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Schensted-Type Correspondences and Plactic Monoids for Types B_n and D_n

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Abstract. We use Kashiwara's theory of crystal bases to study plactic monoids for $U_q(so_{2n+1})$ and $U_q(so_{2n})$. Simultaneously we describe a Schensted type correspondence in the crystal graphs of tensor powers of vector and spin representations and we derive a Jeu de Taquin for type B from the Sheats sliding algorithm.

Keywords: combinatorics, quantum algebra, representation theory

1. Introduction

The Schensted correspondence based on the bumping algorithm yields a bijection between words w of length l on the ordered alphabet $\mathcal{A}_n = \{1 \prec 2 \prec \cdots \prec n\}$ and pairs $(P^A(w), Q^A(w))$ of tableaux of the same shape containing l boxes where $P^A(w)$ is a semi-standard Young tableau on $\mathcal{A}_n$ and $Q^A(w)$ is a standard tableau. By identifying the words w having the same tableau $P^A(w)$, we obtain the plactic monoid $Pl(A_n)$ whose defining relations were determined by Knuth:

$$\begin{aligned} yzx &= yxz \quad \text{and} \quad xzy = zxy \quad \text{if } x \prec y \prec z,\\ xyx &= xxy \quad \text{and} \quad xyy = yxy \quad \text{if } x \prec y. \end{aligned}$$

The Robinson-Schensted correspondence has a natural interpretation in terms of Kashiwara's theory of crystal bases [2, 5, 8]. Let V_n^A denote the vector representation of $U_q(sl_n)$. By considering each vertex of the crystal graph of $\bigoplus_{l\geq 0}(V_n^A)^{\otimes l}$ as a word on $\mathcal{A}_n$, we have for any words w_1 and w_2:

- $P^A(w_1) = P^A(w_2)$ if and only if w_1 and w_2 occur at the same place in two isomorphic connected components of this graph.
- $Q^A(w_1) = Q^A(w_2)$ if and only if w_1 and w_2 occur in the same connected component of this graph.

Replacing V_n^A by the vector representation V_n^C of sp_{2n} whose basis vectors are labelled by the letters of the totally ordered alphabet

$$\mathcal{C}_n = \{1 \prec \cdots \prec n-1 \prec n \prec \bar{n} \prec \overline{n-1} \prec \cdots \prec \bar{1}\},$$

we have obtained in [10] a Schensted type correspondence for type C_n. This correspondence is based on an insertion algorithm for the Kashiwara-Nakashima's symplectic tableaux [4] analogous to the bumping algorithm. It may be regarded as a bijection between words w of length l on $\mathcal{C}_n$ and pairs $(P^C(w), Q^C(w))$ where $P^C(w)$ is a symplectic tableau and $Q^C(w)$ an oscillating tableau of type C and length l, that is, a sequence $(Q_1, \ldots, Q_l)$ of Young diagrams such that two consecutive diagrams differ by exactly one box. Moreover by identifying the words of the free monoid $\mathcal{C}_n^*$ having the same symplectic tableau we also obtain a monoid $Pl(C_n)$. This is the plactic monoid of type C_n in the sense of [12] and [8].

The vector representations V_n^B and V_n^D of $U_q(so_{2n+1})$ and $U_q(so_{2n})$ have crystal graphs whose vertices may be respectively labelled by the letters of

$$\mathcal{B}_n = \{1 \prec \cdots \prec n-1 \prec n \prec 0 \prec \bar{n} \prec \overline{n-1} \prec \cdots \prec \bar{1}\}$$

and

$$\mathcal{D}_n = \left\{1 \prec \cdots \prec n-1 \prec \begin{matrix} n \\ \bar{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \bar{1}\right\}.$$

Let G_n^B and G_n^D be the crystal graphs of $\bigoplus_{l\geq 0}(V_n^B)^{\otimes l}$ and $\bigoplus_{l\geq 0}(V_n^D)^{\otimes l}$. Then it is possible to label the vertices of G_n^B and G_n^D by the words of the free monoids $\mathcal{B}_n^*$ and $\mathcal{D}_n^*$. However the situation is more complicated than in the case of types A and C. Indeed there exist a fundamental representation of $U_q(so_{2n+1})$ and two fundamental representations of $U_q(so_{2n})$ that do not appear in the decompositions of $\bigoplus(V_n^B)^{\otimes l}$ and $\bigoplus_{l\geq 0}(V_n^D)^{\otimes l}$ into their irreducible components. They are called the spin representations and denoted respectively by $V(\Lambda_n^B)$, $V(\Lambda_n^D)$ and $V(\Lambda_{n-1}^D)$. In [4], Kashiwara and Nakashima have described their crystal graphs by using a new combinatorical object that we will call a spin column. Write SP_n for the set of spin columns of height n and set $\mathfrak{B}_n = \mathcal{B}_n \cup SP_n$, $\mathfrak{D}_n = \mathcal{D}_n \cup SP_n$. Then each vertex of the crystal graphs $\mathfrak{G}_n^B$ and $\mathfrak{G}_n^D$ of $\bigoplus_{l\geq 0}(V_n^B \oplus V(\Lambda_n^B))^{\otimes l}$ and $\bigoplus_{l\geq 0}(V_n^D \oplus V(\Lambda_n^D) \oplus V(\Lambda_{n-1}^D))^{\otimes l}$ may be respectively identified with a word on $\mathfrak{B}_n$ or $\mathfrak{D}_n$. We can define two relations $\overset{B}{\sim}$ and $\overset{D}{\sim}$ by:

$w_1 \overset{B}{\sim} w_2$ if and only if w_1 and w_2 occur at the same place in two isomorphic connected components of $\mathfrak{G}_n^B$,

$w_1 \overset{D}{\sim} w_2$ if and only if w_1 and w_2 occur at the same place in two isomorphic connected components of $\mathfrak{G}_n^D$.

In this article, we prove that $Pl(B_n) = \mathcal{B}_n^*/\overset{B}{\sim}$, $Pl(D_n) = \mathcal{D}_n^*/\overset{D}{\sim}$, $\mathfrak{Pl}(B_n) = \mathfrak{B}_n^*/\overset{B}{\sim}$ and $\mathfrak{Pl}(D_n) = \mathfrak{D}_n^*/\overset{D}{\sim}$ are monoids and we undertake a detailed investigation of the corresponding insertion algorithms. We summarize in part 2 the background on Kashiwara's theory of crystals used in the sequel. In part 3, we first recall Kashiwara-Nakashima's notion of orthogonal tableau (analogous to Young tableaux for types B and D) and we relate it to Littelmann's notion of Young tableau for classical types. Then we derive a set of defining relations for $Pl(B_n)$ and $Pl(D_n)$ and we describe the corresponding column insertion algorithms. Using the combinatorial notion of oscillating tableaux (analogous to standard tableaux for types B and D), these algorithms yield the desired Schensted type correspondences in G_n^B and G_n^D. In part 4 we propose an orthogonal Jeu de Taquin for type B based on Sheats' sliding algorithm for type C [16]. Finally in part 5, we bring into the picture the spin representations and extend the results of part 3 to the graphs $\mathfrak{G}_n^B$, $\mathfrak{G}_n^D$ and the monoids $\mathfrak{Pl}(B_n)$, $\mathfrak{Pl}(D_n)$. Note that bounds for the length of the plactic relations are given in [12].

Notation 1.0.1 In the sequel, we often write B and D instead of B_n and D_n to simplify the notation. Moreover, we frequently define similar objects for types B and D. When they are related to type B (respectively D), we attach to them the label ^{B} (respectively the label ^{D}). To avoid cumbersome repetitions, we sometimes omit the labels ^{B} and ^{D} when our statements are true for the two types.

2. Conventions for crystal graphs

2.1. Kashiwara's operators

Let $\mathfrak{g}$ be simple Lie algebra and α_i, $i \in I$ its simple roots. Recall that the crystal graphs of the $U_q(\mathfrak{g})$-modules are oriented colored graphs with colors $i \in I$. An arrow $a \overset{i}{\to} b$ means that $\tilde{f}_i(a) = b$ and $\tilde{e}_i(b) = a$ where $\tilde{e}_i$ and $\tilde{f}_i$ are the crystal graph operators (for a review of crystal bases and crystal graphs see [5]). Let V, V' be two $U_q(\mathfrak{g})$-modules and B, B' their crystal graphs. A vertex $v^0 \in B$ satisfying $\tilde{e}_i(v^0) = 0$ for any $i \in I$ is called a highest weight vertex. The decomposition of V into its irreducible components is reflected into the decomposition of B into its connected components. Each connected component of B contains a unique vertex of highest weight. We write $B(v^0)$ for the connected component containing the highest weight vertex v^0. The crystals graphs of two isomorphic irreducible components are isomorphic as oriented colored graphs. We will say that two vertices b_1 and b_2 of B occur at the same place in two isomorphic connected components Γ_1 and Γ_2 of B if there exist $i_1, \dots, i_r \in I$ such that $w_1 = \tilde{f}_{i_i} \cdots \tilde{f}_{i_r}(w_1^0)$ and $w_2 = \tilde{f}_{i_i} \cdots \tilde{f}_{i_r}(w_2^0)$, where w_1^0 and w_2^0 are respectively the highest weight vertices of Γ_1 and Γ_2.

The action of $\tilde{e}_i$ and $\tilde{f}_i$ on $B \otimes B' = \{b \otimes b'; b \in B, b' \in B'\}$ is given by:

$$\tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i(u) \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v) \\ u \otimes \tilde{f}_i(v) & \text{if } \varphi_i(u) \le \varepsilon_i(v) \end{cases} \tag{1}$$

and

$$\tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i(v) & \text{if } \varphi_i(u) < \varepsilon_i(v) \\ \tilde{e}_i(u) \otimes v & \text{if } \varphi_i(u) \ge \varepsilon_i(v) \end{cases} \tag{2}$$

where $\varepsilon_i(u) = \max\{k; \tilde{e}_i^k(u) \neq 0\}$ and $\varphi_i(u) = \max\{k; \tilde{f}_i^k(u) \neq 0\}$. Denote by Λ_i, $i \in I$ the fundamental weights of $\mathfrak{g}$. The weight of the vertex u is defined by $\mathrm{wt}(u) = \sum_I (\varphi_i(u) -$

$\varepsilon_i(u))\Lambda_i$. Write $s_i = s_{\alpha_i}$ for $i \in I$. The Weyl group W of $\mathfrak{g}$ acts on B by:

$$\begin{aligned} s_i(u) &= (\tilde{f}_i)^{\varphi_i(u)-\varepsilon_i(u)}(u) \quad \text{if } \varphi_i(u) - \varepsilon_i(u) \geq 0, \\ s_i(u) &= (\tilde{e}_i)^{\varepsilon_i(u)-\varphi_i(u)}(u) \quad \text{if } \varphi_i(u) - \varepsilon_i(u) < 0. \end{aligned} \tag{3}$$

We have the equality $\mathrm{wt}(\sigma(u)) = \sigma(\mathrm{wt}(u)$ for any $\sigma \in W$ and $u \in B$. The following lemma is a straightforward consequence of (1) and (2).

Lemma 2.1.1 *Let $u \otimes v \in B \otimes B'$. Then*:

(i) $\varphi_i(u \otimes v) = \begin{cases} \varphi_i(v) + \varphi_i(u) - \varepsilon_i(v) & \text{if } \varphi_i(u) > \varepsilon_i(v) \\ \varphi_i(v) & \text{otherwise}. \end{cases}$.

(ii) $\varepsilon_i(u \otimes v) = \begin{cases} \varepsilon_i(v) + \varepsilon_i(u) - \varphi_i(u) & \text{if } \varepsilon_i(v) > \varphi_i(u) \\ \varepsilon_i(u) & \text{otherwise}. \end{cases}$.

(iii) *$u \otimes v$ is a highest weight vertex of $B \otimes B'$ if and only if for any $i \in I$ $\tilde{e}_i(u) = 0$ (i.e. u is of highest weight) and $\varepsilon_i(v) \leq \varphi_i(u)$.*

For any dominant weight $\lambda \in P_+$, write $B(\lambda)$ for the crystal graph of $V(\lambda)$, the irreducible module of highest weight λ and denote by u_λ its highest weight vertex. Kashiwara has introduced in [6] an embedding of $B(\lambda)$ into $B(m\lambda)$ for any positive integer m. He uses this embedding to obtain a simple bijection between Littlemann's path crystal associated to λ and $B(\lambda)$ [14].

Theorem 2.1.2 (Kashiwara) *There exists a unique injective map*

$$\begin{aligned} S_m : B(\lambda) &\to B(m\lambda) \subset B(\lambda)^{\otimes m} \\ u_\lambda &\mapsto u_\lambda^{\otimes m} \end{aligned}$$

such that for any $b \in B(\lambda)$:

$$\begin{aligned} &\text{(i)} && S_m(\tilde{e}_i(b)) = \tilde{e}_i^m(S_m(b)), \\ &\text{(ii)} && S_m(\tilde{f}_i(b)) = \tilde{f}_i^m(S_m(b)), \\ &\text{(iii)} && \varphi_i(S_m(b)) = m\varphi_i(b), \\ &\text{(iv)} && \varepsilon_i(S_m(b)) = m\varepsilon_i(b), \\ &\text{(v)} && \mathrm{wt}(S_m(b)) = m\,\mathrm{wt}(b). \end{aligned} \tag{4}$$

Corollary 2.1.3 *Let $\lambda_1, \ldots, \lambda_k \in P_+$. Then, the map*:

$$\begin{aligned} S_m : B(\lambda_1) \otimes \cdots \otimes B(\lambda_k) &\to B(m\lambda_1) \otimes \cdots \otimes B(m\lambda_k) \\ b_1 \otimes \cdots \otimes b_k &\mapsto S_m(b_1) \otimes \cdots \otimes S_m(b_k) \end{aligned}$$

is injective and satisfies the relations (4) *with $b = b_1 \otimes \cdots \otimes b_k$. Moreover the image by S_m of a highest weight vertex of $B(\lambda_1) \otimes \cdots \otimes B(\lambda_k)$ is a highest weight vertex of $B(m\lambda_1) \otimes \cdots \otimes B(m\lambda_k)$.*

Proof: By induction, we can suppose $k = 2$. S_m is injective because S_m is injective. Let $u \otimes v \in B(\lambda_1) \otimes B(\lambda_2)$. Suppose that $\varphi_i(u) \leq \varepsilon_i(v)$. We derive the following equalities from Formulas (1) and (2):

$$\mathrm{S}_m \tilde{f}_i(u \otimes v) = \mathrm{S}_m(u \otimes \tilde{f}_i v) = S_m(u) \otimes S_m(\tilde{f}_i v) = S_m(u) \otimes \tilde{f}_i^m S_m(v) \quad \text{and}$$
$$\tilde{f}_i^m(\mathrm{S}_m(u \otimes v)) = \tilde{f}_i^m(S_m(u) \otimes S_m(v)) = S_m(u) \otimes \tilde{f}_i^m S_m(v).$$

Indeed, $\varepsilon_i(S_m(v)) = m\varepsilon_i(v) \geq m\varphi_i(u) = \varphi_i(S_m(u))$ and for $p = 1, \ldots, m$ $\varepsilon_i(\tilde{f}_i^p S_m(v)) > \varepsilon_i(S_m(v))$. Hence $\mathrm{S}_m \tilde{f}_i(u \otimes v) = \tilde{f}_i^m(\mathrm{S}_m(u \otimes v))$. Now suppose $\varepsilon_i(v) < \varphi_i(u)$ i.e. $\varepsilon_i(u) \leq \varphi_i(v) + 1$. We obtain:

$$\mathrm{S}_m \tilde{f}_i(u \otimes v) = \mathrm{S}_m(\tilde{f}_i u \otimes v) = S_m(\tilde{f}_i u) \otimes S_m(v) = \tilde{f}_i^m S_m(u) \otimes S_m(v) \quad \text{and}$$
$$\tilde{f}_i^m(\mathrm{S}_m(u \otimes v)) = \tilde{f}_i^m(S_m(u) \otimes S_m(v)) = \tilde{f}_i^m S_m(u) \otimes S_m(v)$$

because $\varepsilon_i(S_m(v)) = m\varepsilon_i(v) \leq m\varphi_i(u) + m = \varphi_i(S_m u) + m$. Hence we have $\mathrm{S}_m \tilde{f}_i(u \otimes v) = \tilde{f}_i^m(\mathrm{S}_m(u \otimes v))$.

Similarly we prove that $\mathrm{S}_m \tilde{e}_i(u \otimes v = \tilde{e}_i^m(\mathrm{S}_m(u \otimes v))$. So S_m satisfies the formulas (i) and (ii). By Lemma 2.1.1(i) and (ii) we obtain then that S_m satisfies (iii), (iv) and (v).

