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The Shuffle Pasting

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Abstract. The graded set of n-dimensional (p, q)-shuffles is endowed with the structure of well-formed loop-free pasting scheme. In the process, well-formed subpasting schemes and their sources and targets are characterized, using a higher Bruhat type order.

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1. Introduction

Higher Bruhat orders were introduced by Manin and Schechtman [18]. These are posets $B(n, k)$, where $B(n, 1)$ is the symmetric group with its weak Bruhat order and where $B(n, k+1)$ is a quotient of the set $A(n, k+1)$ of maximal chains in $B(n, k)$ by certain *elementary equivalences*. The posets $B(n, k)$ have rank functions and a unique minimal and a unique maximal element.

Another interpretation of higher Bruhat orders was given by Kapranov and Voevodsky [16], in terms of pasting schemes [14] and strict n-categories [21]. The combinatorial n-cube Λ_n has a (more or less canonical) structure of a well-formed loop-free pasting scheme [7, Section 3.3]; denote the free strict n-category on Λ_n by $\mathbb{I}^n$. Because $\mathbb{I}^n$ is the free strict n-category on a well-formed loop-free pasting scheme, it is very structured: the set $\mathrm{Ob}(\mathbb{I}^n)$ is partially ordered by having $x < y$ if and only if there is an arrow $x \to y$ in $\mathbb{I}^n$, and this partial order has unique minimal and maximal elements $x_{\min}$ and $x_{\max}$. Thus there is a strict $(n-1)$-category $\mathbb{I}^n(x_{\min}, x_{\max})$, also denoted $\Omega(\mathbb{I}^n)$; this is precisely (isomorphic to) Manin and Schechtman's strict $(n-1)$-category S_n constructed directly in terms of higher Bruhat orders. Furthermore, $\Omega(\mathbb{I}^n)$ itself also has a partial order with unique minimal and maximal elements, and this continues, giving a sequence of strict $(n-k)$-categories denoted $\Omega^k(\mathbb{I}^n)$. Kapranov and Voevodsky's result is that the set $\mathrm{Ob}(\Omega^k(\mathbb{I}^n))$ with its partial order is precisely (isomorphic to) $B(n, k)$.

The use of the symbols Ω and Ω^k suggests a connection with k-fold loop spaces in topology [5, 19]. At present, this connection is only a tenuous one. One reason is that

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strict n-groupoids and not strict n-categories are used to classify homotopy types. This is actually an advantage: strict n-categories, unlike strict n-groupoids, are sensitive to direction/orientation, so they could give a richer algebraic version of looping. The second reason, which is a very serious disadvantage, is that strict n-groupoids de not classify *all* homotopy types: only those which have trivial Whitehead products [6, p. 114]. It must be admitted that even strict 1-categories do classify all homotopy types [22], but this classification, using the twice iterated barycentric subdivision of a simplicial set, is not very illuminating—for example, one cannot easily recover the homotopy groups from the strict 1-category.

It is known that **Gray**-groupoids classify all homotopy 3-types [4, 15]. **Gray**-categories are similar to strict 3-categories except that instead of the interchange axiom, which implies that horizontal composition of 2-arrows is definable in terms of vertical composition, there is a dimension raising composition $C_2 \times_{C_0} C_2 \to C_3$ satisfying some further axioms. I have defined a 4-dimensional generalization of **Gray**-categories called 4*D teisi*,[1] and I have given some heuristics for the—currently hypothetical—general, higher-dimensional notion of *nD tas* [9]. nD teisi differ from strict n-categories in that all instances of the interchange axiom are replaced by dimension raising compositions $C_p \times_{C_n} C_q \to C_{p+q-n-1}$, that have to satisfy *naturality*, *functoriality*, *associativity*, and more axioms. It must be remarked that there is also the notion of weak n-category, in many different (?) incarnations [1, 2, 12], which are very much in vogue, but these are not of concern for this paper.

Just like the notion of strict n-category, the notion of nD tas is *dimension invariant*: for an nD tas $\mathbb{C}$ and for each pair of m-arrows c and c' in $\mathbb{C}$ with common $(m-1)$-source and -target, the collection of elements $\mathbb{C}$ with m-source c and m-target c' is itself an $(n-m-1)$D tas, denoted $\mathbb{C}(c, c')$. For x an object of $\mathbb{C}$, define $\pi_m(\mathbb{C}, x)$ as the collection of connected components of $\mathbb{C}(\mathrm{id}_x^{m-1}, \mathrm{id}_x^{m-1})$; $\pi_m(\mathbb{C}, x)$ is a monoid for $m \geq 1$, with $\#_{m-1}$ as multiplication, and, by a standard Eckmann-Hilton argument, this monoid is actually commutative for $m \geq 2$. Thus, if $\mathbb{C}$ is an nD iso-tas it is reasonable to interpret $\pi_m(\mathbb{C}, x)$ as the *m-th homotopy group of* $\mathbb{C}$ *based at* x. Unlike for strict n-categories, the dimension raising compositions in an nD tas induce operations $\pi_p(\mathbb{C}, x) \times \pi_q(\mathbb{C}, x) \to \pi_{p+q-n-1}(\mathbb{C}, x)$, which can reasonably be interpreted as (*generalized*) *Whitehead products*. Although it is still unknown whether nD iso-teisi classify all homotopy n-types, and although a rigorous definition of nD tas is still wanting, they thus should reflect topology much better than both strict 1-categories and strict n-categories.

It is now natural to ask whether Kapranov and Voevodsky's interpretation of higher Bruhat orders can be generalized to nD teisi. In other words, one would want to consider the free nD tas on Λ_n, again to be denoted $\mathbb{I}^n$, which one would expect to possess the same remarkable properties of order, allowing one to obtain a sequence of $(n-k)$D teisi again denoted $\Omega^k(\mathbb{I}^n)$. This is both a difficult and interesting combinatorial problem, which is closely related to the absence of a rigorous definition of nD tas. One potentially promising line of attack is to use Kapranov and Voevodsky's idea of derived pasting scheme [16]. If one could show that for *any* well-formed loop-free pasting scheme A, denoting the free nD tas on A by $\mathcal{P}(A)$, that $\Omega(\mathcal{P}(A))$ were free on another well-formed loop-free pasting scheme, which could then reasonably be denoted $\Omega(A)$, the required order properties of $\Omega^k(\mathbb{I}^n)$ would follow immediately by induction. $\Omega(A)$ would be the derived pasting scheme of A, although Kapranov and Voevodsky motivate it as some "free cover" of the strict

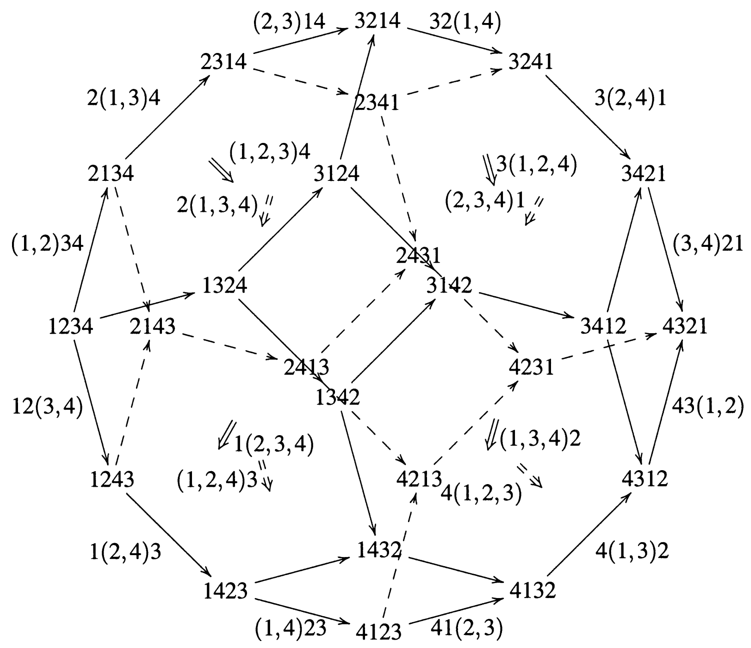


Figure 1. The 4-permutohedron P_4.

$(n-1)$-categorical version of $\Omega(\mathcal{P}(A))$, and define it only as a graded poset, posing the (still open) problem of endowing this graded poset with the structure of well-formed loop-free pasting scheme. From the considerations here, it would perhaps be better to call $\Omega(A)$ the *path pasting scheme of* A.

It is well known that, as a set or as a polytope, the path space of the n-cube is the n-permutohedron P_n [20]. So for $A = \Lambda_n$, the problem would be to endow P_n with the structure of well-formed loop-free pasting scheme. Kapranov and Voevodsky mention that (the graded poset) $\Omega(\Lambda_n) = P_n$ and remark that "for P_n the construction behaves badly" [16, p. 25]. To illustrate the complications involved, consider the 4-permutohedron, pictured in figure 1, where the numbers indicate the four directions of the 4-cube, where higher-dimensional faces of 4-cube are characterized by naming their directions, and where juxtaposition denoted composition. Firstly, the hexagonal faces of the 4-permutohedron come from 3-cubes, whose direction determines the orientation of the hexagons, as indicated. So this already imposes some restrictions on the pasting scheme structure. Secondly, the square faces of the 4-permutohedron come from dimension raising composites of 2-cubes. It turns out that, no matter what convention one would use for the source and target of a dimension raising composite, *some* of the squares run in the 'wrong' direction. This implies several things: one, that in nD teisi dimension raising composition should result in elements running in *both* directions, presumably inverse to one another, two, that one needs to find a way to tell which direction to use when, and three, that one might need something more general than pasting schemes in order to deal with cells running in opposite directions. Thirdly, the squares on the 'top' and the 'bottom' of the 4-permutohedron come from dimension raising composites of 2-cubes in the 2-source and 2-target respectively of the 4-cube, and have no *a priori* preference for being on the front or on the back of the 4-permutohedron. This implies that one needs to find a way to tell which side of the 4-permutohedron to assign these cells to, or that one might need another or a further generalization of pasting schemes allowing such cells to remain 'undecided', so that a 1-source can be 2-dimensional. Fourthly, some square faces of the 4-permutohedron are horizontally composable. This means that in $\Omega(P_4)$ there will be extra 2-cells whose place needs to be determined as well. This phenomenon was already observed by Baues: "[The boundary of $\Omega(P_4)$] is a 2-dimensional complex not a sphere but still homotopy equivalent to the 1-sphere. [$\Omega(P_4)$] is the cone on this complex and so is not a euclidean cell" [3, p. 121].

In this paper I deal with an accessible and usable part of these problems, by not looking at all permutations, restricting attention to shuffles. I show that (p, q)-shuffles are the vertices of a well-formed loop-free pasting scheme $M_{p,q}$. n-cells of this pasting scheme, which are called *n-dimensional (p, q)-shuffles*, are sequences of p 0's and q 1's partitioned into $p + q - n$ parts each of size at most two where the parts of size two must consist of a 0 followed by a 1; the partition will be indicated by brackets around the parts of size two. The example $M_{3,3}$ is given in figure 5. $M_{p,q}$ is the path pasting scheme of the (p, q)-'tablecloth', pictured in figure 2. This makes $M_{p,q}$ the simplest non-trivial double path pasting scheme

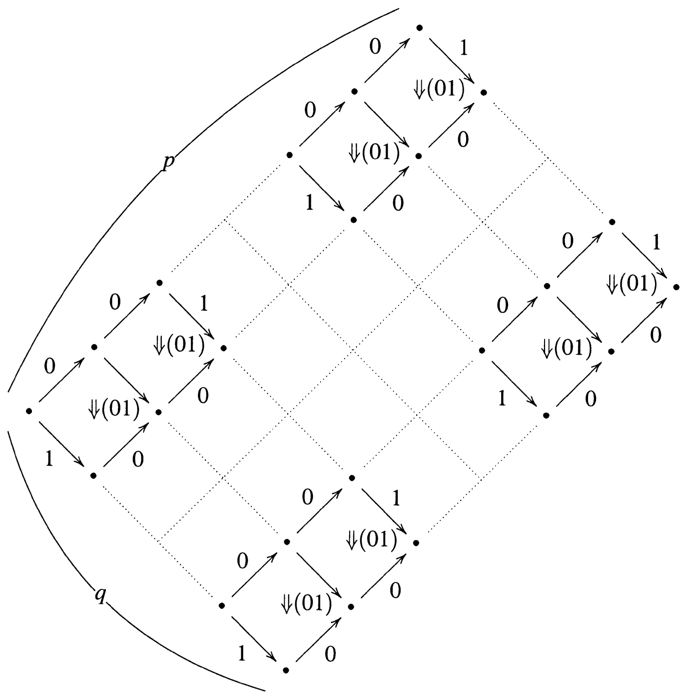


Figure 2. The (p, q)-'tablecloth'.

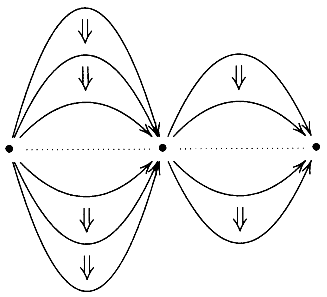


Figure 3.

scheme,[2] because the (p, q)-tablecloth is itself the path pasting scheme of the well-formed loop-free pasting scheme pictured in figure 3, and the first one to be so identified. It must be said that Lawrence has some calculations for $\Omega(P_n)$, but as polytopes not as pasting schemes [17].

$M_{p,q}$ sits inside P_n in $n!(n-p-q+1)/\binom{p+q}{p}$ ways, in particular, in P_{p+q} in $p!\,q!$ ways. For example, the four $M_{2,2}$'s in P_4 are pictured in figure 4. Thus, understanding $M_{p,q}$ is essential in understanding P_n. Because the cells of $M_{p,q}$ themselves have the shape of cubes, $P_{n'}$'s will sit inside $\Omega(M_{p,q})$, hence there will be $M_{p',q'}$'s sitting inside $\Omega(P_n)$ too. It is clear that this compatibility of shapes over dimensions will be useful; it is equally clear that it will only be a small part of the story.

In terms of Manin and Schechtman's original formulation of higher Bruhat orders, replacing strict n-categories by nD teisi in Kapranov and Voevodsky's interpretation means that the elementary equivalences are not factored out, but become, or rather remain, part of the data. For two maximal chains in $B(n, k)$ are elementary equivalent "if they differ by an interchange of two neighbours which do not belong to a common packet", that is, if they are the source and target of the result of a dimension raising composite. The question

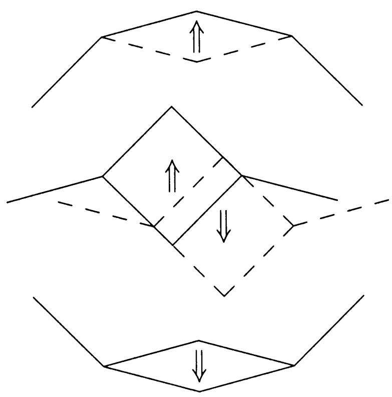


Figure 4. The four $M_{2,2}$'s in P_4.

asked above about generalizing higher Bruhat orders becomes whether there are ranked posets $B'(n,k)$ with unique minimal and maximal elements where $B'(n,1) = B(n,1)$ is the symmetric group with its weak Bruhat order and where $B(n,k+1)$ is the set of maximal chains in $B'(n,k)$, which is (surely) a natural question in its own right. The relevance of the shuffle pasting scheme to this particular phrasing of the question is not that it isolates the interdependences between the elementary equivalences, nor that it covers all elementary equivalences, but that it resolves *some* interaction of *some* elementary equivalences with some 'genuine' elements of $B'(n,k+1)$.

