_K d$ in W_M. Thus

$$\mathrm{Res}_{W_J}(W_M/W_K) = \sum_{d \in W_M \cap X_{KJ}} W_J/W_{J \cap d^{-1}(K)}$$

In view of this formula and Lemma 2.2 we define $\mathrm{Res}_J^M : \sum(W_M) \to \sum(W_J)$ by

$$\mathrm{Res}_J^M(x_K^M) = \sum_{d \in W_M \cap X_{KJ}} x^J_{J \cap d^{-1}(K)} = \sum_{d \in W_M \cap X_{JK}} x^J_{J \cap d(K)}$$

The following proposition is immediate from this discussion.

PROPOSITION 4.6. *Given $J \subseteq M \subseteq \Pi$, let θ_J, and θ_M be the canonical homomorphisms from $\sum(W_J)$ and $\sum(W_M)$ to $\mathcal{PB}(W_J)$ and $\mathcal{PB}(W_M)$ respectively. Then*

$$\theta_M \cdot Ind_J^M = Ind_{W_J}^{W_M} \cdot \theta_J$$

and

$$\theta_J \cdot Res_J^M = Res_{W_J} \cdot \theta_M$$

In this context we see that Theorem 2.5 is the Solomon algebra analogue of the Mackey formula for the product of induced characters (Solomon [14]).

5. Dihedral groups

In this section we give a complete description of the descent algebra of Coxeter group of type $I_2(p)$. That is, we take W to be dihedral group with generating set $S = \{r, s\}$ and relations

$$r^2 = s^2 = (sr)^p = 1$$

The descent algebra of W has dimension 4 and it has a basis $x_\emptyset, x_r, x_s, x_\Pi$, where $\Pi = \{\alpha_r, \alpha_s\}$ is the set of fundamental roots of W and where x_r (resp. x_s) denotes $x_{\{\alpha_r\}}$(resp. $x_{\{\alpha_s\}}$). More explicitly, we have

$$\begin{aligned} x_\Pi &= 1 \\ x_r &= 1 + s + rs + srs + rsrs + \cdots \\ x_s &= 1 + r + sr + rsr + srsr + \cdots \\ x_\emptyset &= \sum_w w \end{aligned}$$

The summation for x_r(resp. x_s) is over the set of all $w \in W$ with only one reduced expression, this unique expression must also end in r(resp. s). The multiplication table for $\sum(W)$ is easy to compute explicitly in this case. When p is even it is

	1	x_r	x_s	$x_\emptyset$
1	1	x_r	x_s	$x_\emptyset$
x_r	x_r	$2x_r + \frac{p-2}{2}x_\emptyset$	$\frac{p}{2}x_\emptyset$	$px_\emptyset$
x_s	x_s	$\frac{p}{2}x_\emptyset$	$2x_s + \frac{p-2}{2}x_\emptyset$	$px_\emptyset$
$x_\emptyset$	$x_\emptyset$	$px_\emptyset$	$px_\emptyset$	$2px_\emptyset$

whereas for p odd it is

	1	x_r	x_s	$x_\emptyset$
1	1	x_r	x_s	$x_\emptyset$
x_r	x_r	$x_r + \frac{p-1}{2}x_\emptyset$	$x_s + \frac{p-1}{2}x_\emptyset$	$px_\emptyset$
x_s	x_s	$x_r + \frac{p-1}{2}x_\emptyset$	$x_s + \frac{p-1}{2}x_\emptyset$	$px_\emptyset$
$x_\emptyset$	$x_\emptyset$	$px_\emptyset$	$px_\emptyset$	$2px_\emptyset$

Using these tables, one can verify that for p even

$$\begin{aligned} e_{\Pi} &= 1 - 1/2x_r - 1/2x_s + \frac{p-1}{2p}x_\emptyset \\ e_r &= 1/2(x_r - 1/2x_\emptyset) \\ e_s &= 1/2(x_s - 1/2x_\emptyset) \\ e_\emptyset &= \frac{1}{2p}x_\emptyset \end{aligned} \tag{5}$$

are mutually orthogonal idempotents whose sum is 1. In this case the algebra $\sum(W)$ is semisimple and each equivalence class of subsets of Π is just a singleton. Thus the idempotents E_λ referred to in (2) can be identified with the idempotents listed above.