Suppose that $u \otimes v$ is a highest weight vertex of $B(\lambda_1) \otimes B(\lambda_2)$. By Lemma 2.1.1(iii), u is the highest weight vertex of $B(\lambda_1)$ and $\varepsilon_i(v) \leq \varphi_i(u)$ for $i \in I$. Then by definition of S_m, $S_m(u)$ is the highest weight vertex of $B(m\lambda_1)$. Moreover for any $i \in I$, $\varepsilon_i(S_m(v)) = m\varepsilon_i(v) \leq m\varphi_i(u) = \varphi_i(S_m(u))$. So $S_m(u) \otimes S_m(v) = \mathrm{S}_m(u \otimes v)$ is of highest weight in $B(m\lambda_1) \otimes B(m\lambda_2)$. □

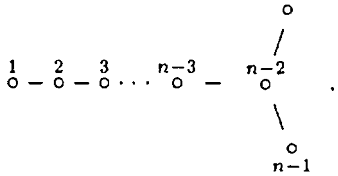
By this corollary, the connected component of $B(\lambda_1) \otimes \cdots \otimes B(\lambda_k)$ of highest weight vertex $u^0 = u_1 \otimes \cdots \otimes u_k$, may be identified with the sub-graph of $B(m\lambda_1) \otimes \cdots \otimes B(m\lambda_k)$ generated by the vertex $S_m(u_1) \otimes \cdots \otimes S_m(u_k)$ and the operators $\tilde{f}_i^m$ for $i \in I$.

2.2. *Tensor powers of the vector representations*

We choose to label the Dynkin diagram of so_{2n+1} by:

$$\overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} \cdots \overset{n-2}{\circ} - \overset{n-1}{\circ} \Rightarrow \overset{n}{\circ}$$

and the Dynkin diagram of so_{2n} by:



Write W_n^B and W_n^D for the Weyl groups of so_{2n+1} and so_{2n}. Denote by V_n^B and V_n^D the vector representations of $U_q(so_{2n+1})$ and $U_q(so_{2n})$. Their crystal graphs are respectively:

$$1 \xrightarrow{1} 2 \cdots \to n-1 \xrightarrow{n-1} n \xrightarrow{n} 0 \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \to \bar{2} \xrightarrow{1} \bar{1} \tag{5}$$

and

$$\begin{array}{ccccccc} & & & \bar{n} & & & \\ & & {\scriptstyle n}\nearrow & & \searrow{\scriptstyle n-1} & & \\ 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-3} n-2 \xrightarrow{n-2} & n-1 & & & & \overline{n-1} & \xrightarrow{n-2} \overline{n-2} \xrightarrow{n-3} \cdots \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}. \\ & & {\scriptstyle n-1}\searrow & & \nearrow{\scriptstyle n} & & \\ & & & n & & & \end{array} \tag{6}$$

By induction, formulas (1), (2) allow to define a crystal graph for the representations $(V_n^B)^{\otimes l}$ and $(V_n^D)^{\otimes l}$ for any l. Each vertex $u_1 \otimes u_2 \otimes \cdots \otimes u_l$ of the crystal graph of $(V_n^B)^{\otimes l}$ will be identified with the word $u_1 u_2 \cdots u_l$ on the totally ordered alphabet

$$\mathcal{B}_n = \{1 \prec \cdots \prec n-1 \prec n \prec 0 \prec \bar{n} \prec \overline{n-1} \prec \cdots \prec \bar{1}\}.$$

Similarly each vertex $v_1 \otimes v_2 \otimes \cdots \otimes v_l$ of the crystal graph of $(V_n^D)^{\otimes l}$ will be identified with the word $v_1 v_2 \cdots v_l$ on the partially ordered alphabet

$$\mathcal{D}_n = \left\{ 1 \prec \cdots \prec n-1 \prec \begin{matrix} n \\ \\ \bar{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \bar{1} \right\}.$$

By convention we set $\bar{0} = 0$ and for $k = 1, \ldots, n$, $\overline{\overline{k}} = k$. The letter x is barred if $x \succeq \bar{n}$ unbarred if $x \preceq n$ and we set:

$$|x| = \begin{cases} x & \text{if } x \text{ is unbarred} \\ \bar{x} & \text{otherwise.} \end{cases}$$

Write $\mathcal{B}_n^*$ and $\mathcal{D}_n^*$ for the free monoids on $\mathcal{B}_n$ and $\mathcal{D}_n$. If w is a word of $\mathcal{B}_n^*$ or $\mathcal{D}_n^*$, we denote by $\mathrm{l}(w)$ its length and by $d(w) = (d_1, \ldots, d_n)$ the n-tuple where d_i is the number of letters i in w minus the number of letters $\bar{i}$. Let G_n^B and $G_{n,l}^B$ be respectively the crystal graphs of $\bigoplus_l (V_n^B)^{\otimes l}$ and $(V_n^B)^{\otimes l}$. Then the vertices of G_n^B are indexed by the words of $\mathcal{B}_n^*$ and those of $G_{n,l}^B$ by the words of $\mathcal{B}_n^*$ of length l. Similarly G_n^D and $G_{n,l}^D$, the crystal graphs of $\bigoplus_l (V_n^B)^{\otimes l}$ and $(V_n^B)^{\otimes l}$ are indexed respectively by the words of $\mathcal{D}_n^*$ and by the words of $\mathcal{D}_n^*$ of length l. If w is a vertex of G_n, write $B(w)$ for the connected component of G_n containing w.

Denote by $\Lambda_1^B, \ldots, \Lambda_n^B$ and $\Lambda_1^D, \ldots, \Lambda_n^D$ the fundamental weights of $U_q(so_{2n+1})$ and $U_q(so_{2n})$. Let P_+^B and P_+^D be the sets of dominant weights of their weight lattices. We set

$$\omega_n^B = 2\Lambda_n^B,$$
$$\omega_i^B = \Lambda_i^B \quad \text{for } i = 1, \ldots, n-1$$

and

$$\omega_n^D = 2\Lambda_n^D,$$
$$\bar{\omega}_n^D = 2\Lambda_{n-1}^D,$$
$$\omega_{n-1}^D = \Lambda_n^D + \Lambda_{n-1}^D,$$
$$\omega_i^D = \Lambda_i^D \quad \text{for } i = 1, \ldots, n-2.$$

Then the weight of a vertex w of G_n is given by:

$$\mathrm{wt}(w) = d_n\omega_n + \sum_{i=1}^{n-1}(d_i - d_{i+1})\omega_i.$$

Thus we recover the well-known fact that there is no connected component of G_n^B isomorphic to $B(\Lambda_n^B)$ and no connected component of G_n^D isomorphic to $B(\Lambda_n^D)$ or $B(\Lambda_{n-1}^D)$. Recall that in the cases of the types A and C, every crystal graph of an irreducible module may be embedded in the crystal graph of a tensor power of the vector representation. For $\lambda \in P_+^B$, $B^B(\lambda)$ may be embedded in a tensor power of the vector representation V_n^B if and only if λ lies in the weight sub-lattice Ω^B generated by the ω_i^B's. Similarly, for $\lambda \in P_+^D$, $B^D(\lambda)$ may be embedded in a tensor power of the vector representation V_n^D if and only if λ lies in the weight sub-lattice Ω^D generated by the ω_i^D's. Set $\Omega_+^B = P_+^B \cap \Omega^B$ and $\Omega_+^D = P_+^D \cap \Omega^D$.

Now we introduce the coplactic relation. For w_1 and $w_2 \in \mathcal{B}_n^*$ (resp. $\mathcal{D}_n^*$), write $w_1 \overset{B}{\leftrightarrow} w_2$ (resp. $w_1 \overset{D}{\leftrightarrow} w_2$) if and only if w_1 and w_2 belong to the same connected component of G_n^B (resp. G_n^D). The proof of the following lemma is the same as in the symplectic case [10].

Lemma 2.2.1 *If $w_1 = u_1v_1$ and $w_2 = u_2v_2$ with $l(u_2)$ and $l(v_1) = l(v_2)$*

$$w_1 \leftrightarrow w_2 \Rightarrow \begin{cases} u_1 \leftrightarrow u_2 \\ v_1 \leftrightarrow v_2 \end{cases}.$$

2.3. *Crystal graphs of the spin representations*

The spin representations of $U_q(so_{2n+1})$ and $U_q(so_{2n})$ are $V(\Lambda_n^B)$, $V(\Lambda_n^D)$ and $V(\Lambda_{n-1}^D)$. Recall that dim $V(\Lambda_n^B) = 2^n$ and dim $V(\Lambda_n^D) =$ dim $V(\Lambda_{n-1}^D) = 2^{n-1}$. Now we review the description of $B(\Lambda_n^B)$, $B(\Lambda_n^D)$ and $B(\Lambda_{n-1}^D)$ given by Kashiwara and Nakashima in [4]. It is based on the notion of spin column. To avoid confusion between these new columns and the classical columns of a tableau that we introduce in the next section, we follow Kashiwara-Nakashima's convention and represent spin columns by column shape diagrams of width $1/2$. Such diagrams will be called K-N diagrams.

Definition 2.3.1 A spin column $\mathfrak{C}$ of height n is a K-N diagram containing n letters of $\mathcal{D}_n$ such that the word $x_1 \cdots x_n$ obtained by reading $\mathfrak{C}$ from top to bottom does not contain a pair $(z, \bar{z})$ and verifies $x_1 \prec \cdots \prec x_n$. The set of spin columns of length n will be denoted SP_n.

- $B(\Lambda_n^B) = \{\mathfrak{C}; \mathfrak{C} \in SP_n\}$ where Kashiwara's operators act as follows:

  if $n \in \mathfrak{C}$ then $\tilde{f}_n\mathfrak{C}$ is obtained by turning n into $\bar{n}$, otherwise $\tilde{f}_n\mathfrak{C} = 0$,
  if $\bar{n} \in \mathfrak{C}$ then $\tilde{e}_n\mathfrak{C}$ is obtained by turning $\bar{n}$ into n, otherwise $\tilde{e}_n\mathfrak{C} = 0$,
  if $(i, \overline{i+1}) \in \mathfrak{C}$ then $\tilde{f}_i\mathfrak{C}$ is obtained by turning $(i, \overline{i+1})$ into $(i+1, \bar{i})$, otherwise $\tilde{f}_i\mathfrak{C} = 0$,
  if $(i+1, \bar{i}) \in \mathfrak{C}$ then $\tilde{e}_i\mathfrak{C}$ is obtained by turning $(i+1, \bar{i})$ into $(i, \overline{i+1})$, otherwise $\tilde{e}_i\mathfrak{C} = 0$.

- $B(\Lambda_n^D) = \{\mathfrak{C} \in SP_n$; the number of barred letters in $\mathfrak{C}$ is even$\}$ and $B(\Lambda_{n-1}^D) = \{\mathfrak{C} \in SP_n$; the number of barred letters in $\mathfrak{C}$ is odd$\}$ where Kashiwara's operators act as follows:

  if $(n-1, n) \in \mathfrak{C}$ then $\tilde{f}_n\mathfrak{C}$ is obtained by turning $(n-1, n)$ into $(\bar{n}, \overline{n-1})$, otherwise $\tilde{f}_n\mathfrak{C} = 0$,
  if $(\bar{n}, \overline{n-1}) \in \mathfrak{C}$ then $\tilde{e}_n\mathfrak{C}$ is obtained by turning $(\bar{n}, \overline{n-1})$ into $(n-1, n)$, otherwise $\tilde{e}_n\mathfrak{C} = 0$, for $i \neq n$, $\tilde{f}_i$ and $\tilde{e}_i$ act like in $B(\Lambda_n^B)$.

In the sequel we denote by $v_{\Lambda_n}^B$ the highest weight vertex of $B(\Lambda_n^B)$, by $v_{\Lambda_n}^D$ and $v_{\Lambda_{n-1}}^D$ the highest weight vertices of $B(\Lambda_n^D)$ and $B(\Lambda_{n-1}^D)$. Note that $v_{\Lambda_n}^B$ and $v_{\Lambda_n}^D$ correspond to the spin column containing the letters of $\{1, \dots, n\}$ and $v_{\Lambda_{n-1}}^D$ corresponds to the spin column containing the letters of $\{1, \dots, n-1, \bar{n}\}$.

3. Schensted correspondences in G_n^B and G_n^D

3.1. Orthogonal tableaux

Let $\lambda \in \Omega_+$. We are going to review the notion of standard orthogonal tableaux introduced by Kashiwara and Nakashima [4] to label the vertices of $B(\lambda)$.

3.1.1. Columns and admissible columns. A column of type B is a Young diagram

$$C = \begin{array}{|c|}\hline x_1 \\ \hline \cdot \\ \hline \cdot \\ \hline x_l \\ \hline\end{array}$$

of column shape filled by letters of $\mathcal{B}_n$ such that C increases from top to bottom and 0 is the unique letter of $\mathcal{B}_n$ that may appear more than once.

A column of type D is a Young diagram C of column shape filled by letters of $\mathcal{D}_n$ such that $x_{i+1} \npreceq x_i$ for $i = 1, \dots, l-1$. Note that the letters n and $\bar{n}$ are the unique letters that may appear more than once in C and if they do, these letters are different in two adjacent boxes.

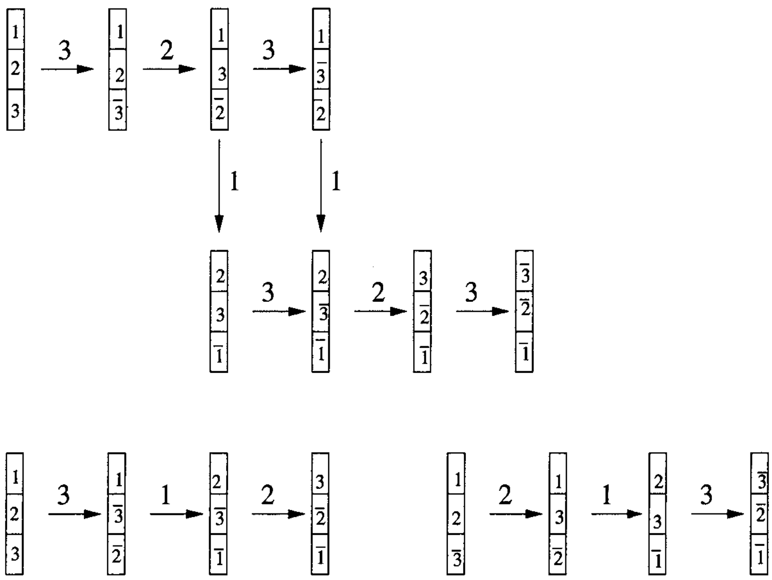


Figure 1. The crystal graphs $B(\Lambda_n^B)$, $B(\Lambda_n^D)$ and $B(\Lambda_{n-1}^D)$ for $U_q(so_7)$ and $U_q(so_6)$.

The height $h(C)$ of the column C is the number of its letters. The word obtained by reading the letters of C from top to bottom is called the reading of C and denoted by w(C). We will say that the column C contains a pair $(z, \bar{z})$ when a letter 0 or the two letters $z \preceq n$ and $\bar{z}$ appear in C.

Definition 3.1.1 (Kashiwara-Nakashima) Let C be a column such that $\mathrm{w}(C) = x_1 \cdots x_{h(C)}$. Then C is admissible if $h(C) \leq n$ and for any pair $(z, \bar{z})$ of letters in C satisfying $z = x_p$ and $\bar{z} = x_q$ with $z \preceq n$ we have

$$|q - p| \geq h(C) - z + 1. \tag{7}$$

(Note that $0 \succ n$ on $\mathcal{B}_n$ and we may have $q - p < 0$ for type D and $z = n$).

Example 3.1.2 For $n = 4, 40\bar{4}\bar{2}$ and $3\bar{4}4\bar{3}$ are readings of admissible columns respectively of type B and D.

Let C be a column of type B or D and $z \preceq n$ a letter of C. We denote by $N(z)$ the number of letters x in C such that $x \preceq z$ or $x \succeq \bar{z}$. Then Condition (7) is equivalent to $N(z) \leq z$.

Suppose that C is non admissible and does not contain a pair $(z, \bar{z})$ with $z \preceq n$ and $N(z) > z$. Then $h(C) > n$. Hence C is of type B and $0 \in C$. Indeed, if $0 \notin C$, C contains a letter z maximal such that $z \preceq n$ and $\bar{z} \in C$. It means that for any $x \in \{z + 1, \ldots, n\}$, there is at most one letter $y \in C$ with $|y| = x$. We have a contradiction because in this case $N(z) > n - (n - z)$. We obtain the

Remark 3.1.3 A column C is non admissible if and only if at least one of the following assertions is satisfied:

(i) C contains a letter $z \preceq n$ and $N(z) > z$
(ii) C is of type B, $0 \in C$ and $h(C) > n$.

If we set $v^B_{\omega_k} = 1 \cdots k$ for $k = 1, \ldots, n$, then $B(v^B_{\omega_k})$ is isomorphic to $B(\omega^B_k)$. Similarly, if we set $v^D_{\omega_k} = 1 \cdots k$ for $k = 1, \ldots, n$ and $v^D_{\bar{\omega}_n} = 1 \cdots (n-1)\bar{n}$, then $B(v^D_{\omega_k})$ and $B(v^D_{\bar{\omega}_n})$ are respectively isomorphic to $B(\omega^D_k)$ and $B(\bar{\omega}^D_n)$.

Proposition 3.1.4 (Kashiwara-Nakashima)

- *The vertices of $B(v^B_{\omega_k})$ are the readings of the admissible columns of type B and length k.*
- *The vertices of $B(v^D_{\omega_k})$ with $k < n$ are the readings of the admissible columns of type D and length k.*
- *The vertices of $B(v^D_{\omega_n})$ are the readings of the admissible columns C of type D such that* $\mathrm{w}(C) = x_1 \cdots x_n$ *and $x_k = n$* (*resp. $x_k = \bar{n}$*) *implies $n - k$ is even* (*resp. odd*).
- *The vertices of $B(v^D_{\bar{\omega}_n})$ are the readings of the admissible columns C of type D such that* $\mathrm{w}(C) = x_1 \cdots x_n$ *and $x_k = \bar{n}$* (*resp. $x_k = n$*) *implies $n - k$ is odd* (*resp. even*).

