Journal of Algebraic Combinatorics, 19, 283–295, 2004

Planar Groups

COLIN L. STARR cstarr@willamette.edu
Department of Mathematics, Willamette University, 900 State Street, Salem, OR 97302, USA

GALEN E. TURNER III
Program of Mathematics and Statistics, Louisiana Tech University, P.O. Box 3189, Ruston, LA 71272, USA

Received September 10, 2002; Revised May 1, 2003

Abstract. In abstract algebra courses, teachers are often confronted with the task of drawing subgroup lattices. For purposes of instruction, it is usually desirable that these lattices be planar graphs (with no crossings). We present a characterization of abelian groups with this property. We also resolve the following problem in the abelian case: if the subgroup lattice is required to be drawn hierarchically (that is, in monotonic order of index within the group), when is it possible to draw the lattice without crossings?

Keywords: graph, subgroup lattice, planar, abelian group

1. Introduction

A master's student, writing a thesis on Galois Theory under the direction of the first author, came to him with an example of a degree 12 Galois extension of $\mathbb{Q}$ that had a Galois group isomorphic to D_6. He was typesetting the example for inclusion in his thesis, and he wanted to know whether he could draw the subfield lattice without having to cross lines. The first author referred the student to the second author (a graph theorist), who reported the bad news that, no, the lattice was not planar.

This sparked the interest of the second author: which groups have lattices that *can* be drawn without crossings? We decided to investigate, as we felt that the topic was interesting not only in terms of algebra and graph theory, but also pedagogically.

Recall that a graph G is **planar** if there is an embedding of the graph in the plane so that no edges intersect, except possibly at their ends. A **subdivision** of a graph G is a graph obtained from G by replacing edges of G with paths that are pairwise disjoint, except possibly at their ends. A **minor** of G is a graph obtained from G by contracting edges of G and/or deleting edges and vertices of G. Contraction of the edge uv results in a new vertex that replaces uv, u, and v. This new vertex is adjacent to exactly those vertices that were adjacent to u or to v in G.

Example 1 The graphs K_5 and $K_{3,3}$, are not planar (figure 1); the graphs C_4 and $K_{2,3}$ are planar. ($K_{2,3}$ can be *redrawn* without crossings.)

The following is a well-known characterization of planar graphs due to Kuratowski [4]. A proof appears in [2].

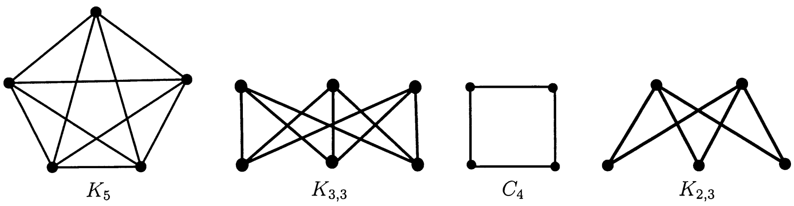

Figure 1. Some common graphs.

Theorem 2 (Kuratowski, [4]) *Let G be a graph. The following are equivalent.*
(1) *G is planar.*
(2) *G does not contain a subdivision of either $K_{3,3}$ or K_5.*
(3) *G contains neither a $K_{3,3}$- nor a K_5-minor.*

Definition 3 Let G be a group. Then the **graph of** G, denoted $\Gamma(G)$, is the (labeled) graph defined as follows:

(1) each vertex corresponds to exactly one subgroup of G
(2) two vertices are joined by an edge if and only if one of the subgroups is a subgroup of the other and there are no intermediate subgroups between them.

The graph of a group is essentially its subgroup lattice, but cast in terms of graph theory.

Example 4 The subgroup lattice of S_3 gives rise to a planar graph (figure 2).

Since we ordinarily draw subgroup lattices so that the orders of subgroups are arranged monotonically from top to bottom, it is natural to ask not only which groups have a planar graph associated with them, but which groups have a subgroup lattice (drawn as a Hasse diagram) that is planar. Furthermore, we generally want the edges of the lattice to be oriented monotonically upward so that the drawing is **upward planar**. These ideas motivate the following definitions.

