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Triple Multiplicities for $s\ell(r+1)$ and the Spectrum of the Exterior Algebra of the Adjoint Representation

A.D. BERENSTEIN AND A.V. ZELEVINSKY
Department of Mathematics, Northeastern University, Boston, MA 02115.

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Abstract. A new combinatorial expression is given for the dimension of the space of invariants in the tensor product of three irreducible finite dimensional $s\ell(r+1)$-modules (we call this dimension the *triple multiplicity*). This expression exhibits a lot of symmetries that are not clear from the classical expression given by the Littlewood–Richardson rule. In our approach the triple multiplicity is given as the number of integral points of the section of a certain "universal" polyhedral convex cone by a plane determined by three highest weights. This allows us to study triple multiplicities using ideas from linear programming. As an application of this method, we prove a conjecture of B. Kostant that describes all irreducible constituents of the exterior algebra of the adjoint $s\ell(r+1)$-module.

Keywords: tensor product multiplicities, systems of linear inequalities.

1. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra of rank r, and V_λ, V_μ, V_ν be three irreducible finite-dimensional $\mathfrak{g}$-modules with highest weights λ, μ, ν. The multiplicity $c_{\lambda\nu}^{\mu}$ of V_μ in the tensor product $V_\lambda \otimes V_\nu$ plays an important part in the representation theory of $\mathfrak{g}$ and its physical applications. For $\mathfrak{g} = s\ell(r+1)$ this multiplicity is given by the classical Littlewood–Richardson rule (see e.g., [10]). The results of this paper provide an alternative to the Littlewood–Richardson rule.

In [2] we suggested the following geometric interpretation of $c_{\lambda\nu}^{\mu}$ (see also [1], [7], [8]). We associate to $\mathfrak{g}$ a convex polyhedral cone $K \subset L$ and a linear map $pr: L \longrightarrow \mathbf{R}^{3r}$, where L is a real vector space of dimension $N = 1/2$ (dim $\mathfrak{g} + 3r$), and $\mathbf{R}^{3r}$ is the space of triples of $\mathfrak{g}$-weights. Then $c_{\lambda\nu}^{\mu}$ is computed as the number of integral points in the section of K by the plane $pr^{-1}(\lambda, \mu, \nu)$.

The choice of K is not unique; in [1], [7], [8] it is constructed by means of the Gelfand–Tsetlin patterns whereas in [2] it is given in terms of partitions of weights into the sum of positive roots. In the present work we give a new construction of K for $\mathfrak{g} = s\ell(r+1)$ that is, in a sense, the most symmetric. More precisely, instead of $c_{\lambda\nu}^{\mu}$ we consider $c_{\lambda\mu\nu} = \dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathfrak{g}}$, the dimension of the space of $\mathfrak{g}$-invariants in the triple tensor product. We call $c_{\lambda\mu\nu}$ the *triple multiplicity*;

evidently $c_{\lambda\mu\nu} = c^{\mu^*}_{\lambda\nu}$, where μ^* is the highest weight of the module V^*_μ dual to V_μ (since there is a natural isomorphism $\mathrm{Hom}_{\mathfrak{g}}(V^*_\mu, V_\lambda \otimes V_\nu) \cong (V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathfrak{g}}$). Clearly $c_{\lambda\mu\nu}$ is invariant under all 6 permutations of (λ, μ, ν) and also under the replacement of (λ, μ, ν) by $(\lambda^*, \mu^*, \nu^*)$. These transformations generate a group of 12 symmetries; our construction of K will make evident 6 of them.

As an application of our approach to multiplicities we determine (for $\mathfrak{g} = s\ell(r+1)$) the spectrum of the $\mathfrak{g}$-module $\Lambda^*(\mathfrak{g})$, the exterior algebra of the adjoint $\mathfrak{g}$-module. The Weyl character formula implies that the $\mathfrak{g}$-module $\Lambda^*(\mathfrak{g})$ is isomorphic to the direct sum of 2^r copies of the $\mathfrak{g}$-module $(V_\rho \otimes V_\rho)$, where ρ is the half-sum of all positive roots of $\mathfrak{g}$. Therefore, V_μ is an irreducible constituent of $\Lambda^*(\mathfrak{g})$ if and only if $c^\mu_{\rho\rho} \neq 0$. B. Kostant conjectured (private communication) that $c^\mu_{\rho\rho} \neq 0$ if and only if the weight $(2\rho - \mu)$ is a nonnegative integral linear combination of simple roots of $\mathfrak{g}$. Using our expression of triple multiplicities, we reduce the proof of this conjecture (for $g = s\ell(r+1)$) to a problem of linear programming, i.e., to the study of compatibility of certain systems of linear inequalities. Such an approach allows us also to obtain more precise information, i.e., to describe all μ such that $c^\mu_{\rho\rho} = 1$.

Our main result on multiplicities is stated and proven in Section 2 (Theorem 1). The results on $c^\mu_{\rho\rho}$ are collected in Section 3; the main results here are Theorems 6 and 14 describing all μ such that $c^\mu_{\rho\rho} > 0$ (respectively, $c^\mu_{\rho\rho} = 1$).

Our proof of Theorems 6 and 14 is based on general criteria for existence and uniqueness of solutions of a system of linear inequalities. For the convenience of the reader we discuss these criteria in the Appendix. Although they must be well known to experts, we were not able to find the statements in the form needed for our purposes in the literature, thus we provide brief sketches of the proofs here.

2. A symmetric expression for triple multiplicity

We fix a natural number r and put $T = T_r = \{(i, j, k) \in \mathbf{Z}^3_+ : i + j + k = 2r - 1\}$. Put also $H = H_r = \{(i, j, k) \in T_r$: all i, j, k are odd$\}$ and $G = G_r = T_r - H_r$. Thus T_r is the set of vertices of a regular triangular lattice filling the regular triangle with vertices $(2r-1, 0, 0)$, $(0, 2r-1, 0)$, and $(0, 0, 2r-1)$; this triangle is decomposed into the union of elementary triangles having all three vertices in G_r and of elementary hexagons centered at points of H_r (see Figure 1).