We can obtain another description of the admissible columns by computing, for each admissible column C, a pair of columns (lC, rC) without pair $(z, \bar{z})$. This duplication was inspired by the description of the admissible columns of type C in terms of De Concini columns used by Sheats in [16].

Definition 3.1.5 Let C be a column of type B and denote by $I_C = \{z_1 = 0, \ldots, z_r = 0 \succ z_{r+1} \succ \cdots \succ z_s\}$ the set of letters $z \preceq 0$ such that the pair $(z, \bar{z})$ occurs in C. We will say that C can be split when there exists (see the example below) a set of s unbarred letters $J_C = \{t_1 \succ \cdots \succ t_s\} \subset \mathcal{B}_n$ such that: t_1 is the greatest letter of $\mathcal{B}_n$ satisfying: $t_1 \prec z_1$, $t_1 \notin C$ and $\bar{t}_1 \notin C$, for $i = 2, \ldots, s$, t_i is the greatest letter of $\mathcal{B}_n$ satisfying: $t_i \prec \min(t_{i-1}, z_i)$, $t_i \notin C$ and $\bar{t}_1 \notin C$.

In this case we write:

- rC for the column obtained first by changing in C $\bar{z}_i$ into $\bar{t}_i$ for each letter $z_i \in I$, next by reordering if necessary.
- lC for the column obtained first by changing in C z_i into t_i for each letter $z_i \in I$, next by reordering if necessary.

Definition 3.1.6 Let C be a column of type D. Denote by $\hat{C}$ the column of type B obtained by turning in C each factor $\bar{n}n$ into 00. We will say that C can be split when $\hat{C}$ can be split. In this case we write $lC = l\hat{C}$ and $rC = l\hat{C}$.

Example 3.1.7 Suppose $n = 9$ and consider the column C of type B such that $\mathrm{w}(C) = 458900\bar{8}\bar{5}\bar{4}$. We have $I_C = \{0, 0, 8, 5, 4\}$ and $J_C = \{7, 6, 3, 2, 1\}$. Hence

$$\mathrm{w}(lC) = 123679\bar{8}\bar{5}\bar{4} \quad \text{and} \quad \mathrm{w}(rC) = 4589\bar{7}\bar{6}\bar{3}\bar{2}\bar{1}.$$

Suppose $n = 8$ and consider the column C' of type D such that $\mathrm{w}(C) = 56\bar{8}8\bar{8}\bar{6}\bar{5}\bar{2}$. Then $\mathrm{w}(\hat{C}') = 5600\bar{8}\bar{6}\bar{5}\bar{2}$, $I_{\hat{C}'} = \{0, 0, 6, 5\}$ and $J_{\hat{C}'} = \{7, 4, 3, 1\}$. Hence

$$\mathrm{w}(lC') = 1347\bar{8}\bar{6}\bar{5}\bar{2} \quad \text{and} \quad \mathrm{w}(rC') = 56\bar{8}\bar{7}\bar{4}\bar{3}\bar{2}\bar{1}.$$

Lemma 3.1.8 *Let C be a column of type B or D which can be split. Then C is admissible.*

Proof: Suppose C of type B. We have $h(C) \leq n$ for C can be split. If there exists a letter $z \prec 0$ in C such that the pair $(z, \bar{z})$ occurs in C and $N(z) \geq z + 1$, C contains at least $z + 1$ letters x satisfying $|x| \preceq z$. So rC contains at least $z + 1$ letters x' satisfying $|x'| \preceq z$. We obtain a contradiction because rC does not contain a pair $(t, \bar{t})$. When C is of type D, by applying the lemma to $\hat{C}$ we obtain that $\hat{C}$ is admissible. So C is admissible. □

The meaning of lC and rC is explained in the following proposition.

Proposition 3.1.9 *Let* $\omega \in \{\omega_1^B, \ldots, \omega_n^B\}$ *or* $\omega \in \{\omega_1^D, \ldots, \omega_{n-1}^D, \omega_n^D, \bar{\omega}_n^D\}$. *The map*

$$S_2 : B(v_\omega) \to B(v_\omega) \otimes B(v_\omega)$$

defined in Theorem 2.1.2 *satisfies for any admissible column* $C \in B(v_\omega)$:

$$S_2(\mathrm{w}(C)) = \mathrm{w}(rC) \otimes \mathrm{w}(lC).$$

Example 3.1.10 Consider $\omega = \omega_2^B$ for $U_q(so_5)$. The following graphs are respectively those of $B(\omega)$ and $S_2(B(\omega))$.

$$\begin{array}{ccccccccc}
12 \xrightarrow{2} & 10 & \xrightarrow{2} 1\bar{2} & \xrightarrow{1} & 2\bar{2} & \xrightarrow{1} & 2\bar{1} & & \\
 & \downarrow 1 & & & & & \downarrow 2 & & \\
 & 20 & \xrightarrow{2} 00 & \xrightarrow{2} & 0\bar{2} & \xrightarrow{1} & 0\bar{1} & \xrightarrow{2} & \bar{2}\bar{1}
\end{array}$$

$$\begin{array}{l}
(12) \otimes (12) \xrightarrow{2^2} (1\bar{2}) \otimes (12) \xrightarrow{2^2} (1\bar{2}) \otimes (1\bar{2}) \xrightarrow{1^2} (2\bar{1}) \otimes (1\bar{2}) \xrightarrow{1^2} (2\bar{1}) \otimes (2\bar{1}) \\
\qquad\qquad\qquad \downarrow 1^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow 2^2 \\
\qquad (2\bar{1}) \otimes (12) \xrightarrow{2^2} (\bar{2}\bar{1}) \otimes (12) \xrightarrow{2^2} (\bar{2}\bar{1}) \otimes (1\bar{2}) \xrightarrow{1^2} (\bar{2}\bar{1}) \otimes (2\bar{1}) \xrightarrow{2^2} (\bar{2}\bar{1}) \otimes (\bar{2}\bar{1})
\end{array}$$

Proof of Proposition 3.1.9: In this proof we identify each column with its reading to simplify the notations. When $C = v_\omega$ is the highest weight vertex of $B(v_\omega)$, $r(v_\omega) = l(v_\omega) = v_\omega$ because v_ω does not contain a pair $(z, \bar{z})$. So $S_2(v_\omega) = rC \otimes lC$. Each vertex C of $B(\omega)$ may be written $C = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(v_\omega)$. By induction on r, it suffices to prove that for any $\mathrm{w}(C) \in B(v_\omega)$ such that $\tilde{f}_i(C) \neq 0$ we have

$$S_2(C) = rC \otimes lC \Rightarrow S_2(\tilde{f}_i C) = r(\tilde{f}_i C) \otimes l(\tilde{f}_i C).$$

For any column D we denote by $[D]_i$ the word obtained by erasing all the letters x of D such that $\tilde{f}_i(x) = \tilde{e}_i(x) = 0$. It is clear that only the letters of $[D]_i$ may be changed in D when we apply $\tilde{f}_i$.

Suppose $\omega \in \{\omega_1^B, \ldots, \omega_n^B)$. Consider $C \in B(v_\omega)$ such that $S_2(C) = rC \otimes lC$ and $\tilde{f}_i(C) \neq 0$.

When $i \neq n$, the letters $x \notin \{\overline{i+1}, \bar{i}, i, i+1\}$ do not interfere in the computation of $\tilde{f}_i$. It follows from the condition $\tilde{f}_i(C) \neq 0$ and an easy computation from (1) and (2) that we need only consider the following cases: (i) $[C]_i = i$, (ii) $[C]_i = \overline{i+1}$, (iii) $[C]_i = (i+1)(\overline{i+1})$, (iv) $[C]_i = (i)(\overline{i+1})$, (v) $[C]_i = i(i+1)(\overline{i+1})$ and (vi) $[C]_i = i(\overline{i+1})\bar{i}$. In the case (i), if $i+1 \notin J_C$, we have $[lC]_i = i$ and $[rC]_i = i$. Then $[\tilde{f}_i(C)]_i = i+1$ and $J_{\tilde{f}_i C} = J_C$ (hence $i \notin J_{\tilde{f}_i C}$). So $[l(\tilde{f}_i C)]_i = i+1$ and $[r(\tilde{f}_i C)]_i = i+1$. That means that $S_2(\tilde{f}_i C) = \tilde{f}_i^2(rC \otimes lC) = \tilde{f}_i(rC) \otimes \tilde{f}_i(lC) = r(\tilde{f}_i C) \otimes l(\tilde{f}_i C)$ by definition of the map S_2. If $i+1 \in J_C$, we can write $[rC]_i = (i)(\overline{i+1})$ and $[lC]_i = (i)(i+1)$. Then $[\tilde{f}_i C)]_i = i+1$ and $J_{\tilde{f}_i C} = J_C - \{i+1\} + \{i\}$. So $[r(\tilde{f}_i C)] = (i+1)(\bar{i})$ and $[l(\tilde{f}_i C)] = (i)(i+1)$. Hence $S_2(\tilde{f}_i C) = \tilde{f}_i^2(rC \otimes lC) = \tilde{f}_i^2(rC) \otimes lC = r(\tilde{f}_i C) \otimes l(\tilde{f}_i C)$. The proof is similar in the cases (ii) to (vi). When $i = n$, only the letters of $\{\bar{n}, 0, n\}$ interfere in the computation of $\tilde{f}_n$. We obtain the proposition by considering the cases: $[C]_n = \underbrace{0 \cdots 0}_{0\ p \text{ times}}$, $[C]_n = n\underbrace{0 \cdots 0}_{0\ p \text{ times}}$ and $[C]_n = n$.

Suppose $\omega \in \{\omega_1^D, \ldots, \omega_{n-1}^D, \bar{\omega}_n^D, \omega_n^D\}$. When $i < n-1$ the proof is the same as above. When $i \in \{n-1, n\}$, the proposition follows by considering successively the cases:

$$\begin{cases} [C]_i = n-1(\bar{n}n)^r, \\ [C]_i = n(\bar{n}n)^r\bar{n}, \\ [C]_i = (n-1)n(\bar{n}n)^r\bar{n}, \\ [C]_i = (\bar{n}n)^r\bar{n}, \\ [C]_i = (n-1)(\bar{n}n)^r\bar{n}, \\ [C]_i = (n-1)(\bar{n}n)^r\bar{n}(\overline{n-1}). \end{cases} \quad \text{if } i = n-1$$

and

$$\begin{cases} [C]_i = n-1(n\bar{n})^r, \\ [C]_i = \bar{n}(n\bar{n})^r n, \\ [C]_i = (n-1)\bar{n}(n\bar{n})^r n, \\ [C]_i = (n\bar{n})^r n, \\ [C]_i = (n-1)(n\bar{n})^r n, \\ [C]_i = (n-1)(n\bar{n})^r n(\overline{n-1}). \end{cases} \quad \text{if } i = n.$$

where $(\bar{n}n)^r$ (resp. $(n\bar{n})^r$) is the word containing the factor $\bar{n}n$ (resp. $n\bar{n}$) repeated r times. □

Using Lemma 3.1.8 we derive immediately the

Corollary 3.1.11 *A column C of type B or D is admissible if and only if it can be split.*

Example 3.1.12 From Example 3.1.7, we obtain that C is admissible for $n = 9$ and C' is admissible for $n = 8$.

Remark 3.1.13 With the notations of the previous proposition, denote by W_n/W_ω the set of cosets of the Weyl group W_n with respect to the stabilizer W_ω of ω in W_n. Then we obtain a bijection τ between the orbit O_ω of v_ω in $B(\omega)$ under the action of W_n defined by (3) and W_n/W_ω. Using Formulas (3) it is easy to prove that O_ω consists of the vertices of $B(v_\omega)$ without the pair $(z, \bar{z})$. Moreover if C_1, C_2 are two columns such that $\mathrm{w}(C_1) = x_1 \cdots x_p$, $\mathrm{w}(C_2) = y_1 \cdots y_p \in O_\omega$, we have

$$C_1 \preceq C_2 \Leftrightarrow \tau_{\mathrm{w}(C_1)} \lhd_\omega \tau_{\mathrm{w}(C_2)}$$

where $C_1 \preceq C_2$ means that $x_i \preceq y_i$, $i = 1, \ldots, p$ and "$\lhd_\omega$" denotes the projection of the Bruhat order on W_n/W_ω. Then Proposition 3.1.9 may be regarded as a version of Littelmann's labelling of $B(v_\omega)$ by pairs $(\tau_{\mathrm{w}(rC)}, \tau_{\mathrm{w}(lC)}) \in W_n/W_\omega \times W_n/W_\omega$ satisfying $\tau_{\mathrm{w}(lC)} \lhd_\omega \tau_{\mathrm{w}(rC)}$ [13].

3.1.2. Orthogonal tableaux. Every $\lambda \in \Omega_+^B$ has a unique decomposition of the form $\lambda = \sum_{i=1}^n \lambda_i \omega_i^B$. Similarly, every $\lambda \in \Omega_+^D$ has a unique decomposition of the form $(*)$ $\lambda = \sum_{i=1}^n \lambda_i \omega_i^D$ or $(**)$ $\lambda = \lambda_n \bar{\omega}_n^D + \sum_{i=1}^{n-1} \lambda_i \omega_i^D$ with $\lambda_n \neq 0$, where $(\lambda_n, \ldots, \lambda_n) \in \mathbb{N}^n$. We will say that $(\lambda_1, \ldots, \lambda_n)$ is the positive decomposition of $\lambda \in \Omega_+$. Denote by Y_λ the Young diagram having λ_i columns of height i for $i = 1, \ldots, n$. If $\lambda \in \Omega_+^D$, Y_λ may not suffice to characterize the weight λ because a column diagram of length n may be associated to ω_n or to $\bar{\omega}_n$. In Section 3.4 we will need to attach to each dominant weight $\lambda \in \Omega_+$ a combinatorial object $Y(\lambda)$. Moreover it will be convenient to distinguish in $(*)$ the cases where $\lambda_n = 0$ or $\lambda_n \neq 0$. This leads us to set:

$$\begin{array}{rl} \text{(i)} & Y(\lambda) = Y_\lambda \text{ if } \lambda \in \Omega_+^B, \\ \text{(ii)} & Y(\lambda) = (Y_\lambda, +) \text{ in case } (*) \text{ with } \lambda_n \neq 0, \\ \text{(iii)} & Y(\lambda) = (Y_\lambda, 0) \text{ in case } (*) \text{ with } \lambda_n = 0, \\ \text{(iv)} & Y(\lambda) = (Y_\lambda, -) \text{ in case } (**). \end{array} \tag{8}$$

When $\lambda \in \Omega_+^D$, $Y(\lambda)$ may be regarded as the generalization of the notion of the shape of type A associated to a dominant weight. Now write

$$\begin{aligned} v_\lambda^B &= \left(v_{\omega_1^B}\right)^{\otimes\lambda_1} \otimes \cdots \otimes \left(v_{\omega_n^B}\right)^{\otimes\lambda_n} \text{ in case (i)}, \\ v_\lambda^D &= \left(v_{\omega_1^D}\right)^{\otimes\lambda_1} \otimes \cdots \otimes \left(v_{\omega_n^D}\right)^{\otimes\lambda_n} \text{ in case (ii)}, \\ v_\lambda^D &= \left(v_{\omega_1^D}\right)^{\otimes\lambda_1} \otimes \cdots \otimes \left(v_{\omega_{n-1}^D}\right)^{\otimes\lambda_{n-1}} \text{ in case (iii) and} \\ v_\lambda^D &= \left(v_{\omega_1^D}\right)^{\otimes\lambda_1} \otimes \cdots \otimes \left(v_{\bar{\omega}_n^D}\right)^{\otimes\lambda_n} \text{ in case (iv)}. \end{aligned}$$

Then v_λ^B and v_λ^D are highest weight vertices of G_n^B and G_n^D. Moreover $B(v_\lambda^B)$ and $B(v_\lambda^D)$ are isomorphic to $B^B(\lambda)$ and $B^D(\lambda)$.

A tabloid τ of type B (resp. D) is a Young diagram whose columns are filled to give columns of type B (resp. D). If $\tau = C_1 \cdots C_r$, we write $\mathrm{w}(T) = \mathrm{w}(C_r) \cdots \mathrm{w}(C_1)$ for the reading of τ.

Definition 3.1.14

- Consider $\lambda \in \Omega_+^B$. A tabloid T of type B is an orthogonal tableau of shape $Y(\lambda)$ and type B if $\mathrm{w}(T) \in B(v_\lambda^B)$.
- Consider $\lambda \in \Omega_+^D$. A tabloid T of type D is an orthogonal tableau of shape $Y(\lambda)$ and type D if $\mathrm{w}(T) \in B(v_\lambda^D)$.

The orthogonal tableaux of a given shape form a single connected component of G_n, hence two orthogonal tableaux whose readings occur at the same place in two isomorphic connected components of G_n are equal. The shape of an orthogonal tableau T of type D may be regarded as a pair $[O_T, \varepsilon_T]$ where O_T is a Young diagram and $\varepsilon_T \in \{-, 0, +\}$. The $\{-, 0, +\}$ part of this shape can be read off directly on T. Indeed $\varepsilon = 0$ if T does not contain a column of height n. Otherwise write $\mathrm{w}(C_1) = x_1 \cdots x_n$ for the reading of the first column of T. Since it is admissible, C_1 contains at least a letter, say x_k of $\{n, \bar{n}\}$. Then ε is given by the parity of $n - k$ according to Proposition 3.1.4.