When p is odd, we obtain idempotents

$$\begin{aligned} e_{\Pi} &= 1 - 1/2x_r - 1/2x_s + \frac{p-1}{2p}x_\emptyset \\ e_r &= x_r - 1/2x_\emptyset \\ e_s &= x_s - 1/2x_\emptyset \\ e_\emptyset &= \frac{1}{2p}x_\emptyset \end{aligned} \tag{6}$$

In this case there are only three equivalence classes of subsets of Π: Π, $\{\{\alpha_r\}, \{\alpha_s\}\}$ and $\{\emptyset\}$. The only nonzero products between distinct e_K's are

$$e_s e_r = e_r \quad \text{and} \quad e_r e_s = e_s$$

Thus the E_λ's of (1.2) can be taken to be

$$e_\Pi, 1/2(e_r + e_s), \quad \text{and} \quad e_\emptyset$$

The radical of $\sum(W)$ is spanned by the nilpotent element $e_r - e_s$.

In preparation for Section 7, we reconsider part of this construction in the context of a general Coxeter system (W, S). For two elements $r, s \in S$ we compute the product $x_s x_r$. (Again, we abbreviate $x_{\{\alpha_r\}}$ to x_r). A direct application of (1) gives

$$x_s x_r = \rho_s^r x_r + \pi_s^r x_\emptyset \tag{7}$$

where $\rho_s^r = |\{w | \alpha_s = w(\alpha_r)\}|$.

Observe that for any $f = \sum_w f_w w$ in $\mathbf{Q}(W)$, we have $f x_\emptyset = x_\emptyset f = (\sum_w f_w) x_\emptyset$. In particular, $x_r x_\emptyset = \frac{1}{2}|W| x_\emptyset$ and multiplying (7) by $x_\emptyset$ gives

$$1/2\rho_s^r + \pi_s^r = 1/4|W|$$

Thus (7) becomes

$$x_s x_r = \rho_s^r (x_r - 1/2x_\emptyset) + 1/4|W| x_\emptyset \tag{8}$$

This identity suggests that for any Coxeter group we set

$$e_s = \frac{1}{\rho_s^s}(x_s - 1/2x_\emptyset)$$

It then follows easily that e_s is an idempotent, and (more generally),

$$e_s e_r = \frac{\rho_s^r}{\rho_s^s} e_r$$

Clearly, if s and r are not conjugate, then $\rho_s^r = 0$. But if they are conjugate, then there exists $d \in W$ such that $\alpha_s = d(\alpha_r)$ and then

$$\rho_s^r = |\{w \in W | \alpha_s = w(\alpha_r)\}| = |\mathrm{Stab}_W(\alpha_s)d| = \rho_s^s$$

Note that $\mathrm{Stab}_W(\alpha_s)$ is a subgroup of index 2 in the centralizer of s. Thus we have

$$e_s e_r = \begin{cases} e_r & \text{if } s \text{ and } r \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}$$

Let $\lambda(s)$ denote the equivalence class of $\alpha_s \in \Pi$. From the calculations just completed we conclude the following.

PROPOSITION 5.5. *In any Coxeter group, for all $s \in S$, the elements*

$$E_{\lambda_s} = \frac{1}{|\lambda(s)|} \sum_{\alpha_r \in \lambda(s)} e_r$$

are idempotents, and if s and r are not conjugate, then

$$E_{\lambda(s)} E_{\lambda(r)} = 0$$

We generalize this result to all equivalence classes λ in Section 7.

6. Idempotents in the parabolic Burnside ring

The **Q**-algebra $\mathcal{PB}(W)$ is isomorphic to an algebra of functions, and therefore it has a basis of idempotent elements. Specifically, if we define

$$\xi_\lambda = \sum_\mu \nu_{\lambda\mu} \varphi_\mu$$

where the coefficient matrix $(\nu_{\lambda\mu})$ is the inverse of the matrix $(\varphi_\lambda(c_\mu))$ that appears in the proof of Theorem 4.5, then

$$\xi_\lambda(c_\mu) = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ 1 & \text{if } \lambda = \mu \end{cases} \tag{9}$$

and it follows that ξ_λ is idempotent. The next theorem shows that (9) holds when c_μ is an arbitrary element of type μ.