Let $\alpha = (1, -1, 0)$, $\beta = (0, 1, -1)$, and $\gamma = (-1, 0, 1)$ so that $\alpha + \beta + \gamma = 0$. Clearly, for any $\eta \in H_r$ six points $\eta \pm \alpha$, $\eta \pm \beta$, $\eta \pm \gamma$ lie in G_r (they are vertices of the elementary hexagon centered at η). Consider the vector space $\mathbf{R}^{G_r}$ consisting of families $(x(\xi))$ $(\xi \in G_r)$ of real numbers. Let $L \subset \mathbf{R}^{G_r}$ be the vector subspace defined by the equations

$$x(\eta + \alpha) - x(\eta - \alpha) = x(\eta + \beta) - x(\eta - \beta) = x(\eta + \gamma) - x(\eta - \gamma) \qquad (1)$$

for any $\eta \in H_r$. Geometrically this means that if $[\xi_1, \xi_2]$ and $[\xi'_1, \xi'_2]$ are two

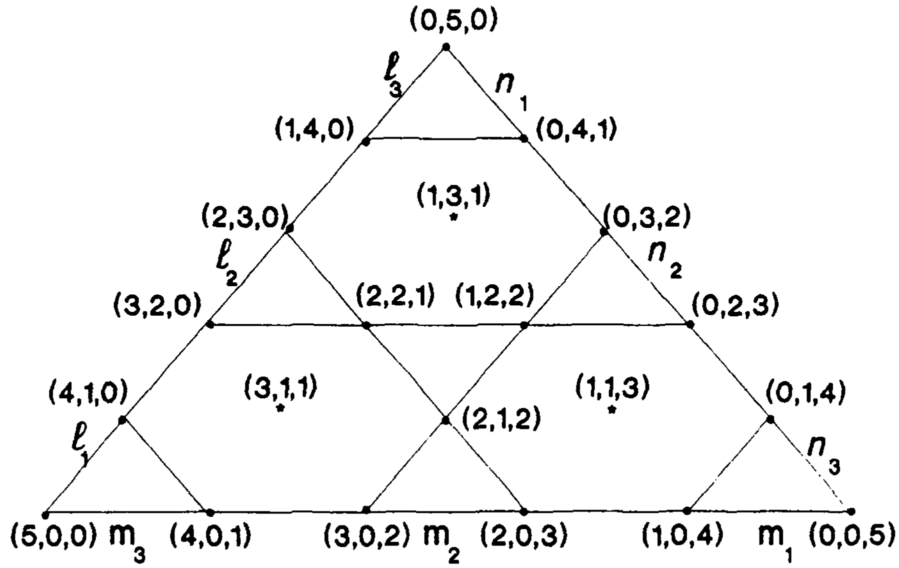

Figure 1. Sets G_r and H_r for $r = 3$. The points of G_r are depicted as circles, and the points of H_r as stars.

opposite edges of an elementary hexagon then $x(\xi_1) + x(\xi_2) = x(\xi_1') + x(\xi_2')$. We define a convex cone $K \subset L$ to be the intersection $L \cap \mathbf{R}_+^{G_r}$, i.e., the set of all points $x \in L$ such that $x(\xi) \geq 0$ for all ξ; let also $K_{\mathbf{Z}} = L \cap \mathbf{Z}_+^{G_r}$.

Now let $\mathfrak{g} = s\ell(r+1)$. We identify the lattice P of integral $\mathfrak{g}$-weights with $\mathbf{Z}^r$ using as a standard basis the family $\omega_1, \ldots, \omega_r$ of fundamental weights of $\mathfrak{g}$ in a standard numeration (see [3]). This identification takes the semigroup P^+ of the highest weights to $\mathbf{Z}_+^r$. We recall that the triple multiplicity $c_{\lambda\mu\nu}$ for$\lambda, \mu, \nu \in P^+$ is defined as $\dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathfrak{g}}$.

We define a linear projection pr: $L \longrightarrow \mathbf{R}^{3r} = (P \otimes \mathbf{R})^3$ by the formulas $\mathrm{pr}(x) = (\ell_1, \ldots, \ell_r; m_1, \ldots, m_r; n_1, \ldots, n_r)$, where

$$\begin{aligned} \ell_p &= x(2(r-p)+1, 2(p-1), 0) + x(2(r-p), 2p-1, 0) \\ n_p &= x(0, 2(r-p)+1, 2(p-1)) + x(0, 2(r-p), 2p-1) \\ m_p &= x(2(p-1), 0, 2(r-p)+1) + x(2p-1, 0, 2(r-p)), p = 1, \ldots, r \end{aligned} \quad (2)$$

Note that the points of G_r contributing to (2) lie on the boundary of our triangle (see Figure 1).

THEOREM 1. *Let $\lambda = \sum \ell_q\omega_q, \mu = \sum m_q\omega_q, \nu = \sum n_q\omega_q$ be the three highest $s\ell(r+1)$-weights. Then the triple multiplicity $c_{\lambda\mu\nu}$ is equal to $\#(K_{\mathbf{Z}} \cap pr_{-1}(\lambda, \mu, \nu))$, i.e., to the number of families $(x(\xi))$, $\xi \in G_r$, of nonnegative integers satisfying (1) and (2).*

Proof. We shall deduce Theorem 1 from the results of [2] expressing $c^{\mu}_{\lambda\nu}$ in terms of so-called $\mathfrak{g}$-partitions. By a *$\mathfrak{g}$-partition of weight θ* we mean a family (m_α) of nonnegative integers indexed by all positive roots α of $\mathfrak{g}$, such that $\sum_\alpha m_\alpha \alpha = \theta$. For $\mathfrak{g} = s\ell(r+1)$ positive roots correspond in a standard way to pairs $1 \le p < q \le r+1$, and we write m_{pq} instead of m_α. Following [2], for any $s\ell(r+1)$-partition $m = (m_{pq})$ we put $\Delta_{pq} = \Delta_{pq}(m) = m_{pq} - m_{p+1,q+1}$ (with the convention $\Delta_{0q} = -m_{1,q+1}, \Delta_{p,r+1} = m_{p,r+1}$).

PROPOSITION 2. (=Theorem 2.1 from [2]). *The multiplicity $c^{\mu}_{\lambda\nu}$ is equal to the number of $s\ell(r+1)$-partitions (m_{pq}) of weight $(\lambda + \nu - \mu)$ satisfying the inequalities*

$$\ell_{q-1} + \sum_{t<p} \Delta_{t,q-1} \ge 0 \tag{3}$$

$$n_p - \sum_{t \ge q} \Delta_{pt} \ge 0 \tag{4}$$

for all (p,q) such that $1 \le p < q \le r+1$.