A more immediate application of the shuffle pasting scheme is in the study of Zamolodchikov equations. Recall that one of the axioms for a braiding on a monoidal 2-category is that the two fillings of the Yang-Baxter hexagon are equal [8, Definition 2-2]. Calling this filling $S_{A,B,C}$ the equation

$$\begin{aligned} &S_{123}S_{124}(R_{34}R_{12})(R_{13}R_{24})^{-1}S_{134}S_{234}(R_{14}R_{23}) \\ &\quad = (R_{23}R_{14})^{-1}S_{234}S_{134}(R_{24}R_{13})(R_{12}R_{34})^{-1}S_{124}S_{123}, \end{aligned} \tag{1.1}$$

which in diagrammatic form just states the commutativity of the realization of the 4-permutohedron in the braided monoidal 2-category, is a consequence of the other axioms for a braiding [10]. In another paper [11] I will use the 2- and 3-dimensional part of the results here to show the converse, that a monoidal 2-category together with a system of hexagonal 2-arrows $S_{A,B,C}$ satisfying Eq. (1.1) gives rise to a braided monoidal 2-category. It must be revealed that Kapranov and Voevodsky's account of this matter [16, Sections 6.10–6.14] fails to take the shuffle issues into account.

Before one can show that a pasting scheme is well formed and loop free, one needs to know its well-formed subpasting schemes and their sources and targets. In order to obtain this knowledge for $M_{p,q}$, I distinguish *n-kinds* of n-dimensional (p,q)-shuffles, where two n-dimensional (p,q)-shuffles of the same n-kind can and should be thought of as being parallel, and I consider a partial order $\lessdot$ which only relates n-dimensional (p,q)-shuffles of the same n-kind. This gives posets that have rank functions, where the rank can and should be thought of as measuring a height, and a unique minimal and a unique maximal element.

I obtain a usable characterization of well-formed subpasting schemes of $M_{p,q}$ by establishing two conditions on collections of n-dimensional (p,q)-shuffles, *filling* and *fitness*. Filling is based on the $\lhd$-relation in $M_{p,q}$; it is the condition most suitable to prove things from. Fitness is based on the $\lessdot$-order; it is the condition most easy to check. I show that given a subpasting scheme of $M_{p,q}$, it is well-formed if and only if all collections of top-dimensional cells of sources and targets fill if and only if all these collections are fit. The most difficult step in the proof, that fitness implies filling, involves a careful analysis of the relation between $\lessdot$ and $\lhd$.

I use the fitness formulation to classify in terms of $\lessdot$ when there is a well-formed subpasting scheme of $M_{p,q}$ with given source and target, to show that $M_{p,q}$ itself is well formed, and in proving that $M_{p,q}$ is loop free.

I hope and expect that the techniques developed here, although fairly particular to $M_{p,q}$, will in some form be useful in future investigations of path pasting schemes.

This paper is organized as follows. Section 2: preliminaries on pasting schemes. Section 3: definition and different representations of n-dimensional (p,q)-shuffles. Section 4: n-dimensional (p,q)-shuffles have the shape of n-cubes, thus $M_{p,q}$ is a pasting scheme. Section 5: $M_{p,q}$ has no direct loops. Section 6: definition of rank and oprank, and various ways of calculating them. Section 7: definition of the $\lessdot$-order, and various characterizations of it. Section 8: filling, and n-stage shuffle collections as representation for subpasting schemes. Section 9: fitness, and that an n-kind is needed if and only if it is properly relevant. Section 10: filling and fitness and well-formedness are equivalent. Section 11: classification of when there is a well-formed subpasting scheme with given n-source and n-target. Section 12: $M_{p,q}$ is well formed. Section 13: $M_{p,q}$ is loop free.

2. Pasting preliminaries

Well-formed loop-free pasting schemes were introduced by Johnson [14] in order to parametrize composable diagrams in an ω-category.

A pasting scheme is a graded set A together with two collections of relations E^i_j and $\mathsf{B}^i_j \subseteq A_i \times A_j$ for $j \leq i$, satisfying certain conditions that will be spelled out below. E^i_j and B^i_j may be thought of as describing which j-cells are at the "end", respectively "beginning" of each of the i-cells. For such a graded set A with relations E^i_j and B^i_j, let R^i_j be the relation between A_i and A_j given by $x\mathsf{R}^i_j y$ when there exists a sequence $x = x_1, \ldots, x_k = y$ of cells of A satisfying $x_{k'}\mathsf{D}^{i_{k'}}_{i_{k'+1}} x_{k'+1}$ for all $1 \leq k' \leq k$ with $\mathsf{D}^i_j = \mathsf{E}^i_j$ or B^i_j. If $x\mathsf{R}^i_j y$ then y is said to be a *face* of x.

If X is a subgraded set of A of dimension n then the graded set $\mathsf{E}^i(X)$ is defined by $\mathsf{E}^i(X)_j = \{y \in A_j \mid \text{there exists } x \in X_i \text{ with } x\mathsf{E}^i_j y\}$. The graded set $\mathsf{E}^n(X)$ will be denoted $\mathsf{E}(X)$. Graded sets $\mathsf{B}(X)$ and $\mathsf{R}(X)$ are defined analogously. The grading will often be placed on the relation, thus denoting the set $\mathsf{E}(X)_j$ of j-dimensional elements of $\mathsf{E}(X)$ by $\mathsf{E}_j(X)$. The relation E^i_j is called *finitary* when, for any $x \in A_i$, the set $\mathsf{E}^i_j(x)$ is finite.

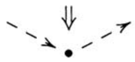
There is a duality between the "end" and "beginning" relations: for every proposition P and every i, the i-th dual of P is obtained from P by replacing all occurrences of E^i by B^i and vice versa.

Definition 2.1 A *pasting scheme* is a graded set A together with finitary relations E^i_j and B^i_j for $j \leq i$ such that

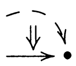
(i) E^i_j is a relation between A_i and A_j;
(ii) E^i_i is the identity relation on A_i;
(iii) for $i > 0$ and any $x \in A_i$ there exists $y \in A_{i-1}$ with $x\mathsf{E}^i_{i-1}y$;
(iv) for $j < i$, $w\mathsf{E}^i_j x$ if and only if there exists u and v such that $w\mathsf{E}^i_{i-1}u\mathsf{E}^{i-1}_j x$ and $w\mathsf{E}^i_{i-1}v\mathsf{B}^{i-1}_j x$;
(v) if $w\mathsf{E}^i_{i-1}u\mathsf{E}^{i-1}_j x$, then either $w\mathsf{E}^i_j x$ or there exists v such that $w\mathsf{B}^i_{i-1}v\mathsf{E}^{i-1}_j x$

and dually (notice that there are four dual forms of condition (v)).

Informally, condition (iii) says that every i-cell ends in at least one $(i-1)$-cell, and dually begins in at least one $(i-1)$-cell. Condition (iv) ensures that low dimensional ends occur between higher dimensional ends:

 .

Finally, condition (v) ensures that a cell's ends close up:

 or  .

In a pasting scheme A, define $\lhd_A$, written as $\lhd$ when there is no danger of confusion, as follows: for any i, and for any $a, b \in A_i$, say $a \lhd b$ if there is a sequence $a = a_0, \ldots a_k = b$, $k > 0$, of elements of A_i with, for all $k' < k$, $\mathsf{E}_{i-1}(a_{k'}) \cap \mathsf{B}_{i-1}(a_{k'+1}) \neq \varnothing$.

Definition 2.2 A pasting scheme A has *no direct loops* when, for any i and any $a, b \in A_i$, $\mathsf{B}(a) \cap \mathsf{E}(a) = \{a\}$ and $a \lhd b$ implies $\mathsf{B}(a) \cap \mathsf{E}(b) = \varnothing$.

If A is a pasting scheme and X is a finite subgraded set of A, define its domain $\mathrm{dom}(X)$ by $X - \mathsf{E}(X)$ and its codomain $\mathrm{cod}(X)$ by $X - \mathsf{B}(X)$.

Lemma 2.3 (*Johnson*) *If A is a finite, n-dimensional pasting scheme with no direct loops, then* $\mathrm{dom}(A)$ *is a* $(n-1)$*-dimensional graded set.*

Theorem 2.4 (*Johnson*) *If A is a finite pasting scheme with no direct loops then* $\mathrm{dom}\,\mathrm{dom}(A) = \mathrm{dom}\,\mathrm{cod}(A)$.

Thus, finite pasting schemes with no direct loops have sensible notions of domain and codomain. If A is a finite n-dimensional pasting scheme with no direct loops, write

$$\begin{aligned} s_m(A) &= A && \text{if } m \geq n \\ &= \mathrm{dom}^{n-m}(A) && \text{if } m < n \end{aligned}$$

and

$$\begin{aligned} t_m(A) &= A && \text{if } m \geq n \\ &= \mathrm{cod}^{n-m}(A) && \text{if } m < n. \end{aligned}$$

Notice that if $m < n$ then $s_m(A)$ and $t_m(A)$ are m-dimensional by Lemma 2.3. $s_n(A)$ is called the n-source of A, and $t_n(A)$ the n-target of A.

Definition 2.5 A pasting scheme A of dimension $n > 0$ is *compatible* when for any $x, y \in A_n$, if $x \neq y$, then $\mathsf{B}_{n-1}(x) \cap \mathsf{B}_{n-1}(y) = \varnothing$ and dually. A zero-dimensional pasting scheme is compatible if it is a singleton.

A subgraded set X of a pasting scheme A is a *subpasting scheme* of A if $y \in \mathsf{R}(X)$ implies $y \in X$.

A finite pasting scheme A with no direct loops is *well formed* if

(i) A is compatible;
(ii) for all $m \geq 0$, both $s_m(A)$ and $t_m(A)$ are compatible subpasting schemes of A.

Loop-freeness is a technical condition serving to eliminate more subtle looping behaviour.

Definition 2.6 A pasting scheme A is *loop free* if

(i) A has no direct loops;
(ii) for any $x \in A$, $\mathsf{R}(x)$ is well formed;
(iv) for any well-formed j-dimensional subpasting scheme Y of A and any $x \in A$ with $s_j(\mathsf{R}(x)) \subseteq Y$, if $u, u' \in s_j(\mathsf{R}(x))$ and, for some $v \in Y_j, u \lhd_Y v \lhd_Y u'$, then $v \in s_j(\mathsf{R}(x))$

and dually.

Condition (iii) is omitted here since it is a consequence of the other three conditions and the pasting axioms, see [13].

3. n-Dimensional shuffles

For any $n \in \mathbb{N}$, let $\underline{n}$ denote the ordered set $\{1, 2, \ldots, n\}$.

Definition 3.1 A (p, q)-*shuffle* is a function $f : \underline{p+q} \to \{0, 1\}$ such that $\# f^{-1}(0) = p$.

Thus, a (p, q)-shuffle is a sequence of p 0's and q 1's.

There are three alternative combinatorial representations of (p, q)-shuffles:

- (p, q)-shuffles are usually defined as permutations $\sigma : \underline{p+q} \to \underline{p+q}$ which are order preserving on $\{1, 2, \ldots, p\}$ and on $\{p+1, \ldots, p+q\}$. The image under σ of $\{1, 2, \ldots, p\}$ is $f^{-1}(0)$ and the image under σ of $\{p+1, p+2, \ldots, p+q\}$ is $f^{-1}(1)$. I will not use this way to represent a (p, q)-shuffle in the sequel.
- Another way to represent a (p, q)-shuffle is as a partition of $\underline{p+q}$ in two parts, one of size p and one of size q, that is, as a pair of strictly order preserving functions $\alpha : \underline{p} \to \underline{p+q}$ and $\beta : \underline{q} \to \underline{p+q}$ which have disjoint images. For $i_0 \in \underline{p}, \alpha(i_0)$ gives the place of the i_0-th 0 in the sequence represented by f, and for $i_1 \in \underline{q}$, $\beta(i_1)$ gives the place of the i_1-th 1.
- A fourth way to represent a (p, q)-shuffle is as a pair of surjective order preserving functions $\zeta_0 : \underline{p+q} \to \{0, \ldots, p\}$ and $\zeta_1 : \underline{p+q} \to \{0, \ldots, q\}$ satisfying $\zeta_0(i) + \zeta_1(i) = i$ for all $i \in \underline{p+q}$. For $i \in \underline{p+q}$, $\zeta_0(i)$ gives the number of 0's in the sequence represented by f up to and including position i, and $\zeta_1(i)$ gives the corresponding number of 1's.

Table 1. Representation of (p, q)-shuffles.

	f	α, β	ζ_0, ζ_1	σ
f		via ζ_0, ζ_1	$f(i) = 0$ if $\zeta_0(i) > \zeta_0(i-1)$ $= 1$ if $\zeta_1(i) > \zeta_1(i-1)$	$f(i) = 0$ if $\sigma^{-1}(i) \leq p$ $= 1$ if $\sigma^{-1}(i) > p$
α, β	via ζ_0, ζ_1		$\alpha(i_0) = \min\{i \mid \zeta_0(i) = i_0\}$ $\beta(i_1) = \min\{i \mid \zeta_1(i) = i_1\}$	$\alpha(i_0) = \sigma(i_0)$ $\beta(i_1) = \sigma(p + i_1)$
ζ_0, ζ_1	$\zeta_0(i) = \#(f^{-1}(0) \cap \underline{i})$ $\zeta_1(i) = \#(f^{-1}(1) \cap \underline{i})$	$\zeta_0(i) = \max\{i_0 \mid \alpha(i_0) \leq i\}$ $\zeta_1(i) = \max\{i_1 \mid \beta(i_1) \leq i\}$		via α, β
σ	via ζ_0, ζ_1	$\sigma(i) = \alpha(i)$ if $i \leq p$ $= \beta(i - p)$ if $i > p$	via α, β	

Table 1 summarizes how to go from one representation to another, for $i \in \underline{p+q}$, $i_0 \in \underline{p}$, $i_1 \in \underline{q}$ (take $\zeta_0(-1) = \zeta_1(-1) = 0$).

It is perhaps worth mentioning that α and $\zeta_0|_{f^{-1}(0)}$ give the bijection between $\underline{p}$ and $f^{-1}(0)$, and similarly β and $\zeta_1|_{f^{-1}(1)}$ between $\underline{q}$ and $f^{-1}(1)$. Also, the above allows one to express β in terms of α, which at the present point does not seem to be very useful, but it will be of use in the proof of Lemma 11.6.

Definition 3.2 Let $f : \underline{p+q} \to \{0, 1\}$ be a (p, q)-shuffle. Its *opposite* is the (q, p)-shuffle f^{op} given by $f^{\mathrm{op}}(i) = 1 - f(i)$.

Definition 3.3 An *n-dimensional* (p, q)*-shuffle* consists of a (p, q)-shuffle f together with an order preserving surjective function $g : \underline{p+q} \to \underline{p+q-n}$ such that if $g(i) = g(i+1)$ then $f(i) = 0$ and $f(i+1) = 1$.

Thus, an n-dimensional (p, q)-shuffle is a sequence of p 0's and q 1's partitioned into $p + q - n$ parts each of size at most two. A pair $(i, i+1)$ for which $g(i) = g(i+1)$ is called a *swap*, of which there are precisely n. In the sequence represented by f a swap must be a 0 followed by a 1, and I will indicate the swap by bracketing the 01 together. In particular, a 0-dimensional (p, q)-shuffle has no swaps, and is just a (p, q)-shuffle.

There are three other—equivalent—ways to represent where the swaps are:

- One is by means of a function $h : \underline{p+q} \to \{1, 2\}$ such that $\#h^{-1}(2) = 2n$, and $h(i) = 2$ and $f(i) = 0$ if and only if $h(i+1) = 2$ and $f(i+1) = 1$. For $i \in \underline{p+q}$, $h(i)$ tells whether—if $h(i) = 2$—or not the position i is part of a swap.
- Another one is by an order preserving surjective function $\pi : \underline{p+q} \to \underline{n}$ such that if $\pi(i+1) = \pi(i) + 1$ then $f(i) = 0$ and $f(i+1) = 1$. For $i \in \underline{p+q}$, $\pi(i)$ is the number of first halves of swaps in the sequence represented by f before position i—so for i not part of a swap this is just the number of swaps before position i; if i is part of a swap the swap itself only counts if i is the position of the second half of the swap.