THEOREM 6.2. *Let $J, K \subseteq \Pi$ and let $c \in W$ be any element of type J. Then $\varphi_K(c) = a_{KJJ}$, the number of $d \in X_{KJ}$ such that $d(J) \subseteq K$. In particular, a_{KJJ} depends only on the equivalence classes of K and J.*

Proof. Without loss of generality we may suppose that $c \in W_J$. By Mackey's formula, the restriction of φ_K to W_J is

$$\mathrm{Res}_{W_J}\left(\mathrm{Ind}_{W_K}^{W} 1\right) = \sum_{d \in X_{KJ}} \mathrm{Ind}_{W_{d^{-1}(K)\cap J}}^{W_J} 1$$

But since c is not contained in any proper parabolic subgroup of W_J, the character $\mathrm{Ind}_{W_{d^{-1}(K)\cap J}}^{W_J} 1$ vanishes on c unless $d^{-1}(K) \cap J = J$, in which case it takes the value 1. □

For $J \subseteq \Pi$, let $N_J = \{w \in W | w(J) = J\}$. Then N_J is the intersection of X_J and the normalizer of W_J, whereas $|N_J| = a_{JJJ}$ is the index of W_J in its normalizer.

For convenience we define $\xi_J = \xi_\lambda$ and $\nu_{JK} = \nu_{\lambda\mu}$ whenever $J \in \lambda$ and $K \in \mu$. For $J \subseteq K \subseteq \Pi$, let ξ_J^K be the primitive idempotent of $\mathcal{PB}(W_K)$ that takes the value 1 on elements of type J relative to W_K.

The next two propositions descrive the effect of the restriction and induction maps on these idempotents.

PROPOSITION 6.3. *Let $J, K \subseteq \Pi$ and let $J_1, J_2, \ldots, J_h$ be representatives of the W_K-equivalence classes of subsets of K that are W-equivalent to J. Then*

$$Res_{W_K}\xi_J = \sum_{i=1}^{h} \xi_{J_i}^K$$

In particular, $Res_{W_K}\xi_J = 0$ if J is not equivalent to any subset of K.

PROPOSITION 6.4. *If $J \subseteq K \subseteq \Pi$, then*

$$Ind_{W_K}^{W}\xi_J^K = \frac{|N_J|}{|W_K \cap N_J|}\xi_J$$

Proof. Suppose at first that $J = K$ and that $c \in W_J$ is an element of type J. Then $A(c) = W_J$ and therefore $x^{-1}cx \in W_J$ if and only if x is in the normalizer of W_J. So $(\mathrm{Ind}_{W_J}^{W}\xi_J^J)(c) = |N_J|$. It is clear that $\mathrm{Ind}_{W_J}^{W}\xi_J^J$ vanishes everywhere except at elements of type J, and therefore $\mathrm{Ind}_{W_J}^{W}\xi_J^J = |N_J|\xi_J$.

In general, we have

$$\mathrm{Ind}_{W_K}^{W}\xi_J^K = \mathrm{Ind}_{W_K}^{W}\left(\frac{1}{|W_K \cap N_J|}\mathrm{Ind}_{W_J}^{W_K}\xi_J^J\right) = \frac{|N_J|}{|W_K \cap N_J|}\xi_J \qquad \square$$

For the purposes of calculation, the following theorem is sometimes more useful than Theorem 6.2. The quantities $|N_J|/|W_K \cap N_J|$ can be obtained from the tables in Howlett [11].

THEOREM 6.5. *Let $J \preceq K \subseteq \Pi$ and let $J_1, J_2, \ldots, J_h$ be representatives of the W_K-equivalence classes of subsets of K that are W-equivalent to J. If $c \in W$ is of type J, then*

$$\varphi_K(c) = \sum_{i=1}^{h} \frac{|N_J|}{|W_K \cap N_{J_i}|}$$

Proof. By definition, $\sum \xi_L^K = 1$, where L runs through representatives of the W_K-equivalence classes of subsets of K. Inducing to W and using Proposition 6.4 gives

$$\varphi_J = \sum_L \frac{|N_L|}{|W_K \cap N_L|}\xi_L$$

Since $\xi_L(c) = 1$ if and only if $L \sim J_i$ for some i, evaluation at c completes the proof. $\square$

This theorem is also a consequence of the fact that $|N_J|/|W_K \cap N_{J_i}|$ is the number of $d \in X_{KJ}$ such that $d(J) \subseteq K$ and $d(J)$ is W_K-equivalent to J_i.

Let $C(J)$ be the set of elements of type J and note that $C(J)$ depends only on the equivalence class of J.