Definition 5 A **planar group** is a group whose graph is planar, and we will call a group whose graph is non-planar a **non-planar group**. If the subgroup lattice of a planar group

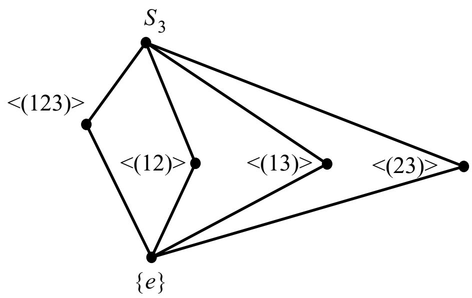

Figure 2. Subgroup lattice of S_3.

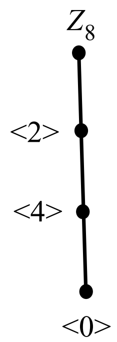

Figure 3. $\Gamma(\mathbb{Z}_8)$.

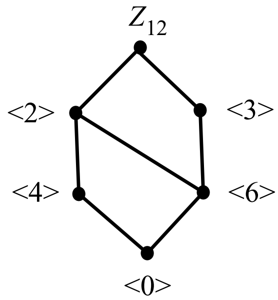

Figure 4. $\Gamma(\mathbb{Z}_{12})$.

is arranged so that it is a Hasse diagram on the orders of the subgroups and this diagram is planar, we will call the group **Hasse-planar**. If the subgroup lattice of a planar group is upward planar, we will say that the group is **upward planar**.

Note that the graph of $K_{2,3}$ in figure 1 (with edges oriented upward) is planar and Hasse-planar, but it is not upward planar.

Example 6 We saw above that S_3 is a planar group. In addition, both $\mathbb{Z}_8$ and $\mathbb{Z}_{12}$ are planar groups; their subgroup lattices are shown in figures 3 and 4. These groups are also Hasse-planar and upward planar.

2. Basic observations

The following theorem contains readily apparent results that nevertheless should be stated.

Theorem 7 *Let G be a group, and let H be a subgroup of G. If G is planar, then H is planar. If G is Hasse-planar, then H is Hasse-planar. If G is upward planar, then H is upward planar. If, in addition, H is normal in G, then $\Gamma(G/H)$ is isomorphic (as a graph)* *to an induced subgraph of* $\Gamma(G)$. *If* G' *is isomorphic to* G, *then* G *is planar if and only if* G' *is planar.*

Proof: Certainly all sublattices of a planar lattice must be planar, and likewise for Hasse-planarity and upward planarity. In G/H, all subgroups have the form K/H, where K is a subgroup of G that contains H; thus $\Gamma(G/H)$ is isomorphic to the subgraph of $\Gamma(G)$ induced by all such K. □

Corollary 8 *If* G/H *is nonplanar*, *then* G *is nonplanar.*

In addition, the following theorem of Platt [5] will be essential in characterizing Hasse-planar groups from our characterization of planar groups.

Theorem 9 (Platt [5]) *A finite lattice* L *is Hasse-planar if and only if the graph obtained from* L *by adjoining an edge between its greatest and least elements is itself planar.*

3. Cyclic groups

In the next section, we make heavy use of the Fundamental Theorem of Finite Abelian Groups to analyze which finite abelian groups are planar. Since cyclic groups are the building blocks of finite abelian groups, we begin with a few results about cyclic groups.

Throughout this section, G is a cyclic group of order $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes, and $k, e_1, e_2, \dots, e_k \in \mathbb{Z}^+$.

Theorem 10 *Let* G *be a cyclic group of order* $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, *where* $p_1, p_2, \dots, p_k$ *are distinct primes*, *and* $k, e_1, e_2, \dots, e_k \in \mathbb{Z}^+$.

Then G *is planar if and only if*

(1) $k \leq 2$ *or*

(2) $k = 3$ *and at most one of* e_1, e_2, e_3 *is greater than* 1.

In addition, G *is Hasse-planar if and only if* $k \leq 2$, *and* G *is upward planar if and only if* $k \leq 2$.

Proof: The proof addresses the following cases.

(1) If $k \leq 2$, then G is planar, Hasse-planar, and upward planar.
(2) If $k = 3$ and $e_i > 1$ for exactly one of $i = 1, 2, 3$, then G is planar.
(3) If $k \geq 3$ and for some $i \neq j$, $e_i \geq 2$ and $e_j \geq 2$, then G is not planar.
(4) If $k \geq 3$, then G is neither Hasse-planar nor upward planar.
(5) If $k \geq 4$, then G is not planar.