Table 2. Representation of swaps for (p, q)-shuffles.

	g	h	π	w
g		via π	$g(i)=i-\pi(i)$	via π
h	$h(i)=\#g^{-1}(g(i))$		$h(i)=1$ if $\pi(i-1)=\pi(i)$ $=\pi(i+1)$ $=2$ if $\pi(i)>\pi(i-1)$ or $\pi(i+1)>\pi(i)$	$h(i)=1$ if $w^{-1}(\{i-1,i\})=\varnothing$ $=2$ if $w^{-1}(\{i-1,i\})\neq\varnothing$
π	$\pi(i)=i-g(i)$	$\pi(i)=\left\lfloor\frac{\#(h^{-1}(2)\cap\underline{i})}{2}\right\rfloor$		$\pi(i)=\max\{k \mid w(k)<i\}$
w	via π	via π	$w(k)=\max\{i \mid \pi(i)<k\}$	

- A fourth one is by an order-preserving injective function $w : \underline{n} \to \underline{p+q}$ such that if $i = w(k)$ then $f(i) = 0$ and $f(i+1) = 1$. For $k \in \underline{n}$, $w(k)$ and $w(k)+1$ comprise the k-th swap in the sequence.

Table 2 summarizes how to go from one representation to another, for $i \in \underline{p+q}$, $k \in \underline{n}$ (take $\pi(-1) = 0$ and $\pi(n) = n$).

Definition 3.4 Let (f, g) be an n-dimensional (p, q)-shuffle. Its *opposite* is the n-dimensional (q, p)-shuffle $(f^{\mathrm{op}}, g^{\mathrm{op}})$ given by

$$\begin{aligned} f^{\mathrm{op}}(i) &= 1 - f(i) && \text{if } i \text{ is not part of a swap} \\ f^{\mathrm{op}}(i) &= f(i) && \text{if } i \text{ is part of a swap} \\ g^{\mathrm{op}}(i) &= g(i). \end{aligned}$$

Denote the graded set of n-dimensional (p, q)-shuffles by $M_{p,q}$, so that $(M_{p,q})_n$ consists of all n-dimensional (p, q)-shuffles, for $n \leq \min(p, q)$.

Definition 3.5 An *n-kind* is a pair of injective order preserving functions $\vartheta_0 : \underline{n} \to \underline{p}$ and $\vartheta_1 : \underline{n} \to \underline{q}$.

To each n-dimensional (p, q)-shuffle (f, g) is associated its n-kind, given by: $\vartheta_0(k) = \zeta_0(w(k))$, $\vartheta_1(k) = \zeta_1(w(k)+1)$. Thus, the n-kind of an n-dimensional (p, q)-shuffle tells which swaps occur in (f, g), the k-th swap swapping the $\vartheta_0(k)$-th 0 in the sequence represented by f with the $\vartheta_1(k)$-th 1. Notice that the n-kind of an n-dimensional (p, q)-shuffle determines w, by $w(k) = \vartheta_0(k) + \vartheta_1(k) - 1$, but not conversely. Thus, given the n-kind, the swaps occur at the same places. However, the n-kind does not determine the n-dimensional (p, q)-shuffle, because in between the swaps anything can happen.

4. Faces

The principle is that the cell (01) has source 01 and target 10, with a sign convention for sources and targets of higher-dimensional (p, q)-shuffles, depending on the parity of $\pi(i)$.

Let (f, g) be an n-dimensional (p, q)-shuffle. Define, for each $\varepsilon = \pm$ and $k \in \underline{n}$ an $(n-1)$-dimensional (p, q)-shuffle $(f^{k\varepsilon}, g^{k\varepsilon})$ by:

$$\begin{aligned} f^{k\varepsilon}(i) &= f(i) && \text{if } i \text{ is not part of the } k\text{-th swap} \\ &= f(i) && \text{if } i \text{ is part of the } k\text{-th swap and } \varepsilon = (-)^k \\ &= 1 - f(i) && \text{if } i \text{ is part of the } k\text{-th swap and } \varepsilon = (-)^{k+1} \\ g^{k\varepsilon}(i) &= g(i) && \text{if } \pi(i) < k \\ &= g(i) + 1 && \text{if } \pi(i) \geq k. \end{aligned}$$

Define relations E^n_{n-1} and B^n_{n-1} between $(M_{p,q})_n$ and $(M_{p,q})_{n-1}$ by

$$\begin{aligned} &(f, g)\mathsf{E}^n_{n-1}(f^{k\varepsilon}, g^{k\varepsilon}) && \text{iff } \varepsilon = + \\ &(f, g)\mathsf{B}^n_{n-1}(f^{k\varepsilon}, g^{k\varepsilon}) && \text{iff } \varepsilon = -. \end{aligned}$$

This suffices to define the data for a pasting scheme. To show it is one, I will give a bijection between $\mathsf{R}((f, g))$, the 'closure' of (f, g), and Λ_n, the n-cube as a well-formed loop-free pasting scheme, see [7, Sections 3.2–3.3]. The bijection will basically disregard everything between the swaps, and will map each swap to an interval; thus a k-dimensional (p, q)-shuffle will correspond to a product of k intervals.

Let (f, g) be an n-dimensional (p, q)-shuffle and let (f', g') be an m-dimensional (p, q)-shuffle, $m \leq n$. Say (f', g') *refines* (f, g) if there exist $1 \leq k_1 < \cdots < k_{n-m} \leq n$ such that

$$\begin{aligned} f'(i) &= f(i) && \text{if } i \text{ is not in a } k_\ell\text{-th swap of } f \text{ for any } \ell \\ g'(i) &= g(i) && \text{if } \pi(i) < k_1 \\ g'(i) &= g(i) + \ell && \text{if } k_\ell \leq \pi(i) < k_{\ell+1} \\ g'(i) &= g(i) + n - m && \text{if } k_{n-m} \leq \pi(i). \end{aligned}$$

Thus, (f', g') refines (f, g) if the set of swaps for (f', g') is contained in that for (f, g) and (f, g) and (f', g') are identical outside the set of swaps for (f, g).

Obviously, $k_1 < \cdots < k_{n-m}$ are determined uniquely by (f', g') if they exist.

Lemma 4.1 $\mathsf{R}((f, g))$ *consists of all* (f', g') *refining* (f, g).

Define a map $\mathsf{R}((f, g)) \to \Lambda_n$ by sending (f', g') to the function $x : \underline{n} \to \Lambda$ given by

$$\begin{aligned} x(k) &= 0 && \text{if } k \neq k_\ell \text{ for all } \ell \\ &= - && \text{if } k = k_\ell \text{ and } f'(w(k_\ell)) = f(w(k_\ell)) \\ &= + && \text{if } k = k_\ell \text{ and } f'(w(k_\ell)) = 1 - f(w(k_\ell)). \end{aligned}$$

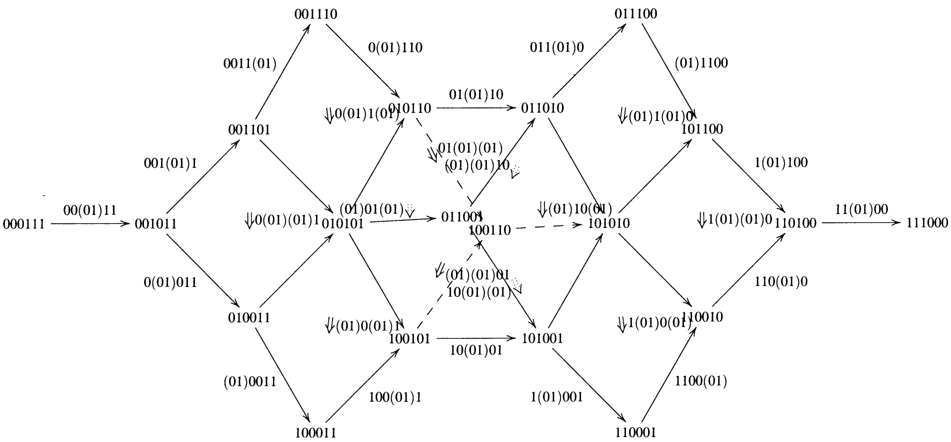


Figure 5. $M_{3,3}$.

Lemma 4.2 *The map just defined is a bijection and preserves the relations* E^k_{k-1} *and* B^k_{k-1}.

Proposition 4.3 $M_{p,q}$ *is a pasting scheme.*

Proof: The axioms for a pasting scheme are all local, so can be checked in the $\mathsf{R}((f, g))$'s. These are cubes, which are known to satisfy them. □

The example $M_{3,3}$ is given in figure 5, where (01)(01)(01) has as 2-source the front side and as 2-target the back side of the 3-cube in the middle.

For $\vartheta = (\vartheta_0, \vartheta_1)$ an n-kind for an n-dimensional (p, q)-shuffle, define $(n-1)$-kinds $\vartheta^k = (\vartheta^k_0, \vartheta^k_1)$, for $k \in \underline{n}$, by:

$$\begin{aligned}
\vartheta^k_0(k') &= \vartheta_0(k') && \text{if } k' < k\\
\vartheta^k_0(k') &= \vartheta_0(k'+1) && \text{if } k' \geq k\\
\vartheta^k_1(k') &= \vartheta_1(k') && \text{if } k' < k\\
\vartheta^k_1(k') &= \vartheta_1(k'+1) && \text{if } k' \geq k.
\end{aligned}$$

So ϑ^k is obtained from ϑ by removing the k-th swap.

Definition 4.4 An $(n-1)$-kind ϑ' *bounds* an n-kind ϑ if there exists a $k \in \underline{n}$ for which $\vartheta' = \vartheta^k$.

The following is an immediate consequence of Definition 4.4 and the definition of faces of n-dimensional (p, q)-shuffles:

Lemma 4.5 *Let* (f, g) *be an* n*-dimensional* (p, q)*-shuffle. If* $(f', g') \in \mathsf{D}((f, g))$ *then the* $(n-1)$*-kind of* (f', g') *bounds the* n*-kind of* (f, g).

For $\vartheta = (\vartheta_0, \vartheta_1)$ an n-kind for an n-dimensional (p, q)-shuffle, define $(n-2)$-kinds $\vartheta^{k,k'}$, for $k, k' \in \underline{n}$, $k \neq k'$, by

$$\begin{aligned} \vartheta^{k,k'} &= (\vartheta^{k'})^{k} &= (\vartheta^{k})^{k'-1} \quad &\text{if } k < k' \\ &= (\vartheta^{k'})^{k-1} &= (\vartheta^{k})^{k'} \quad &\text{if } k > k'. \end{aligned}$$

Lemma 4.6 *Let (f, g) be an n-dimensional (p, q)-shuffle. If $(f', g') \in \mathsf{D}((f, g))$ of kind ϑ^k, $(f'', g'') \in \mathsf{D}((f, g))$ of kind $\vartheta^{k'}$ and $(f''', g''') \in \mathsf{D}((f', g')) \cap \mathsf{D}((f'', g''))$, then (f''', g''') has kind $\vartheta^{k,k'}$. Moreover, (f''', g''') is determined by k and k′ and the choices for the* D*’s.*

Proof: This happens in a single n-cube, and says that any two $(n-1)$-faces meet in at most one $(n-2)$-‘edge’. □

5. No direct loops

Proposition 5.1 *For each p, q, the pasting scheme $M_{p,q}$ has no direct loops.*

Proof: For any $a \in M_{p,q}$, $\mathsf{B}(a) \cap \mathsf{E}(a) = \{a\}$ because this is the case for a cell which has the shape of a cube.

Consider a, b both n-dimensional (p, q)-shuffles, and suppose that $a \lhd b$, i.e., there is a sequence $a = a_0, a_1, \ldots, a_r = b$ and a sequence $a'_1, \ldots, a'_r$ with $a_\ell \mathsf{B}^n_{n-1} a'_\ell$ and $a_{\ell-1} \mathsf{E}^n_{n-1} a'_\ell$ for all $1 \leq \ell \leq r$. Let $a'_0 \in \mathsf{B}(a)$ and $a'_{r+1} \in \mathsf{E}(b)$, which can be assumed to be via only B’s and only E’s respectively. I need to show that $a'_0 \neq a'_{r+1}$. To this end, say $a_\ell = (f_{(a_\ell)}, g_{(a_\ell)})$ and $a'_\ell = (f_{(a'_\ell)}, g_{(a'_\ell)})$, and let i be the smallest number for which $h_{(a_\ell)}(i)$ or $h_{(a'_\ell)}(i)$ differs from $h_{(a'_0)}(i)$. Assume without loss of generality that in a_0, and hence in all a_ℓ and a'_ℓ, there is an even number of swaps before position i. Now if $h_{(a'_0)}(i) \neq h_{(a'_{r+1})}(i)$ then $a'_0 \neq a'_{r+1}$. If $h_{(a'_0)}(i) = h_{(a'_{r+1})}(i) = 1$ then for some ℓ it must be that $h_{(a_\ell)}(i) = 2$—perhaps also some $h_{(a'_\ell)}(i) = 2$ but then $h_{(a_\ell)}(i) = 2$ anyway—and then, by the formulae for source and target, the even number of swaps before position i, and the fact that nothing changes before position i, only one change at position i has occurred before a_ℓ, namely the introduction of a swap, and only one after a_ℓ, namely the removal of that swap; hence, $f_{(a'_0)}(i) = 0$ and $f_{(a'_{r+1})}(i) = 1$, so $a'_0 \neq a'_{r+1}$. Finally, $h_{(a'_0)}(i) = h_{(a'_{r+1})}(i) = 2$ implies for some ℓ it must be that $h_{(a'_\ell)}(i) = 1$, but then, by the same argument, $f_{(a'_\ell)}(i) = 0$ and $f_{(a'_\ell)}(i) = 1$ at the same time, a contradiction. □

So $M_{p,q}$ has meaningful notions of domain and codomain, and hence of n-source and n-target.

6. Rank

The idea of the rank is that it measures how far a shuffle is from having all 0’s at the front and all 1’s at the back, in terms of how many swaps are needed to get to it.

Definition 6.1 Let $f : \underline{p+q} \to \{0, 1\}$ be a (p, q)-shuffle, with corresponding $\zeta_0 : \underline{p+q} \to \underline{p}$ and $\zeta_1 : \underline{p+q} \to \underline{q}$. Then

$$\mathrm{rk}(f) = \sum_{i \in f^{-1}(0)} \zeta_1(i)$$
$$\mathrm{oprk}(f) = \sum_{i \in f^{-1}(1)} \zeta_0(i).$$

Another way to calculate the rank of a (p, q)-shuffle f, with corresponding $\alpha : \underline{p} \to \underline{p+q}$ and $\beta : \underline{q} \to \underline{p+q}$, would be:

$$\mathrm{rk}(f) = \sum_{i_0 \in \underline{p}} (\alpha(i_0) - i_0)$$
$$\mathrm{oprk}(f) = \sum_{i_1 \in \underline{q}} (\beta(i_1) - i_1).$$

Indeed, the rank counts the number of swaps:

Lemma 6.2 $\mathrm{rk}(f) = \#\{(i_0, i_1) \mid \alpha(i_0) > \beta(i_1)\}$ *and* $\mathrm{oprk}(f) = \#\{(i_0, i_1) \mid \alpha(i_0) < \beta(i_1)\}$.

Proof: $\sum_{i \in f^{-1}(0)} \zeta_1(i) = \sum_{i \in f^{-1}(0)} \#(f^{-1}(1) \cap \underline{i}) = \#\{(i, i') \mid i \in f^{-1}(0),\ i' \in f^{-1}(1), i > i'\} = \#\{(i_0, i_1) \mid \alpha(i_0) > \beta(i_1)\}$. □

Corollary 6.3 $\mathrm{rk}(f) + \mathrm{oprk}(f) = p \cdot q$.

Corollary 6.4 $\mathrm{oprk}(f) = \mathrm{rk}(f^{\mathrm{op}})$.