The main result of this section yields a remarkable formula for the coefficients ν_{JK} in the case $K = \emptyset$.

THEOREM 6.6. *If $m_1, m_2, \ldots, m_n$ are the exponents of W, then*

$$\nu_{\Pi\emptyset} = (-1)^n \frac{m_1 m_2 \cdots m_n}{|W|}$$

Proof. If ε is the sign character of W, then by Frobenius reciprocity

$$(\varphi_J, \varepsilon) = \begin{cases} 1 & J = \emptyset \\ 0 & J \neq \emptyset \end{cases}$$

and therefore $\nu_{\Pi\emptyset} = (\xi_\Pi, \varepsilon)$. By definition of the inner product,

$$\begin{aligned}(\xi_\Pi, \varepsilon) &= |W|^{-1} \sum_{w \in W} (-1)^{\tau(w)} \xi_\Pi(w) \\ &= (-1)^n |C(\Pi)|/|W|\end{aligned}$$

A well-known formula of Shephard and Todd [12] (see also Solomon [13]) states that

$$\sum_{\omega \in W} t^{\tau(\omega)} = (1 + m_1 t)(1 + m_2 t) \cdots (1 + m_n t)$$

Lemma 4.3 (2) shows that $\tau(\omega) = n$ if and only if ω is of type Π. Thus $m_1 m_2 \cdots m_k$ is the number of elements of type Π in W. This completes the proof. □

COROLLARY 6.7. *Let $J \subset \Pi$ with $|J| = k$ and let $m_1, m_2, \ldots, m_k$ be the exponents of W_J. Then*

$$\nu_{J\emptyset} = (-1)^k \frac{m_1 m_2 \cdots m_k}{|N_W(W_J)|}$$

where $N_W(W_J)$ is the normalizer in W of W_J.

Proof. To see this, apply Theorem 6.6 to W_J and then use Proposition 6.4. □

It is also interesting to observe that

$$|W| \sum_\mu \nu_{J\mu} = |C(J)|$$

The proof is obtained by taking the inner product of $\xi_J = \sum \nu_{J\mu} \varphi_\mu$ with the trivial character and using the fact that $(\varphi_\mu, 1) = 1$ for all μ.

A similar calculation, but taking the inner product of ξ_J with the sign character of W_L induced to W gives

$$\sum_\mu \nu_{J\mu} a_{L\mu\emptyset} = (-1)^{|J|} \frac{|C(J) \cap W_L|}{|W_L|}$$

7. Idempotents in the Solomon algebra

Our main aim in this section is to construct the elements $e_J \in \sum(W)$ mentioned in Section 1. These elements are mapped by $\theta : \sum(W) \to \mathcal{PB}(W)$ to scalar multiples of the idempotents ξ_J defined in Section 6, and are themselves scalar multiples of idempotents. Moreover, the e_J and the analogous elements $e_J^K \in \sum(W_K)$ are related by analogues of Propositions 6.3 and 6.4.

We begin by defining certain positive constants μ_K^J for all $K \subseteq J \subseteq \Pi$. For $K \subseteq \Pi$ we choose μ_K^Π arbitrarily, and then we put

$$\mu_K^J = \sum_{\substack{\omega \in X_J \\ \omega(K) \subseteq \Pi}} \mu_{\omega(K)}^\Pi \tag{10}$$

For convenience, we define $\mu_K^J = 0$ if $K \not\subseteq J$.

Inverting the upper triangular matrix (μ_K^J) yields constants β_K^J such that $\beta_K^J = 0$ if $J \not\subseteq K$ and

$$\sum_K \mu_K^J \beta_L^K = \delta_{JL} = \sum_K \beta_L^K \mu_K^J$$

where δ_{JL} is the Kronecker delta. For $J \subseteq M \subseteq \Pi$ we define $e_J^M \in \sum(W_M)$ by

$$e_J^M = \sum_{K \subseteq M} \beta_K^J x_K^M \tag{11}$$

noting that the coefficient of x_K^M in e_J^M is zero unless $K \subseteq J$. Let $e_J = e_J^\Pi$.

From Lemma 2.1 and this definition we immediately derive the analogue of Proposition 6.4.