If $k = 1$, then $\Gamma(G)$ is simply a path. Assume now that $k = 2$. The lattice for the cyclic group $\mathbb{Z}_{p^e q^f}$ ($p \neq q$, p, q prime) is shown in figure 5, and it is clearly planar, Hasse-planar, and upward planar.

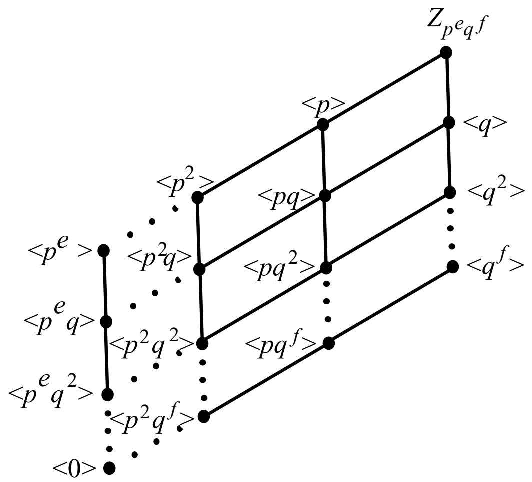

Figure 5. Subgroup lattice for $\mathbb{Z}_{p^e q^f}$.

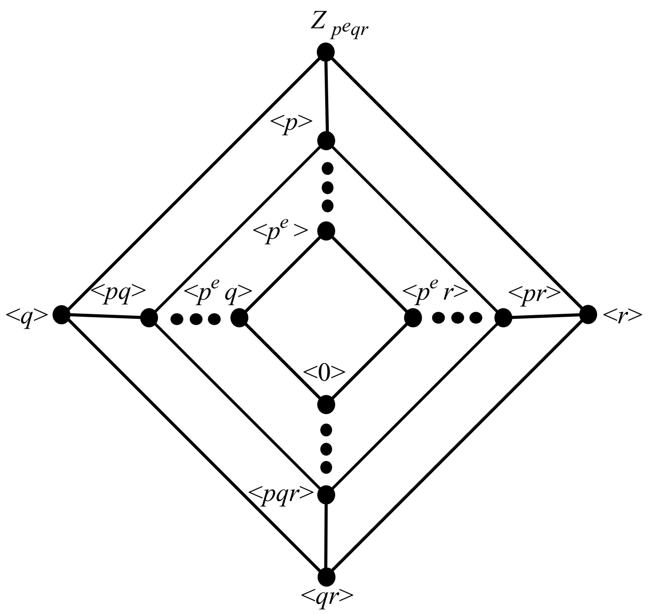

Figure 6. Subgroup lattice of $\mathbb{Z}_{p^e qr}$.

For Case (2), we may assume that G has order $p^e qr$, where p, q, and r are distinct primes and e is an integer greater than 1.

Given the subgroup lattice for $\mathbb{Z}_{p^e qr}$, arranged as in figure 6, we can obtain the lattice for $\mathbb{Z}_{p^{e+1}qr}$ in the following manner. We have four new vertices with labels $\langle p^{e+1}\rangle$, $\langle p^{e+1}q\rangle$,

$\langle p^{e+1}r\rangle$, and $\langle p^{e+1}qr\rangle$, which form a square that we will place inside the innermost square of figure 6. Their neighbors are those vertices with labels that differ by exactly one factor. In the case of $\langle p^{e+1}\rangle$, these are $\langle p^{e}\rangle$, $\langle p^{e+1}q\rangle$, and $\langle p^{e+1}r\rangle$. Thus, we will orient the new square with $\langle p^{e+1}\rangle$ on top, just below p^e, and $\langle p^{e+1}q\rangle$ to the left, next to $\langle p^{e}q\rangle$.

The neighbors of the other new vertices are found similarly. The result is that the subgroup lattice of such a group can be drawn as a nested sequence of squares that are only joined to each other at "closest corners," as in figure 6.

Now consider Case (3). Cyclic groups contain a subgroup of each order dividing the order of the group, and a group with a nonplanar subgroup is nonplanar. Thus, we may focus on a subgroup that all groups of the type specified have in common; in particular, it suffices to show that $\mathbb{Z}_{p^2q^2r}$ is nonplanar for any distinct primes p, q, and r. Figure 7 shows a K_5-subdivision in the subgroup lattice of this group. Therefore, G is nonplanar.