Definition 6.5 Let (f, g) be an n-dimensional (p, q)-shuffle. Then

$$\mathrm{rk}(f, g) = \sum_{i \text{ not part of a swap},\ f(i)=0,\ \pi(i) \text{ even}} (\zeta_1(i) - \vartheta_1(\pi(i))) + \sum_{i \text{ not part of a swap},\ f(i)=1,\ \pi(i) \text{ odd}} (\zeta_0(i) - \vartheta_0(\pi(i)))$$
$$\mathrm{oprk}(f, g) = \sum_{i \text{ not part of a swap},\ f(i)=0,\ \pi(i) \text{ odd}} (\zeta_1(i) - \vartheta_1(\pi(i))) + \sum_{i \text{ not part of a swap},\ f(i)=1,\ \pi(i) \text{ even}} (\zeta_0(i) - \vartheta_0(\pi(i)))$$

In order to explain these formulae, define, for an n-dimensional (p, q)-shuffle (f, g), or more generally for an n-kind $\vartheta = (\vartheta_0, \vartheta_1)$, numbers p_k and q_k, for each $0 \leq k \leq n$, by:

$$\begin{aligned} p_0 &= \vartheta_0(1) - 1 \\ q_0 &= \vartheta_1(1) - 1 \\ p_k &= \vartheta_0(k+1) - \vartheta_0(k) - 1 \quad \text{for } 1 \leq k \leq n-1 \\ q_k &= \vartheta_1(k+1) - \vartheta_1(k) - 1 \quad \text{for } 1 \leq k \leq n-1 \\ p_n &= p - \vartheta_0(n) \\ q_n &= q - \vartheta_1(n). \end{aligned}$$

Thus, there are p_k 0's and q_k 1's between the k-th and $(k+1)$-th swaps.

Given (f, g), define 0-dimensional (p_k, q_k)-shuffles $f_k^{\mathfrak{r}}$ by:

$$\begin{aligned} f_0^{\mathfrak{r}}(i) &= f(i) \\ f_k^{\mathfrak{r}}(i) &= f(w(k) + 1 + i) \quad \text{for } 1 \leq k \leq n. \end{aligned}$$

Thus, $f_k^{\mathfrak{r}}$ is just the part of f between the k-th and $(k+1)$-th swaps.

Lemma 6.6 *Let (f, g) be an n-dimensional (p, q)-shuffle. Then*

$$\begin{aligned} \operatorname{rk}(f, g) &= \sum_{0 \leq k \leq n,\, k \text{ even}} \operatorname{rk}\left(f_k^{\mathfrak{r}}\right) + \sum_{0 \leq k \leq n,\, k \text{ odd}} \operatorname{oprk}\left(f_k^{\mathfrak{r}}\right) \\ \operatorname{oprk}(f, g) &= \sum_{0 \leq k \leq n,\, k \text{ odd}} \operatorname{rk}\left(f_k^{\mathfrak{r}}\right) + \sum_{0 \leq k \leq n,\, k \text{ even}} \operatorname{oprk}\left(f_k^{\mathfrak{r}}\right). \end{aligned}$$

Ranks can be calculated by splitting at swaps. To this end, for an n-dimensional (p, q)-shuffle (f, g), or more generally for an n-kind $\vartheta = (\vartheta_0, \vartheta_1)$, define $p_k^{\mathfrak{l}}$ and $q_k^{\mathfrak{l}}$ and $p_k^{\mathfrak{r}}$ and $q_k^{\mathfrak{r}}$, for each $1 \leq k \leq n$, by

$$\begin{aligned} p_k^{\mathfrak{l}} &= \vartheta_0(k) - 1 \\ q_k^{\mathfrak{l}} &= \vartheta_1(k) - 1 \\ p_k^{\mathfrak{r}} &= p - \vartheta_0(k) \\ q_k^{\mathfrak{r}} &= q - \vartheta_1(k). \end{aligned}$$

Define $(k-1)$-dimensional $(p_k^{\mathfrak{l}}, q_k^{\mathfrak{l}})$-shuffles $(f_k^{\mathrm{i}\mathfrak{l}}, g_k^{\mathrm{i}\mathfrak{l}})$ and $(n-k)$-dimensional $(p_k^{\mathfrak{r}}, q_k^{\mathfrak{r}})$-shuffles $(f_k^{\mathrm{i}\mathfrak{r}}, g_k^{\mathrm{i}\mathfrak{r}})$, for each $1 \leq k \leq n$, by:

$$\begin{aligned} f_k^{\mathrm{i}\mathfrak{l}}(i) &= f(i) \\ g_k^{\mathrm{i}\mathfrak{l}}(i) &= g(i) \\ f_k^{\mathrm{i}\mathfrak{r}}(i) &= f(w(k) + 1 + i) \\ g_k^{\mathrm{i}\mathfrak{r}}(i) &= g(w(k) + 1 + i) - k. \end{aligned}$$

Thus, $(f_k^{\mathrm{il}}, g_k^{\mathrm{il}})$ and $(f_k^{\mathrm{ir}}, g_k^{\mathrm{ir}})$ are the parts of f up to the k-th swap and after the k-th swap, respectively.

Lemma 6.7 *Let (f, g) be an n-dimensional (p, q)-shuffle. Then*

$$\mathrm{rk}(f, g) = \frac{\sum_{1\le k\le n}\left(\mathrm{rk}\left(f_k^{\mathrm{il}}, g_k^{\mathrm{il}}\right) + \mathrm{oprk}\left(f_k^{\mathrm{ir}}, g_k^{\mathrm{ir}}\right)\right)}{n}.$$

Introduce the notation that $x <^k y$ if $x < y$ for k even and $x > y$ for k odd. Similarly, $x >^k y$ if $x >^{k+1} y$, that is, if $y <^k x$. One should think of the relation $<^k$ as $(-)^k \cdot <$ where $-<$ equals $>$.

Corollary 6.8

$$\begin{aligned}\mathrm{rk}(f, g) &= \#\{(i_0, i_1) \mid \alpha(i_0) \textit{ and } \beta(i_1) \textit{ both not part of a swap}, \pi(\alpha(i_0)) \\ &= \pi(\beta(i_1)) = k, \textit{ and } \alpha(i_0) >^k \beta(i_1)\} \\ \mathrm{oprk}(f, g) &= \#\{(i_0, i_1) \mid \alpha(i_0) \textit{ and } \beta(i_1) \textit{ both not part of a swap}, \pi(\alpha(i_0)) \\ &= \pi(\beta(i_1)) = k, \textit{ and } \alpha(i_0) <^k \beta(i_1)\}.\end{aligned}$$

Cells go from lower rank to higher rank:

Proposition 6.9 *Let (f, g) be an n-dimensional (p, q)-shuffle. Then* $\mathrm{rk}(f^{k+}, g^{k+}) = \mathrm{rk}(f^{k-}, g^{k-}) + 1$ *and* $\mathrm{oprk}(f^{k+}, g^{k+}) = \mathrm{oprk}(f^{k-}, g^{k-}) - 1$.

Proof: Consider the k-th swap in (f, g), say with k even. In the formula of Lemma 6.7 for $n\,\mathrm{rk}(f, g)$, if one replaces this swap by 01, then $(f_k^{\mathrm{il}}, g_k^{\mathrm{il}})$ and $(f_k^{\mathrm{ir}}, g_k^{\mathrm{ir}})$ disappear from the sum, and for all $k' \neq k$, the k-th swap of (f, g) gets also replaced by 01 in all appropriate $(f_{k'}^{\mathrm{il}}, g_{k'}^{\mathrm{il}})$ and $(f_{k'}^{\mathrm{ir}}, g_{k'}^{\mathrm{ir}})$. Renumbering the $k' > k$, this gives precisely the formula of the lemma for $(n-1)\,\mathrm{rk}(f^{k-}, g^{k-})$. Similarly for k odd and/or replacing this swap by 10, in the 'or' case giving $(n-1)\,\mathrm{rk}(f^{k+}, g^{k+})$. Now by induction the statement is true for each appropriate $(f_{k'}^{\mathrm{il}}, g_{k'}^{\mathrm{il}})$ and $(f_{k'}^{\mathrm{ir}}, g_{k'}^{\mathrm{ir}})$, so the difference between $(n-1)\,\mathrm{rk}(f^{k+}, g^{k+})$ and $(n-1)\,\mathrm{rk}(f^{k-}, g^{k-})$ is $n-1$, giving the conclusion. □

The ranks for 0-cells and 1-cells of $M_{3,3}$ are given in figure 6; all 2-cells of $M_{3,3}$ have rank 0 except the ones on the back side of the cube which have rank 1, and the 3-cell (01)(01)(01) has rank 0.

7. The $\lessdot$-order

The rank defined in the previous section is the rank function for a partial order. This partial order will only relate n-dimensional (p, q)-shuffles of the same n-kind.

Definition 7.1 Let f and f' be two (p, q)-shuffles. Then $f \lessdot f'$ if and only if $\alpha(i_0) \le \alpha'(i_0)$ for all $i_0 \in \underline{p}$.

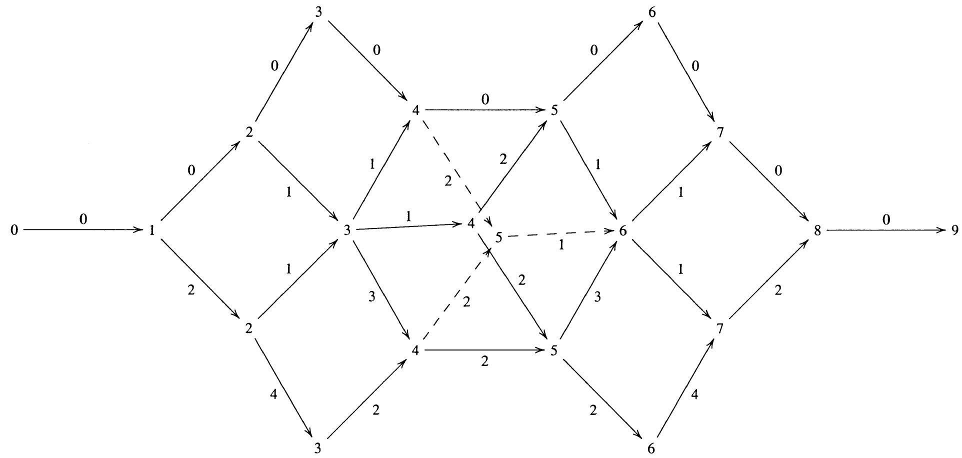


Figure 6. Ranks of cells in $M_{3,3}$.

Lemma 7.2 *$f \lessdot f'$ if and only if $\alpha(i_0) > \beta(i_1)$ implies $\alpha'(i_0) > \beta'(i_1)$ for every $i_0 \in \underline{p}$ and $i_1 \in \underline{q}$.*

Proof: On the one hand, $\zeta_1(\alpha(i_0)) = \alpha(i_0) - \zeta_0(\alpha(i_0)) = \alpha(i_0) - i_0$, so $\alpha(i_0) \leq \alpha'(i_0)$ if and only if $\zeta_1(\alpha(i_0)) \leq \zeta_1'(\alpha'(i_0))$. On the other hand, $\zeta_1(\alpha(i_0)) = \max\{i_1 \mid \beta(i_1) < \alpha(i_0)\}$, so $\zeta_1(\alpha(i_0)) \leq \zeta_1'(\alpha'(i_0))$ if and only if $\alpha(i_0) > \beta(i_1)$ implies $\alpha'(i_0) > \beta'(i_1)$. □

Corollary 7.3 *$f \lessdot f'$ if and only if $\beta(i_1) \geq \beta'(i_1)$ for all $i_1 \in \underline{q}$.*

Proof (from lemma)**:** $\alpha(i_0) > \beta(i_1)$ implies $\alpha'(i_0) > \beta'(i_1)$ if and only if $\alpha'(i_0) < \beta'(i_1)$ implies $\alpha(i_0) < \beta(i_1)$, and so by calculating $\zeta_0(\beta(i_1))$, it follows as in the lemma that $\beta(i_1) \geq \beta'(i_1)$. □

Proof (direct)**:** Let i_1 be the smallest element of $\underline{q}$ for which $\beta(i_1) < \beta'(i_1)$. Now if $f'(i) = 1$ for some $\beta(i_1) \leq i < \beta'(i_1)$ then $\zeta_0'(i) < i_1$ and $\beta(\zeta_0'(i)) < \beta(i_1) \leq i = \beta'(\zeta_0'(i))$, contradicting minimality of i_1. And if $f(i) = 0$ for all $\beta(i_1) \leq i < \beta'(i_1)$ then there are $i_1 - 1$ 1's up to position $\beta(i_1) - 1$ in both f and f', and consequently $\beta(i_1) - i_1$ 0's. So take $i_0 = \beta(i_1) - i_1 + 1$, then $\alpha(i_0) > \beta(i_1) = \alpha'(i_0)$, contradicting $f \lessdot f'$. □

Definition 7.4 Let (f, g) and (f', g') be two n-dimensional (p, q)-shuffles of the same n-kind ϑ. Then $(f, g) \lessdot (f', g')$ if and only if $\alpha(i_0) \leq^k \alpha'(i_0)$ for every k and $\vartheta_0(k) < i_0 < \vartheta_0(k + 1)$.

The relation $\lessdot$ only depends on what is in between the swaps—because two n-dimensional shuffles of the same n-kind only differ there.

Lemma 7.5 *$(f, g) \lessdot (f', g')$ if and only if $f_k^{\mathrm{r}} \lessdot^k (f')_k^{\mathrm{r}}$ for every k.*

Combining Lemmas 7.3 and 7.5 immediately gives

Corollary 7.6 *$(f, g) \lessdot (f', g')$ if and only if $\beta(i_1) \geq^k \beta'(i_1)$ for every k and $\vartheta_1(k) < i_1 < \vartheta_1(k+1)$.*

Another consequence, from Lemmas 7.2 and 7.5 this time, which will be used extensively in the sequel:

Corollary 7.7 *$(f, g) \lessdot (f', g')$ if and only if $\alpha(i_0) >^k \beta(i_1)$ implies $\alpha'(i_0) >^k \beta'(i_1)$ for every k and $\vartheta_0(k) < i_0 < \vartheta_0(k+1)$ and $\vartheta_1(k) < i_1 < \vartheta_1(k+1)$.*

Lemma 7.5 also implies that

Corollary 7.8 *For any (f, g) and any k, $(f^{k-}, g^{k-}) \lessdot (f^{k+}, g^{k+})$.*

Proposition 7.9 *If $(f, g) \lessdot (f', g')$ then $\mathrm{rk}(f, g) \leq \mathrm{rk}(f', g')$, with equality only when $(f, g) = (f', g')$.*

Proof: By Lemmas 6.6 and 7.5 I only need to consider 0-dimensional (p, q)-shuffles. And then the statement is trivial. □

8. Filling

The *filling* condition will be based on the $\lhd$-relation in pasting schemes, but restricted to faces of one given $(n-1)$-kind.

Definition 8.1 Let ϑ'' be an $(n-1)$-kind for (p, q)-shuffles, and let (f, g) and (f', g') be two n-dimensional (p, q)-shuffles whose n-kinds have bound ϑ'', say (f, g) has n-kind ϑ and (f', g') has n-kind ϑ' and $\vartheta^k = \vartheta'' = (\vartheta')^{k'}$. Then $(f, g) <^{\vartheta''} (f', g')$ if and only if $(f^{k+}, g^{k+}) = ((f')^{k'-}, (g')^{k'-})$.

Let $\lhd^{\vartheta''}$ be the transitive closure of the relation $<^{\vartheta''}$.

Where $\lessdot$ relates n-dimensional (p, q)-shuffles of the same n-kind, $\lhd^{\vartheta''}$ relates n-dimensional (p, q)-shuffles with same bounding $(n-1)$-kind. These two relations are connected:

Lemma 8.2 *Let $(f, g) \lhd^{\vartheta''} (f', g')$, say (f, g) has n-kind ϑ and (f', g') has n-kind ϑ' and $\vartheta^k = \vartheta'' = (\vartheta')^{k'}$. Then $(f^{k-}, g^{k-}) \lessdot ((f')^{k'+}, (g')^{k'+})$.*

Proof: This immediately follows from Corollary 7.8 and transitivity of $\lessdot$. □

Definition 8.3 Let ${}^n\gamma$ be a collection of n-dimensional (p, q)-shuffles. $K({}^n\gamma)$ is the collection of n-kinds of elements in ${}^n\gamma$.