PROPOSITION 7.3. *If $K \subseteq L \subseteq M$, then $e_K^M = x_L^M e_K^L$; that is, $e_K^M = Ind_L^M(e_K^L)$.*

Observe that if $K \subseteq J$ and $L \sim K$, then

$$\{\omega \in X_J | \omega(K) = L\} = \{\omega \in X_{LJ} | \omega^{-1}(L) \cap J = K\}$$

and so (10) can be restated as

$$\mu_K^J = \sum_{L \sim K} \mu_L^\Pi a_{LJK} \tag{12}$$

Our first lemma shows that (12) remains true when Π is replaced by $M \subseteq \Pi$.

LEMMA 7.5. *If $J, K \subseteq M \subseteq \Pi$ then*

$$\mu_K^J = \sum_L \mu_L^M a_{LJK}^M$$

where the a_{LJK}^M are the structure constants of $\sum(W_M)$, and L runs through subsets of M that are W_M-equivalent to K.

Proof.

$$\sum_{\substack{L \subseteq M \\ L \underset{M}{\sim} K}} \mu_L^M a_{LJK}^M = \sum_{\substack{L \subseteq M \\ L \underset{M}{\sim} K}} \sum_{N \sim L} \mu_N^\Pi a_{NML} a_{LJK}^M$$

$$= \sum_{N\sim L} \mu_N^{\Pi} \left(\sum_{\substack{L\subseteq M\\ L\underset{M}{\sim}K}} a_{NML} a_{LJK}^M \right)$$

Now $a_{NML}a_{LJK}^M = 0$ unless $L \preceq N$ and K is W_M-equivalent to a subset of L, and when $K \sim N$ this forces K to be W_M-equivalent to L. Hence the condition $L \underset{M}{\sim} K$ on the inner sum is superfluous, and so Proposition 2.6 gives

$$\begin{aligned} \sum_L \mu_L^M a_{LJK}^M &= \sum_{N\sim K} \mu_N^{\Pi} a_{NJK} \\ &= \mu_K^J \qquad \text{by (7.4).} \end{aligned}$$

□

Associativity in $\sum(W_M)$ gives the following relation on the structure constants:

$$\sum_{J\subseteq M} a_{LJK}^M a_{PNJ}^M = \sum_{Q\subseteq M} a_{LPQ}^M a_{QNK}^M \tag{13}$$

for all subsets L, P, N, K, of M. If L and K are W_M-equivalent then the only nonzero terms on the right-hand side come from Q which are in this same W_M-equivalence class. So multiplying (13) by μ_L^M and summing over $L \subseteq M$ that are W_M-equivalent to K gives (by Lemma 7.5)

$$\begin{aligned} \sum_{J\subseteq M} a_{PNJ}^M \mu_K^J &= \sum_{Q\underset{M}{\sim}K} \left(\sum_{L\underset{M}{\sim}Q} \mu_L^M a_{LPQ}^M \right) a_{QNK}^M \\ &= \sum_{Q\sim K} \mu_Q^P a_{QNK}^M \end{aligned} \tag{14}$$

for all P, N, $K \subseteq M$.

We can now prove the main facts concerning multiplication of the e_J^M.

THEOREM 7.8. *If $N, J \subseteq M \subseteq \Pi$ then*

$$e_J^M x_N^M = \sum_{\substack{K\subseteq N\\ K\underset{M}{\sim}J}} a_{JNK}^M e_K^M$$

Note that the coefficient a_{JNK}^M is the number of $v \in W_M \cap X_N$ such that $v(K) = J$.

Proof. If $J, L \subseteq M$ then $\sum_{K\subseteq M} \mu_K^J \beta_L^K = \delta_{JL}$. Hence multiplying (14) through by β_L^K and summing over $K \subseteq M$ gives

$$a_{PNL}^M = \sum_{K\subseteq M} \sum_{Q\underset{M}{\sim}K} \mu_Q^P a_{QNK}^M \beta_L^K$$

$$= \sum_{Q\subseteq M} \mu_Q^P \left(\sum_{K \underset{M}{\sim} Q} a_{QNK}^M \beta_L^K \right)$$

Multiplying through by β_P^J, where $J \subseteq M$, and summing over $P \subseteq M$ gives

$$\sum_{P\subseteq M} \beta_P^J a_{PNL}^M = \sum_{K \underset{M}{\sim} J} a_{JNK}^M \beta_L^K \tag{15}$$