Consider $\tau = C_1 C_2 \cdots C_r$ a tabloid whose columns are admissible. The split form of τ is the tabloid obtained by splitting each column of τ. We write $\mathrm{spl}(\tau) = (lC_1 rC_1)(lC_2 rC_2) \cdots (lC_r rC_r)$. With the notations of Proposition 3.1.9, we will have $\mathrm{w}(\mathrm{spl}(T)) = S_2\mathrm{w}(C_r) \cdots S_2\mathrm{w}(C_1)$. Kashiwara-Nakashima's combinatorial description [4] of an orthogonal tableau T is based on the enumeration of configurations that should not occur in two adjacent columns of T. Considering its split form $\mathrm{spl}(T)$, this description becomes more simple because the columns of $\mathrm{spl}(T)$ does not contain any pair $(z, \bar{z})$.

Lemma 3.1.15 *Let $T = C_1 C_2 \cdots C_r$ be a tabloid whose columns are admissible. Then T is an orthogonal tableau if and only if* $\mathrm{spl}(T)$ *is an orthogonal tableau.*

Proof: Suppose first that $\mathrm{w}(T)$ is a highest weight vertex of weight λ. Then, by Corollary 2.1.3, $\mathrm{w}(\mathrm{spl}(T))$ is a highest weight vertex of weight 2λ. If T is an orthogonal tableau, $\mathrm{w}(T) = v_\lambda$ and we have $\mathrm{w}(\mathrm{spl}(T)) = v_{2\lambda}$. So $\mathrm{spl}(T)$ is an orthogonal tableau. Conversely, if $\mathrm{spl}(T)$ is an orthogonal tableau, $\mathrm{w}(\mathrm{spl}(T)) = S_2\mathrm{w}(C_r) \cdots S_2\mathrm{w}(C_1)$ is a highest weight vertex of weight 2λ by Corollary 2.1.3. Hence we have $\mathrm{w}(\mathrm{spl}(T)) = v_{2\lambda}$ because there exists only one orthogonal tableau of highest weight 2λ. So $\mathrm{w}(T) = v_\lambda$. In the general case, denote by T_0 the tableau such that $\mathrm{w}(T_0)$ is the highest weight vertex of the connected component of G_n containing $\mathrm{w}(T)$. Then $\mathrm{w}(\mathrm{spl}(T_0))$ is the highest weight vertex of the connected component containing $\mathrm{w}(\mathrm{spl}(T))$ and the following assertions are equivalent:

(i) $\mathrm{spl}(T)$ is an orthogonal tableau,
(ii) $\mathrm{spl}(T_0)$ is orthogonal tableau,
(iii) T_0 is orthogonal tableau,
(iv) T is orthogonal tableau. □

Definition 3.1.16 Let $\tau = C_1 C_2$ be a tabloid with two admissible columns C_1 and C_2. We set:

- $C_1 \preceq C_2$ when $h(C_1) \geq h(C_2)$ and the rows of C_1C_2 are weakly increasing from left to right,
- $C_1 \trianglelefteq C_2$ when $rC_1 \preceq lC_2$.

Definition 3.1.17 (Kashiwara-Nakashima)

Let $C_1 = \begin{array}{|c|}\hline x_1 \\ \hline \cdot \\ \hline \cdot \\ \hline x_N \\ \hline \end{array}$ and $C_2 = \begin{array}{|c|}\hline y_1 \\ \hline \cdot \\ \hline \cdot \\ \hline y_N \\ \hline \end{array}$ be admissible columns of type D and p, q, r, s integers satisfying $1 \leq p \leq q < r \leq s \leq M$.

C_1C_2 contains an a-odd-configuration (with $a \notin \{\bar{n}, n\}$) when:

- $a = x_p, \bar{n} = x_r$ are letters of C_1 and $\bar{a} = y_s, n = y_q$ letters of C_2 such that $r - q + 1$ is odd or
- $a = x_p, n = x_r$ are letters of C_1 and $\bar{a} = y_s, \bar{n} = y_q$ letters of C_2 such that $r - q + 1$ is odd C_1C_2 contains an a-even-configuration (with $a \notin \{\bar{n}, n\}$) when:
- $a = x_p, n = x_r$ are letters of C_1 and $\bar{a} = y_s, n = y_q$ letters of C_2 such that $r - q + 1$ is even or
- $a = x_p, \bar{n} = x_r$ are letters of C_1 and $\bar{a} = y_s, \bar{n} = y_q$ letters of C_2 such that $r - q + 1$ is even

Then we denote by $\mu(a)$ the positive integer defined by:

$$\mu(a) = s - p$$

Theorem 3.1.18

(i) *Consider $C_1, C_2, \ldots, C_r$ some admissible columns of type B. Then the tabloid $T = C_1C_2 \cdots C_r$ is an orthogonal tableau if and only if $C_i \trianglelefteq C_{i+1}$ for $i = 1, \ldots, r-1$.*

(ii) *Consider $C_1, C_2, \ldots, C_r$ some admissible columns of type D. Then the tabloid $T = C_1C_2 \cdots C_r$ is an orthogonal tableau if and only if, $C_i \trianglelefteq C_{i+1}$ for $i = 1, \ldots, r-1$, and rC_ilC_{i+1} does not contain an a-configuration (even or odd) such that $\mu(a) = n - a$.*

Proof: Kashiwara and Nakashima describe an orthogonal tableau T by listing the configurations that should not occur in two adjacent columns of T. If we except the a-configurations even or odd, these configurations disappear in spl(T) because spl(T) does not contain a column with a pair $(z, \bar{z})$. Hence the theorem follows from Lemma 3.1.15 and Theorems 5.7.1 and 6.7.1 of [4]. □

Example 3.1.19 Suppose $n = 4$. Then $T = \begin{array}{|c|c|c|}\hline 3 & 3 & 4 \\ \hline 4 & 0 & \bar{4} \\ \hline 0 & \bar{2} \\ \cline{1-2} 0 \\ \cline{1-1}\end{array}$ is an orthogonal tobleau of type B because $\mathrm{spl}(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 3 & 3 & 3 & 4 \\ \hline 2 & 4 & 4 & \bar{4} & \bar{4} & \bar{3} \\ \hline 3 & \bar{2} & \bar{2} & \bar{2} \\ \cline{1-4} 4 & \bar{1} \\ \cline{1-2}\end{array}$. But $\begin{array}{|c|c|}\hline 3 & \bar{4} \\ \hline \bar{4} & \bar{3} \\ \hline\end{array}$ is not orthogonal of type D because it contains a 3-even configuration with $\mu(3) = 1$.

3.2. Plactic monoids for types B_n and D_n

Definition 3.2.1 Let w_1 and w_2 be two words on $\mathcal{B}_n$ (resp. $\mathcal{D}_n$). We write $w_1 \overset{B}{\sim} w_2$ (resp. $w_1 \overset{D}{\sim} w_2$) when these two words occur at the same place in two isomorphic connected components of the crystal G_n^B (resp. G_n^D).

The definition of the orthogonal tableaux implies that for any word $w \in \mathcal{B}_n^*$ (resp. $w \in \mathcal{D}_n^*$) there exists a unique orthogonal tableau $P^B(w)$ (resp. $P^D(w)$) such that $w \sim \mathrm{w}(P(w))$. So the sets $\mathcal{B}_n^*/\overset{B}{\sim}$ and $\mathcal{D}_n^*/\overset{D}{\sim}$ can be identified respectively with the sets of orthogonal tableaux of type B and D. Our aim is now to show that $\overset{B}{\sim}$ and $\overset{D}{\sim}$ are in fact congruencies $\overset{B}{\equiv}$ and $\overset{D}{\equiv}$ so that $\mathcal{B}_n^*/\overset{B}{\sim}$ and $\mathcal{D}_n/\overset{D}{\sim}$ are in a natural way endowed with a multiplication.

Definition 3.2.2 The monoid $Pl(B_n)$ is the quotient of the free monoid $\mathcal{B}_n^*$ by the relations:

R_1^B: if $x \neq \bar{z}$ and $x \prec y \prec z$:

$$yzx \overset{B}{\equiv} yxz \quad \text{and} \quad xzy \overset{B}{\equiv} zxy.$$

R_2^B: If $x \neq \bar{y}$ and $x \prec y$:

$$xyx \overset{B}{\equiv} xxy \quad \text{for} \quad x \neq 0 \quad \text{and} \quad xyy \overset{B}{\equiv} yxy \quad \text{for} \quad y \neq 0.$$

R_3^B: If $1 \prec x \preceq n$ and $x \preceq y \preceq \bar{x}$:

$$y\overline{(x-1)}(x-1) \overset{B}{\equiv} yx\bar{x}, \quad \text{and} \quad x\bar{x}y \overset{B}{\equiv} \overline{(x-1)}(x-1)y,$$
$$0\bar{n}n \equiv \bar{n}n0.$$

R_4^B: If $x \preceq n$:

$$00x \overset{B}{\equiv} 0x0 \quad \text{and} \quad 0\bar{x}0 \overset{B}{\equiv} \bar{x}00.$$

R_5^B: Let $w = \mathrm{w}(C)$ be a non admissible column word each strict factor of which is admissible. When C satisfies the assertion (i) of Remark 3.1.3, let z be the lowest unbarred letter of w such that the pair $(z, \bar{z})$ occurs in w and $N(z) > z$, otherwise set $z = 0$. The $w \overset{B}{\equiv} \tilde{w}$ where $\tilde{w}$ is the column word obtained by erasing the pair $(z, \bar{z})$ in w if $z \preceq n$, by erasing 0 otherwise.

Definition 3.2.3 The monoid $Pl(D_n)$ is the quotient of the free monoid $\mathcal{D}_n^*$ by the relations:

R_1: If $x \neq \bar{z}$

$$yzx \overset{D}{\equiv} yxz \quad \text{for} \quad x \preceq y \prec z \quad \text{and} \quad xzy \overset{D}{\equiv} zxy \quad \text{for} \quad x \prec y \preceq z.$$

R_2: If $1 \prec x \preceq n$ and $x \preceq y \preceq \bar{x}$

$$y\overline{(x-1)}(x-1) \overset{D}{\equiv} yx\bar{x} \quad \text{and} \quad x\bar{x}y \overset{D}{\equiv} \overline{(x-1)}(x-1)y.$$

R_3^D: If $x \preceq n-1$:

$$\begin{cases} \bar{n}\bar{x}n \overset{D}{\equiv} \bar{x}\bar{n}n \\ n\bar{x}\bar{n} \overset{D}{\equiv} \bar{x}n\bar{n} \end{cases} \quad \text{and} \quad \begin{cases} \bar{n}nx \overset{D}{\equiv} \bar{n}xn \\ n\bar{n}x \overset{D}{\equiv} nx\bar{n} \end{cases}.$$

R_4^D:

$$\begin{cases} n\bar{n}\bar{n} \overset{D}{\equiv} \overline{(n-1)}(n-1)\bar{n} \\ \bar{n}nn \overset{D}{\equiv} \overline{(n-1)}(n-1)n \end{cases} \quad \text{and} \quad \begin{cases} \bar{n}\overline{(n-1)}(n-1) \overset{D}{\equiv} \bar{n}\bar{n}n \\ n\overline{(n-1)}(n-1) \overset{D}{\equiv} nn\bar{n} \end{cases}.$$

R_5^D: Consider w a non admissible column word each strict factor of which is admissible. Let z be the lowest unbarred letter such that the pair $(z, \bar{z})$ occurs in w and $N(z) > z$ (see Remark 3.1.3). Then $w \overset{D}{\equiv} \tilde{w}$ where $\tilde{w}$ is the column word obtained by erasing the pair $(z, \bar{z})$ in w if $z \prec n$, by erasing a pair $(n, \bar{n})$ of consecutive letters otherwise.

The relations R_5^B and R_5^D are called the contraction relations. When the letter 0 or a pair $(n, \bar{n})$ disappears, we have $l(C) = n + 1$ and in R_5^D the word $\tilde{w}$ does not depend on the factor $n\bar{n}$ or $\bar{n}n$ erased. Moreover $\tilde{w}$ is an admissible column word. Note that $w_1 \equiv w_2$ implies $d(w_1) = d(w_2)$, that is, $\equiv$ is compatible with the grading given by d.

Theorem 3.2.4 *Given two words w_1 and w_2*

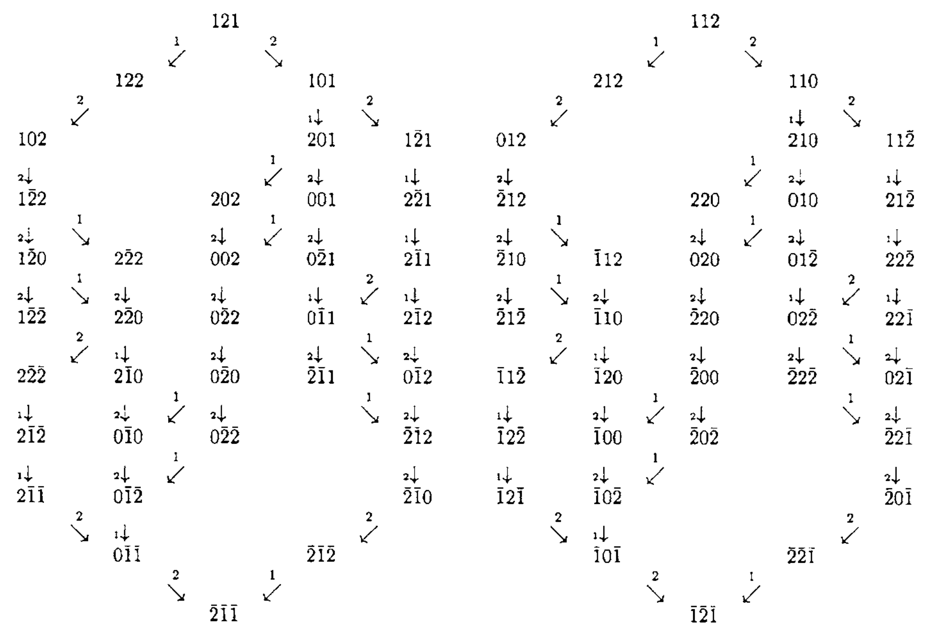
$$w_1 \sim w_2 \Leftrightarrow w_1 \equiv w_2 \Leftrightarrow P(w_1) = P(w_2) \tag{9}$$

This theorem is proved in the same way as in the symplectic case [10], and we will only sketch the arguments. Note first that we have

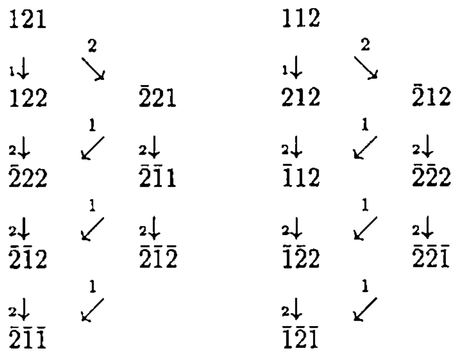
$$w_1 \sim w_2 \Leftrightarrow P(w_1) = P(w_2)$$

immediately from the definition of P. For any word w occurring in the left hand side of a relation $R_1^B, \dots, R_4^B$ (resp. $R_1^D, \dots, R_4^D$), write $\xi^B(w)$ (resp. $\xi^D(w)$) for the word occurring in the right hand side of this relation. Similarly for $p = 1, \dots, n$ and w a word of length $p+1$ occurring in the left hand side of R_5^B (resp. R_5^D), denote by $\xi_p^B(w)$ (resp. $\xi_p^D(w)$) the word occurring in the right hand side of this relation. By using similar arguments to those of [10], we obtain the following assertions:

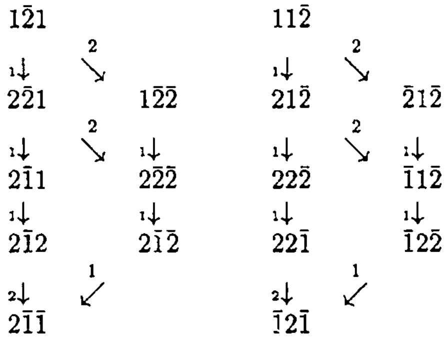
- The map $\xi^B : w \mapsto \xi(w)$ is the crystal isomorphism from $B^B(121)$ to $B^B(112)$.
- If $n > 2$, the map $\xi^D : w \mapsto \xi(w)$ is the crystal isomorphism from $B^D(121)$ to $B^D(112)$ otherwise ξ^D is the crystal isomorphism from $B^D(121) \cup B^D(1\bar{2}1)$ to $B^D(112) \cup B^D(11\bar{2})$.
- For $p = 2, \dots, n-1$, ξ_p: $\mapsto \xi_p(w)$ is the crystal isomorphism from B $(12\cdots p\bar{p})$ to $B(12\cdots p-1)$.
- The map ξ_n^B:$w \mapsto \xi_n^B(w)$ is the crystal isomorphism from B^B $(12\cdots n\bar{n}) \cup B^B(12\cdots n0)$ to $B^B(12\cdots n-1) \cup B^B(12\cdots n)$.
- The words w of length $n+1$ occurring in the left hand side of R_5^D are the vertices of $B^D(12\cdots n\bar{n}) \cup B^D(12\cdots \bar{n}n)$. Moreover the restriction of the map $\xi_n^D : w \mapsto \xi_n^D(w)$ to $B^D(12\cdots n\bar{n})$ (resp. to $B^D(12\cdots \bar{n}n)$) is the crystal isomorphism from $B^D(12\cdots n\bar{n})$ (resp. $B^D(12\cdots \bar{n}n)$) to $B^D(12\cdots n-1)$.