To prove Theorem 1 it remains to establish a bijective correspondence between $K_{\mathbf{Z}} \cap pr^{-1}(\lambda, \mu^*, \nu)$ and the set of partitions from Proposition 2. Notice that the weight μ^* has the form $\mu^* = m_r\omega_1 + m_{r-1}\omega_2 + \ldots + m_1\omega_r$. To any $(x(\xi)) \in K_{\mathbf{Z}} \cap pr^{-1}(\lambda, \mu^*, \nu)$ we associate an $s\ell(r+1)$-partition (m_{pq}) by the rule

$$m_{pq} = x(2(r+1-q), 2(q-p)-1, 2(p-1)) \tag{5}$$

Conversely, to any partition (m_{pq}) of weight $\lambda + \nu - \mu$ satisfying (3) and (4) we associate a point $(x(\xi))$ defined by (5) and

$$x(2(r+1-q)+1, 2(q-1-p), 2(p-1)) = \ell_{q-1} + \sum_{t<p} \Delta_{t,q-1} \tag{6}$$

$$x(2(r+1-q), 2(q-1-p), 2p-1) = n_p - \sum_{t \ge q} \Delta_{pt} \tag{7}$$

It is straightforward to verify that these formulas give the desired correspondence. This proves Theorem 1. □

Remarks.

(a) Obviously, the sets G_r and H_r are invariant under all permutations of indices (i,j,k). Therefore, we have a natural action of S_3 on $\mathbf{R}^{G_r}$, and it is evident that L, K, and $K_{\mathbf{Z}}$ are invariant under this action. Let $s_1 = (1,2)$ and $s_2 = (2,3)$ be two standard generators of S_3. Equations (2) imply at once that if $pr(x) = (\lambda, \mu, \nu)$ then $pr(s_1(x)) = (\lambda^*, \nu^*, \mu^*)$, and $pr(s_2(x)) = (\mu^*, \lambda^*, \nu^*)$. We see that our expression of triple multiplicities makes evident the following symmetries: $c_{\lambda\mu\nu} = c_{\lambda^*\nu^*\mu^*} = c_{\mu^*\lambda^*\nu^*} = c_{\nu\lambda\mu} = c_{\mu\nu\lambda} = c_{\nu^*\mu^*\lambda^*}$. The remaining symetries also can be derived from Theorem 1, e.g., we can prove that $c_{\lambda\mu\nu} = c_{\mu\lambda\nu}$ by constructing an explicit bijection between $K_{\mathbf{Z}} \cap pr^{-1}(\lambda,\mu,\nu)$ and $K_{\mathbf{Z}} \cap pr^{-1}(\mu,\lambda,\nu)$ but this requires some work.

(b) Consider the set $\Lambda = \{(\lambda,\mu,\nu) \in \mathbf{Z}_+^{3r} : c_{\lambda\mu\nu} \neq 0\}$. One can prove that Λ is a finitely generated semigroup (this can be done by means of invariant-theoretic arguments (F. Knop, August 1989, private communication)). Theorem 1 makes this statement for $\mathfrak{g} = s\ell(r+1)$ evident. Indeed, in this case $\Lambda = pr(K_{\mathbf{Z}})$, and it is enough to show that $K_{\mathbf{Z}}$ is a finitely generated semigroup but this is a consequence of the following general statement: if $K \subset \mathbf{R}^N$ is the set of solutions of a finite system of homogeneous linear equations and inequalities with rational coefficients, then $K_{\mathbf{Z}} = K \cap \mathbf{Z}^N$ is a finitely generated semigroup.

In [2] we suggested an analogous (conjectural) expression for $c_{\lambda\nu}^{\mu}$ for other classical Lie algebras. This would also imply at once that Λ is a finitely generated semigroup. It would be very interesting to describe this semigroup explicitly.

(c) We hope that the expression for triple multiplicities given by Theorem 1 will be useful for the explicit computation of so-called fusion coefficients (see [11]).

(d) In the classical Littlewood–Richardson rule [10] an $s\ell(r+1)$-weight λ is represented by a partition $\lambda = (\lambda_1,\ldots,\lambda_{r+1})$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1}$ are nonnegative integers. The coordinates $\ell_1,\ldots,\ell_r$ of λ used in Theorem 1 are given by $\ell_q = \lambda_q - \lambda_{q+1}$. A direct relationship between our Theorem 1 and the classical Littlewood–Richardson rule has been recently established by C. Carré [4].

We conclude this section by several equivalent versions of Theorem 1. First, we construct an isomorphism $\Delta : L \longrightarrow \mathbf{R}^N$, where $N = r(r+5)/2$. We extend H_r to the set $\overline{H}_r = \{(i,j,k) : i,j,k$ are odd integers, $-1 \leq i,j,k \leq 2r-1, i+j+k = 2r-1\}$. Definitions imply at once the following.

LEMMA 3. *For any $\xi \in G_r$ the set $\{\xi-\alpha, \xi-\beta, \xi-\gamma\}$ meets $\overline{H}_r$ at a unique point (see Figure 2).*

It is easy to see that $\#\overline{H}_r = N$, and we write $y \in \mathbf{R}^N$ as $(y(\eta)), \eta \in \overline{H}_r$. We put $\Delta(x) = y$, where $y(\eta)$ is defined as follows. If $\eta \in H_r$ then $y(\eta)$ is equal to each of the three expressions in (1), and if $\eta \in \overline{H}_r - H_r$, i.e., exactly one coordinate of η is (-1) then $y(\eta) = x(\xi)$, where ξ is the unique point among $\eta+\alpha, \eta+\beta, \eta+\gamma$ lying in G_r (see Lemma 3).

PROPOSITION 4.

(a) Δ is a vector space isomorphism $L \cong \mathbf{R}^N$.

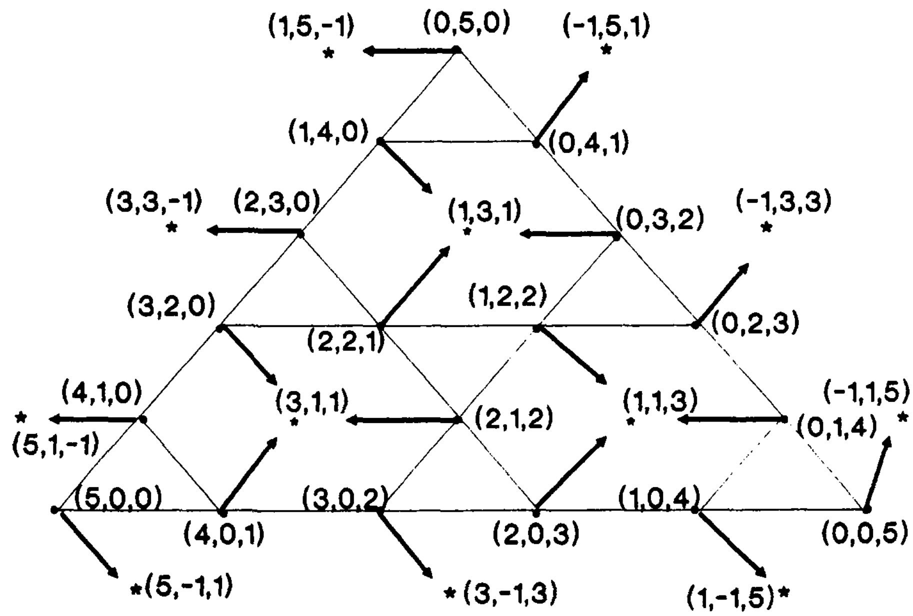

Figure 2. Sets G_r and H_r for $r = 3$. The points of G_r are depicted as circles, and the points of H_r as stars. The map $G_r \to \overline{H}_r$ from Lemma 3 is shown by arrows.