Next, if $k \geq 3$, then G contains a subgroup isomorphic to $\mathbb{Z}_{pqr}$, where p, q, and r are distinct primes. Joining the vertex $\mathbb{Z}_{pqr}$ to the vertex $\langle 0\rangle$ (see figure 8) results in a graph containing a K_5-minor, so $\mathbb{Z}_{pqr}$ is not Hasse-planar by Platt's Theorem. Therefore, by Theorem 7, G is also not Hasse-planar.

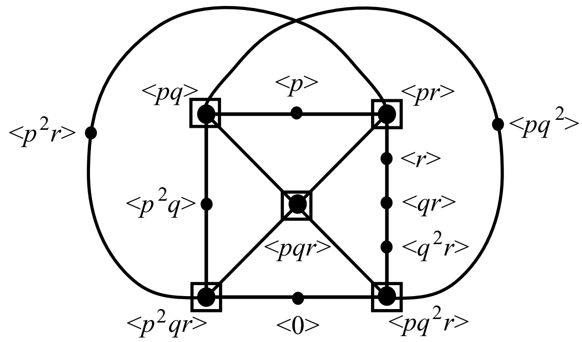

Figure 7. K_5-subdivision in the subgroup lattice for $\mathbb{Z}_{p^2q^2r}$.

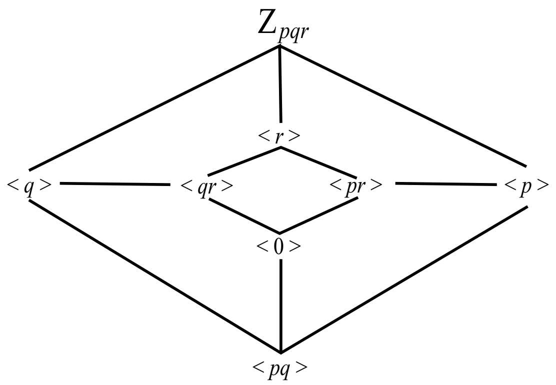

Figure 8. Subgroup lattice of $\mathbb{Z}_{pqr}$.

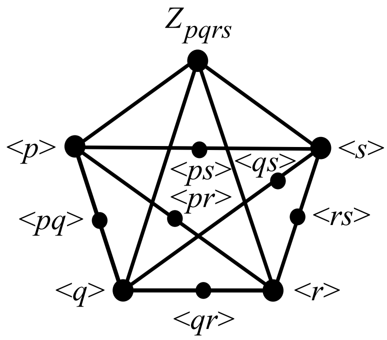

Figure 9. Subgraph of $G(\mathbb{Z}_{pqrs})$.

Suppose now that $k \geq 4$. Such a group must contain a subgroup isomorphic to $H = \mathbb{Z}_{pqrs}$, where p, q, r, and s are distinct primes. We will show that $\Gamma(H)$ contains a K_5-subdivision. This subdivision is illustrated in figure 9; we also offer the following edge descriptions. Certainly H contains the subgroups $\langle 1\rangle$, $\langle p\rangle$, $\langle q\rangle$, $\langle r\rangle$, and $\langle s\rangle$.

(1) The subgroup $\langle 1\rangle$ is joined (directly) to $\langle p\rangle$, $\langle q\rangle$, $\langle r\rangle$, and $\langle s\rangle$ by the definition of $\Gamma(G)$.
(2) For each $x, y \in \{p, q, r, s\}(x \neq y)$, $\langle x\rangle$ is joined to $\langle y\rangle$ through $\langle xy\rangle$; these give distinct edges for the subdivision.

Therefore, H has a K_5-subdivision, so G cannot be planar. ☐

We digress momentarily to generalize part of the preceding Theorem (Case (5) of the proof).

Theorem 11 *Let G be an abelian group. If $|G|$ is divisible by k distinct primes, then $\Gamma(G)$ has a K_{k+1}-subdivision.*

Proof: The construction is the same: each prime generates a subgroup, and if p and q are distinct primes, then $\langle p\rangle$ and $\langle q\rangle$ are joined through $\langle pq\rangle$. Since the paths $\langle p\rangle - \langle pq\rangle - \langle q\rangle$ are all distinct, this accounts for a K_k-subdivision. Additionally, $\langle 1\rangle$ is joined directly to each subgroup generated by a prime. ☐

4. Finite abelian groups

Having characterized the planar cyclic groups, we now consider general finite abelian groups. We begin with a Lemma.