Multiplying this by x_L^M and summing over $L \subseteq M$ gives

$$\sum_{P\subseteq M} \beta_P^J x_P^M x_N^M = \sum_{K \underset{M}{\sim} J} a_{JNK}^M \left(\sum_{L\subseteq M} \beta_L^K x_L^M \right)$$

So, by (11), $e_J^M x_N^M = \sum_{K\sim J} a_{JNK}^M e_K^M$, as required (since $a_{JNK}^M = 0$ for $K \not\subseteq N$). □

COROLLARY 7.10. *If $L, J \subseteq M \subseteq \Pi$, then*

$$e_J^M e_L^M = \sum_{\substack{K\subseteq L \\ K \underset{M}{\sim} J}} \left(\sum_N a_{JNK}^M \beta_N^L \right) e_K^M$$

where the inner sum is over N such that $K \subseteq N \subseteq L$.

Proof. Simply write $e_L^M = \sum_{N\subseteq L} \beta_N^L x_N^M$ and use Theorem 7.8. □

Corollary 7.10 shows that for each W_M-equivalence class λ, the elements e_J^M, for $J \in \lambda$, span a right ideal $I(M, \lambda)$ of $\sum(W_M)$. Furthermore, $\sum(W_M)$ is the direct sum of these right ideals, since the e_J^M, for $J \subseteq M$, form a basis of $\sum(W_M)$. Our next proposition shows that $\theta_M : \sum(W_M) \to \mathcal{PB}(W_M)$ maps the $I(M, \lambda)$'s to the simple components of $\mathcal{PB}(W_M)$.

PROPOSITION 7.11. *If $J \subseteq M$ and λ is the W_M equivalence class containing J, then*

$$\theta_M(e_J^M) = \frac{|W_M \cap N_J|}{\mu_J^J} \xi_\lambda^M$$

Proof. We have

$$1 = x_M^M = \sum_{J\subseteq M} \mu_J^M e_J^M = \sum_\lambda \sum_{J\in\lambda} \mu_J^M e_J^M$$

where λ runs through all W_M-equivalence classes. Hence the elements $E_\lambda^M = \sum_{J\in\lambda} \mu_J^M e_J^M$ are the orthogonal idempotents corresponding to the decomposition $\sum(W_M) = \bigoplus_\lambda I(M, \lambda)$. Applying the homomorphism θ_M we see that

$\sum_\lambda \theta_M(E_\lambda^M) = 1$ and $\theta_M(E_\lambda^M)\theta_M(E_\mu^M) = 0$ whenever $\lambda \neq \mu$. It will follow that $\theta_M(E_\lambda^M)$ is the primitive idempotent ξ_λ^M, provided we can prove that $\theta_M(E_\lambda^M)$ is nonzero on elements of type λ.

We do this first in the case $\lambda = \{M\}$. Observe that if $K \subseteq M$ and $K \neq M$ then the character $\theta_M(e_K^M)$ vanishes on elements of type M, since

$$\theta_M(e_K^M) = \theta_M(\mathrm{Ind}_K^M(e_K^K)) = \mathrm{Ind}_{W_K}^{W_M}(\theta_K(e_K^K))$$

by Proposition 4.6. Since

$$1_{W_M} = \theta_M(1) = \sum_{K \subseteq M} \mu_K^M \theta_M(e_K^M)$$

we deduce that $\theta_M(E_{\{M\}}^M) = \mu_M^M \theta_M(e_M^M)$ takes the value 1 on elements of type M.

Now consider an arbitrary W_M-equivalence class λ. Since $\theta_M(E_\lambda^M) = \sum_{K\in\lambda} \mu_K^M \mathrm{Ind}_{W_K}^{W_M}(\theta_K(e_K^K))$, and since $\theta_K(e_K^K)$ takes a positive value on elements of type K, it follows that $\theta_M(E_\lambda^M)$ is nonzero on elements of type λ, as required.