(b) *The image $\Delta(K)$ of the cone $K \subset L$ is defined by the inequalities $y(\eta) \geq 0$ for all $\eta \in \overline{H}_r - H_r$, and $\sum_{p\geq 0} y(\eta - 2p\alpha) \geq 0, \sum_{p\geq 0} y(\eta - 2p\beta) \geq 0, \sum_{p\geq 0} y(\eta - 2p\gamma) \geq 0$ for all $\eta \in H_r$, the summations over $p \in \mathbf{Z}_+$ such that the corresponding point lies in $\overline{H}_r$. Furthermore, $\Delta(K_{\mathbf{Z}}) = \Delta(K) \cap \mathbf{Z}^N$.*

(c) *The isomorphism Δ takes $pr : L \longrightarrow \mathbf{R}^{3r}$ to the linear map $\mathbf{R}^N \longrightarrow \mathbf{R}^{3r}$ (denoted also by pr) given by*

$$\ell_q = \sum_{i=2(r-q)+1} y(i,j,k), \quad n_q = \sum_{j=2(r-q)+1} y(i,j,k)$$
$$m_q = \sum_{k=2(r-q)+1} y(i,j,k) \tag{8}$$

the summations over all $(i,j,k) \in \overline{H}_r$ indicated above.

Proof. Let us construct the inverse isomorphism $\sum : \mathbf{R}^N \longrightarrow L$. Let $y = (y(\eta)) \in \mathbf{R}^N$. By Lemma 3, for any $\xi \in G_r$ there is the unique element of $\{\alpha, \beta, \gamma\}$, say

α, such that $(\xi-\alpha)\in\overline{H}_r$. Then we put $\sum(y)(\xi)=\sum y(\xi-\alpha-2p\alpha)$, the sum over all $p\in\mathbf{Z}_+$ such that $(\xi-\alpha-2p\alpha)\in\overline{H}_r$ (for $(\xi-\beta)\in\overline{H}_r$ or $(\xi-\gamma)\in\overline{H}_r$ the definition is similar). The facts that $\sum(y)$ belongs to L, and that $\sum$ and Δ are mutually inverse, are verified directly. The proof of b) and c) is also straightforward. □

Using this proposition we can reformulate Theorem 1 as follows: $c_{\lambda\mu\nu}$ is equal to $\#(\Delta(K_{\mathbf{Z}})\cap pr^{-1}(\lambda,\mu,\nu))$, i.e., to the number of families of integers $(y(\eta)),\eta\in\overline{H}_r$, satisfying (8) and the inequalities of Proposition 4 b).

In the next section we use one more equivalent version of Theorem 1. Let $\alpha_1,\ldots,\alpha_r$ be the simple roots of $\mathfrak{g}=s\ell(r+1)$, and $\omega_1,\ldots,\omega_r$ the fundamental weights of $\mathfrak{g}$ in a standard numeration. The transition matrix between these two bases of the space of $\mathfrak{g}$-weights is the Cartan matrix of $\mathfrak{g}$ [3] i.e., if $\mu=m_1\omega_1+\cdots+m_r\omega_r=g_1\alpha_1+\cdots+g_r\alpha_r$ then the coefficients m_p and g_p are related by $m_p=-g_{p+1}+2g_p-g_{p-1}$ (with the convention $g_0=g_{r+1}=0$).

PROPOSITION 5. *Let $\lambda=\sum\ell_p\omega_p$, $\nu=\sum n_p\omega_p$ be two highest weights for $s\ell(r+1)$, and suppose a highest weight μ is expressed as $\mu=\lambda+\nu-\sum g_p\alpha_p$. Then $c^{\mu}_{\lambda\nu}$ is equal to the number of families $(g^q_p)(1\le p<q\le r+1)$ of nonnegative integers satisfying the following conditions for all p, q:*

$$\begin{aligned} g_p^{r+1} &= g_p && (9)\\ \ell_{r+1-q+p}+(g^q_{p-1}-g^q_p)-(g^{q-1}_{p-1}-g^{q-1}_p) &\ge 0 && (10)\\ n_p-(g^q_p-g^q_{p+1})+(g^{q-1}_{p-1}-g^{q-1}_p) &\ge 0 && (11)\\ g^q_p-g^{q-1}_{p-1}-g^{q-1}_p+g^{q-2}_{p-1} &\ge 0 && (12)\end{aligned}$$

(with the convention $g^q_p=0$ unless $1\le p<q\le r+1$).

Proof. It suffices to establish a bijective correspondence between the set of all (g^q_p) from our proposition and the set $K_{\mathbf{Z}}\cap pr^{-1}(\lambda,\mu^*,\nu)$ (see Theorem 1). To any family (g^q_p) we associate a point $(x(\xi))$, $\xi\in G_r$ so that $x(2(q-p)-1,2(r+1-q),2(p-1))$ is the left-hand side of (10), $x(2(q-1-p),2(r+1-q),2p-1)$ is the left-hand side of (11), and $x(2(q-1-p),2(r+1-q)+1,2(p-1))$ is the left-hand side of (12). We have to verify that these values of $x(\xi)$ satisfy (1) and (2) (with m_p replaced by m_{r+1-p}). This can be done by straightforward calculations. The inverse map is given by the formula $g^q_p=\sum x(2(q'-1-p'),2(r+1-q')+1,2(p'-1))$, the sum over all (p',q') such that $1\le p'\le p$ and $1\le q'-p'\le q-p$. □

Note that the index p plays the same part in (5) to (7) and in (9) to (12) but this is not so for q.

3. Kostant conjecture for $s\ell(r+1)$

The main goal of this section is to prove the following theorem.

THEOREM 6. *Let $\mathfrak{g} = s\ell(r+1)$, μ be a highest $\mathfrak{g}$-weight, and ρ be the half-sum of all positive roots of $\mathfrak{g}$. Then the following conditions are equivalent:*

(a) V_μ is an irreducible constituent of $\Lambda^*(\mathfrak{g})$
(b) $c^\mu_{\rho\rho} > 0$
(c) μ is a weight of $V_{2\rho}$
(d) $(2\rho - \mu)$ is a linear combination of simple roots with coefficients in $\mathbf{Z}_+$.