Lemma 12 *If p is prime and a is any nonnegative integer, then $G = \mathbb{Z}_{p^a} \times \mathbb{Z}_p$ is planar.*

Proof: We will argue that the complete subgroup lattice has the form shown in figure 10.

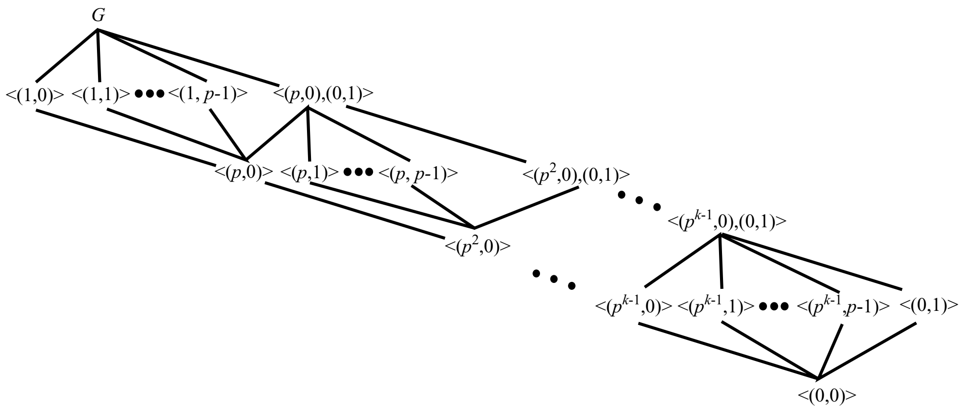

Figure 10. Subgroup lattice of $\mathbb{Z}_{p^a} \times \mathbb{Z}_p$.

First, note that every cyclic subgroup is generated by an element of the form (kp^b, c) for some $b \in \{0, 1, \ldots, a\}$ and $c \in \{0, 1, \ldots, p-1\}$, where $k \in \mathbb{Z}$ is relatively prime to p. However, since $\langle(kp^b, c)\rangle = \langle(p^b, k^{-1}c)\rangle$, we may assume that each cyclic subgroup is generated by an element of the form (p^b, c).

Suppose that $\langle(p^b, c)\rangle = \langle(p^b, d)\rangle$ for some $b \in \{0, 1, \ldots, a-1\}$ and $c, d \in \{0, 1, \ldots, p-1\}$. Then for some $k \in \mathbb{Z}$, $(p^b, c) = k(p^b, d)$. That is, $p^b \equiv kp^b (\text{mod } p^a)$ and $c \equiv kd (\text{mod } p)$. The first congruence implies that $1 \equiv k (\text{mod } p^{a-b})$, so in fact $1 \equiv k (\text{mod } p)$ since $b < a$. Thus $c \equiv d (\text{mod } p)$. Therefore, $\langle(p^b, 0)\rangle, \langle(p^b, 1)\rangle, \ldots, \langle(p^b, p-1)\rangle$ are distinct subgroups when $b < a$. If $b = a$, then the only cyclic subgroups are $\langle(0, 0)\rangle$ and $\langle(0, 1)\rangle$.

In addition, $\langle(p^b, c)\rangle$ and $\langle(p^d, e)\rangle$ are clearly distinct if $b \neq d$ since one will have an element of greater order than any element in the other.

If $\langle(p^b, c)\rangle$ contains (p^d, e) (where $d > b$), then $(p^d, e) = r(p^b, c)$ for some integer r. But this implies that p divides r, so $e \equiv 0 (\text{mod } p)$. Thus, the lattice shown in figure 10 includes all cyclic subgroups of G.

Now suppose that H is a noncyclic subgroup. We claim that H must have the form $\langle(p^b, 0), (0, 1)\rangle$ for some $b \in \{0, 1, \ldots, a\}$.

Certainly, the generators of H have the form (kp^b, c) and (lp^d, e), where, without loss of generality, $b \leq d$ and k and l are relatively prime to p. However,

$$\langle(kp^b, c), (lp^d, e)\rangle = \langle(p^b, k^{-1}c), (p^d, l^{-1}e)\rangle,$$

so we may assume that the generators have the form (p^b, c) and (p^d, e).