Definition 8.4 Let ${}^{n-1}K$ be a collection of $(n-1)$-kinds for (p, q)-shuffles. An n-kind ϑ is *relevant for* ${}^{n-1}K$ if ${}^{n-1}K$ contains all $(n-1)$-kinds bounding it.

Definition 8.5 Let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles. A collection ${}^{n}\gamma$ of n-dimensional (p, q)-shuffles *fills* ${}^{n-1}\gamma, {}^{n-1}\gamma'$ if any n-kind in $K({}^{n}\gamma)$ is relevant for $K({}^{n-1}\gamma)$ and for $K({}^{n-1}\gamma')$, and for every $(n-1)$-kind ϑ' and every $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ of $(n-1)$-kind ϑ',

- $({}^{n-1}f, {}^{n-1}g) = ({}^{n-1}f', {}^{n-1}g')$ if and only if $K({}^{n}\gamma)$ contains no n-kind with bound ϑ'
- if $({}^{n-1}f, {}^{n-1}g) \neq ({}^{n-1}f', {}^{n-1}g')$ then the relation $\lhd^{\vartheta'}$ is a total order on the collection of elements of ${}^{n}\gamma$ whose n-kind has bound ϑ', and for the $\lhd^{\vartheta'}$-first and $\lhd^{\vartheta'}$-last element in this collection respectively, say (f, g) of n-kind ϑ and $\vartheta^k = \vartheta'$, one has $(f^{k-}, g^{k-}) = ({}^{n-1}f, {}^{n-1}g)$ and $(f^{k+}, g^{k+}) = ({}^{n-1}f', {}^{n-1}g')$ respectively.

A collection ${}^{0}\gamma$ of 0-dimensional (p, q)-shuffles *fills* $\varnothing$ if it is a singleton.

One should think of filling as a *dynamic* condition—it sort of says that one can ‘move’ from ${}^{n-1}\gamma$ to ${}^{n-1}\gamma'$ via elements of ${}^{n}\gamma$ in an orderly manner.

To compare filling to well-formedness I need to be able to make a subpasting scheme out of collections of cells.

Definition 8.6 An *n-stage shuffle collection* γ consists of collections of j-dimensional (p, q)-shuffles ${}^{j}\gamma, {}^{j'}\gamma$ for each $0 \leq j < n$ and a collection of n-dimensional (p, q)-shuffles ${}^{n}\gamma$.

Definition 8.7 Let $\gamma = \{{}^{j}\gamma, {}^{j'}\gamma \mid 0 \leq j < n\} \cup \{{}^{n}\gamma\}$ be an n-stage shuffle collection. Then $\mathrm{dom}(\gamma) = \{{}^{j}\gamma, {}^{j'}\gamma \mid 0 \leq j < n-1\} \cup \{{}^{n-1}\gamma\}$ and $\mathrm{cod}(\gamma) = \{{}^{j}\gamma, {}^{j'}\gamma \mid 0 \leq j < n-1\} \cup \{{}^{n-1}\gamma'\}$.

This follows the usual pattern of globular collections, except that the cells are not globular here.

Definition 8.8 Let $\gamma = \{{}^{j}\gamma, {}^{j'}\gamma \mid j \in \underline{n}\} \cup \{{}^{n}\gamma\}$ be an n-stage shuffle collection. Then $R(\gamma) = \mathsf{R}(\bigcup \gamma)$.

So $R(\gamma)$ is a subpasting scheme of $M_{p,q}$.

Domains and codomains of subpasting schemes and of n-stage shuffle collections are related:

Lemma 8.9 *Let γ be an n-stage shuffle collection. If ${}^{n}\gamma$ fills ${}^{n-1}\gamma, {}^{n-1}\gamma'$ then* $\mathrm{dom}(R(\gamma)) = R(\mathrm{dom}(\gamma))$ *and* $\mathrm{cod}(R(\gamma)) = R(\mathrm{cod}(\gamma))$.

Proof: By elementary pasting scheme arguments, it suffices to consider $(n-1)$-dimensional cells.

First consider $({}^{n-1}f, {}^{n-1}g)$ of $(n-1)$-kind ϑ' for which there are no elements in ${}^{n}\gamma$ having bound ϑ'. Then $({}^{n-1}f, {}^{n-1}g) \in \mathrm{dom}(R(\gamma))$ if and only if $({}^{n-1}f, {}^{n-1}g) \in R(\gamma)$ if and only if $({}^{n-1}f, {}^{n-1}g) \in \mathsf{R}({}^{n-1}\gamma)$.

For $({}^{n-1}f, {}^{n-1}g)$ of $(n-1)$-kind ϑ' for which there is an element in ${}^{n}\gamma$ having bound ϑ', $({}^{n-1}f, {}^{n-1}g) \in \mathrm{dom}(R(\gamma))$ if and only if there is an (f, g) beginning in $({}^{n-1}f, {}^{n-1}g)$ but no $(f', g') \in {}^{n}\gamma$ ending in $({}^{n-1}f, {}^{n-1}g)$ if and only if (f, g) is the first element in the $\lhd^{\vartheta'}$-sequence if and only if $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$. □

Conversely, given an n-dimensional subpasting scheme A of $M_{p,q}$ define an n-stage shuffle collection γ by ${}^{j}\gamma = s_j(A)_j$, ${}^{j'}\gamma = t_j(A)_j$, ${}^{n}\gamma = A_n$.

Proposition 8.10 *Let γ be an n-stage shuffle collection. If γ fills ${}^{n-1}\gamma, {}^{n-1}\gamma'$ then $R(\gamma)$ is compatible.*

Proof: If $z \in \mathsf{B}_{n-1}(x) \cap \mathsf{B}_{n-1}(y)$ then $x \lhd^{\vartheta'} y$ say where ϑ' the $(n-1)$-kind of z. But then, replacing the last step in this $\lhd^{\vartheta'}$-sequence, which is possible because y and ϑ' determine z uniquely, $x \lhd^{\vartheta'} x$, contradicting that $\lhd^{\vartheta'}$ is a total order.

${}^{0}\gamma$ filling means it is a singleton which means that $\mathsf{R}({}^{0}\gamma)$ is compatible. □

9. Fitness

The *fitness* condition will be based on the $\lessdot$-order, extended to collections of cells.

Definition 9.1 Let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles. ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$ if whenever $(f, g) \in {}^{n-1}\gamma$ and $(f', g') \in {}^{n-1}\gamma'$ of the same $(n-1)$-kind then $(f, g) \lessdot (f', g')$.

I apologize for the (over)abundance of terminology to follow, but the reward will be a tidy formulation of fitness.

Definition 9.2 Let ϑ be an n-kind and let (f, g) and (f', g') be two $(n-1)$-dimensional (p, q)-shuffles of kind ϑ^k. Then $(f, g) \lessdot_\vartheta (f', g')$ if $(f, g) \lessdot (f', g')$ and $\alpha(\vartheta_0(k)) <^{k-1} \beta(\vartheta_1(k))$ and $\alpha'(\vartheta_0(k)) >^{k-1} \beta'(\vartheta_1(k))$.

For (f, g) and (f', g') of $(n-1)$-kind ϑ', say that $(f, g), (f', g')$ *needs* ϑ if $\vartheta' = \vartheta^k$ and $(f, g) \lessdot_\vartheta (f', g')$.

Lemma 9.3 *Let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles, and let ${}^{n}\gamma$ be a collection of n-dimensional (p, q)-shuffles filling ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$. Let ϑ be an n-kind and let $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ of $(n-1)$-kind ϑ^k. If $({}^{n-1}f, {}^{n-1}g), ({}^{n-1}f', {}^{n-1}g')$ needs ϑ then ${}^{n}\gamma$ contains an element of kind ϑ.*

Proof: Among the elements of ${}^{n}\gamma$ whose n-kind has bound ϑ^k the only ones that change the relative positions of the $\vartheta_0(k)$-th 0 and the $\vartheta_1(k)$-th 1 are the ones of n-kind ϑ. These relative positions in $({}^{n-1}f, {}^{n-1}g)$ and $({}^{n-1}f', {}^{n-1}g')$ respectively are actually different, so

$({}^{n-1}f, {}^{n-1}g) \neq ({}^{n-1}f', {}^{n-1}g')$, and (at least) one of the elements in the total order $\lhd^{\vartheta}$ must actually swap them. □

Definition 9.4 Let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles and let ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$. An n-kind ϑ is *proper to* ${}^{n-1}\gamma, {}^{n-1}\gamma'$ if for every k and for every $(f, g) \in {}^{n-1}\gamma$ and $(f', g') \in {}^{n-1}\gamma'$ both of $(n-1)$-kind ϑ^k, $(f, g) \lessdot_{\vartheta} (f', g')$.

An n-kind ϑ is *properly relevant to* ${}^{n-1}\gamma, {}^{n-1}\gamma'$ if it is relevant to both $K({}^{n-1}\gamma)$ and $K({}^{n-1}\gamma')$ and proper to ${}^{n-1}\gamma, {}^{n-1}\gamma'$.

Definition 9.5 Let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles. An n-dimensional (p, q)-shuffle (f, g) is *fit for* ${}^{n-1}\gamma, {}^{n-1}\gamma'$ if whenever $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ with equal $(n-1)$-kind bounding the n-kind of (f, g), say of $(n-1)$-kind ϑ^k, then $({}^{n-1}f, {}^{n-1}g) \lessdot (f^{k-}, g^{k-})$ and $(f^{k+}, g^{k+}) \lessdot ({}^{n-1}f', {}^{n-1}g')$.

This definition is not very useful when not ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$, but as that is not its intended use, this is not a problem.

Lemma 9.6 *If (f, g) of n-kind ϑ is fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ then ϑ is proper to ${}^{n-1}\gamma, {}^{n-1}\gamma'$.*

Proof: Let $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ both of $(n-1)$-kind ϑ^k. Then $({}^{n-1}f, {}^{n-1}g) \lessdot (f^{k-}, g^{k-}) \lessdot (f^{k+}, g^{k+}) \lessdot ({}^{n-1}f', {}^{n-1}g')$, and moreover ${}^{n-1}\alpha(\vartheta_0(k)) \leq^{k-1} \alpha^{k-}(\vartheta_0(k)) <^{k-1} \beta^{k-}(\vartheta_1(k)) \leq^{k-1} {}^{n-1}\beta(\vartheta_1(k))$ and similarly ${}^{n-1}\alpha'(\vartheta_0(k)) > {}^{n-1}\beta'(\vartheta_1(k))$, proving $({}^{n-1}f, {}^{n-1}g) \lessdot_{\vartheta} ({}^{n-1}f', {}^{n-1}g')$, as required. □

Definition 9.7 Let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles, and let ${}^{n}\gamma$ be a collection of n-dimensional (p, q)-shuffles. ${}^{n}\gamma$ is *fit for* ${}^{n-1}\gamma, {}^{n-1}\gamma'$ if

(i) every n-kind in $K({}^{n}\gamma)$ is relevant to both $K({}^{n-1}\gamma)$ and $K({}^{n-1}\gamma')$;
(ii) every n-kind occurs at most once as n-kind of an element of ${}^{n}\gamma$;
(iii) every element of ${}^{n}\gamma$ is fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$;
(iv) for every n-kind properly relevant to ${}^{n-1}\gamma, {}^{n-1}\gamma'$ there exists an element of ${}^{n}\gamma$ of that n-kind;
(v) (continuation) if $({}^{n-1}f, {}^{n-1}g) \notin {}^{n-1}\gamma$ and $(f', g')\mathsf{B}^{n}_{n-1}({}^{n-1}f, {}^{n-1}g)$ for some $(f', g') \in {}^{n}\gamma$ then there exists an $(f, g) \in {}^{n}\gamma$ with $(f, g)\mathsf{E}^{n}_{n-1}({}^{n-1}f, {}^{n-1}g)$, and dually.

A collection ${}^{0}\gamma$ of 0-dimensional (p, q)-shuffles *fits* $\varnothing$ if it is a singleton.

The first three conditions restrict what ${}^{n}\gamma$ can contain, the other two say what ${}^{n}\gamma$ must contain. The conditions completely determine what kinds can occur in a fitting collection: only properly relevant ones by Lemma 9.6, and all properly relevant ones precisely once by conditions (ii) and (iv).

One should think of fitness as a *static* condition—it sort of ‘drops down’ the elements of ${}^{n}\gamma$ with only condition (v) providing some cohesion.

To compare fitness to filling I will show that if an n-kind ϑ is needed by elements of ${}^{n-1}\gamma, {}^{n-1}\gamma'$ of $(n-1)$-kind ϑ^k then for any other $(n-1)$-kind bounding ϑ there are elements of ${}^{n-1}\gamma, {}^{n-1}\gamma'$ of that $(n-1)$-kind needing ϑ. In other words, if ϑ is needed by ${}^{n-1}\gamma, {}^{n-1}\gamma'$ for one reason, then it is properly relevant to ${}^{n-1}\gamma, {}^{n-1}\gamma'$. Of course, one needs assumptions on ${}^{n-1}\gamma, {}^{n-1}\gamma'$. The following lemma prepares the way, and will also be used extensively in Section 11.

Lemma 9.8 *Let ${}^{n-2}\gamma, {}^{n-2'}\gamma$ be two collections of $(n-2)$-dimensional (p, q)-shuffles, and let ${}^{n-1}\gamma$ be a collection of $(n-1)$-dimensional (p, q)-shuffles filling and fitting ${}^{n-2}\gamma, {}^{n-2'}\gamma$. Let ϑ be an n-kind, and let $(f, g) \in {}^{n-1}\gamma$ of kind ϑ^k and $(f'', g'') \in {}^{n-1}\gamma$ of $(n-1)$-kind $\vartheta^{k'}$. Then*

$$(f, g) \lhd^{\vartheta^{k,k'}} (f'', g'') \quad \textit{if and only if for} \quad \begin{array}{ll} k < k': & \alpha(\vartheta_0(k)) <^{k-1} \beta(\vartheta_1(k)) \\ k > k': & \alpha(\vartheta_0(k)) <^{k} \beta(\vartheta_1(k)). \end{array}$$

Proof: As in the proof of Lemma 9.3, among the elements of ${}^{n-1}\gamma$ whose $(n-1)$-kind has bound $\vartheta^{k,k'}$ the only ones that swap the $\vartheta_0(k)$-th 0 with the $\vartheta_1(k)$-th 1 are the ones of $(n-1)$-kind $\vartheta^{k'}$. There is only one such because ${}^{n-1}\gamma$ is fitting. Thus, $\lhd^{\vartheta^{k,k'}}$-between (f, g) and (f', g') this swap does not occur.

Now in (f'', g'') the parity of the swap $\vartheta_0(k), \vartheta_1(k)$ depends not only on the parity of k but also on whether or not $k' < k$. So in $((f'')^{k-}, (g'')^{k-})$, by the formula for this face, one has for $k' < k$ that $\alpha''(\vartheta_0(k)) >^{k-1} \beta''(\vartheta_1(k))$, and in $((f'')^{k+}, (g'')^{k+})$ the other way around, and the other way around for $k' > k$. So if $(f, g) \lhd^{\vartheta^{k,k'}} (f'', g'')$ then the conclusion follows, and if not then assuming the conclusion would lead to a contradiction. □

Proposition 9.9 *Let ${}^{n-2}\gamma, {}^{n-2'}\gamma$ be two collections of $(n-2)$-dimensional (p, q)-shuffles, and let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles both filling and fit for ${}^{n-2}\gamma, {}^{n-2'}\gamma$. Let ϑ be an n-kind, and let $(f, g) \in {}^{n-1}\gamma$ of kind ϑ^k and $(f', g') \in {}^{n-1}\gamma'$ also of $(n-1)$-kind ϑ^k. Then $(f, g), (f', g')$ needs ϑ if and only if ϑ is properly relevant to ${}^{n-1}\gamma, {}^{n-1}\gamma'$.*

Proof: One direction is trivial, so assume $(f, g) \lessdot_\vartheta (f', g')$, i.e., $\alpha(\vartheta_0(k)) <^{k-1} \beta(\vartheta_1(k))$ and $\alpha'(\vartheta_0(k)) >^{k-1} \beta'(\vartheta_1(k))$.