Note that the equivalences $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$ are well known. Furthermore, the implication $(b) \Rightarrow (d)$ is evident. In fact, for any $\mathfrak{g}$ the multiplicity $c^\mu_{\lambda\nu}$ can be nonzero only if $(\lambda + \nu - \mu)$ is a linear combination of simple roots with coefficients in $\mathbf{Z}_+$. It remains to prove $(d) \Rightarrow (b)$.

We shall use Proposition 5 in the case when $\lambda = \nu = \rho$. So from now on we assume that $\ell_p = n_p = 1$ for all p.

PROPOSITION 7. *Let $\mu = 2\rho - \sum g_p\alpha_p$, where $g_1, \ldots, g_r \in \mathbf{Z}$. Then μ is a highest weight if and only if*

$$2 + g_{p-1} - 2g_p + g_{p+1} \geq 0 \quad \text{for} \quad p = 1, \ldots, r \tag{13}$$

(with the convention $g_0 = g_{r+1} = 0$).

Proof. The left-hand side of (13) is the coefficient m_p in the decomposition $\mu = \sum m_p\omega_p$. □

Propositions 5 and 7 reduce the proof of the remaining part of Theorem 6 to the next statement concerning integral programming.

PROPOSITION 8. *Let $g_1, \ldots, g_r \in \mathbf{Z}_+$. Then the following conditions are equivalent:*

(a) $(g_1, \ldots, g_r)$ satisfies (13);
(b) there exists a family (g^q_p) of nonnegative integers satisfying conditions (9) to (12) (with all n_p, ℓ_p equal to 1).

Proof. First we observe that by induction on q we can deduce from (12) the inequalities

$$g^q_p - g^{q-1}_p \geq 0\ , \quad g^q_p - g^{q-1}_{p-1} \geq 0 \tag{14}$$

Now let us introduce some terminology. A vector $(g_1, \ldots, g_r) \in \mathbf{Z}^r_+$ satisfying (13) will be called *admissible*. Denote the vector $(g^q_1, \ldots, g^q_{q-1})$ by $g^{(q)}$. We shall say that a pair $(g^{(q)}, g^{(q-1)})$ is *admissible* if $g^{(q)}$ and $g^{(q-1)}$ satisfy (10), (11), and (14).

LEMMA 9. *If a pair $(g^{(q)}, g^{(q-1)})$ is admissible then both $g^{(q)}$ and $g^{(q-1)}$ are admissible.*

Proof of Lemma 9. The inequality (13) for $g^{(q)}$ is simply the sum of (10) and (11) with the same p and q. To get (13) for $g^{(q-1)}$ it suffices to add (10) and (11) with the same q and p shifted by 1. □

The implication $(b) \Rightarrow (a)$ in Proposition 8 follows at once from Lemma 9. To prove $(a) \Rightarrow (b)$ choose an admissible $g^{(r+1)}$; we have to construct $g^{(r)}, g^{(r-1)}, \dots, g^{(2)}$ satisfying (10) to (12). This can be done by repeated application of the next two lemmas.

LEMMA 10. *For any admissible $g^{(q)} \in \mathbf{Z}_+^{q-1}$ there exists $g^{(q-1)} \in \mathbf{Z}_+^{(q-2)}$ such that $(g^{(q)}, g^{(q-1)})$ is admissible.*

LEMMA 11. *For any admissible pair $(g^{(q)}, g^{(q-1)})$ there exists $g^{(q-2)} \in \mathbf{Z}_+^{(q-3)}$ such that $(g^{(q-1)}, g^{(q-2)})$ is admissible, and the triple $(g^{(q)}, g^{(q-1)}, g^{(q-2)})$ satisfies (12).*

Our proof of both lemmas is based on the following.

LEMMA 12. *Suppose we are given real numbers g_p^+, $g_p^-(0 \le p \le k)$, $d_p^+, d_p^-(0 \le p \le k-1)$. Then the system of linear inequalities*

$$g_p^- \le g_p \le g_p^+, d_p^- \le g_p - g_{p+1} \le d_p^+ \tag{15}$$

in real variables $g_0, \dots, g_k$ has a solution if and only if $(g_p^+, g_p^-, d_p^+, d_p^-)$ satisfy

$$d_p^- \le d_p^+ \; , \quad g_p^- \le g_t^+ + \sum_{p \le s < t} d_s^+ \; , \quad g_p^+ \ge g_t^- + \sum_{p \le s < t} d_s^- \tag{16}$$

for all $0 \le p \le t \le k$. Moreover, if all $g_p^+, g_p^-, d_p^+, d_p^-$ are integers satisfying (16), then (15) has an integral solution.

The proof will be given in the Appendix.

Proof of Lemma 10. It is easy to see that the conditions on $g^{(q-1)}$ assuring the admissibility of $(g^{(q)}, g^{(q-1)})$ can be written in the form (15) with $k = q-1$, $g_p^- = 0$, $g_p^+ = \min(g_p^q, g_{p+1}^q)$, $d_p^- = g_{p+1}^q - g_{p+2}^q - 1$, $d_p^+ = g_p^q - g_{p+1}^q + 1$ (recall that we use the convention $g_0^q = g_q^q = 0$). It remains to verify that the inequalities (13) for $g^{(q)}$ imply that the data just defined satisfy (16).

The inequality $d_p^- \le d_p^+$ is just (13), and the last two inequalities in (16) can be written as

$$g_t^q \;\; \le \;\; \min(g_t^q, g_{t+1}^q) + g_p^q + t - p \quad (0 \le p \le t \le q-1) \tag{17}$$

$$g_{p+1}^q \le \min(g_p^q, g_{p+1}^q) + g_{t+1}^q + t - p \quad (0 \le p \le t \le q-1) \tag{18}$$

In fact, the stronger inequalities hold:

$$g_t^q \le t - p + [(g_p^q + (t-p)g_{t+1}^q)/(t-p+1)] \tag{17'}$$

$$g_{p+1}^q \le t - p + [(g_{t+1}^q + (t-p)g_p^q)/(t-p+1)] \tag{18'}$$

Indeed, it is easy to verify that (right-hand side of (17')) - (left-hand side of (17')) $= \frac{1}{t-p+1}\sum_{p<s\le t}(s-p)(2+g_{s-1}^q - 2g_s^q + g_{s+1}^q)$, which proves (17'), the proof of (18') is analogous. This completes the proof of Lemma 10. □

Proof of Lemma 11. We see again that the conditions on $g^{(q-2)}$ to be satisfied can be written in the form (15) with $k = q-2$, $g_p^- = \max(g_p^{q-1} + g_{p+1}^{q-1} - g_{p+1}^q, 0)$, $g_p^+ = \min(g_p^{q-1}, g_{p+1}^{q-1})$, $d_p^- = g_{p+1}^{q-1} - g_{p+2}^{q-1} - 1$, $d_p^+ = g_p^{q-1} - g_{p+1}^{q-1} + 1$. It remains to verify that the inequalities (10), (11), and (14) for $(g^{(q)}, g^{(q-1)})$ imply that these data satisfy (16).