The crystals $B^B(121)$ and $B^B(112)$ in G_2^B



The crystals $B^D(121)$ and $B^D(112)$ in G_2^D



The crystals $B^D(1\bar{2}1)$ and $B^D(11\bar{2})$ in G_2^D

By (1) and (2), this implies that the plactic relations above are compatible with Kashiwara's operators, that is, for any words w_1 and w_2 such that $w_1 \equiv w_2$ one has:

$$\begin{cases} \tilde{e}_i(w_1) \equiv \tilde{e}_i(w_2) & \text{and } \varepsilon_i(w_1) = \varepsilon_i(w_2) \\ \tilde{f}_i(w_1) \equiv \tilde{f}_i(w_2) & \text{and } \varphi_i(w_1) = \varphi_i(w_2). \end{cases} \tag{10}$$

Hence:

$$w_1 \equiv w_2 \Rightarrow w_1 \sim w_2.$$

To obtain the converse we show that for any highest weight vertex w^0

$$\mathrm{w}(P(w^0)) \equiv w^0. \tag{11}$$

This follows by induction on $l(w^0)$. When $l(w^0) = 1$, $\mathrm{w}(P(w^0)) = w^0$. By writing $w^0 = v^0x^0$, it is possible (see the proof of Lemma 3.2.6 in [10]) to show that $\mathrm{w}(P(w^0))$ may be obtained from the word $\mathrm{w}(P(v^0))x^0$ by applying only Knuth relations and contraction relations of type $12\cdots r\bar{p} \equiv 12\cdots \hat{p}\cdots r$ with $p \leq r \leq n$ (the hat means removal the letter p).

From (11), we obtain that two highest weight vertices w_1^0 and w_2^0 with the same weight λ verify $w_1^0 \equiv w_2^0$. Indeed there is only one orthogonal tableau whose reading is a highest vertex of weight λ. Now suppose that $w_1 \sim w_2$ and denote by w_1^0 and w_2^0 the highest weight vertices of $B(w_1)$ and $B(w_2)$. We have $w_1^0 \equiv w_2^0$. Set $w_1 = \tilde{F}w_1^0$ where $\tilde{F}$ is a product of Kashiwara's operators $\tilde{f}_i, i = 1, \ldots, n$. Then $w_2 = \tilde{F}w_2^0$ because $w_1 \sim w_2$. So by (10) we obtain

$$w_1^0 \equiv w_2^0 \Rightarrow \tilde{F}w_1^0 \equiv \tilde{F}w_2^0 \Rightarrow w_1 \equiv w_2.$$

3.3. A bumping algorithm for types B and D

Now we are going to see how the orthogonal tableau $P(w)$ may be computed for each vertex w by using an insertion scheme analogous to bumping algorithm for type A. As a first step, we describe $P(w)$ when $w = \mathrm{w}(C)x$, where x and C are respectively a letter and an admissible column. This will be called "the insertion of the letter x in the admissible column C" and denoted by $x \to C$. Then we will be able to obtain $P(w)$ when $w = \mathrm{w}(T)x$ with x a letter and T an orthogonal tableau. This will be called "the insertion of the letter x in the orthogonal tableau T" and denoted by $x \to T$. Our construction of P will be recursive, in the sense that if $P(u) = T$ and x is a letter, then $P(ux) = x \to T$.

3.3.1. Insertion of a letter in an admissible column. Consider a word $w = \mathrm{w}(C)x$, where x and C are respectively a letter and an admissible column of height p. When $w = \mathrm{w}(C^x)$ is the reading of a column C^x, we have:

$x \to C = C^x$ if C^x is admissible or
$x \to C = \widetilde{C^x}$ where $\widetilde{C^x}$ is the column whose reading correspondsto $\tilde{w}$ otherwise.

Indeed, $x \to C$ must be an orthogonal tableau such that $\mathrm{w}(x \to C) \equiv w$.

When w is not a column word, by Lemma 2.1.1 the highest weight vertex w^0 of $B(w)$ may be written $w^0 = v^0 1$ where $v^0 \in \{b_{\omega_p}; p = 1, \ldots, n\} \cup \{b_{\bar{\omega}_n}\}$. Then $u^0 = 1v^0$ is the reading of an orthogonal tableau and $u^0 \equiv w^0$. So u^0 is the highest weight vertex of the connected component containing $\mathrm{w}(x \to C)$. Moreover there exists a unique sequence of highest weight vertices $w_1^0, \ldots, w_p^0$ such that $w_1^0 = w^0$, $w_p = u^0$ and for $i = 2, \ldots, p w_i^0$ differs from w_{i-1}^0 by applying one relation R_1 from left to right. This implies that there exists a unique sequence of vertices $w_1, \ldots, w_p$ such that $w_1 = w$ and for $i = 2, \ldots, p-1$ $B(w_i) = B(w_i^0)$. Each w_i differs from w_{i-1} by applying one relation R_1, R_2, R_3 or R_4 from left to right. The word w_p is the reading of an orthogonal tableau and can be factorized as $w_p = v'x'$ where $v' = \mathrm{w}(C')$ is a column word an x' a letter. We will have $x \to C = C'x'$.

Example 3.3.1

Suppose $n = 7$. Let $\mathrm{w}(C) = 6700\bar{7}\bar{6}$ be an admissible column word of type B. Choose $x = 6$. Then by applying relations $R_i^B i = 1, \ldots, 4$ we obtain successively:

$$6700\mathbf{\bar{7}\bar{6}6} \equiv 6700\mathbf{\bar{7}7\bar{7}} \equiv 6\mathbf{70\bar{7}7}0\mathbf{\bar{7}} \equiv 6\mathbf{7\bar{7}\bar{7}}00\bar{7} \equiv \mathbf{6\bar{6}67}00\bar{7} \equiv \bar{5}567007\bar{7}$$

Suppose $n = 7$. Let $\mathrm{w}(C) = 67\bar{7}7\bar{7}\bar{6}$ be an admissible column word of type D. Choose $x = 6$. Then by applying relations R_i^D $i = 1, \ldots, 4$ we obtain successively:

$$67\bar{7}7\mathbf{\bar{7}\bar{6}6} \equiv 67\bar{7}\mathbf{7\bar{7}\bar{7}}7 \equiv 67\mathbf{\bar{7}\bar{6}6}77 \equiv 6\mathbf{7\bar{7}\bar{7}}7\bar{7}7 \equiv \mathbf{6\bar{6}6}\bar{7}7\bar{7}7 \equiv \bar{5}567\bar{7}7\bar{7}.$$

Hence

$$6 \to \begin{array}{|c|}\hline 6\\\hline 7\\\hline 0\\\hline 0\\\hline \bar{7}\\\hline \bar{6}\\\hline\end{array} = \begin{array}{|c|c|}\hline 5 & \bar{5}\\\hline 6\\\cline{1-1} 7\\\cline{1-1} 0\\\cline{1-1} 0\\\cline{1-1} \bar{7}\\\cline{1-1}\end{array} \quad \text{and} \quad 6 \to \begin{array}{|c|}\hline 6\\\hline 7\\\hline \bar{7}\\\hline 7\\\hline \bar{7}\\\hline \bar{6}\\\hline\end{array} = \begin{array}{|c|c|}\hline 5 & \bar{5}\\\hline 6\\\cline{1-1} 7\\\cline{1-1} \bar{7}\\\cline{1-1} 7\\\cline{1-1} \bar{7}\\\cline{1-1}\end{array}$$

3.3.2. Insertion of a letter in an orthogonal tableau. Consider an orthogonal tableau $T = C_1 C_2 \cdots C_r$. We can prove as in [10] that the insertion $x \to T$ is characterized as follows:

- If $\mathrm{w}(C_1)x$ is an admissible column word, then $x \to T = C_1^x C_2 \cdots C_r$ where C_1^x is the column of reading $\mathrm{w}(C_1)x$.
- If $\mathrm{w}(C_1)x$ is a non admissible column word each strict factor of which is admissible and such that $x\widetilde{\mathrm{w}}(C_1) = x_1 \cdots x_s$, then $x \to T = x_s \to (x_{s-1} \to (\cdots x_1 \to T'))$ where $T' = C_2 \cdots C_r$. Moreover the insertion of $x_1, \ldots, x_s$ in T' does not cause a new contraction.
- If $\mathrm{w}(C_1)x$ is not a column word, the insertion of x in C_1 gives a column C_1' and a letter x' (with the notation of 3.3.1). Then $x \to T = C_1'(x' \to T')$, that is, $x \to T$ is the tableau defined by C_1' and the columns of $x' \to T'$.

Notice that the algorithm terminates because in the last two cases we are reduced to the insertion of a letter in a tableau whose number of boxes is strictly less than that of T. Finally for any vertex $w \in G_n$, we will have:

$$P(w) = \boxed{w} \quad \text{if } w \text{ is a letter,}$$
$$P(w) = x \to P(u) \quad \text{if } w = ux \text{ with } u \text{ a word and } x \text{ a letter.}$$

3.4. Schensted-type correspondences

In this section a bijection is established between words w of length l on $\mathcal{B}_n$ and pairs $(P^B(w), Q^B(w))$ where $P^B(w)$ is the orthogonal tableau defined above and $Q^B(w)$ is an oscillating tableau of type B. Similarly we obtain a bijection between words w of length l on $\mathcal{D}_n$ and pairs $(P^D(w), Q^D(w))$ where $P^D(w)$ is an oscillating tableau of type D. For type B, such a one-to-one correspondence has already been obtained by Sundaram [17] using another definition of orthogonal tableaux and an appropriate insertion algorithm. Unfortunately it is not known if this correspondence is compatible with a monoid structure. Our bijection based on the previous insertion algorithm for admissible orthogonal tableaux of type B will be different from Sundaram's one but compatible with the plactic relations defining $Pl(B_n)$.

Definition 3.4.1 An oscillating tableau Q of type B and length l is a sequence of Young diagrams $(Q_1, \dots, Q_l)$ whose columns have height $\leq n$ and such that any two consecutive diagrams are equal or differ by exactly one box (i.e. $Q_{k+1} = Q_k$, $Q_{k+1}/Q_k = (\square)$ or $Q_k/Q_{k+1} = (\square)$).

An oscillating tableau Q of type D and length l is a sequence $(Q_1, \dots, Q_l)$ of pairs $Q_k(O_k, \varepsilon_k)$ where O_k is a Young diagram whose columns have height $\leq n$ and $\varepsilon_k \in \{-, 0, +\}$, satisfying for $k = 1, \dots, l$

- $O_{k+1}/O_k = (\square)$ or $O_k/O_{k+1} = (\square)$,
- $\varepsilon_{k+1} \neq 0$ and $\varepsilon_k \neq 0$ implies $\varepsilon_{k+1} = \varepsilon_k$.
- $\varepsilon_k = 0$ if and only if O_k has no columns of height n.

Let $w = x_1 \cdots x_l$ be a word. The construction of $P(w)$ involves the construction of the l orthogonal tableaux defined by $P_i = P(x_1 \cdots x_i)$. For $w \in \mathcal{B}_n^*$ (resp. $w \in \mathcal{D}_n^*$) we denote by $Q^B(w)$ (resp. $Q^D(w)$) the sequence of shapes of the orthogonal tableaux $P_1, \dots, P_l$.

Proposition 3.4.2 *$Q_B(w)$ and $Q_D(w)$ are respectively oscillating tableaux of type B and D.*

Proof: Each Q_i is the shape of an orthogonal tableau so it suffices to prove that for any letter x and any orthogonal tableau T, the shape of $x \to T$ differs from the shape of T by at most one box according to Definition 3.4.1.

The highest weight vertex of the connected component containing $\mathrm{w}(T)x$ may be written $\mathrm{w}(T^0)x^0$ where T^0 is an orthogonal tableau. It follows from Lemma 2.2.1(ii) that $\mathrm{w}(T) \leftrightarrow \mathrm{w}(T^0)$. So $\mathrm{wt}(\mathrm{w}(T^0))$ is given by the shape of T. Then the shape of $x \to T$ is given by the coordinates of $\mathrm{wt}(\mathrm{w}(T^0)x^0)$ on the basis $(\omega_1^B, \dots, \omega_n^B)$ for type B, on the base $(\omega_1^D, \dots, \omega_n^D)$ or $(\omega_1^D, \dots, \omega_{n-1}^D, \bar{\omega}_n^D)$ for type D.

Suppose that $x \in \mathcal{B}_n^*$ and T is orthogonal of type B. Let $(\lambda_1, \dots, \lambda_n)$ be the coordinates of $\mathrm{wt}(T^0)$ on the basis of the ω_i^B's. If $x^0 = \bar{i} \succ 0$ then $\mathrm{wt}(x^0) = \omega_{i-1}^B - \omega_i^B$. So $\lambda_i > 0$ and $\mathrm{wt}(\mathrm{w}(T^0)x^0) = (\lambda_1, \dots, \lambda_{i-1} + 1, \lambda_i - 1, \dots, \lambda_{n-1})$. Hence during the insertion of the letter x in T, a column of height i (corresponding to the weight ω_i) is turned into a column of height $i - 1$ (corresponding to the weight ω_{i-1}). So the shape of $x \to T$ is obtained by erasing one box to the shape of T. If $x^0 = i \prec 0$, then we can prove by similar arguments that the shape of $x \to T$ is obtained by adding one box to the shape of T. When $x^0 = 0$, $\mathrm{wt}(x^0) = 0$, so $\mathrm{wt}(\mathrm{w}(T^0)x^0) = \mathrm{wt}(\mathrm{w}(T^0))$. Hence the shapes of T and $x \to T$ are the same.

Suppose $x \in \mathcal{D}_n^*$ and T orthogonal of type D. When $|x^0| \neq n$, the proof is the same as above. If $x^0 = n$, $\mathrm{wt}(x^0) = \Lambda_n - \Lambda_{n-1} = \omega_n - \omega_{n-1} = \omega_{n-1} - \bar{\omega}_n$. We have to consider three cases, (i) $\varepsilon_T = -$; (ii) $\varepsilon_T = 0$ and (iii) $\varepsilon_T = +$. Denote by $(\lambda_1, \ldots, \lambda_n)$ the positive decomposition of $\mathrm{wt}(\mathrm{w}(T^0))$ on the basis $(\omega_1^D, \ldots, \omega_n^D)$ or on the basis $(\omega_1^D, \ldots, \bar{\omega}_n^D)$.

In the first case, $\lambda_n > 0$ and the positive decomposition of $\mathrm{wt}(x^0\mathrm{w}(T^0))$ on the base $(\omega_1^D, \ldots, \bar{\omega}_n^D)$ is $(\lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1} + 1, \lambda_n - 1)$. It means that during the insertion of x in T a column of height n (corresponding to $\bar{\omega}_n$) is turned into a column of height $n - 1$ (corresponding to ω_{n-1}). Moreover $\varepsilon_{x\to T} = \varepsilon_T$ if $\lambda_n > 1$ and $\varepsilon_{x\to T} = 0$ otherwise.

In the second case, $\lambda_{n-1} > 0$, $\lambda_n = 0$ and the positive decomposition of $\mathrm{wt}(x^0\mathrm{w}(T^0))$ on the base $(\omega_1^D, \ldots, \omega_n^D)$ is $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1} - 1, 1)$. It means that during the insertion of x in T a column of height $n - 1$ (corresponding to ω_{n-1}) is turned into a column of height n (corresponding to ω_n). Moreover $\varepsilon_{x\to T} = +$.

In the last case, $\lambda_{n-1} > 0$, $\lambda_n > 0$ and the positive decomposition of $\mathrm{wt}(x^0\mathrm{w}(T^0))$ on $(\omega_1^D, \ldots, \omega_n^D)$ is $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1} - 1, \lambda_n + 1)$. It means that during the insertion of x in T a column of height $n - 1$ (corresponding to ω_{n-1}) is turned into a column of height n (corresponding to ω_n). Moreover $\varepsilon_{x\to T} = \varepsilon_T$.

When $x^0 = \bar{n}$, the proof is similar. □

Theorem 3.4.3 *For any vertices w_1 and w_2 of G_n:*

$$w_1 \leftrightarrow w_2 \Leftrightarrow Q(w_1) = Q(w_2).$$

Proof: The proof is analogous to that of Proposition 5.2.1 in [10]. □

Corollary 3.4.4 *Let $\mathcal{B}_{n,l}^*$ and $\mathcal{O}_l^B$ (resp. $\mathcal{D}_{n,l}^*$ and $\mathcal{O}_l^D$) be the set of words of length l on $\mathcal{B}_n$ (resp. $\mathcal{D}_n$) and the set of pairs (P, Q) where P is an orthogonal tableau of type B (resp. D) and Q an oscillating tableau of type B (resp. D) and length l such that P has shape Q_l (Q_l is the last shape of Q). Then the maps:*

$$\begin{array}{l} \Psi^B : B_{n,l}^* \to O_l^B \\ w \mapsto (P^B(w), Q^B(w)) \end{array} \quad \text{and} \quad \begin{array}{l} \Psi^D : \mathcal{D}_{n,l}^* \to \mathcal{O}_l^D \\ w \mapsto (P^D(w), Q^D(w)) \end{array}$$

are bijections.