An element of the form $r(p^b, c) + s(p^d, e)$ may be rewritten as $(p^b(r + sp^{d-b}), rc + se)$, which is clearly a member of $H' = \langle(p^b, 0), (0, 1)\rangle$. The order of H' is p^{a-b+1}. The order of $\langle(p^b, c)\rangle$ (a subgroup of H) is p^{a-b}. Since H is non-cyclic, it must contain at least one more element. On the other hand, since H is a subgroup of H', its order must divide that of H', and so the orders of H' and H are in fact equal. Thus, $\langle(p^b, c), (p^d, e)\rangle = \langle(p^b, 0), (0, 1)\rangle$, as claimed.

In summary, the cyclic subgroups all have the form $\langle (p^b, c)\rangle$, and distinct values of b or c generate distinct subgroups. The non-cyclic subgroups all have the form $\langle (p^b, 0), (0, 1)\rangle$. These are shown in figure 10. $\square$

Corollary 13 *The group $G = \mathbb{Z}_{p^a} \times \mathbb{Z}_p$ is Hasse-planar and upward planar.*

Proof: By Theorem 9 (Platt's Theorem), to show that G is Hasse planar it suffices to observe that G and $\langle (0, 0)\rangle$ can be joined by an edge that crosses none of the other edges in figure 10. Alternatively, it is not hard to see that figure 10 is already a Hasse diagram. This figure is also upward planar. $\square$

Assume now that G is an abelian group of order $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ with $e_i \geq 1$ for $i = 1, \dots, k$. From the Fundamental Theorem of Finite Abelian Groups, we know that if $k \geq 4$, then G contains a cyclic subgroup isomorphic to $\mathbb{Z}_{p_1 p_2 p_3 p_4}$. Since this is not planar by Theorem 10, G is also not planar. Therefore, we now need only consider $k \leq 3$.

If $k = 3$, we have $n = p^a q^b r^c$, where p, q, and r are distinct primes. If G contains subgroups isomorphic to $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_{q^2}$, then G contains a subgroup isomorphic to $\mathbb{Z}_{p^2 q^2 r}$ and is not planar by Theorem 10. We now consider the remaining possibilities in which G is not cyclic. Since the order of the group can be divisible by at most three distinct primes, and the group requires two or more generators, the following are all the cases we must consider.

(1) If the order is divisible by only one prime p, then G must contain a subgroup isomorphic to one of the following.
  (a) $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$
  (b) $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ for some positive integers a and b

(2) If the order is divisible by two primes p and q, then without loss of generality G must contain a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$ since G is not cyclic.

Theorem 14 *Let G be a finite abelian group. Then G is planar if and only if G is isomorphic to a planar cyclic group or to $\mathbb{Z}_{p^a} \times \mathbb{Z}_p$, where p is prime and a is a positive integer.*

The group G is Hasse-planar if and only if G is isomorphic to a Hasse-planar cyclic group or $\mathbb{Z}_{p^a} \times \mathbb{Z}_p$, and G is upward planar if and only if G is isomorphic to an upward planar cyclic group or $\mathbb{Z}_{p^a} \times \mathbb{Z}_p$.

Proof: Again, the proof is by cases. Let p and q be distinct primes. The following groups are all nonplanar.

(1) $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$
(2) $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_p \times \mathbb{Z}_{pq}$
(3) $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$

The partial subgroup lattices in figures 11–13 show that $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2})$, respectively, contain subgraphs that are subdivisions of $K_{3,3}$.

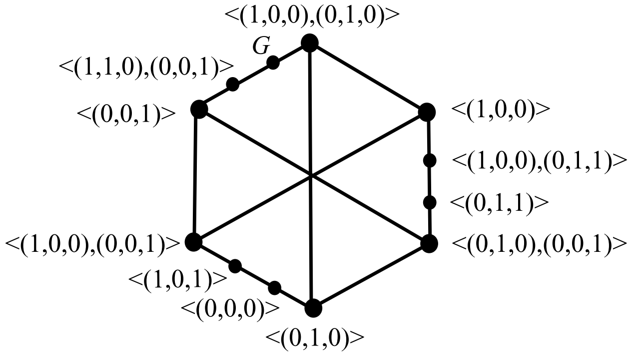

Figure 11. Subgraph of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$.