The inequality $d_p^- \le d_p^+$ again coincides with (13). The inequality $g_p^- \le g_t^+ + \sum_{p\le s<t} d_s^+$ can be rewritten as

$$\min(g_t^{q-1}, g_{t+1}^{q-1}) + \min(g_p^{q-1}, g_{p+1}^q - g_{p+1}^{q-1}) + t - p \ge g_t^{q-1}. \tag{19}$$

If $g_t^{q-1} \le g_{t+1}^{q-1}$ then (19) is a consequence of (14). It remains to prove

$$g_t^{q-1} \le t - p + g_p^{q-1} + g_{t+1}^{q-1} \tag{19'}$$

$$(g_{p+1}^q - g_{p+1}^{q-1}) - (g_t^{q-1} - g_{t+1}^{q-1}) + t - p \ge 0 \tag{19''}$$

But (19') follows at once from (17') (wiht q replaced by $q-1$), and the left-hand side of (19'') is equal to

$$(g_{t+1}^q - g_{t+1}^{q-1}) + \sum_{p<s\le t}(1 + (g_s^q - g_{s+1}^q) - (g_s^{q-1} - g_{s+1}^{q-1})),$$

which is nonnegative by (10) and (14). The last inequality in (16) is proven the same way. □

Proposition 8 and hence Theorem 6 are proven.

By our method we can obtain more information about the multiplicities $c_{\rho\rho}^\mu$, namely to find all μ such that $c_{\rho\rho}^\mu = 1$. To formulate the result consider the convex polytope $P(2\rho) \subset \mathbf{R}^r$ which is the convex hull of all dominant weights μ of the form $\mu = 2\rho - \sum g_p\alpha_p$, where $g_p \in \mathbf{Z}_+$ for $p = 1, \dots, r$. The following result is due to B. Kostant (private communication).

PROPOSITION 13. *The polytope $P(2\rho)$ is combinatorially equivalent to an r–cube. Vertices of $P(2\rho)$ are in a bijective correspondence with subsets $I \subset \{1,\ldots,r\}$: a vertex v_I corresponding to I is equal to $\rho + w_I\rho$, where w_I is the maximal element in the Weyl group generated by reflections $s_{\alpha_i}(i \in I)$.*

Remarks.

(a) The result of Proposition 13 can be extended to any regular dominant weight λ instead of 2ρ.
(b) For $g = s\ell(r+1)$ the polytope $P(2\rho)$ is naturally identified with the Newton polytope of the discriminant of a polynomial of degree $(r+1)$ in one variable (see [9]).

THEOREM 14. *For $g = s\ell(r+1)$ the multiplicity $c^{\mu}_{\rho\rho}$ is equal to 1 if and only if μ is a vertex of $P(2\rho)$.*

Proof. The "if" part follows at once from the well known inequality $c^{\mu}_{\lambda\nu} \leq K_{\lambda,\mu-\nu}$, where $K_{\lambda,\mu-\nu}$ is the multiplicity of weight $(\mu-\nu)$ in V_λ (see, e.g., [2]). Indeed, we have $c^{\rho+w_I\rho}_{\rho\rho} \leq K_{\rho,w_I\rho} = K_{\rho,\rho} = 1$.

To prove the "only if" part we shall use the following criterion for uniqueness of a solution of a system (15). Suppose we are given real numbers $g^+_p, g^-_p (0 \leq p \leq k), d^+_p, d^-_p (0 \leq p \leq k-1)$ satisfying (16). Consider the set $[0,k] = \{0,1,\ldots,k\}$. Define a subset Γ of $[0,k]$ as follows: $u \in \Gamma$ if and only if for some p and t with $0 \leq p \leq u \leq t \leq k$ we have either $g^-_p = g^+_t + \sum_{p\leq s<t} d^+_s$ or $g^+_p = g^-_t + \sum_{p\leq s<t} d^-_s$. Denote also by E the set of pairs $\{p,p+1\} \subset [0,k]$ such that $d^-_p = d^+_p$; we represent $\{p,p+1\}$ as an edge connecting points p and $p+1$. □

LEMMA 15. *In the conditions of Lemma 12 the system (15) has a unique solution if and only if any $p \in [0,k]$ can be connected with a point from Γ by a sequence of edges from E. Furthermore, if all $g^+_p, g^-_p, d^+_p, d^-_p$ are integers then the same conditions are necessary and sufficient for the system to have a unique integer solution.*

Lemma 15 will be proven in the Appendix.

Now let $g^{(r+1)} = (g^{r+1}_1, \ldots, g^{r+1}_r) \in \mathbf{Z}^r_+$ and $\mu = 2\rho - \sum g^{r+1}_p \alpha_p$. By Proposition 7, $g^{(r+1)}$ is admissible if and only if μ is dominant. The next lemma follows at once from Proposition 5 and Lemmas 10 and 11.

LEMMA 16. *If $c^{\mu}_{\rho\rho} = 1$ then there is a unique vector $g^{(r)} \in \mathbf{Z}^{r-1}_+$ such that $(g^{(r+1)}, g^{(r)})$ is admissible.*

LEMMA 17. *Suppose there is a unique vector $g^{(r)} \in \mathbf{Z}^{r-1}_+$ such that $(g^{(r+1)}, g^{(r)})$ is admissible. Then at least one of the following two conditions is satisfied: (a) $g^{(r+1)}_p = 0$ for some $p \in [1,r]$; (b) $\mu = m_u\omega_u$ for some $u \in [1,r]$ and $m_u \in \mathbf{Z}_+$.*

Proof of Lemma 17 . Recall that $(g^{(r+1)}, g^{(r)})$ is admissible if and only if $g^{(r)}$ satisfies the system of type (15) with $k = r$, $g_p^- = 0$, $g_p^+ = \min(g_p^{r+1}, g_{p+1}^{r+1})$, $d_p^- = g_{p+1}^{r+1} - g_{p+2}^{r+1} - 1$, $d_p^+ = g_p^{r+1} - g_{p+1}^{r+1} + 1$ (see the proof of Lemma 10). Let us apply Lemma 15 to these data.