Proof: For type Ψ^B the proof is analogous to that of Theorem 5.2.2 in [10]. By Theorems 3.2.4 and 3.4.3, we obtain that Ψ^D is injective. Consider an oscillating tableau Q of length l and type D. Set $x_1 = 1$ and for $i = 2, \ldots, l$

- $x_i = k$ if O_i differs from O_{i-1} by adding a box in row k of height $<n$,
- $x_i = \bar{k}$ if Q_i differs from Q_{i-1} by removing a box in row k of height $<n$,
- $x_i = n$ if O_i differs from O_{i-1} by adding a box in row n and $\varepsilon_i = +$,
- $x_i = \bar{n}$ if Q_i differs from Q_{i-1} by adding a box in row n and $\varepsilon_i = -$,
- $x_i = \bar{n}$ if O_i differs from O_{i-1} by removing a box in row n and $\varepsilon_i = +$,
- $x_i = n$ if O_i differs from O_{i-1} by removing a box in row n and $\varepsilon_i = -$,

– Consider $w_Q = x_l \cdots x_2 1$. Then $Q(w_Q) = Q$. By Theorem 3.1.18, the image of $B(w_Q)$ by Ψ^D consists in the pairs (P, Q) where P is a symplectic tableau of shape Q_l. We deduce immediately that Ψ is surjective. □

3.5. Jeu de Taquin for type B

In [16], J.T. Sheats has developed a sliding algorithm for type C acting on the skew admissible symplectic tableaux. This algorithm is analogous to the classical Jeu de Taquin of Lascoux and Schützenberger for type A [9]. Each inner corner of the skew tableau considered is turned into an outside corner by applying vertical and horizontal moves. We have shown in [10] how to extend it to take into account the contraction relation of the plactic monoid $Pl(C_n)$ (analogous to $Pl(B_n)$ and $Pl(D_n)$ for type C). Then we have proved that the tableau obtained does not depend on the way the inner corners disappear. In this section we propose a sliding algorithm for type B. The main idea is that the split form of any skew orthogonal tableau T of type B may be regarded as a symplectic skew tableau.

Set $\mathcal{C}_n = \{1 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{1}\} \subset \mathcal{B}_n$. The symplectic tableaux are, for type C, the combinatorial objects analogous to the orthogonal tableaux. They can be regarded as orthogonal tableaux of type B on the alphabet $\mathcal{C}_n$ instead of $\mathcal{B}_n$. The plactic monoid $Pl(C_n)$ is the quotient of the free monoid $\mathcal{C}_n^*$ by relations R_1^B, R_2^B and R_5^B. We denote by $\overset{C}{\equiv}$ the congruence relation in $Pl(C_n)$. Then for w_1 and w_2 two words of $\mathcal{C}_n^*$ we have:

$$w_1 \overset{C}{\equiv} w_2 \Rightarrow w_1 \overset{B}{\equiv} w_2.$$

A skew orthogonal tableau of type B is a skew Young diagram filled by letters of $\mathcal{B}_n$ whose columns are admissible of type B and such that the rows of its split form (obtained by splitting its columns) are weakly increasing from left to right. Skew orthogonal tableaux are the combinatorial objects analogous to the admissible skew tableaux introduced by Sheats in [16] for type C. Note that two different skew tableaux may have the same reading.

Example 3.5.1 For $n = 3$,

$$T = \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 3 & 0 \\ \hline 0 & \bar{3} & \bar{1} \\ \hline 0 & & \\ \hline \end{array}$$ is a skew orthogonal tableau of type B because

$$\mathrm{spl}(T) = \begin{array}{|c|c|c|c|c|c|} \hline & & & & 2 & 2 \\ \hline & & 2 & 3 & 3 & \bar{3} \\ \hline 2 & \bar{3} & \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline 3 & \bar{2} & & & & \\ \hline \end{array}.$$

The relation $0\bar{n}n \equiv \bar{n}n0$ has no natural interpretation in terms of horizontal or vertical slidings in skew orthogonal tableaux. To overcome this problem we are going to work on the split form of the skew tableaux instead of the skew tableaux themselves that is, we are going to obtain a Jeu de Taquin for type B by applying the symplectic Jeu de Taquin on the split form of the skew orthogonal tableaux of type B.

Lemma 3.5.2 *Let T and T' be two skew orthogonal tableaux of type B. Then*:

$$\mathrm{w}(T) \overset{B}{\equiv} \mathrm{w}(T') \Leftrightarrow \mathrm{w}[\mathrm{spl}(T)] \overset{B}{\equiv} \mathrm{w}[\mathrm{spl}(T')].$$

Proof: We can write $\mathrm{w}(T) = \mathrm{w}(C_1)\cdots\mathrm{w}(C_r)$ and $\mathrm{w}(T') = \mathrm{w}(C_1')\cdots\mathrm{w}(C_s')$ where C_k and C_k', $k = 1, \dots, r$ are admissible columns. All the vertices $w \in B(\mathrm{w}(T))$ and $w' \in B(\mathrm{w}(T'))$ can be respectively written on the form $w = c_\tau \cdots c_1$ and $w' = c_s' \cdots c_1'$ where $c_i, i = 1, \dots, r$ and $c_j', j = 1, \dots, s$ are readings of admissible columns of type B. Consider the maps:

$$\theta_2\colon \begin{cases} B(\mathrm{w}(T)) \to B(\mathrm{spl}(\mathrm{w}(T)) \\ w = c_\tau \cdots c_1 \mapsto S_2(c_\tau)\cdots S_2(c_1) \end{cases} \quad \text{and}$$

$$\theta_2'\colon \begin{cases} B(\mathrm{w}(T')) \to B(\mathrm{spl}(\mathrm{w}(T)) \\ w' = c_s' \cdots c_1 \mapsto S_2(c_\tau')\cdots S_2(c_1') \end{cases}$$

where S_2 is the map defined in Proposition 3.1.9. We have $\mathrm{w}[\mathrm{spl}(T)] = \theta_2(\mathrm{w}(T))$ and $\mathrm{w}[\mathrm{spl}(T')] = \theta_2'(\mathrm{w}(T'))$. By using Corollary 2.1.3 we obtain

$$\mathrm{w}(T) \overset{B}{\equiv} \mathrm{w}(T') \Leftrightarrow \mathrm{w}(T) \overset{B}{\sim} \mathrm{w}(T') \Leftrightarrow \mathrm{w}[\mathrm{spl}(T)] \overset{B}{\sim} \mathrm{w}[\mathrm{spl}(T')]$$
$$\Leftrightarrow \mathrm{w}[\mathrm{spl}(T)] \overset{B}{\equiv} \mathrm{w}[\mathrm{spl}(T')]. \qquad \square$$

If T is a skew orthogonal tableau of type B with r columns, then $\mathrm{spl}(T)$ is a symplectic skew tableau with $2r$ columns. We can apply the symplectic Jeu de taquin to $\mathrm{spl}(T)$ to obtain a symplectic tableau $\mathrm{spl}(T)'$. We will have $\mathrm{w}[\mathrm{spl}(T)'] \overset{C}{\equiv} \mathrm{w}[\mathrm{spl}(T)]$ so $\mathrm{w}[\mathrm{spl}(T)'] \overset{B}{\equiv} \mathrm{w}[\mathrm{spl}(T)]$.

Proposition 3.5.3 $\mathrm{spl}(T)'$ *is the split form of the orthogonal tableau $P^B(T)$.*

Proof: It follows from $\mathrm{w}(T) \overset{B}{\equiv} \mathrm{w}(P_B(T))$ and the lemma above that $\mathrm{w}[\mathrm{spl}(T)] \overset{B}{\equiv} \mathrm{w}[\mathrm{spl}(P^B(T))]$. So we obtain $\mathrm{w}[\mathrm{spl}(T)'] \overset{B}{\equiv} \mathrm{w}[\mathrm{spl}(P^B(T))]$. But $\mathrm{spl}(T')$ and $\mathrm{spl}(p^B(T))$ are orthogonal tableaux, hence $\mathrm{spl}(T)' = \mathrm{spl}(P^B(T))$. $\square$

The columns of the split form of a skew orthogonal tableau T of type B contain no letters 0 and no pairs of letters $(x, \bar{x})$ with $x \preceq n$. In this particular case most of the elementary steps of the symplectic Jeu de Taquin applied on T are simple slidings identical to those of the original Jeu de Taquin of Lascoux and Schützenberger (that is complications of the symplectic Jeu de taquin are not needed in these slidings).

Example 3.5.4 From $\mathrm{spl}\left(\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 1 & 0 & \bar{3} \\ \hline 3 & \bar{3} & \bar{2} \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|c|c|}\hline * & * & 1 & 1 & 1 & 2 \\ \hline 1 & 1 & 2 & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{3} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}$ we compute successively:

$$\begin{array}{|c|c|c|c|c|c|}\hline * & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & * & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{3} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|}\hline * & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{2} & * & \bar{2} & \bar{1} \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|}\hline * & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{2} & \bar{2} & * & \bar{1} \\ \hline \end{array},$$

$$\begin{array}{|c|c|c|c|c|c|}\hline * & 1 & 1 & 1 & 2 & 2 \\ \hline 1 & 2 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{2} & \bar{2} & \bar{2} & * \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & * & 3 & 3 & 3 & \bar{3} \\ \hline 3 & 3 & \bar{2} & \bar{2} & \bar{2} & * \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & \bar{2} & * & \bar{2} & \bar{2} & * \\ \hline \end{array},$$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & \bar{2} & \bar{2} & * & \bar{2} & * \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & \bar{2} & \bar{2} & \bar{2} & * & * \\ \hline \end{array} = \mathrm{spl}\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & \bar{3} & \bar{3} \\ \hline 0 & \bar{2} & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array}\right).$$

Note that the sliding applied in the fourth duplicated tableau above is the unique sliding which is not identical to an original Jeu de taquin step.

The split form of a skew orthogonal tableau of type D (defined in the same way than for type B) is still a symplectic skew tableau. But

$$w_1 \overset{C}{\equiv} w_2 \not\Longrightarrow w_1 \overset{D}{\equiv} w_2$$

so we can not use the same idea to obtain an Jeu de Taquin for type D. Moreover the examples (computed by using P^D with $n = 3$)

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline \bar{3} & \bar{2} \\ \hline * & \bar{1} \\ \hline \end{array} \equiv \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \bar{3} & \bar{2} \\ \hline \bar{2} & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & \bar{3} \\ \hline \bar{3} & \bar{2} \\ \hline * & \bar{1} \\ \hline \end{array} \not\equiv \begin{array}{|c|c|}\hline \bar{3} & \bar{3} \\ \hline 3 & \bar{2} \\ \hline \bar{3} & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

show that it is not enough to know what letter x slides from the second column C_2 to the first C_1 to be able to compute an horizontal sliding. Indeed the result depends on the whole column C_2. Thus, to give a combinatorial description of a sliding algorithm for type D would probably be very complicated.

4. Plactic monoid for $\mathfrak{G}_n$

Write $\mathfrak{G}_n^B$ and $\mathfrak{G}_n^D$ for the crystal graphs of the direct sums

$$\bigoplus_{l \geq 0} \left(V\left(\Lambda_1^B\right) \oplus V\left(\Lambda_n^B\right)\right)^{\otimes l} \quad \text{and} \quad \bigoplus_{l \geq 0} \left(V\left(\Lambda_1^D\right) \oplus V\left(\Lambda_n^D\right) \oplus V\left(\Lambda_{n-1}^D\right)\right)^{\otimes l}.$$

We call $\mathfrak{B}_n = \mathcal{B}_n \cup SP_n$ and $\mathfrak{D}_n = \mathcal{D}_n \cup SP_n$ the sets of generalized letters of type B and D. Then we identify the vertices of $\mathfrak{G}_n^B$ and $\mathfrak{G}_n^D$ respectively with the words of the free monoid $\mathfrak{B}_n^*$ and $\mathfrak{D}_n^*$. If w is a vertex of $\mathfrak{G}_n$, we write $\mathrm{wt}(w)$ for the weight of w. The spin representations are minuscule, hence every spin column is determined by its weight.

We can extend the Definition 3.2.1 to vertices of $\mathfrak{G}_n$. Consider two vertices $\mathfrak{b}_1$ and $\mathfrak{b}_2$ of $\mathfrak{G}_n^B$ (resp. $\mathfrak{G}_n^D$). We write $\mathfrak{b}_1 \overset{B}{\sim} \mathfrak{b}_2$ (resp. $\mathfrak{b}_1 \overset{D}{\sim} \mathfrak{b}_2$) when these vertices occur at the same place in two isomorphic connected components of $\mathfrak{G}_n^B$ (resp. $\mathfrak{G}_n^D$). Our aim is now to extend the results of Section 3.2 to the vertices of $\mathfrak{G}_n$.

4.1. *Tensor products of spin representations*

Write $B(0)$ for the connected component of $\mathfrak{G}_n$ containing only the empty word. Let $\mathfrak{C}_0$ be the spin column containing only barred letters. For $p = 1, \dots, n$, denote by $\mathfrak{C}_p$ the spin column containing exactly the unbarred letters $x \preceq p$. For any admissible column C, set $|C| = \{x \preceq n, x \in lC \text{ or } \bar{x} \in lC\} = \{x \preceq n, x \in rC \text{ or } \bar{x} \in rC\}$.

Lemma 4.1.1

1. *There exists a unique crystal isomorphism* S^B

$$B(0) \cup B\big(v_{\omega_n^B}\big) \cup \left(\bigcup_{i=1}^{n-1} B\big(v_{\omega_i^B}\big)\right) \overset{S^B}{\to} B\big(v_{\Lambda_n^B}\big)^{\otimes 2}.$$

2. *Let w be the reading of an admissible column C of type B. Write*
   - *$l\mathfrak{C}$ for the spin column of height n obtained by adding to lC the barred letters $\bar{x}$ such that $x \notin |C|$,*
   - *$r\mathfrak{C}$ for the spin column of height n obtained by adding to rC the unbarred letters x such that $x \notin |C|$.*

Then

$$S^B(w) = r\mathfrak{C} \otimes l\mathfrak{C}.$$

Proof:

1. From Lemma 2.1.1 we obtain that the highest weight vertices of $B(v_{\Lambda_n^B})^{\otimes 2}$ are the vertices $v_p^B = \mathfrak{C}_n \otimes \mathfrak{C}_p$ with $p = 0, \dots, n$. We have $\mathrm{wt}(v_p^B) = \omega_p^B$ for $p = 1, \dots, n$ and $\mathrm{wt}(v_0^D) = 0$. Hence S^B is the crystal isomorphism which sends $B(v_{\omega_p^B})$ on $B(v_p^B)$ for $p = 1, \dots, n$ and $B(0)$ on $B(v_0^B)$.
2. When $w = v_{\omega_p^B}$, the equality $S^B(w) = r\mathfrak{C} \otimes l\mathfrak{C}$ is true. Consider $w \in B(v_{\omega_p^B})$ and $i = 1, \dots, n$ such that $w' = \tilde{f}_i(w) \neq 0$. Write $w = \mathrm{w}(C)$ and $w' = \mathrm{w}(C')$ where C and C' are two admissible columns of height p. The lemma will be proved if we show the implication

$$S^B(w) = r\mathfrak{C} \otimes l\mathfrak{C} \Rightarrow S^B(w') = r\mathfrak{C}' \otimes l\mathfrak{C}'$$

where $r\mathfrak{C}'$ and $l\mathfrak{C}'$ are defined from C' in the same manner than $r\mathfrak{C}$ and $l\mathfrak{C}$ from C. This is equivalent to

$$\tilde{f}_i(r\mathfrak{C} \otimes l\mathfrak{C}) = r\mathfrak{C}' \otimes l\mathfrak{C}'. \tag{12}$$

Suppose $i \neq n$. Set $E_i = \{i, i+1, \overline{i+1}, \bar{i}\}$.

(i) If $\{i, i+1\} \subset |C|$, lC and $l\mathfrak{C}$ coincide on E_i. Similarly rC and $r\mathfrak{C}$, lC' and $l\mathfrak{C}'$, rC' and $l\mathfrak{C}'$ coincide on E_i. By Proposition 3.1.9, we know that

$$\tilde{f}_i^2(rC \otimes lC) = rC' \otimes lC'.$$

The action of $\tilde{f}_i^2$ on $rC \otimes lC$ is analogous to that of $\tilde{f}_i$ on $r\mathfrak{C} \otimes l\mathfrak{C}$. It means that $\tilde{f}_i$ changes a pair $(i, \overline{i+1})$ of $r\mathfrak{C}$ (resp $l\mathfrak{C}$) into a pair $(i+1, \bar{i})$ if and only if $\tilde{f}_i^2$ changes a pair $(i, \overline{i+1})$ of rC (resp. lC) into a pair $(i+1, \bar{i})$. So (12) is true because only the letters of E_i may be modified when we apply $\tilde{f}_i$.