To show ϑ is relevant to $K({}^{n-1}\gamma)$ I need to show that ${}^{n-1}\gamma$ contains an element of $(n-1)$-kind $\vartheta^{k'}$, for each k'. Now because $(f, g) \in {}^{n-1}\gamma$ of $(n-1)$-kind ϑ^k and ${}^{n-1}\gamma$ fits ${}^{n-2}\gamma, {}^{n-2'}\gamma$, ϑ^k is relevant to $K({}^{n-2}\gamma)$ and $K({}^{n-2'}\gamma)$, and so ${}^{n-2}\gamma, {}^{n-2'}\gamma$ both contain an element of $(n-2)$-kind $\vartheta^{k,k'}$, say $({}^{n-2}f, {}^{n-2}g) \in {}^{n-2}\gamma$ and $({}^{n-2'}f, {}^{n-2'}g) \in {}^{n-2'}\gamma$. Moreover, (f, g) and (f', g') are fit for ${}^{n-2}\gamma, {}^{n-2'}\gamma$, so $({}^{n-2}f, {}^{n-2}g) \lessdot (f^{k'-}, g^{k'-})$, $(f^{k'+}, g^{k'+}) \lessdot ({}^{n-2'}f, {}^{n-2'}g)$, $({}^{n-2}f, {}^{n-2}g) \lessdot ((f')^{k'-}, (g')^{k'-})$ and $((f')^{k'+}, (g')^{k'+}) \lessdot ({}^{n-2'}f, {}^{n-2'}g)$. If $k' > k$ then the first one of these says ${}^{n-2}\alpha(\vartheta_0(k)) \le^{k-1} \alpha(\vartheta_0(k))$ and $\beta(\vartheta_1(k)) \le^{k-1} {}^{n-2}\beta(\vartheta_0(k))$, while the fourth one says ${}^{n-2'}\alpha(\vartheta_0(k)) \ge^{k-1} \alpha'(\vartheta_0(k))$ and $\beta'(\vartheta_1(k)) \ge^{k-1} {}^{n-2'}\beta(\vartheta_0(k))$. Combined with the first couple of inequalities in this proof it follows that $({}^{n-2}f, {}^{n-2}g)$ to $({}^{n-2'}f, {}^{n-2'}g)$ needs $\vartheta^{k'}$. For $k' < k$ one uses the second and third one of these similarly, and again $({}^{n-2}f, {}^{n-2}g)$ to $({}^{n-2'}f, {}^{n-2'}g)$ needs $\vartheta^{k'}$. By Lemma 9.3, this implies that ${}^{n-1}\gamma$, which fills ${}^{n-2}\gamma, {}^{n-2'}\gamma$, contains an element of $(n-1)$-kind $\vartheta^{k'}$.

To show ϑ is proper to ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$, I have to show that for any k' and any $(f'', g'') \in {}^{n-1}\gamma$ of kind $\vartheta^{k'}$ and $(f''', g''') \in {}^{n-1}\gamma'$ also of $(n-1)$-kind $\vartheta^{k'}$, $(f'', g'') \lessdot_\vartheta (f''', g''')$. But this follows from Lemma 9.8: if $k > k'$ then $\alpha(\vartheta_0(k)) <^{k-1} \beta(\vartheta_1(k))$ implies $(f, g) \lhd^{\vartheta^{k,k'}} (f'', g'')$ in ${}^{n-1}\gamma$ implies, using the lemma in the other direction, $\alpha(\vartheta_0(k')) <^{k'-1} \beta(\vartheta_1(k'))$, and the other way around in ${}^{n-1}\gamma'$, as required. $\square$

Lemma 9.10 *Let (f, g) and (f', g') be $(n-1)$-dimensional (p, q)-shuffles both of $(n-1)$-kind ϑ' with $(f, g) \lessdot (f', g')$. Then* $\mathrm{rk}(f', g') - \mathrm{rk}(f, g)$ *is equal to the number of $(n-1)$-kinds needed by (f, g), (f', g').*

Proof: Immediate from Corollary 6.8. $\square$

Proposition 9.11 *Let ${}^{n-2}\gamma$, ${}^{n-2'}\gamma$ be two collections of $(n-2)$-dimensional (p, q)-shuffles, let ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles both filling and fit for ${}^{n-2}\gamma$, ${}^{n-2'}\gamma$, and let ${}^{n}\gamma$ be a collection of n-dimensional (p, q)-shuffles. If ${}^{n}\gamma$ is fit for ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$ then it fills ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$.*

Proof: Consider an $(n-1)$-kind ϑ', $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ of kind ϑ', and the collection of elements of ${}^{n}\gamma$ whose n-kind has bound ϑ'. By Lemma 9.10 and the fact that each n-kind needed occurs and occurs only once, there are as many of those as the difference in rank between $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$.

So firstly, $({}^{n-1}f, {}^{n-1}g) = ({}^{n-1}f', {}^{n-1}g')$ if and only if they have equal rank if and only if there are no elements of ${}^{n}\gamma$ whose n-kind has bound ϑ'.

Secondly, if $({}^{n-1}f, {}^{n-1}g) \neq ({}^{n-1}f', {}^{n-1}g')$ then there is an n-kind needed by $({}^{n-1}f, {}^{n-1}g)$, $({}^{n-1}f', {}^{n-1}g')$ so by Lemma 9.9 and fitness ${}^{n}\gamma$ contains an element, (f, g) say, of this n-kind which has bound ϑ'. By Lemma 9.6 $({}^{n-1}f, {}^{n-1}g) \lessdot ({}^{n-1}f', {}^{n-1}g')$, and Lemma 9.10 can be applied. Now by condition (v) of the definition of fitness, starting from (f, g), one finds a descending $\lhd^{\vartheta'}$-sequence of elements of ${}^{n}\gamma$ of n-kinds with bound ϑ', and dually an ascending $\lhd^{\vartheta'}$-sequence of elements of ${}^{n}\gamma$ of n-kinds with bound ϑ'. This whole sequence cannot contain a loop, because each element in the sequence increases rank by 1, and must reach $({}^{n-1}f, {}^{n-1}g)$ and $({}^{n-1}f', {}^{n-1}g')$ on either side because ${}^{n}\gamma$ is finite. Now again because each element in the sequence increases rank by 1 it must contain precisely $\mathrm{rk}(f', g') - \mathrm{rk}(f, g)$ elements, i.e., precisely all elements of ${}^{n}\gamma$ whose n-kind has bound ϑ'. Consequently, this collection is totally $\lhd^{\vartheta'}$-ordered. $\square$

10. Characterization

Finally, to compare well-formedness to fitness:

Proposition 10.1 *Let γ be an n-stage shuffle collection. If $s_{n-1}(R(\gamma))_{n-1} = {}^{n-1}\gamma$ and $t_{n-1}(R(\gamma))_{n-1} = {}^{n-1}\gamma'$, both ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$ are filling and fit for ${}^{n-2}\gamma$, ${}^{n-2'}\gamma$ and $R(\gamma)$ is compatible then ${}^{n}\gamma$ is fit for ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$.*

Proof: (v) if $({}^{n-1}f, {}^{n-1}g) \notin \mathrm{dom}(R(\gamma))$ but in $R(\gamma)$ then there exists an $(f, g) \in {}^{n}\gamma$ with $(f, g)\mathsf{E}^{n}_{n-1}({}^{n-1}f, {}^{n-1}g)$, by definition of domain.

(i) if $(f, g) \in {}^n\gamma$ of n-kind ϑ and $(f^{k-}, g^{k-}) \notin \mathrm{dom}(R(\gamma))$ then there is an element of ${}^n\gamma$ ending in (f^{k-}, g^{k-}); repeat, there will not be a loop by no direct loops and this will not go on forever because $M_{p,q}$ is finite. So eventually an element of ${}^{n-1}\gamma$ of $(n-1)$-kind ϑ^k will be found.

(ii) suppose a kind occurs twice, then the previous process will give the same element of ${}^{n-1}\gamma$ because that is fit for ${}^{n-2}\gamma, {}^{n-2'}\gamma$ itself. But at some point these processes will start to coincide, at which point one would have a situation contradicting compatibility.

(iii) suppose a non-fit element occurs, then the previous process will go on indefinitely because of repeated application of Corollary 7.8.

(iv) suppose ϑ properly relevant, i.e., needed by $({}^{n-1}f, {}^{n-1}g)$, $({}^{n-1}f', {}^{n-1}g')$ both of $(n-1)$-kind ϑ^k. Again, use the same process; if n-kind ϑ does not occur then it will always be that $\alpha(\vartheta_0(k)) <^{k-1} \beta(\vartheta_1(k))$, so $({}^{n-1}f', {}^{n-1}g')$ won't be hit and the process will go on indefinitely again.

□

It remains to collect the three comparisons together.

Definition 10.2 An n-stage shuffle collection γ *fills* if ${}^0\gamma$ and ${}^{0'}\gamma$ fill $\varnothing$, all ${}^j\gamma$ and ${}^{j'}\gamma$ fill ${}^{j-1}\gamma, {}^{j-1'}\gamma$, and ${}^n\gamma$ fills ${}^{n-1}\gamma, {}^{n-1}\gamma'$.

Definition 10.3 An n-stage shuffle collection γ *is fit* if ${}^0\gamma$ and ${}^{0'}\gamma$ fit $\varnothing$, all ${}^j\gamma$ and ${}^{j'}\gamma$ are fit for ${}^{j-1}\gamma, {}^{j-1'}\gamma$, and ${}^n\gamma$ is fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$.

Theorem 10.4 *Let γ be an n-stage shuffle collection. The following are equivalent*:

(i) *γ fills*

(ii) *γ is fit*

(iii) *$R(\gamma)$ is well formed with $s_j(R(\gamma)) = R(s_j(\gamma))$ and $t_j(R(\gamma)) = R(t_j(\gamma))$ for all $j \in \underline{n}$.*

Proof: By induction, using the induction hypothesis that the three statements are equivalent in one dimension lower. Then: (i) implies (iii) by Proposition 8.10, (ii) implies (i) by Proposition 9.11, and (iii) implies (ii) by Proposition 10.1. □

11. Classification

Corollary 11.1 *If ${}^n\gamma$ fills ${}^{n-1}\gamma, {}^{n-1}\gamma'$ then ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$.*

Proof: This immediately follows from Lemma 8.2. □

I will establish the converse of this corollary, that if ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$ then they are source and target for a fitting ${}^n\gamma$. Of course, I will have the same assumptions on ${}^{n-1}\gamma, {}^{n-1}\gamma'$ as before.

First, another way to see whether an n-dimensional (p, q)-shuffle is fit, which only works when ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$.

Lemma 11.2 *Let* $^{n-1}\gamma, {}^{n-1}\gamma'$ *be two collections of* $(n-1)$*-dimensional* (p, q)*-shuffles with* $^{n-1}\gamma \lessdot {}^{n-1}\gamma'$*, let* ϑ *be properly relevant to* $^{n-1}\gamma, {}^{n-1}\gamma'$*, and let* (f, g) *be an n-dimensional* (p, q)*-shuffle of kind* ϑ*.* (f, g) *is fit for* $^{n-1}\gamma, {}^{n-1}\gamma'$ *if and only if for every* $(n-1)$*-kind* ϑ' *bounding* ϑ *and every* $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ *and* $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ *both of* $(n-1)$*-kind* ϑ' *and for every* $i_0 \in \underline{p}$ *and* $i_1 \in \underline{q}$*,* ${}^{n-1}\alpha(i_0) \lessgtr {}^{n-1}\beta(i_1)$ *and* ${}^{n-1}\alpha'(i_0) \lessgtr {}^{n-1}\beta'(i_1)$ *imply* $\alpha(i_0) \lessgtr \beta(i_1)$*.*

Proof: Say $({}^{n-1}f, {}^{n-1}g)$ and $({}^{n-1}f', {}^{n-1}g')$ are of kind ϑ^k.

Suppose (f, g) is fit for $^{n-1}\gamma, {}^{n-1}\gamma'$ and suppose ${}^{n-1}\alpha(i_0) \lessgtr {}^{n-1}\beta(i_1)$ and ${}^{n-1}\alpha'(i_0) \lessgtr {}^{n-1}\beta'(i_1)$. Then the conclusion follows from Corollary 7.7 applied to either $({}^{n-1}f, {}^{n-1}g) \lessdot (f^{k-}, g^{k-})$ or to $(f^{k+}, g^{k+}) \lessdot ({}^{n-1}f', {}^{n-1}g')$, depending on whether the sign is $>^{k'}$ or $<^{k'}$ where $\vartheta^k(k') < (i_0, i_1) < \vartheta^k(k'+1)$.

For the converse, suppose the conclusion and, in order to prove $({}^{n-1}f, {}^{n-1}g) \lessdot (f^{k-}, g^{k-})$, that ${}^{n-1}\alpha(i_0) >^{k'} {}^{n-1}\beta(i_1)$ where $\vartheta^k(k') < (i_0, i_1) < \vartheta^k(k'+1)$. Then because $({}^{n-1}f, {}^{n-1}g) \lessdot ({}^{n-1}f', {}^{n-1}g')$ Corollary 7.7 gives ${}^{n-1}\alpha'(i_0) >^{k'} {}^{n-1}\beta'(i_1)$ and the conclusion gives $\alpha(i_0) >^{k'} \beta(i_1)$ as required. $(f^{k+}, g^{k+}) \lessdot ({}^{n-1}f', {}^{n-1}g')$ is proven similarly using the conclusion with the opposite sign. □

In defining $^n\gamma$ I will pick one n-dimensional (p, q)-shuffle for each n-kind properly relevant for $^{n-1}\gamma, {}^{n-1}\gamma'$. The one I will pick will satisfy the following condition:

Definition 11.3 Let $^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles, and let (f, g) be an n-dimensional (p, q)-shuffle of n-kind ϑ. (f, g) is *minimally fit for* $^{n-1}\gamma, {}^{n-1}\gamma'$ if it is fit for $^{n-1}\gamma, {}^{n-1}\gamma'$ and for every (i_0, i_1) with $\vartheta(k-1) < (i_0, i_1) < \vartheta(k)$, such that $\alpha(i_0) >^{k-1} \beta(i_1)$ there is a k'' and elements $({}^{n-1}f'', {}^{n-1}g'') \in {}^{n-1}\gamma$ and $({}^{n-1}f''', {}^{n-1}g''') \in {}^{n-1}\gamma'$ both of kind $\vartheta^{k''}$ such that ${}^{n-1}\alpha''(i_0) >^{k-1} {}^{n-1}\beta''(i_1)$ and ${}^{n-1}\alpha'''(i_0) >^{k-1} {}^{n-1}\beta'''(i_1)$.

To explain the use of the word 'minimal':

Lemma 11.4 *Let* $^{n-1}\gamma, {}^{n-1}\gamma'$ *be two collections of* $(n-1)$*-dimensional* (p, q)*-shuffles with* $^{n-1}\gamma \lessdot {}^{n-1}\gamma'$*, and let* (f, g) *be an n-dimensional* (p, q)*-shuffle of n-kind* ϑ *properly relevant to* $^{n-1}\gamma, {}^{n-1}\gamma'$*.* (f, g) *is minimally fit for* $^{n-1}\gamma, {}^{n-1}\gamma'$ *if and only if it is fit for* $^{n-1}\gamma, {}^{n-1}\gamma'$ *and* $(f, g) \lessdot (f', g')$ *for every n-dimensional* (p, q)*-shuffle* (f', g') *of kind* ϑ *fit for* $^{n-1}\gamma, {}^{n-1}\gamma'$*.*

Proof: By Lemma 11.2, fitness of (f, g) and (f', g') fixes whether $\alpha(i_0) \lessgtr \beta(i_1)$ as soon as these are in the same order in $^{n-1}\gamma, {}^{n-1}\gamma'$. So I only have to consider those (i_0, i_1) for which this is not the case.