By our conventions, $g_p^- = g_p^+ = 0$ for $p = 0,\ r$, so $\{0, r\} \subset \Gamma$. Consider two cases: a) $\Gamma \cap [1, r-1] \neq \phi$. Analysing the proof of Lemma 10 we can easily understand when the inequalities (17) and (18) become equalities, and this implies that either $g^{(r+1)}$ satisfies (a), or $\mu = 0$ (which is a special case of (b)).

b) $\Gamma = \{0, r\}$. By Lemma 15, E must contain all edges $\{p-1, p\}$ except maybe one edge $\{u-1, u\}$. But we have $d_{p-1}^+ - d_{p-1}^- = m_p$, so $m_p = 0$ for $p \neq u$, which is exactly (b). □

We shall treat cases (a) and (b) in Lemma 17 separately. In the case (a) we shall reduce the statement of Theorem 14 to the smaller values of r. To do this we need some notation. Denote by P_r the polytype $\{(g_1, \ldots, g_r) \in \mathbf{R}^r : 2\rho - \sum g_p\alpha_p \in P(2\rho)\}$. For $(g_1, \ldots, g_r) \in \mathbf{Z}_+^r \cap P_r$ put $c(g_1, \ldots, g_r) = c_{\rho\rho}^{\mu}$, where $\mu = 2\rho - \sum g_p\alpha_p$.

PROPOSITION 18. *A point $(g_1, \ldots, g_r)$ with $g_p = 0$ lies in P_r if and only if $(g_1, \ldots, g_{p-1}) \in P_{p-1}$ and $(g_{p+1}, \ldots, g_r) \in P_{r-p}$. If in addition all g_k are nonnegative integers then $c(g_1, \ldots, g_r) = c(g_1, \ldots, g_{p-1})c(g_{p+1}, \ldots, g_r)$.*

Proof. The first statement follows at once from definitions. The second one follows readily from Proposition 5 (this is also a special case of a general result, see [2], Proposition 1.3). □

PROPOSITION 19. *A point $v = \sum m_p\omega_p = 2\rho - \sum g_p\alpha_p \in P(2\rho)$ is a vertex of $P(2\rho)$ if and only if $\min(g_p, m_p) = 0$ for all p.*

Proof. It is easy to see that for $v = \rho + w_I\rho$ we have $g_p = 0$ for $p \notin I$, and $m_p = 0$ for $p \in I$. Conversely, suppose that $v \in P(2\rho)$ is such that $\min(g_p, m_p) = 0$ for all p. It is clear that g_p and m_p cannot be both equal to 0. Therefore, if $I = \{p : m_p = 0\}$ then $\{p : g_p = 0\} = [1, r] - I$. Then we have $2\rho = \sum_{p \in I} g_p\alpha_p + \sum_{p \notin I} m_p\omega_p$. But it is easy to see that the vectors $\alpha_p (p \in I)$,$\omega_p (p \notin I)$ are linearly independent, hence $v = \rho + w_I\rho$ as required. □

Propositions 18 and 19 imply the statement of Theorem 14 in the case (a) of Lemma 17 by induction on r. It remains to consider the case (b), when μ is proportional to a fundamental weight.

Let $\mu = m\omega_u$, where $m \in \mathbf{Z}_+, u \in [1, r]$. Put $n = mu/(r+1)$. Direct calculation shows that the vectors $g^{(r+1)}$ and $g^{(r)}$ from Lemma 17 take the form:

$$g_p^{r+1} = p(r + 1 - p - m + n)$$

$$\begin{aligned} g_t^{(r+1)} &= (t-n)(r+1-t) \\ & \quad 0 \le p \le u \le t \le r+1 \end{aligned} \tag{20}$$

$$\begin{aligned} g_p^r &= p(r-p-m+n) \\ g_t^r &= (t-n)(r-t) \\ & \quad 0 \le p < u \le t \le r \end{aligned} \tag{21}$$

(it is easy to see that the expressions in (20) coincide for $p = t = u$). It follows that n must be an integer and $0 \le n \le u$. Futhermore, if $n = 0$ or $n = u$ then μ is a vertex of $P(2\rho)$ according to Proposition 19. It remains to verify

LEMMA 20. *Let $g^{(r+1)}$ and $g^{(r)}$ be defined by (20) and (21), where $u \in [1, r], n \in [1, u-1]$ are such that $m = n(r+1)/u \in \aleph$. Then there are at least two different vectors $g^{(r-1)}$ such that the triple $(g^{(r+1)}, g^{(r)}, g^{(r-1)})$ satisfies the conditions of Lemma 11.*

Proof. The vectors $g^{(r-1)}$ in question are integral solutions of the system (15) with $k = r-1$, $g_p^- = \max(g_p^r + g_{p+1}^r - g_{p+1}^{r+1}, 0)$, $g_p^+ = \min(g_p^r, g_{p+1}^r)$, $d_p^- = g_{p+1}^r - g_{p+2}^r - 1$, $d_p^+ = g_p^r - g_{p+1}^r + 1$ (see the proof of Lemma 11). By Lemma 15 it suffices to verify the inequalities $d_p^+ > d_p^-$ for $p = u-2$, $u-1$ and

$$g_p^- < g_t^+ + \sum_{p \le s < t} d_s^+ , \qquad g_p^+ > g_t^- + \sum_{p \le s < t} d_s^- \tag{22}$$

for $p \le u-1 \le t$.

It follows from (21) that $d_{u-2}^+ - d_{u-2}^- = n$, $d_{u-1}^+ - d_{u-1}^- = m-n$, both obviously positive. To prove the first inequality in (22) it suffices to examine the proof of (19) given above and to check that in our case the inequality in (19) is strict. The second part of (22) is proven in the same way. □

Theorem 14 is proven.

Remark. Let $\gamma = (g_1, ..., g_r) \in \mathbf{Z}_+^r$ be an admissible vector. Denote by $G(\gamma)$ the set of all families $g = (g^{(q)} = (g_p^q))$, $2 \le q \le r+1$ from Proposition 8. We supply $G(\gamma)$ with the following partial order: $g = (g^{(q)}) > h = (h^{(q)})$ if there exists some t such that $g^{(q)} = h^{(q)}$ for $q > t$, $g^{(t)} \ne h^{(t)}$ and $g_p^t \ge h_p^t$ for all p. One can show that for any γ the set $G(\gamma)$ has a unique maximal element $g_{\max}(\gamma)$ and a unique minimal element $g_{\min}(\gamma)$. This follows from the fact that any system of type (15) has the unique maximal and the unique minimal solution with respect to the componentwise partial order, which can be proven directly.