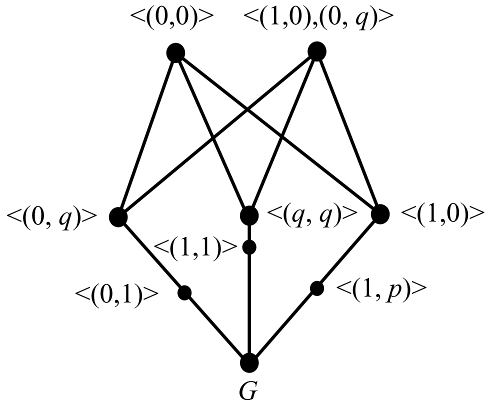

Figure 12. $K_{3,3}$-subdivision in $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq})$.

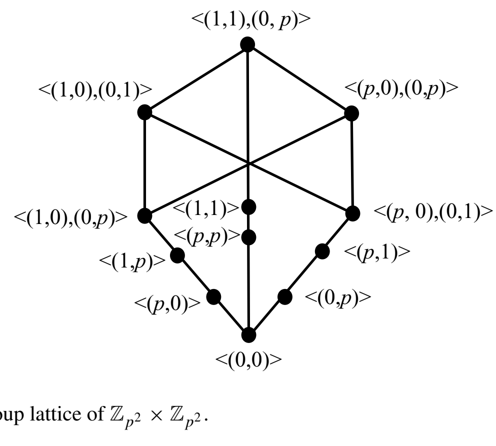

Figure 13. Partial subgroup lattice of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$.

We have shown that $\mathbb{Z}_{p^a} \times \mathbb{Z}_p$ is planar. Now suppose that G is planar. If G is cyclic the theorem is automatically satisfied, so assume also that G is not cyclic.

Suppose that the order of G is $p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ for some distinct primes $p_1, \ldots, p_k$ and positive integers $e_1, \ldots, e_k$. We have seen already that $k \leq 3$.

Suppose that $k = 1$ and $|G| = p^e$, p prime. Since G is noncyclic, G must be isomorphic to a group of the form $\mathbb{Z}_{p^{a_1}} \times \mathbb{Z}_{p^{a_2}} \times \cdots \times \mathbb{Z}_{p^{a_m}}$, where $a_1 + \cdots + a_m = e$. If $m \geq 3$, then G contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and is therefore nonplanar by Case (1). Thus, $m \leq 2$. If both a_1 and a_2 are greater than 1, then G contains a subgroup isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and is again nonplanar by Case (3). Therefore, at most one of a_1 and a_2 is greater than 1, and the theorem holds.

Now suppose that $k = 2$ and $|G| = p^e q^f$ for some primes p and q and positive integers e and f. Since G is noncyclic, it must contain a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$ or $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$; in either case, G is nonplanar by Case (2).

Finally, assume $k = 3$ and $|G| = p^e q^f r^g$ for some primes p, q, and r and positive integers e, f, and g. Since G is noncyclic, it contains (without loss of generality) a subgroup of the form $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, which in turn contains a subgroup of the form $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$. Again, G is nonplanar by Case (2). $\square$

5. Infinite abelian groups

For completeness, we characterize those infinite abelian groups that are planar. First, recall that a *primary group* is a group all of whose elements have orders that are a power of a fixed prime p.

Let P be the additive group of rational numbers whose denominators are all powers of a fixed prime p. The quotient group $P/\mathbb{Z}$ is denoted by $\mathbb{Z}_{p^\infty}$. The lattice of subgroups of $\mathbb{Z}_{p^\infty}$ is shown in figure 14.

This lattice is very much like that of $\mathbb{Z}_{p^n}$ except that it is infinite. Notice that every proper subgroup of $\mathbb{Z}_{p^\infty}$ is finite, and $\langle 1/p^n \rangle \cong \mathbb{Z}_{p^n}$.

Theorem 15 *An infinite abelian group G is planar if and only if it is isomorphic to one of $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_p$, $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_{q^b}$, $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_{q^\infty}$, or $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p, q, and r are distinct primes and b is a positive integer.*

Proof: It is easy to see that $\mathbb{Z}$ is not a planar group. Thus, no planar infinite abelian group can contain an element of infinite order, so such a group must be a torsion group. By Theorem 1 of [3, p. 5], a torsion group is a direct sum of primary groups. If G contains elements of distinct prime orders p, q, r, and s, then G will contain a subgroup isomorphic to $\mathbb{Z}_{pqrs}$, which is nonplanar by Theorem 5. Therefore, we can assume that we have one of the following cases.