Thus we have proved that $\theta_M(E_\lambda^M) = \xi_\lambda^M$ for all λ, and in particular, $\theta_M(e_M^M) = (1/\mu_M^M)\xi_M^M$. Applying this with M replaced by J we find that

$$\theta_M(e_J^M) = \mathrm{Ind}_{W_J}^{W_M}(\frac{1}{\mu_J^J}\xi_J^J) = \frac{|W_M \cap N_J|}{\mu_J^J}\xi_J^M$$

by Proposition 6.4. □

COROLLARY 7.12. *Let λ be a W_M-equivalence class of subsets of M and let $L, J \in \lambda$. Then*

$$e_J^M e_L^M = \frac{|W_M \cap N_J|}{\mu_J^J} e_L^M$$

Proof. The only subset K of L such that $K \underset{M}{\sim} J$ is $K = L$, and so Corollary 7.10 shows that $e_J^M e_L^M$ is a scalar multiple of e_L^M. Applying θ_M and using Proposition 7.11 determines the scalar. □

Note also that the scalar $|W_M \cap N_J|/\mu_J^J$ depends only on the W_M-equivalence class λ containing J. Indeed, it is easily shown (by use of Lemma 7.5, for example) that

$$\frac{|W_M \cap N_J|}{\mu_J^J} = \left(\sum_{N\in\lambda} \mu_N^M\right)^{-1} = a_{JLL}^M \beta_L^L \tag{16}$$

for all $L \in \lambda$.

It is immediate from Corollary 7.12 that the right ideal $I(M,\lambda)$ is generated by an element $\sum_{J\in\lambda} \rho_J x_J^M$ such that $\sum_{J\in\lambda} \rho_J \neq 0$. In particular, $I(M,\lambda)$ is an indecomposable. By standard theory (see [1]), it follows that the $I(M,\lambda)$'s form a full set of projective indecomposable modules for $\sum(W_M)$. Since $\theta_M(I(M,\lambda))$ is one-dimensional, we deduce that $I(M,\lambda)\cap \ker\theta_M$ has codimension 1 in $I(M,\lambda)$, and is the unique maximal submodule of $I(M,\lambda)$. Hence the kernel of θ_M, being the direct sum of the subspaces $I(M,\lambda)\cap\ker\theta_M$, is the radical of $\sum(W_M)$.

The following proposition provides the analogue of Proposition 6.3.

PROPOSITION 7.14. *Let $J,K \subseteq M \subseteq \Pi$. Then*

$$Res_K^M(e_J^M) = \sum_{\substack{Q\subseteq K\\ Q\underset{M}{\sim} J}} a_{JKQ}^M e_Q^K$$

Proof. Since $e_J^M = \sum_{L\subseteq M}\beta_L^J x_L^M$, the definition of Res_K^M gives

$$\begin{aligned}\mathrm{Res}_K^M(e_J^M) &= \sum_{L\subseteq M}\beta_L^J\left(\sum_{d\in W_M\cap X_{LK}} x^K_{d^{-1}(L)\cap K}\right)\\ &= \sum_{L\subseteq M}\sum_{N\subseteq M}\beta_L^J a_{LKN}^M x_N^K\\ &= \sum_{N\subseteq M}\sum_{Q\underset{M}{\sim}J} a_{JKQ}^M\beta_N^Q x_N^K\end{aligned}$$

by (15). By definition we have $e_Q^K = \sum_{N\subseteq M}\beta_N^Q x_N^K$, and the result follows. □

As our final result we determine the dimension of the right ideal H_λ of $\mathbf{Q}(W)$ generated by the idempotent E_λ^Π.

THEOREM 7.15. *The dimension of H_λ is the number of elements of W of type λ.*

Proof. The dimension of H_λ is the trace of E_λ^Π in the regular representation of W. This can be determined by expressing E_λ^Π as a linear combination of elements of W and multiplying the coefficient of 1 by $|W|$.

Now $E_\lambda^\Pi = \sum_{J\in\lambda}\mu_J^\Pi e_J^\Pi = \sum_{J\in\lambda}\sum_{K\subseteq\Pi}\mu_J^\Pi\beta_K^J x_K$, and since 1 occurs in each x_K with coefficient 1, we deduce that

$$\dim H_\lambda = |W|\sum_{J\in\lambda}\sum_{K\subseteq\Pi}\mu_J^\Pi\beta_K^J$$

Let $C(\lambda)$ be the set of elements of W of type λ. Then

$$|C(\lambda)| = |W|(\xi_\lambda, 1)$$

$$= |W|(\theta(E_\lambda^\Pi),1)$$
$$= |W|\sum_{J\in\lambda}\sum_{K\subseteq\Pi}\mu_J^\Pi\beta_K^J(\theta(x_K),1)$$

Since $\theta(x_K) = \phi_K$ and $(\phi_K,1) = 1$, we see that $|C(\lambda)| = \dim H_\lambda$, as required. □

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