Appendix. Systems of linear inequalities and proofs of Lemmas 12 and 15

We shall deduce Lemmas 12 and 15 from general existence and uniqueness criteria for systems of linear inequalities. Suppose we are given a set $F = (f_1, \ldots, f_N)$ of linear forms on $\mathbf{R}^\ell$ of rank ℓ. Consider the system of linear inequalities

$$f_j(x) \leq c_j \qquad j = 1, \ldots, N \tag{A1}$$

with real parameters $c = (c_1, \ldots, c_N)$. Denote by $\Delta(F, c) \subset \mathbf{R}^\ell$ the polyhedral set of solutions of (A1). Since $rk(F) = \ell$ is is easy to see that if $\Delta(F, c)$ is nonempty then it has a vertex.

A subset $C \subset F$ is called a *circuit* if the forms $f_j \in C$ are linearly dependent, and C is minimal with this property (the terminology comes from the matroid theory). Clearly, for any circuit C a linear relation of the form $\sum_{f_j \in C} a_j f_j = 0$ is unique up to a scalar multiple. We say that the relation is *positive* if all $a_j > 0$, and that C is *positive* if it admits a positive relation.

THEOREM A.1.

(a) $\Delta(F, c) \neq \emptyset$ *if and only if for any positive circuit* C *and a positive relation* $\sum_{f_j \in C} a_j f_j = 0$ *we have*

$$\sum a_j c_j \geq 0. \tag{A2}$$

(b) Under the assumptions of a), $\Delta(F, c)$ *consists of one point if and only if the union of all positive circuits* C *such that (A2) becomes an equality has rank* ℓ.

(c) Suppose that all $f_j \in (\mathbf{Z}^\ell)^*$*, i.e., have integral coefficients, and* $c_j \in \mathbf{Z}$ *for all* j. *Suppose also that each subset of* F *which is a basis of* $(\mathbf{R}^\ell)^*$ *is also a* $\mathbf{Z}$*-basis of* $(\mathbf{Z}^\ell)^*$. *Then all vertices of* $\Delta(F, c)$ *lie in* $\mathbf{Z}^\ell$.

All statements of Theorem A.1 are well known in linear programming. Since we have not found the statements exactly in this form in literature let us give a brief sketch of the proofs. First note that if $\Delta(F, c) \neq \emptyset$ then obviously $\sum b_j c_j \geq 0$ for any linear relation $\sum b_j f_j = 0$ with all $b_j \geq 0$. The converse statement is also true (see, e.g., [6]). Therefore, a) is a consequence of the following.

PROPOSITION A.2. [5]. *Any linear relation* $\sum b_j f_j = 0$ *with all* $b_j \geq 0$ *can be represented as a sum of positive relations* $\sum_{f_j \in C} a_j f_j = 0$ *corresponding to positive circuits* C.

Now suppose that $\Delta(F, c) \neq \emptyset$ and consider the statement A.1 b). If (A.2) becomes an equality for a positive circuit C then $f_j(x) = c_j$ for any $f_j \in C$, $x \in \Delta(F, c)$. Therefore, if the union of such circuits has full rank then x satisfies a system of linear equations of full rank hence is unique. Let us outline the proof of the converse statement. So we suppose that $\Delta(F, c)$ consists of one point x. Consider the subset $S = \{f_j \in F : f_j(x) = c_j\}$ and let K be the convex cone generated by S. Let $K^\vee = \{y : f_j(y) \geq 0 \text{ for all } f_j \in S\}$ be the dual cone. We claim that $K^\vee = \{0\}$. Indeed, if $K^\vee$ contains some $y \neq 0$ then $(x - \varepsilon y) \in \Delta(F, c)$

for sufficiently small $\varepsilon > 0$. It follows that $K = (K^\vee)^\vee = (\mathbf{R}^\ell)^*$. So A.1 b) is a consequence of Proposition A.2 and the following.

PROPOSITION A.3. [5] *Let $S \subset \mathbf{R}^\ell$ be a finite subset of rank ℓ. Then the convex cone generated by S coincides with $\mathbf{R}^\ell$ if and only if there is a subset $S' \subset S$ such that $rk(S') = \ell$ and some linear combination of the elements of S' with positive coefficients is equal to 0.*

Finally, part A.1 c) is evident since each vertex of $\Delta(F, c)$ is uniquely determined from a system of linear equation of the form $\{f_j(x) = c_j\}$, where f_j runs over a certain subset of F, which forms a basis in $(\mathbf{R}^\ell)^*$.

Proofs of Lemmas 12 and 15 . We apply Theorem A.1. in the following situation: $\mathbf{R}^\ell = \mathbf{R}^{k+1}$ with coordinates $g_0, \ldots, g_k$; $F = (g_0, -g_0, \ldots, g_k, -g_k, g_0 - g_1, -(g_0 - g_1), \ldots, g_{k-1} - g_k, -(g_{k-1} - g_k))$; $c = (g_0^+, -g_0^-, \ldots, g_k^+, -g_k^-, d_0^+, -d_0^-, \ldots, d_{k-1}^+, -d_{k-1}^-)$. For any subset $S \subset F$ put $\Gamma(S) = \{p \in [0,k]: S \text{ contains } g_p \text{ or } (-g_p)\} \subset [0,k]$ and let $E(S)$ be the set of all pairs $\{p, p+1\} \subset [0,k]$ such that S contains $(g_p - g_{p+1})$ or $-(g_p - g_{p+1})$. As in Lemma 15 we represent $\{p, p+1\}$ as an edge connecting the points p and $p+1$. □

LEMMA A.4. *A subset $S \subset F$ is of rank $k+1$ if and only if any $p \in [0,k]$ can be connected with a point from $\Gamma(S)$ by a sequence of edges from $E(S)$. Any such S of cardinality $k+1$ is a $\mathbf{Z}$-basis of $\mathbf{Z}^{k+1}$.*

The proof is straightforward.

Now let us list all positive circuits of F. For any $p \in [0, k-1]$ put $C_p = \{g_{p-1} - g_p, -(g_{p-1} - g_p)\}$. For any p, t such that $0 \le p \le t \le k$ put $C_{pt} = \{g_p, -(g_p - g_{p+1}), \ldots, -(g_{t-1} - g_t), -g_t\}$, and $C_{tp} = \{-g_p, g_p - g_{p+1}, \ldots, g_{t-1} - g_t, g_t\}$. Clearly, all C_p, C_{pt}, and C_{tp} are positive circuits; the corresponding positive linear relations have all coefficients equal to 1.

LEMMA A.5. *The circuits C_p, C_{pt}, and C_{tp} exhaust all positive circuits of F.*

This follows easily from Lemma A.4.

Taking into account Lemmas A.4 and A.5 we see that Lemma 12 is a special case of Theorem A.1 a), c), and Lemma 15 is a special case of Theorem A.1 b), c).

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