An (f, g) fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ is not minimally fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ if there exists such an (i_0, i_1) with $\vartheta(k-1) < (i_0, i_1) < \vartheta(k)$, such that $\alpha(i_0) >^{k-1} \beta(i_1)$.

There is an (f', g') with $(f, g) \not\lessdot (f', g')$ if, by Corollary 7.7, there exists such an (i_0, i_1) with $\vartheta(k-1) < (i_0, i_1) < \vartheta(k)$, such that $\alpha(i_0) >^{k-1} \beta(i_1)$ but $\alpha'(i_0) <^{k-1} \beta'(i_1)$.

So one direction is trivial, and for the other direction (f', g') can be chosen by swapping such i_0 and i_1 for which $\alpha(i_0) \pm 1 = \beta(i_1)$, which must exist because α and β are order preserving. □

Checking whether something is minimally fit can be simplified somewhat when ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$ are as wanted:

Lemma 11.5 *Let ${}^{n-2}\gamma, {}^{n-2'}\gamma$ be two collections of $(n-2)$-dimensional (p, q)-shuffles, and let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles both filling and fit for ${}^{n-2}\gamma, {}^{n-2'}\gamma$, and let ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$. Then (f, g) of n-kind properly relevant to ${}^{n-1}\gamma, {}^{n-1}\gamma'$ is minimally fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ precisely when: for every (i_0, i_1) with $\vartheta(k-1) < (i_0, i_1) < \vartheta(k), \alpha(i_0) >^{k-1} \beta(i_1)$ if and only if there is a k'' and elements $({}^{n-1}f'', {}^{n-1}g'') \in {}^{n-1}\gamma$ and $({}^{n-1}f''', {}^{n-1}g''') \in {}^{n-1}\gamma'$ both of kind $\vartheta^{k''}$ such that ${}^{n-1}\alpha''(i_0) >^{k-1} {}^{n-1}\beta''(i_1)$ and ${}^{n-1}\alpha'''(i_0) >^{k-1} {}^{n-1}\beta'''(i_1)$.*

Proof: Clearly, this condition is necessary for minimally fit, partly by Lemma 11.2. For sufficiency, the only case to consider is when ${}^{n-1}\alpha(i_0) <^{k-1} {}^{n-1}\beta(i_1)$ and ${}^{n-1}\alpha'(i_0) <^{k-1} {}^{n-1}\beta'(i_1)$ but $\alpha(i_0) >^{k-1} \beta(i_1)$. Then the condition gives a k'' and elements $({}^{n-1}f'', {}^{n-1}g'') \in {}^{n-1}\gamma$ and $({}^{n-1}f''', {}^{n-1}g''') \in {}^{n-1}\gamma'$ both of kind $\vartheta^{k''}$ such that ${}^{n-1}\alpha''(i_0) >^{k-1} {}^{n-1}\beta''(i_1)$ and ${}^{n-1}\alpha'''(i_0) >^{k-1} {}^{n-1}\beta'''(i_1)$. But then, by repeated application of Lemma 9.8, $({}^{n-1}f, {}^{n-1}g)$ and $({}^{n-1}f'', {}^{n-1}g'')$ are in the *same* $\lhd^{k,k''}$-order in ${}^{n-1}\gamma$ as are $({}^{n-1}f', {}^{n-1}g')$ and $({}^{n-1}f''', {}^{n-1}g''')$ in ${}^{n-1}\gamma'$, from looking at i_0 and i_1, but also in the *opposite* $\lhd^{k,k''}$-order, from looking at ϑ being properly relevant, contradiction. □

So the proof of this lemma is basically saying that for a properly relevant n-kind one cannot get contradictory prescriptions from ${}^{n-1}\gamma, {}^{n-1}\gamma'$ on whether $\alpha(i_0) \lessgtr \beta(i_1)$.

Lemma 11.6 *Let ${}^{n-2}\gamma, {}^{n-2'}\gamma$ be two collections of $(n-2)$-dimensional (p, q)-shuffles, and let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles both filling and fit for ${}^{n-2}\gamma, {}^{n-2'}\gamma$, and let ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$. For every n-kind ϑ properly relevant to ${}^{n-1}\gamma, {}^{n-1}\gamma'$ there exists precisely one n-dimensional (p, q)-shuffle of n-kind ϑ that is minimally fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$.*

Proof: Uniqueness is immediate from Lemma 11.4.

To show that there exists one, notice first that the n-kind determines the places of all swaps, as observed at the end of Section 3. So I only have to define positions between swaps. For i_0 not part of a swap, (i.e., $\neq \vartheta_0(k)$ for any k), define for (i_0, i_1) with $\vartheta(k-1) < (i_0, i_1) < \vartheta(k)$

and $k-1$ *even*

$$\begin{aligned}\alpha(i_0) = i_0 + \max\{i_1' \mid\ & {}^{n-1}\alpha(i_0) >^{k-1}\ {}^{n-1}\beta(i_1') \text{ and } {}^{n-1}\alpha'(i_0) >^{k-1}\ {}^{n-1}\beta'(i_1') \text{ for}\\ & \text{some } ({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma \text{ and } ({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma' \text{ of}\\ & (n-1)\text{-kind bounding } \vartheta\}\\ \beta(i_1) = i_1 + \max\{i_0' \mid\ & \text{there are no } ({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma \text{ and } ({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'\\ & \text{of } (n-1)\text{-kind bounding } \vartheta \text{ with } {}^{n-1}\alpha(i_0') >^{k-1}\ {}^{n-1}\beta(i_1)\\ & \text{and } {}^{n-1}\alpha'(i_0') >^{k-1}\ {}^{n-1}\beta'(i_1)\}\end{aligned}$$

and for $k-1$ *odd*

$$\begin{aligned}\alpha(i_0) = i_0 + \max\{i_1' \mid\ & \text{there are no } ({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma \text{ and } ({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'\\ & \text{of } (n-1)\text{-kind bounding } \vartheta \text{ with } {}^{n-1}\alpha(i_0) >^{k-1}\ {}^{n-1}\beta(i_1')\\ & \text{and } {}^{n-1}\alpha'(i_0) >^{k-1}\ {}^{n-1}\beta'(i_1')\}\\ \beta(i_1) = i_1 + \max\{i_0' \mid\ & {}^{n-1}\alpha(i_0') >^{k-1}\ {}^{n-1}\beta(i_1) \text{ and } {}^{n-1}\alpha'(i_0') >^{k-1}\ {}^{n-1}\beta'(i_1) \text{ for}\\ & \text{some } ({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma \text{ and } ({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma' \text{ of}\\ & (n-1)\text{-kind bounding } \vartheta\}.\end{aligned}$$

I need first to show that (f, g) is an n-dimensional (p, q)-shuffle, i.e., that α and β are strictly order preserving with disjoint images. But strictly order preserving is almost immediate from ${}^{n-1}\alpha, {}^{n-1}\beta, {}^{n-1}\alpha, {}^{n-1}\beta'$ being strictly order preserving. And take i to be the smallest for which $i = \alpha(i_0) = \beta(i_1)$, then because up to $i-1$ α and β are strictly order preserving with disjoint images one has $i-1 = i_0-1+i_1-1$ and hence $\alpha(i_0) = i_0+(i_1-1)$. For $k-1$ even, by maximality of i_1' in the definition of α, there are no $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ of $(n-1)$-kind bounding ϑ with ${}^{n-1}\alpha(i_0) >^{k-1}\ {}^{n-1}\beta(i_1)$ and ${}^{n-1}\alpha'(i_0) >^{k-1}\ {}^{n-1}\beta'(i_1)$. But similarly $\beta(i_1) = i_1 + (i_0 - 1)$, and by maximality of i_0' in the definition of β there *are* such $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$. Contradiction, so the above i does not exist, so α and β have disjoint images. The case $k-1$ odd is completely analogous.

To show that (f, g) is minimally fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$, using Lemma 11.5, $\alpha(i_0) >^{k-1} \beta(i_1)$ if and only if $\zeta_1(\alpha(i_0)) \geq i_1$ if and only if for $k-1$ even $\max\{i_1' \mid {}^{n-1}\alpha(i_0) >^{k-1}\ {}^{n-1}\beta(i_1')$ and ${}^{n-1}\alpha'(i_0) >^{k-1}\ {}^{n-1}\beta'(i_1')$ for some $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ of $(n-1)$-kind bounding $\vartheta\} \geq i_1$ if and only if ${}^{n-1}\alpha(i_0) >^{k-1}\ {}^{n-1}\beta(i_1)$ and ${}^{n-1}\alpha'(i_0) >^{k-1}\ {}^{n-1}\beta'(i_1)$ for some $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ and $({}^{n-1}f', {}^{n-1}g') \in {}^{n-1}\gamma'$ of $(n-1)$-kind bounding ϑ. □

Among the fit n-dimensional (p, q)-shuffles of fixed n-kind, the minimally fit one is now the one with minimal rank:

Lemma 11.7 *Let ${}^{n-2}\gamma, {}^{n-2'}\gamma$ be two collections of $(n-2)$-dimensional (p, q)-shuffles, let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles both filling and fit for ${}^{n-2}\gamma, {}^{n-2'}\gamma$, and let ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$, and let (f, g) be an n-dimensional (p, q)-shuffle of n-kind ϑ. (f, g) is minimally fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ if and only if it is fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ and* $\mathrm{rk}(f, g) \leq \mathrm{rk}(f', g')$ *for every n-dimensional (p, q)-shuffle (f', g') of kind ϑ fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$.*

Proof: One direction is immediate from Lemma 11.4. For the other direction, if there is an n-dimensional (p, q)-shuffle (f', g') of kind ϑ fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ with $\mathrm{rk}(f, g) \not\leq \mathrm{rk}(f', g')$ then (f, g) is not the by Lemma 11.6 unique n-dimensional (p, q)-shuffle of n-kind ϑ that is minimally fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$. □

Lemma 11.8 *Let ${}^{n-2}\gamma, {}^{n-2'}\gamma$ be two collections of $(n-2)$-dimensional (p, q)-shuffles, and let ${}^{n-1}\gamma, {}^{n-1}\gamma'$ be two collections of $(n-1)$-dimensional (p, q)-shuffles both filling and fit for ${}^{n-2}\gamma, {}^{n-2'}\gamma$, and let ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$. Let (f, g) of n-kind ϑ be minimally fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$. Let $({}^{n-1}f, {}^{n-1}g) \in {}^{n-1}\gamma$ of $(n-1)$-kind ϑ^k. If $(f^{k-}, g^{k-}) \neq ({}^{n-1}f, {}^{n-1}g)$ then there exists an (f', g') minimally fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ and for which $((f')^{\kappa+}, (g')^{\kappa+}) = (f^{k-}, g^{k-})$, for some κ.*

Proof: Because (f, g) is fit for ${}^{n-1}\gamma, {}^{n-1}\gamma'$ one has $({}^{n-1}f, {}^{n-1}g) \lessdot (f^{k-}, g^{k-})$. Call (i_0, i_1) with $\vartheta^k(\kappa-1) < (i_0, i_1) < \vartheta^k(\kappa)$ *considerable* if $\alpha^{k-}(i_0) >^{\kappa-1} \beta^{k-}(i_1)$ and ${}^{n-1}\alpha(i_0) <^{\kappa-1} {}^{n-1}\beta(i_1)$. Because $({}^{n-1}f, {}^{n-1}g) \neq (f^{k-}, g^{k-})$ there must be, using Corollary 7.7, a considerable (i_0, i_1). Call (i_0, i_1) with $\vartheta^k(\kappa-1) < (i_0, i_1) < \vartheta^k(\kappa)$ *inconsiderable* if $\alpha^{k-}(i_0) <^{\kappa-1} \beta^{k-}(i_1)$ and ${}^{n-1}\alpha'(i_0) >^{\kappa-1} {}^{n-1}\beta'(i_1)$, and *unremarkable* if it is neither considerable nor inconsiderable, i.e., when ${}^{n-1}\alpha(i_0) \lesseqgtr {}^{n-1}\beta(i_1)$ if and only if ${}^{n-1}\alpha'(i_0) \lesseqgtr {}^{n-1}\beta'(i_1)$.

For any (i_0, i_1), if $\min\{\alpha^{k-}(i_0), \beta^{k-}(i_1)\} + 1 = \min\{\alpha^{k-}(i_0), \beta^{k-}(i_1)\}$ say that (i_0, i_1) is *neighbouring* (even though properly speaking this refers to $\alpha^{k-}(i_0)$ and $\beta^{k-}(i_1)$). Because α^{k-} and β^{k-} are order preserving there actually must be a considerable neighbouring (i_0, i_1).

For any (i_0, i_1) with $\vartheta^k(\kappa-1) < (i_0, i_1) < \vartheta^k(\kappa)$, define an n-kind ϑ' by:

$$\begin{aligned} \vartheta'(k') &= \vartheta^k(k') && \text{if } k' < \kappa \\ &= (i_0, i_1) && \text{if } k' = \kappa \\ &= \vartheta^k(k'-1) && \text{if } k' > \kappa. \end{aligned}$$

For any considerable neighbouring (i_0, i_1) one can define an n-dimensional (p, q)-shuffle (f', g') by:

$$\begin{aligned} f'(i) &= f^{k-}(i) && \text{if } i \neq \alpha^{k-}(i_0), \beta^{k-}(i_1) \\ &= f^{k-}(i) && \text{if } i \neq \alpha^{k-}(i_0), \beta^{k-}(i_1), \kappa-1 \text{ even} \\ &= 1 - f^{k-}(i) && \text{if } i \neq \alpha^{k-}(i_0), \beta^{k-}(i_1), \kappa-1 \text{ odd} \\ g'(i) &= g^{k-}(i) && \text{if } i \leq \min\{\alpha^{k-}(i_0), \beta^{k-}(i_1)\} \\ &= g^{k-}(i) - 1 && \text{if } i \geq \max\{\alpha^{k-}(i_0), \beta^{k-}(i_1)\}. \end{aligned}$$

It is clear from this definition that $((f')^{\kappa+}, (g')^{\kappa+}) = (f^{k-}, g^{k-})$, and that (f', g') has n-kind ϑ'.

Now it might be that for a given considerable neighbouring (i_0, i_1), there is a considerable (i_0', i_1') with $(i_0, i_1) < (i_0', i_1')$ such that there exists a k'' and $({}^{n-1}f'', {}^{n-1}g'') \in {}^{n-1}\gamma$ and $({}^{n-1}f''', {}^{n-1}g''') \in {}^{n-1}\gamma'$ both of kind $(\vartheta')^{k''}$ such that ${}^{n-1}\alpha''(i_0') >^{\kappa'-1} {}^{n-1}\beta''(i_1')$ and ${}^{n-1}\alpha'''(i_0') >^{\kappa'-1} {}^{n-1}\beta'''(i_1')$. Call (i_0, i_1) *desirable* if this is *not* the case.