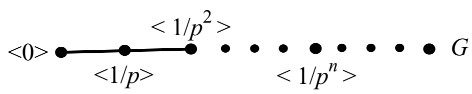

Figure 14. $\Gamma(\mathbb{Z}_{p^\infty})$.

(1) There is a prime p such that every element of G has order a power of p.
(2) There are primes p and q such that every element of G has order $p^a q^b$ for some nonnegative integers a and b.
(3) There are primes p, q, and r such that every element of G has order $p^a q^b r^c$ for some nonnegative integers a, b and c.

We consider each case in turn.

(1) Suppose that G is indecomposable; that is, G cannot be written as a nontrivial direct sum. Then by Theorem 10 of [3, p. 22], since G is an infinite torsion group, it is isomorphic to $\mathbb{Z}_{p^\infty}$. If G is decomposable, then G can be written as $H_1 \times H_2 \times \cdots \times H_k$ for some integer k and subgroups $H_1, \ldots, H_k$.

   Each H_i must contain an element of order p, so if $k \geq 3$, then G contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, which is nonplanar by Theorem 14. Therefore, we may assume that G is isomorphic to $H_1 \times H_2$ for some *indecomposable* subgroups H_1 and H_2. Since one of them, say H_1, must be an infinite indecomposable torsion group, by Theorem 10 of [3] it is isomorphic to $\mathbb{Z}_{p^\infty}$.

   If H_2 has order p^2 or greater, then G contains a subgroup isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$, which is nonplanar by Theorem 14. Therefore, H_2 has order p, and G is isomorphic to $Z_{p^\infty} \times \mathbb{Z}_p$. The lattice for this group is nearly identical to that in figure 10 except that it is infinite.
(2) We may write G as the direct product $G_p \times G_q$, where G_p is a primary group of order p^a and G_q is a primary group of order q^b, since p and q are the only primes that divide the orders of elements of G. If G_p is decomposable, then G contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$, which is nonplanar by Theorem 14. Similarly, G_q must be indecomposable. Again using Theorem 10 of [3], we see that G must be one of $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_{q^\infty}$, $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_{q^b}$, or $\mathbb{Z}_{p^a} \times \mathbb{Z}_{q^\infty}$, where a and b are nonnegative integers. Each of these lattices is similar to the lattice in figure 5, the only difference being that the lattice has infinitely many vertices.
(3) We may write G as $G_p \times G_q \times G_r$, where G_p, G_q, and G_r are primary groups. As above, we may assume that G_p is infinite and indecomposable and that G_q and G_r are indecomposable, so that $G_p \cong \mathbb{Z}_{p^\infty}$. If G_q has order greater than q, then G contains a cyclic subgroup isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$, which is nonplanar by Theorem 10. Therefore, $G \cong \mathbb{Z}_{p^\infty} \times \mathbb{Z}_q \times \mathbb{Z}_r$. The lattice for G is nearly identical to that in figure 6 except that it extends inward infinitely.

□

Corollary 16 *An infinite abelian group G is Hasse-planar if and only if G is isomorphic to one of $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_p$, $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_{q^b}$, or $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_{q^\infty}$. G is upward planar if and only if G is isomorphic to one of these groups as well.*

Proof: By Platt's Theorem and an examination of figures 5, 6, and 10, we see that these and only these are still planar if G is joined to $\langle 0 \rangle$. Since an upward planar group must be Hasse-planar as well, only these groups are candidates for upward planarity. It is clear from the figures that these are indeed upward planar. □

6. Conclusion

There is still much work to be done here. A characterization of nonabelian planar groups is proving substantially more difficult; the authors of this paper have partial results, but are still nowhere near a complete characterization.

Acknowledgments

The authors are indebted to the reviewers and Dr. David B. Leep for their excellent suggestions.

References

1. A. Barlow, "Galois theory and applications," Master's Thesis at Stephen F. Austin State University, 2000.
2. R. Diestel, *Graph Theory*, 2nd edition, Springer Graduate Texts in Mathematics, 2000.
3. I. Kaplansky, *Infinite Abelian Groups*, revised edition, University of Michigan Press, 1969.
4. K. Kuratowski, "Sur le problème des courbes gauches en topologie," *Fund. Math.* **15** (1930), 271–283.
5. C.R. Platt, "Planar lattices and planar graphs," *J. Comb. Theory Series B.* **21** (1976), 30–39.