(ii) If $\{i, i+1\} \cap |C| = \{i+1\}$, we have $[rC]_i = [lC]_i = \overline{i+1}$ with the notation of the proof of Proposition 3.1.9. Then $r\mathfrak{C} \cap E_i = \{\overline{i+1}, i\}$ and $l\mathfrak{C} \cap E_i = \{\overline{i+1}, \bar{i}\}$. Moreover $[C']_i = \bar{i}$, $r\mathfrak{C}' \cap E_i = \{\bar{i}, i+1\}$ and $l\mathfrak{C}' \cap E_i = \{\overline{i+1}, \bar{i}\}$. Hence $\tilde{f}_i(r\mathfrak{C} \otimes l\mathfrak{C})$ and $r\mathfrak{C}' \otimes l\mathfrak{C}'$ coincide on E_i. So they are equal because $\tilde{f}_i$ does not modify the letters $x \notin E_i$.

(iii) If $\{i, i+1\} \cap |C| = \{i\}$, the proof is analogous to case (ii).

Suppose $i = n$. Set $E_n = \{n, \bar{n}\}$. Then $n \in |C|$ because $\tilde{f}_i(w) \neq 0$. We obtain (12) by using similar arguments to those of (i). □

Lemma 4.1.2

1. *There exists two crystal isomorphisms* S_n^D *and* S_{n-1}^D

$$B(0) \cup B(v_{\omega_n^D}) \cup \left(\bigcup_{i=1}^{n-1} B(v_{\omega_i^D})\right) \overset{S_n^D}{\to} B(v_{\Lambda_n^D}) \otimes \left(B(v_{\Lambda_n^D}) \cup B(v_{\Lambda_{n-1}^D})\right),$$

$$B(0) \cup B(v_{\bar{\omega}_n^D}) \cup \left(\bigcup_{i=1}^{n-1} B(v_{\omega_i^D})\right) \overset{S_{n-1}^D}{\to} B(v_{\Lambda_{n-1}^D}) \otimes \left(B(v_{\Lambda_{n-1}^D}) \cup B(v_{\Lambda_n^D})\right).$$

2. *Let w be the reading of an admissible column C of type D. If $h(C) \prec n$, denote by t the greatest unbarred letter such that $t \notin |C|$. Write*
   - *$l\mathfrak{C}$ for the spin column of height n obtained by adding to lC the barred letters $\bar{x}$ such that $x \notin |C|$.*
   - *$r\mathfrak{C}$ for the spin column of height n obtained by adding to rC the unbarred letters x such that $x \notin |C|$.*
   - *$l_t\mathfrak{C}$ for the spin column of height n obtained by adding to lC the letter t and the barred letters $\bar{x}$ such that $x \notin |C| \cup \{t\}$.*
   - *$r_t\mathfrak{C}$ for the spin column of height n obtained by adding to rC the letter $\bar{t}$ and the unbarred letters x such that $x \notin |C| \cup [t\}$.*

Then we have

$$\text{(i)}\quad \begin{cases} S_n^D(w) = r\mathfrak{C} \otimes l\mathfrak{C} & \text{if } r\mathfrak{C} \in B\big(v_{\Lambda_n^D}\big) \\ S_n^D(w) = r_t\mathfrak{C} \otimes l_t\mathfrak{C} & \text{otherwise} \end{cases} \quad \text{and}$$

$$\text{(ii)}\quad \begin{cases} S_{n-1}^D(w) = r\mathfrak{C} \otimes l\mathfrak{C} & \text{if } r\mathfrak{C} \in B\big(v_{\Lambda_{n-1}^D}\big) \\ S_{n-1}^D(w) = r_t\mathfrak{C} \otimes l_t\mathfrak{C} & \text{otherwise} \end{cases}$$

(*recall that* $r\mathfrak{C} \in B(v_{\Lambda_n^D})$ *if and only if it contains an even number of barred letters*).

Proof: We only sketch the proof for S_n^D, the arguments are analogous for S_{n-1}^D.

1. The highest weight vertices of $B(v_{\Lambda_n^D}) \otimes (B(v_{\Lambda_n^D}) \cup B(v_{\Lambda_{n-1}^D}))$ are the vertices $v_p^D = \mathfrak{C}_n \otimes \mathfrak{C}_p$ with $p = 0, \ldots, n$. We have $\mathrm{wt}(v_p^D) = \omega_p^D$ for $p = 1, \ldots, n$ and $\mathrm{wt}(v_0^D) = 0$. Hence S_n^D is the crystal isomorphism which sends $B(v_{\omega_p^D})$ on $B(v_p^D)$ for $p = 1, \ldots, n$ and $B(0)$ on v_0^D.
2. When $w = v_{\omega_p^D}$, the equality $S_n^D(w) = r\mathfrak{C} \otimes l\mathfrak{C}$ is true. Consider $w \in B(v_{\omega_p^D})$ and $i = 1, \ldots, n$ such that $w' = \tilde{f}_i(w) \neq 0$. Write $w = \mathrm{w}(C)$ and $w' = \mathrm{w}(C')$ where C and C' are two admissible columns of height p. Let t' be the greatest unbarred letter such that $t' \notin |C'|$. If the number of barred letters of C is equal to that of C', $r\mathfrak{C}$ and $r\mathfrak{C}'$ belongs together in $B(v_{\Lambda_n^D})$ or in $B(v_{\Lambda_{n-1}^D})$. In these cases we can prove that

$$\begin{aligned} &S_n^D(w) = r\mathfrak{C} \otimes l\mathfrak{C} \Rightarrow S_n^D(w') = r\mathfrak{C}' \otimes l\mathfrak{C}' \quad \text{and} \\ &S_n^D(w) = r_t\mathfrak{C} \otimes l_t\mathfrak{C} \Rightarrow S_n^D(w') = r_{t'}\mathfrak{C}' \otimes l_{t'}\mathfrak{C}' \end{aligned} \tag{13}$$

as we have done for S^B. Otherwise we have $i = n$ and $rC \cap E_n = \{n-1\}$ or $rC \cap E_n = \{n\}$.

Suppose $i = n$ and $n \in |C|$. Then $n-1$ is the unique letter of $E_n = \{n-1, n, \bar{n}, \overline{n-1}\}$ that occurs in C. We have $t = n$ and $t' = n-1$ because $lC' \cap E_n = \bar{n}$. So $r\mathfrak{C} \cap E_n = \{n, n-1\}$, $r_t\mathfrak{C} \cap E_n = \{\bar{n}, n-1\}$, $l\mathfrak{C} \cap E_n = \{\bar{n}, n-1\}$ and $l_t\mathfrak{C} \cap E_n = \{n, n-1\}$. Similarly $r\mathfrak{C}' \cap E_n = \{\bar{n}, n-1\}$, $r_t\mathfrak{C}' \cap E_n = \{\bar{n}, \overline{n-1}\}$, $l\mathfrak{C}' \cap E_n = \{\bar{n}, \overline{n-1}\}$ and $l_t\mathfrak{C}' \cap E_n = \{\bar{n}, n-1\}$. Hence $\tilde{f}_i(r\mathfrak{C} \otimes l\mathfrak{C}) = r_{t'}\mathfrak{C}' \otimes l_{t'}\mathfrak{C}'$ and $\tilde{f}_i(r_t\mathfrak{C} \otimes l_t\mathfrak{C}) = r\mathfrak{C}' \otimes l\mathfrak{C}'$. We have

$$\begin{aligned} &S_n^D(w) = r\mathfrak{C} \otimes l\mathfrak{C} \Rightarrow S_n^D(w') = r_{t'}\mathfrak{C}' \otimes l_{t'}\mathfrak{C}' \quad \text{and} \\ &S_n^D(w) = r_t\mathfrak{C} \otimes l_t\mathfrak{C} \Rightarrow S_n^D(w') = r\mathfrak{C}' \otimes l\mathfrak{C}'. \end{aligned} \tag{14}$$

When $i = n$ and $n-1 \in |C|$, we obtain (14) by similar arguments. Finally (i) follows from (13) and (14). □

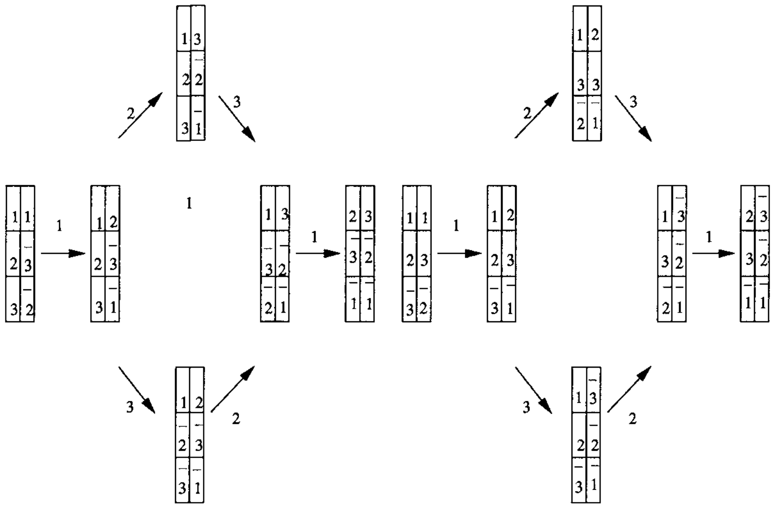


Figure 2. The connected components of $V(\Lambda_3^D)^{\otimes 2}$ and $V(\Lambda_2^D)^{\otimes 2}$ isomorphic to $V(\omega_1^D)$ for $U_q(so_6)$.

Example 4.1.3 Suppose $n = 7$ and consider the admissible column C of type D such that $\mathrm{w}(C) = 67\bar{7}\bar{7}\bar{6}$. Then $\mathrm{w}(lC) = 3457\bar{6}$, $\mathrm{w}(rC) = 67\bar{5}\bar{4}\bar{3}$. So $(t, \bar{t}) = (2, \bar{2})$ and, by identifying the spin columns with the set of letters that they contain, we have $l\mathfrak{C} = \{3457\bar{6}\bar{2}\bar{1}\}$, $r\mathfrak{C} = \{1267\bar{5}\bar{4}\bar{3}\}$, $l_t\mathfrak{C} = \{23457\bar{6}\bar{1}\}$, $r_t\mathfrak{C} = \{167\bar{5}\bar{4}\bar{3}\bar{2}\}$. We have $S_n^D(\mathrm{w}(C)) = r_t\mathfrak{C} \otimes l_t\mathfrak{C}$ and $S_{n-1}^D(\mathrm{w}(C)) = r\mathfrak{C} \otimes l\mathfrak{C}$ for $r\mathfrak{C} \notin B(v_{\Lambda_n^D})$.

Although C must be the empty column in Lemmas 4.1.1 and 4.1.2, we only use these lemmas with $h(C) \geq 1$ in the sequel. Figure 2 below describe the connected components of $V(\Lambda_3^D)^{\otimes 2}$ and $V(\Lambda_2^D)^{\otimes 2}$ isomorphic to the vector representation $V(\Lambda_1^D)$ of $U_q(so_6)$ (see also (5)).

Note that it is possible to describe explicitly the isomorphisms $(S^B)^{-1}, (S_n^D)^{-1}$ and $(S_{n-1}^D)^{-1}$. The reader interested by this subject is referred to [11].

4.2. *Plactic monoid for $\mathfrak{G}_n$*

Let λ be a dominant weight such that $\lambda \notin \Omega_+$. If $\lambda \in P_+^B$ then λ has a unique decomposition $\lambda = \Lambda_n^B + \lambda'$ with $\lambda' \in \Omega_+^B$. We set $v_\lambda^B = v_{\lambda'} \otimes v_{\Lambda_n^B}$. Then v_λ^B is the highest weight vector of $B(v_\lambda^B)$, a connected component of $\mathfrak{G}_n^B$ isomorphic to $B^B(\lambda)$. Denote by $Y(\lambda)$ the diagram obtained by adding a K.N-diagram of height n to $Y(\lambda')$.

When $\lambda \in P_+^D$, λ has a unique decomposition of type $\lambda = \Lambda_n^D + \lambda'$ with $\lambda' \in \Omega_-^D$ and $\bar{\omega}_n^D$ not appearing in λ' or $\lambda = \Lambda_{n-1}^D + \lambda'$ with $\lambda' \in \Omega_+^D$ and ω_n^D not appearing in λ'. According to this decomposition we set $v_\lambda^D = v_{\lambda'} \otimes v_{\Lambda_n^D}$ or $v_\lambda = v_{\lambda'} \otimes v_{\Lambda_{n-1}^D}$. Then v_λ^D is the highest weight vector of $B(v_\lambda^D)$, a connected component of $\mathfrak{G}_n^D$ isomorphic to $B^D(\lambda)$. If $Y(\lambda') = (Y', \varepsilon)$ (see 8) with $\varepsilon \in \{-, 0, +\}$, we set $Y(\lambda) = (Y, \varepsilon)$ where Y is the diagram obtained by adding a K.N diagram of height n to Y'.

Given a tabloid τ and a spin column $\mathfrak{C}$, the spin tabloid $[\mathfrak{C}, T]$ is obtained by adding $\mathfrak{C}$ in front of τ. The reading of the spin tabloid $[\mathfrak{C}, \tau]$ is $w([\mathfrak{C}, \tau]) = \mathrm{w}(\tau) \otimes \mathfrak{C} = \mathrm{w}(\tau)\mathfrak{C}$. Note that the vertices of $B(v_\lambda)$ are readings of spin tabloids.

Definition 4.2.1

- Let $\lambda \in P_+^B$ such that $\lambda \notin \Omega_+^B$. A spin tabloid is a spin tableau of type B and shape $Y(\lambda)$ if its reading is a vertex of $B(v_\lambda^B)$.
- Let $\lambda \in P_+^D$ such that $\lambda \notin \Omega_+^D$. A spin tabloid is a spin tableau of type D and shape $Y(\lambda)$ if its reading is a vertex of $B(v_\lambda^D)$.

It follows from this definition that for $\mathfrak{T}_1$ and $\mathfrak{T}_2$ two spin tableaux $\mathfrak{T}_1 \sim \mathfrak{T}_2 \Leftrightarrow \mathfrak{T}_1 = \mathfrak{T}_2$. It is possible to extend Definition 3.1.17 to a spin tableau $[\mathfrak{C}, C]$ of type D with C an admissible column of type D. We will say that $[\mathfrak{C}, C]$ contains an a-configuration even or odd when this configuration appears in the tableau of two columns $C_{\mathfrak{C}}C$ where $C_{\mathfrak{C}}$ is the admissible column of type D and height n containing the letters of $\mathfrak{C}$. Kashiwara and Nakashima have obtained in [4] a combinatorial description of the orthogonal spin tableaux equivalent to the following:

Theorem 4.2.2

- $\mathfrak{T} = [\mathfrak{C}, T]$ *is a spin tableau of type B if and only if T is a tableau of type B and the rows of* $[\mathfrak{C}, lC_1]$ *weakly increase from left to right.*
- $\mathfrak{T} = [\mathfrak{C}, T]$ *is a spin tableau of type D if and only if T is a tableau of type D, the rows of* $[\mathfrak{C}, lC_1]$ *weakly increase from left to right and* $[\mathfrak{C}, lC_1]$ *does not contain an a-configuration (even or odd) with* $q(a) = n - a$.

It follows from the definition above that for any spin tableau $[\mathfrak{C}, T]$ of type D

$$\mathfrak{C} \in B\left(\Lambda_n^D\right) \text{ implies that the shape of } T \text{ is } (Y, \varepsilon) \text{ with } \varepsilon \neq -,$$
$$\mathfrak{C} \in B\left(\Lambda_{n-1}^D\right) \text{ implies that the shape of } T \text{ is } (Y, \varepsilon) \text{ with } \varepsilon \neq +.$$

A generalized tableau is an orthogonal tableau or a spin orthogonal tableau. Similarly to Section 3.2, the quotient sets $\mathfrak{G}_n/\overset{B}{\sim}$ and $\mathfrak{G}_n/\overset{D}{\sim}$ can be respectively identified with the sets of generalized tableaux of type B and D. For x a letter of $\mathcal{B}_n$ or $\mathcal{D}_n$ and $\mathfrak{C}$ a spin column of height n whose greatest letter is z, we write $x \triangle \mathfrak{C}$ when $x \npreceq z$.

Definition 4.2.3 The monoid $\mathfrak{Pl}(B_n)$ is the quotient set of $\mathfrak{B}_n^*$ by the relations:

- R_i^B, $i = 1, \ldots, 5$ defining $Pl(B_n)$,
- R_6^B: for $x \in \mathcal{B}_n$ and $\mathfrak{C}$ a spin column such that $x \triangle \mathfrak{C}$; $\mathfrak{C}x \equiv \mathfrak{C}'$ where $\mathfrak{C}'$ is the spin column such that $\mathrm{wt}(\mathfrak{C}') = \mathrm{wt}(\mathfrak{C}) + \mathrm{wt}(x)$,
- R_7^B: for $x \in \mathcal{B}_n$ and $\mathfrak{C}$ a spin column such that $x \not\triangle \mathfrak{C}$; $\mathfrak{C}x \equiv x'\mathfrak{C}'$ where

$$\begin{cases} x' = \min\{t \in \mathfrak{C};\ t \succeq x\} & \text{if } x \succeq 0 \\ x' = \min\{t \in \mathfrak{C};\ t \succeq x\} \cup \{0\} & \text{if } x \preceq n \end{cases}$$

  and $\mathfrak{C}'$ is the spin column such that $\mathrm{wt}(\mathfrak{C}') = \mathrm{wt}(\mathfrak{C}) + \mathrm{wt}(x) - \mathrm{wt}(x')$,
- R_8^B: for C an admissible column of type B, $S^B(\mathrm{w}(C)) \equiv \mathrm{w}(C)$.