The first claim is that if (i_0, i_1) is desirable then (f', g') satisfies the "$\Leftarrow$" condition of Lemma 11.5. So assume that (i'_0, i'_1) with $\vartheta'(k'-1) < (i'_0, i'_1) < \vartheta'(k')$ and that there is a k'' and elements $({}^{n-1}f'', {}^{n-1}g'') \in {}^{n-1}\gamma$ and $({}^{n-1}f''', {}^{n-1}g''') \in {}^{n-1}\gamma'$ both of kind $(\vartheta')^{k''}$ such that ${}^{n-1}\alpha''(i'_0) >^{k-1} {}^{n-1}\beta''(i'_1)$ and ${}^{n-1}\alpha'''(i'_0) >^{k-1} {}^{n-1}\beta'''(i'_1)$; I will want to show that $\alpha'(i'_0) >^{k-1} \beta'(i'_1)$. There are the following situations:

- (i'_0, i'_1) is unremarkable. Then k'' can be taken to be κ because if that would not be possible k'' and κ would give contradictory prescriptions for the order of i'_0 and i'_1 for n-kind ϑ'. (f, g) is fit so $\alpha(i'_0) >^{k-1} \beta(i'_1)$ which implies the same in (f', g').
- $(i'_0, i'_1) > (i_0, i_1)$, (i'_0, i'_1) is considerable. Immediate from desirability of (f', g').
- $(i'_0, i'_1) > (i_0, i_1)$, (i'_0, i'_1) is inconsiderable, *or* $(i'_0, i'_1) < (i_0, i_1)$, (i'_0, i'_1) is considerable. In both these instances the assumptions already give $\alpha'(i'_0) >^{k-1} \beta'(i'_1)$, without any need for the k'' etc.
- $(i'_0, i'_1) < (i_0, i_1)$, (i'_0, i'_1) is inconsiderable. This situation actually does not occur. For this, there are twelve cases to consider, depending on where $\vartheta(k)$ and $\vartheta(k'')$ are with respect to (i'_0, i'_1) and (i_0, i_1) and to each other (some more if one counts 'overlaps', but these are easily subsumed under the other cases). In all these cases Lemma 9.8 is used, in four of them to show that the specifications for $({}^{n-1}f'', {}^{n-1}g'')$ and $({}^{n-1}f''', {}^{n-1}g''')$ are not possible, in the other eight to show a contradiction with (i'_0, i'_1) being inconsiderable.

The second claim is that if (i_0, i_1) is the *first* desirable then (f', g') also satisfies the "$\Rightarrow$" condition of Lemma 11.5. So assume (i'_0, i'_1) with $\vartheta'(k'-1) < (i'_0, i'_1) < \vartheta'(k')$ and that $\alpha'(i'_0) >^{k-1} \beta'(i'_1)$; I will want to show that there is a k'' and elements $({}^{n-1}f'', {}^{n-1}g'') \in {}^{n-1}\gamma$ and $({}^{n-1}f''', {}^{n-1}g''') \in {}^{n-1}\gamma'$ both of kind $(\vartheta')^{k''}$ such that ${}^{n-1}\alpha''(i'_0) >^{k-1} {}^{n-1}\beta''(i'_1)$ and ${}^{n-1}\alpha'''(i'_0) >^{k-1} {}^{n-1}\beta'''(i'_1)$. There are the following situations:

- (i'_0, i'_1) is unremarkable. Take $k'' = \kappa$.
- $(i'_0, i'_1) > (i_0, i_1)$, (i'_0, i'_1) is considerable, *or* $(i'_0, i'_1) < (i_0, i_1)$, (i'_0, i'_1) is inconsiderable. In both these instances the assumptions contradict $\alpha'(i'_0) >^{k-1} \beta'(i'_1)$.
- $(i'_0, i'_1) > (i_0, i_1)$, (i'_0, i'_1) is inconsiderable. This situation actually does not occur. For this, there are twelve cases to consider, as before, but slightly differently divided: in four, namely where $\vartheta(k) > (i'_0, i'_1)$ and where one cannot use minimally fitness of (f, g) with respect to (i'_0, i'_1), take k'' from minimally fitness of (f, g) with respect to (i_0, i_1). In all these cases Lemma 9.8 is used, in five of them to show that the specifications for $({}^{n-1}f'', {}^{n-1}g'')$ and $({}^{n-1}f''', {}^{n-1}g''')$ are not possible, in four others to show a contradiction with (i'_0, i'_1) being inconsiderable. The remaining three cases, which all have k'' obtained from minimally fitness of (f, g) with respect to (i'_0, i'_1), are now done by taking a further k''' from minimally fitness of (f, g) with respect to (i_0, i_1) and an extensive use of Lemma 9.8 to show that this also contradicts (i'_0, i'_1) being inconsiderable.
- $(i'_0, i'_1) < (i_0, i_1)$, (i'_0, i'_1) is considerable. (i'_0, i'_1) may be assumed to be neighbouring: if it isn't there must also be one such, and the conclusion for that one will, again because α's and β's are order preserving, hold for (i'_0, i'_1) too. (i_0, i_1) is assumed to be the first desirable, so (i'_0, i'_1) is not desirable, giving a considerable (i''_0, i''_1). If $(i''_0, i''_1) > (i_0, i_1)$ this is done via a by now simple application of Lemma 9.8. If $(i''_0, i''_1) = (i_0, i_1)$ then it is an even more straightforward application of said lemma. So this takes care of the *last*

considerable (i_0', i_1') before (i_0, i_1). For the before-last considerable (i_0', i_1') before (i_0, i_1), there is again an (i_0'', i_1''); $(i_0'', i_1'') \geq (i_0, i_1)$ is already done, which leaves as the only other possibility that (i_0'', i_1'') is not desirable because of (i_0, i_1). But this is the mirror image of the previous situation, with mirror image proof. The conclusion can be paraphrased as that (i_0', i_1') is not desirable because of (i_0, i_1), and so this argument takes care of all earlier (i_0', i_1') too.

Further details, which are quite instructive, are left to the reader. □

Proposition 11.9 *Let* ${}^{n-2}\gamma$, ${}^{n-2'}\gamma$ *be two collections of* $(n-2)$*-dimensional* (p, q)*-shuffles, and let* ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$ *be two collections of* $(n-1)$*-dimensional* (p, q)*-shuffles both filling and fit for* ${}^{n-2}\gamma$, ${}^{n-2'}\gamma$. *If* ${}^{n-1}\gamma \lessdot {}^{n-1}\gamma'$ *then there is a collection of* n*-dimensional* (p, q)*-shuffles fit for* ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$.

Proof: For each n-kind ϑ properly relevant to ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$, take the unique minimally fit n-dimensional (p, q)-shuffle of this kind established in Lemma 11.6.

The collection of n-dimensional (p, q)-shuffles thus defined is fit for ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$. Indeed, (i) every element has properly relevant n-kind, so in particular relevant, (ii) for every n-kind only one n-dimensional (p, q)-shuffle has been picked, (iii) this n-dimensional (p, q)-shuffle is minimally fit for ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$ hence fit for ${}^{n-1}\gamma$, ${}^{n-1}\gamma'$, (iv) for every properly relevant n-kind an n-dimensional (p, q)-shuffle has indeed been picked, and (v) continuation holds by Lemma 11.8. □

Theorem 11.10 *Let* A *and* A' *be* $(n-1)$*-dimensional well-formed subpasting schemes of* $M_{p,q}$ *such that* $s_{n-2}(A) = s_{n-2}(A')$ *and* $t_{n-2}(A) = t_{n-2}(A')$. *Then there is an* n*-dimensional well-formed subpasting scheme of* $M_{p,q}$ *with* $(n-1)$*-source* A *and* $(n-1)$*-target* A' *if and only if* $A_{n-1} \lessdot A'_{n-1}$.

Proof: Direct from the previous proposition, whose conditions are satisfied because of Theorem 10.4. □

12. Well formed

Fix p and q, and let ${}^{i}\mu$ and ${}^{i'}\mu$ consist of the i-dimensional (p, q)-shuffles of rank 0 and oprank 0 respectively.

Lemma 12.1 $R(\mu) = M_{p,q}$.

Proof: This is because every n-dimensional (p, q)-shuffle is in the boundary of a higher-dimensional one of rank zero: if (f, g) does not have rank 0 there must be an f_k^{r} with rank, for k even say, not equal to 0, so this cannot consist of only 0's or only 1's, so somewhere a 0 and a 1 must be next to one another—introducing a swap here gives an $(n+1)$-dimensional (p, q)-shuffle of which (f, g) is in the boundary, it need not have rank zero but repeat. □

Lemma 12.2 μ *is fit.*

Proof: First observe that for every n-kind there is precisely one n-dimensional (p, q)-shuffle of rank 0: the kind determines the places of the swaps and rank zero says that for f_k^{r} it's all 0's followed by all 1's for k even and the other way around for k odd, and the other way around for oprank. Hence $K({}^{n-1}\mu)$ and $K({}^{n-1}\mu')$ contain *all* $(n-1)$-kinds.

Secondly, it is obvious from Corollary 7.7 that for an $(n-1)$-dimensional (p, q)-shuffle (f, g) of rank 0 one has that for any (f', g') of the same $(n-1)$-kind $(f, g) \lessdot (f', g')$. This implies immediately that any n-dimensional (p, q)-shuffle is fit for ${}^{n-1}\mu, {}^{n-1}\mu'$, and so, using Lemma 9.6, that every n-kind is proper hence properly relevant to ${}^{n-1}\mu, {}^{n-1}\mu'$, and, using Lemma 11.7, that any n-dimensional (p, q)-shuffle of rank 0 is minimally fit for ${}^{n-1}\mu, {}^{n-1}\mu'$.

This proves most of fitness, except (v) continuation for which one can again use Lemma 11.8, or, if one wants to avoid this overkill, prove it directly, for which one only needs simple versions of three of the cases in the proof of Lemma 11.8. □

Theorem 12.3 *For each p, q, the pasting scheme $M_{p,q}$ is well formed. Moreover, $s_n(M_{p,q})$ and $t_n(M_{p,q})$ have as n-cells precisely the n-dimensional (p, q)-shuffles of rank* 0 *and oprank* 0 *respectively.*

Proof: This follows immediately from the previous, Theorem 10.4 together with Lemmas 12.1 and 12.2. □

13. Loop free

Theorem 13.1 *For each p, q, the pasting scheme $M_{p,q}$ is loop free.*

Proof: Let Y be a well-formed j-dimensional subpasting scheme of $M_{p,q}$, let $x = (f, g)$ be an n-dimensional (p, q)-shuffle with $s_j(\mathsf{R}(x)) \subseteq Y$, and let $u, u' \in s_j(\mathsf{R}(x))$ and $v \in Y_j$ with $w \in \mathsf{E}_{j-1}(u) \cap \mathsf{B}_{j-1}(v)$ and $v \lhd_Y u'$.

First, it suffices to show $v \in \mathsf{R}(x)$: Y, being well-formed, contains each j-kind at most once, and, because $s_j(\mathsf{R}(x)) \subseteq Y$, it does contain each j-kind bounding the n-kind of x, so if v has such j-kind it must be the element of Y of this j-kind, which is in $s_j(\mathsf{R}(x))$.

$v = (f', g')$ being in $\mathsf{R}(x)$ means that v has j-kind bounding the n-kind of x and that for positions outside this n-kind, where v cannot have any swaps, f' agrees with f. Assume $v \notin \mathsf{R}(x)$, then $w \in t_{j-1}(\mathsf{R}(x))$ because if not it would have an outgoing cell in $s_j(\mathsf{R}(x))$ which together with v would contradict well-formedness of Y. Thus, from w one cannot introduce a swap from the n-kind of x.

For $v \notin \mathsf{R}(x)$, going from w to v introduces a swap outside the n-kind of x. In order to get back to $\mathsf{R}(x)$ at some point in $v \lhd_Y u'$ this swap needs to be removed, exactly as it was introduced, which can be done if at that point the parity is different. For this to happen, at some earlier point in $v \lhd_Y u'$ a swap needs to be introduced or removed at a position before the swap introduced from w to v. Looking at the $\lhd$-first time that happens, the j-dimensional (p, q)-shuffle at that point agrees with w up to the swap introduced from w to v, so only a swap outside the n-kind of x can be introduced, and then the argument can be repeated for this swap. Removing a swap, necessarily from the n-kind of x because those are the only swaps appearing before the swap introduced from w to v, means removing to the *target*, so in order to get back to $s_j(\mathsf{R}(x))$, this swap needs to be removed exactly as it was introduced. This needs another parity change, hence another swap needs to be introduced or removed at a position before this swap, and again the argument repeats. Because j-dimensional (p, q)-shuffles are finite sequences there is no room to introduce or remove swaps at earlier positions indefinitely, as would be necessary, hence it must be that $v \in \mathsf{R}(x)$. □

Notes

1. *Tas*, plural teisi (pronounced TAY-see), is Welsh for "stack".
2. This is not a typing error (or pasting error ⓣ): P_4 can easily be made into a well-formed loop-free pasting scheme, but that is only one, not a scheme of those.

References

1. J.C. Baez and J. Dolan, "Higher dimensional algebra III. n-categories and the algebra of opetopes," *Adv. Math.* **135** (1998), 145–206.
2. M.A. Batanin, "Monoidal globular categories as natural environment for the theory of weak n-categories," *Adv. Math.* **136** (1998), 39–103.
3. H.J. Baues, "Geometry of loop spaces and the cobar construction," *Mem. Amer. Math. Soc.* **25**(230) (1980).
4. C. Berger, "Double loop spaces, braided monoidal categories and algebraic 3-type of space," in *Higher Homotopy Structures in Topology and Mathematical Physics* (Poughkeepsie, 1996), J. McCleary (Ed.), vol. 227 of *Contemp. Math.*, pp. 49–66, Amer. Math. Soc., Providence, RI, 1999.
5. J.M. Boardman and R.M. Vogt, "Homotopy invariant algebraic structures on topological spaces," vol. 347 of *Lecture Notes in Math.*, Springer Verlag, Berlin, 1973.
6. R. Brown and P.J. Higgins, "The classifying space of a crossed complex," *Math. Proc. Cambridge Philos. Soc.* **110** (1991), 95–120.
7. S.E. Crans, "On combinatorial models for higher dimensional homotopies," Ph.D. thesis, Utrecht University, 1995.
8. S.E. Crans, "Generalized centers of braided and sylleptic monoidal 2-categories," *Adv. Math.* **136** (1998), 183–223.
9. S.E. Crans, "On braidings, syllepses, and symmetries," *Cahiers Topologie Géom. Différentielle Catég.* **XLI**(1) (2000).
10. S.E. Crans, "Higher-dimensional Mac Lane's pentagon and Zamolodchikov equations," *J. Pure Appl. Algebra* **168** (2002), 347–365.
11. S.E. Crans, "Braided 2-categories and Zamolodchikov systems," in preparation.
12. C. Hermida, M. Makkai, and J. Power, "On weak higher dimensional categories I," *J. Pure Appl. Algebra* **154** (2000), 221–246, **157** (2001), 247–277 and **166** (2002), 83–104.
13. M. Johnson, "Pasting diagrams in n-categories with applications to coherence theorems and categories of paths," Preprint 88-14, University of Sydney.
14. M. Johnson, "The combinatorics of n-categorical pasting," *J. Pure Appl. Algebra* **62** (1989), 211–225.
15. A. Joyal and M. Tierney, "Algebraic homotopy types," unpublished.
16. M.M. Kapranov and V.A. Voevodsky, "Combinatorial-geometric aspects of polycategory theory: Pasting schemes and higher Bruhat orders (list of results)," *Cahiers Topologie Géom. Différentielle Catég.* **32** (1991), 11–27.
17. R.J. Lawrence, "Yang-Baxter type equations and posets of maximal chains," *J. Combin. Theory Ser. A* **79** (1997), 68–104.
18. Y.I. Manin and V.V. Schechtman, "Arrangements of hyperplanes, higher braid groups and higher Bruhat orders," in *Algebraic Number Theory*, J. Coates, R. Greenberg, B. Mazur, and I. Satake (Eds.), vol. 17 of *Adv. Stud. Pure Math.*, pp. 289–308, Academic Press, Boston, 1989.

19. J.P. May, “The geometry of iterated loop-spaces,” vol. 271 of *Lecture Notes in Math.*, Springer Verlag, New York, 1972.
20. R.J. Milgram, “Iterated loop spaces,” *Ann. of Math.* **84**(2) (1966), 386–403.
21. R. Street, “The algebra of oriented simplexes,” *J. Pure Appl. Algebra* **49** (1987), 283–335.
22. R.W. Thomason, “*Cat* as a closed model category,” *Cahiers Topologie Géom. Différentielle Catég.* **XXI** (1980), 305–324.