Lemma 2.1.1 implies that the highest weight vertex of the connected component containing a word $\mathfrak{C}x$ with $x \in \mathcal{B}_n$ and $\mathfrak{C}$ a spin column may be written $\mathfrak{C}_n x_0$ where $x_0 \in \{0, 1\}$. So $\mathfrak{C}x \in B(v_{\Lambda_n^B} \otimes 0)$ or $\mathfrak{C}x \in B(v_{\Lambda_n^B} \otimes 1)$. The following lemma gives the interpretation of relations R_6^B and R_7^B in terms of crystal isomorphisms.

Lemma 4.2.4

1. *The vertices of* $B(v_{\Lambda_n^B} \otimes 0)$ *are the words of the form* $\mathfrak{C}x$ *where* $\mathfrak{C}$ *is a spin column and* $x \in \mathcal{B}_n$ *such that* $x \vartriangle \mathfrak{C}$.
2. *The vertices of* $B(v_{\Lambda_n^B} \otimes 1)$ *are the words of the form* $\mathfrak{C}x$ *where* $\mathfrak{C}$ *is a spin column and* $x \in \mathcal{B}_n$ *such that* $x \not\vartriangle \mathfrak{C}$.
3. *Denote by* Ψ *and* Ψ' *the crystal isomorphisms*:

$$\Psi\colon B\big(v_{\Lambda_n^B} \otimes 0\big) \to B\big(v_{\Lambda_n^B}\big)$$
$$\Psi'\colon B\big(v_{\Lambda_n^B} \otimes 1\big) \to B\big(1 \otimes v_{\Lambda_n^B}\big).$$

Then if the word $\mathfrak{C}x$ *occurs in the left hand side a relation* R_6^B *(resp. of* R_7^B*),* $\Psi(\mathfrak{C}x)$ *(resp.* $\Psi'(\mathfrak{C}x)$*) is the word occurring in the right hand side of this relation.*

Proof:

1. Consider a word $\mathfrak{C}x$ such that $x \vartriangle \mathfrak{C}$ and $\tilde{f}_i(\mathfrak{C}x) \neq 0$. Let y be the greatest letter of $\mathfrak{C}$. Set $\tilde{f}_i(\mathfrak{C}x) = \mathfrak{U}t$ where $\mathfrak{U}$ is a spin column and t a letter of $\mathcal{B}_n$. We are going to show that $t \vartriangle \mathfrak{U}$. If y is the greatest letter of $\mathfrak{U}$ then $t \succeq x \succ y$, hence $t \vartriangle \mathfrak{U}$. Otherwise $\tilde{f}_i(\mathfrak{C}x) = \tilde{f}_i(\mathfrak{C})x$ thus $\varepsilon_i(x) = 0$ by (1). When $i \neq n$, we must have $y = \overline{i+1}$, $x \succ y$ and $x \notin \{\bar{i}, i+1\}$ because $\varepsilon_i(x) = 0$. Hence $x \succ \bar{i}$ and $x = t \vartriangle \mathfrak{U}$ for $\bar{i}$ is the greatest letter of $\mathfrak{U}$. When $i = n$, $y = n$ and $x \succ \bar{n}$ because $\varepsilon_n(x) = 0$. We obtain similarly $t \vartriangle \mathfrak{U}$. Hence the set of words $\mathfrak{C}x$ such that $x \vartriangle \mathfrak{C}$ is closed under the action of the $\tilde{f}_i$. By similar arguments we can prove that this set is also closed under the action of the $\tilde{e}_i$. Moreover $v_{\Lambda_n^B} \otimes 0$ is the unique highest weight vertex among these words $\mathfrak{C}x$. Hence $B(v_{\Lambda_n^B} \otimes 0)$ contains exactly the words of the form $\mathfrak{C}x$ such that $x \vartriangle \mathfrak{C}$.
2. Follows immediately from 1.
3. If $x \vartriangle \mathfrak{C}$, $\Psi(\mathfrak{C}x)$ is the unique spin column of weight $\mathrm{wt}(\mathfrak{C}x)$, that is $\Psi(\mathfrak{C}x) = \mathfrak{C}'$ with the notation of R_6^B. When $x \not\vartriangle \mathfrak{C}$, we consider the following cases:

   (i) $x \in \mathfrak{C}$. Set $\Psi(\mathfrak{C}x) = y\mathfrak{D}$. Then we deduce from the equality $\mathrm{wt}(y\mathfrak{D}) = \mathrm{wt}(\mathfrak{C}x)$ that $y = x$ and $\mathfrak{D} = \mathfrak{C}$. Indeed $x\mathfrak{C}$ is the unique vertex of $B(1) \otimes B(v_{\Lambda_n^B})$ of weight $\mathrm{wt}(\mathfrak{C}x)$. Hence $y = x = t$ and $\mathfrak{D} = \mathfrak{C}'$ with the notation of R_6^B.

   (ii) $x \notin \mathfrak{C}$. When $x \succ 0$, set $x = \bar{p}$ and $\bar{k} = \min\{t \in \mathfrak{C}; t \succeq x\}$. Then $\{p, p-1, \ldots, k+1\} \subset \mathfrak{C}$. By using the formulas (1) and (2) we obtain

   $$\tilde{f}_k \cdots \tilde{f}_{p-2}\tilde{f}_{p-1}(\mathfrak{C}\bar{p}) = \mathfrak{C}\bar{k}$$

   So, by (i), $\mathfrak{C}\bar{k} \sim \bar{k}\mathfrak{C}$ which implies

   $$\mathfrak{C}\bar{p} \sim \tilde{e}_{p-1} \cdots \tilde{e}_k(\bar{k}\mathfrak{C}) = \bar{k}\tilde{e}_{p-1} \cdots \tilde{e}_k(\mathfrak{C}) = \bar{k}\mathfrak{C}'$$

with the notation of R_7^B. It means that $\Psi(\mathfrak{C}x) = \bar{k}\mathfrak{C}'$. When $x = 0$, we have $\tilde{f}_{x'-1}\cdots\tilde{f}_1\tilde{f}_n(\mathfrak{C}0) = \mathfrak{C}\bar{k}$. Because $\{n, n-1, \dots, k+1\} \subset \mathfrak{C}$ and we terminate as above. When $x = p \prec 0$ and $\min\{t \in \mathfrak{C}; t \succeq x\} \cup \{0\} = k \prec 0$, we have $\{\bar{p}, \overline{p+1}, \dots, \overline{k-1}\} \subset \mathfrak{C}$. So $\tilde{f}_{\overline{k-1}}\cdots\tilde{f}_{p+1}\tilde{f}_p(\mathfrak{C}p) = \mathfrak{C}k$ and the proof is similar. If $\min\{t \in \mathfrak{C}; t \succeq p\} \cup \{0\} = 0$, $\{\bar{p}, \overline{p+1}, \dots, \bar{n}\} \subset \mathfrak{C}$. Then $\tilde{f}_n\cdots\tilde{f}_{p+1}\tilde{f}_p(\mathfrak{C}p) = \mathfrak{C}0 \sim \bar{n}\mathfrak{C}^\circ$ with $\mathfrak{C}^\circ = \mathfrak{C} - \{\bar{n}\} + \{n\}$ by the case $x = 0$. So formulas (1) and (2) imply that $\mathfrak{C}x \sim \tilde{e}_p\cdots\tilde{e}_n(\bar{n}\mathfrak{C}^\circ) = \tilde{e}_n(\bar{n})\tilde{e}_p\cdots\tilde{e}_{n-1}(\mathfrak{C}^\circ) = 0\mathfrak{C}'$ with the notation of R_7^B. It means that $\Psi(\mathfrak{C}x) = 0\mathfrak{C}'$. □

Definition 4.2.5 The monoid $\mathfrak{Pl}(D_n)$ is the quotient set of $\mathfrak{D}_n^*$ by the relations:

- R_i^D, $i = 1, \dots, 5$ defining $Pl(D_n)$,
- R_6^D: for $x \in \mathcal{D}_n$ and $\mathfrak{C}$ a spin column such that $x \triangle \mathfrak{C}$; $\mathfrak{C}x \equiv \mathfrak{C}'$ where $\mathfrak{C}'$ is the spin column such that $\mathrm{wt}(\mathfrak{C}') = \mathrm{wt}(\mathfrak{C}) + \mathrm{wt}(x)$,
- R_7^D: for $x \in \mathcal{D}_n$ and $\mathfrak{C}$ a spin column such that $x \ntriangle \mathfrak{C}$; $\mathfrak{C}x \equiv x'\mathfrak{C}'$ where $x' = \min\{t \in \mathfrak{C}; t \succeq x\}$ and $\mathfrak{C}'$ is the spin column such that $\mathrm{wt}(\mathfrak{C}') = \mathrm{wt}(\mathfrak{C}) + \mathrm{wt}(x) - \mathrm{wt}(x')$,
- R_8^D: for C an admissible column of type D, $S_n^D(\mathrm{w}(C)) \equiv \mathrm{w}(C)$ and $S_{n-1}^D(\mathrm{w}(C)) \equiv \mathrm{w}(C)$.

We can prove by using similar arguments to those of Lemma 4.2.4 that the relations R_6^D and R_7^D read from left to right describe respectively the crystal isomorphisms

$$\begin{cases} B\big(v_{\Lambda_n^D} \otimes \bar{n}\big) \to B\big(v_{\Lambda_{n-1}^D}\big) \\ B\big(v_{\Lambda_{n-1}^D} \otimes n\big) \to B\big(v_{\Lambda_n^D}\big) \end{cases} \quad \text{and} \quad \begin{cases} B\big(v_{\Lambda_n^D} \otimes 1\big) \to B\big(1 \otimes v_{\Lambda_n^D}\big) \\ B\big(v_{\Lambda_{n-1}^D} \otimes 1\big) \to B\big(1 \otimes v_{\Lambda_{n-1}^D}\big) \end{cases}. \tag{15}$$

Lemma 4.2.6 *Let w_1 and w_2 be two vertices of $\mathfrak{G}_n$ such that $w_1 \equiv w_2$. Then for $i = 1, \dots, n$:*

$$\tilde{e}_i(w_1) \equiv \tilde{e}_i(w_2) \quad \textit{and} \quad \varepsilon_i(w_1) = \varepsilon_i(w_2),$$
$$\tilde{f}_i(w_1) \equiv \tilde{f}_i(w_2) \quad \textit{and} \quad \varphi_i(w_1) = \varphi_i(w_2).$$

Proof: By induction we can suppose that w_2 is obtained from w_1 by applying only one plactic relation. In this case we write $w_1 = u\hat{w}_1v$ and $w_2 = u\hat{w}_2v$ where $u, v, \hat{w}_1, \hat{w}_2$ are factors of w_1 and w_2 such that $\hat{w}_1 \equiv \hat{w}_2$ by one of the relations R_i. Formulas (1) and (2) imply that it is enough to prove the lemma for $\hat{w}_1$ and $\hat{w}_2$. This last point is immediate because we have seen that each plactic relation may be interpreted in terms of a crystal isomorphism. □

So we obtain $w_1 \equiv w_2 \Rightarrow w_1 \sim w_2$. To establish the implication $w_1 \sim w_2 \Rightarrow w_1 \equiv w_2$, it suffices, as in Section 3.2 to prove that two highest weight vertices of $\mathfrak{G}_n^B$ (resp. $\mathfrak{G}_n^D$) with the same weight are congruent in $\mathfrak{Pl}(B_n)$ (resp. $\mathfrak{Pl}(D_n)$). Given a vertex $w \in \mathfrak{G}_n$, we know by Theorems 4.2.2 and 3.1.18 that there exists a unique generalized tableau $\mathfrak{P}(w)$ such that

$$\mathrm{w}(\mathfrak{P}(w)) \sim w.$$

Lemma 4.2.7 *Let w be a highest weight vertex of $\mathfrak{G}_n$. Then* $\mathrm{w}(\mathfrak{P}(w)) \equiv w$.

Proof: By using relations R_6 and R_7, w is congruent to a word $u\mathfrak{U}$ such that $u \in G_n$ and $\mathfrak{U} \in \mathfrak{G}_n$. Relation R_8 implies that any word consisting in an even number of spin columns is congruent to a vertex of G_n. If $\mathfrak{U}$ contains an even number of spin columns, there exists $v \in G_n$ such that $w \equiv v$. We have $\mathfrak{P}(w) = P(v)$ because $w \equiv v \Rightarrow w \sim v$. Thus $\mathrm{w}(\mathfrak{P}(w)) = \mathrm{w}(P(v)) \equiv v \equiv w$ and the lemma is proved. If w contains an odd number of spin columns, there exists a vertex $v \in G_n$ and a spin column $\mathfrak{C}$ such that $w \equiv v\mathfrak{C}$. Set $P(v) = T$. Then $w \equiv \mathrm{w}(T)\mathfrak{C}$. Write $T = C\hat{T}$ where C is the first column of T and $\hat{T}$ the tableau obtained by erasing C in T. By Lemma 2.1.1, $\mathrm{w}(T)$ is a highest weight vertex because w is a highest weight vertex of $\mathfrak{G}_n$. In particular, $\mathrm{w}(C)$ is a highest weight vertex. Set $p = h(C)$.

Suppose first $w \in \mathfrak{G}_n^B$. We have $S^B(\mathrm{w}(C)) = \mathfrak{C}_n\mathfrak{C}_p$ (see Lemma 4.1.1). So $w \equiv \mathrm{w}(\hat{T})\mathfrak{C}_n\mathfrak{C}_p\mathfrak{C}$. By Lemma 2.1.1 we must have $\varepsilon_i(\mathfrak{C}) = 0$ for $i = p+1, \ldots, n$. This implies that the letters of $\{\overline{p+1}, \ldots, \bar{n}\}$ do not appear in $\mathfrak{C}$. Indeed $\bar{n} \notin \mathfrak{C}$ otherwise $\varepsilon_n(\mathfrak{C}) \neq 0$ and if $\bar{q} \succ \bar{n}$ is the lowest barred letter of $\{\overline{p+1}, \ldots, \bar{n}\}$ appearing in $\mathfrak{C}$ we obtain $\varepsilon_q(\mathfrak{C}) = 1 \neq 0$ because $q+1 \in \mathfrak{C}$. So $\mathfrak{C}$ contains the letters of $\{p+1, \ldots, n\}$. Let $\{x_1 \prec \cdots \prec x_s\}$ be the set of unbarred letters $\preceq p$ that occur in $\mathfrak{C}$. By Lemma 4.1.1, we have

$$S^B(x_1 \cdots x_s \underbrace{0 \cdots 0}_{n-p \text{ times}}) = \mathfrak{C}_p\mathfrak{C}.$$

Hence

$$w \equiv \mathrm{w}(\hat{T})\mathfrak{C}_n(x_1 \cdots x_s \underbrace{0 \cdots 0}_{n-p \text{ times}})$$

and by applying relations R_6^B and R_7^B we have $w \equiv \mathrm{w}(\hat{T})(x_1 \cdots x_s)\mathfrak{C}_n$. Write $T' = x_s \to (\to \cdots x_1 \to \hat{T})$. Then $[\mathfrak{C}_n, T']$ is a spin orthogonal tableau and $\mathrm{w}(T')\mathfrak{C}_n \equiv w$. So $T' = \mathfrak{P}(w)$ and the lemma is true.

Suppose now $w \in \mathfrak{G}_n^D$. If the shape of $\hat{T}$ is (Y, ε) with $\varepsilon \neq -$, we consider $S_n^D(\mathrm{w}(C)) = \mathfrak{C}_n\mathfrak{C}_p$. Then $[\mathfrak{C}_n, \hat{T}]$ is a spin tableau and the proof is similar to that of the type B case. If the shape of $\hat{T}$ is (Y, ε) with $\varepsilon = -$, it suffices to consider $S_{n-1}^D(\mathrm{w}(C)) = \mathfrak{C}_{n-1}\mathfrak{C}_{n-1}$ where instead of $S_n^D(\mathrm{w}(C))$. □

Now if w_1 and w_2 are two highest weight vertices of $\mathfrak{G}_n$ with the same weight λ, we have $\mathfrak{P}(w_1) = \mathfrak{P}(w_2)$ because there is only one orthogonal tableau of highest weight λ. Then the lemma above implies that $w_1 \equiv w_2$. We can state the

Theorem 4.2.8 *Let w_1 and w_2 be two vertices of $\mathfrak{G}_n$. Then $w_1 \sim w_2$ if and only if $w_1 \equiv w_2$.*

For any vertex $w \in \mathfrak{G}_n$, it is possible to obtain $\mathfrak{P}(w)$ by using an insertion algorithm analogous to that described in Section 3. Considering the sequence of shape of the intermediate generalized tableaux appearing during the computation of $\mathfrak{P}(w)$, we obtain a $\mathfrak{Q}$-symbol $\mathfrak{Q}(w)$. Then for w_1 and w_2 two vertices of $\mathfrak{G}_n$ we have:

$$w_1 \leftrightarrow w_2 \Leftrightarrow \mathfrak{Q}(w_1) = \mathfrak{Q}(w_1)$$

where $w_1 \leftrightarrow w_2$ means that w_1 and w_2 occur in the same connected component of $\mathfrak{G}_n$. The reader interested in this subject is referred to [11].

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