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1-Homogeneous Graphs with Cocktail Party μ-Graphs

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Abstract. Let Γ be a graph with diameter $d \geq 2$. Recall Γ is 1-homogeneous (in the sense of Nomura) whenever for every edge xy of Γ the distance partition

$$\{\{z \in V(\Gamma) \mid \partial(z, y) = i, \partial(x, z) = j\} \mid 0 \leq i, j \leq d\}$$

is equitable and its parameters do not depend on the edge xy. Let Γ be 1-homogeneous. Then Γ is distance-regular and also locally strongly regular with parameters (v', k', λ', μ'), where $v' = k, k' = a_1, (v' - k' - 1)\mu' = k'(k' - 1 - \lambda')$ and $c_2 \geq \mu' + 1$, since a μ-graph is a regular graph with valency μ'. If $c_2 = \mu' + 1$ and $c_2 \neq 1$, then Γ is a Terwilliger graph, i.e., all the μ-graphs of Γ are complete. In [11] we classified the Terwilliger 1-homogeneous graphs with $c_2 \geq 2$ and obtained that there are only three such examples. In this article we consider the case $c_2 = \mu' + 2 \geq 3$, i.e., the case when the μ-graphs of Γ are the Cocktail Party graphs, and obtain that either $\lambda' = 0$, $\mu' = 2$ or Γ is one of the following graphs: (i) a Johnson graph $J(2m, m)$ with $m \geq 2$, (ii) a folded Johnson graph $\bar{J}(4m, 2m)$ with $m \geq 3$, (iii) a halved m-cube with $m \geq 4$, (iv) a folded halved $(2m)$-cube with $m \geq 5$, (v) a Cocktail Party graph $K_{m\times 2}$ with $m \geq 3$, (vi) the Schläfli graph, (vii) the Gosset graph.

Keywords: distance-regular graph, 1-homogeneous, Cocktail Party graph, Johnson graph

1. Introduction

We study 1-homogeneous graphs in the sense of Nomura [16] (defined later in this section). Some examples of such graphs are distance-regular graphs with at most one i, such that $a_i \neq 0$ (e.g. bipartite graphs, complete multipartite graphs $K_{m\times t}$ and generalized Odd graphs, in particular triangle free strongly regular graphs), regular near $(2n)$-gons (i.e., distance-regular graphs with $a_i = c_i a_1$ for all i and no induced $K_{1,2,1}$), Taylor graphs (antipodal distance-regular 2-covers of complete graphs with diameter three), the Johnson graphs $J(2d, d)$, the folded Johnson graphs $\bar{J}(4d, 2d)$, the halved n-cubes, the folded halved $(4n)$-cubes and 3-valent distance-regular graphs [11, Proposition 3.5].

Let Γ be a graph with diameter at least 2, and let x, y be vertices of Γ at distance 2. Then the **μ-graph** of x and y is the subgraph of Γ induced by their common neighbours. Let Γ be 1-homogeneous. Then Γ is distance-regular, locally strongly regular and, by [11, Proposition 2.1], the local graphs have the same parameters. Let us denote them by (v', k', λ', μ'). Obviously, we have $v' = k, k' = a_1, (v' - k' - 1)\mu' = k'(k' - 1 - \lambda')$

and $c_2 \geq \mu' + 1$, since a μ-graph is a regular graph with valency μ' by [9, Theorem 3.1]. The case $c_2 = \mu' + 1 \geq 2$, i.e., the case when Γ is a Terwilliger graph, was classified in [11]. The μ-graphs of Terwilliger graphs are complete graphs. Since many of the above mentioned examples of 1-homogeneous graphs have the property that their μ-graphs are complete multipartite graphs, it is natural to study 1-homogeneous graphs or even some more general graphs satisfying this property. Alternative motivation comes from the study of extended generalized quadrangles, see for example [5] and [22].

We establish some general properties of distance-regular graphs with certain local structure, parameters and eigenvalues. There are some families of 1-homogeneous graphs for which we can show that their μ-graphs are complete multipartite. One such (obvious) example is the case $c_2 = \mu' + 2 \geq 3$, i.e., the case when the μ-graphs are Cocktail Party graphs. In this case we show that either $\lambda' = 0$ and $\mu' = 2$ or the smallest eigenvalue of each local graph is -2 and so, by Seidel's classification [17], [3, Theorem 3.12.4], either $\lambda' = 0$ and $\mu' = 2$ or each local graph of Γ is one of the well known strongly regular graphs. In the latter case we show that Γ must be one of the well known distance-regular graphs. Before we state the precise statement of our main result, we establish some notation and review basic definitions, for more details see Brouwer, Cohen and Neumaier [3], and Godsil [8]. At the end of this section we describe the organization of this paper.

Let us first recall that an **equitable partition** of a graph is a partition $\pi = \{P_1, \ldots, P_s\}$ of its vertices into cells, such that for all $i, j \in \{1, \ldots, s\}$ the number c_{ij} of neighbours, which a vertex in the cell P_i has in the cell P_j, is independent of the choice of the vertex in P_i. Let Γ be a connected graph with diameter d. For a vertex x of Γ we define $\Gamma_i(x)$ to be the set of vertices at distance i from x, and set $\Gamma(x) = \Gamma_1(x)$. For $y \in \Gamma_i(x)$ and integers j and h we define $D_j^h(x, y) = \Gamma_j(x) \cap \Gamma_h(y)$ and $p_{jh}^i(x, y) = |D_j^h(x, y)|$. Then Γ is i-**homogeneous** in the sense of Nomura [16] when the distance partition corresponding to any pair x, y of vertices at distance i, i.e., the collection of nonempty sets $D_h^j(x, y)$, is an equitable partition, and the parameters corresponding to all such equitable partitions are independent of vertices x and y at distance i. Note that the graph Γ is 0-homogeneous if and only if it is **distance-regular**, and that if Γ is 1-homogeneous then it is distance-regular.

Let Γ be a graph. As usually, we denote the distance between vertices x and y of Γ by $\partial(x, y)$. If x, y and z are vertices of Γ such that $\partial(x, y) = 1$, $\partial(x, z) = \partial(y, z) = 2$, then we define a (triple) intersection number $\alpha(x, y, z) = |\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)|$ (see figure 2.1(b) and (c) and also figure 1.1). We say that the parameter α of Γ **exists** when $\alpha = \alpha(x, y, z)$ for all triples of vertices (x, y, z) of Γ such that $\partial(x, y) = 1$, $\partial(x, z) = \partial(y, z) = 2$. If Γ is 1-homogeneous graph with diameter $d \geq 2$ and $a_2 \neq 0$, then α exists. A strongly regular graph with $a_2 \neq 0$, that is locally strongly regular is 1-homogeneous if and only if α exists (see figure 1.1(a)). Similarly, we say that the intersection number p_{jh}^i **exists** in a graph Γ if $p_{jh}^i(x, y) = p_{jh}^i$ for all pairs of vertices x and y at distance i. Of course, if Γ is distance-regular, then for all i, j and h the numbers p_{jh}^i exist. Let $a_i(x, y) := p_{1i}^i(x, y)$, $b_i(x, y) := p_{1,i+1}^i(x, y)$ and $c_i(x, y) := p_{1,i-1}^i(x, y)$.

For a vertex x of a graph Γ we define the **local graph** $\Delta(x)$ as the subgraph of Γ, induced by the neighbours of x. If Γ is distance-regular, then $\Delta(x)$ has $k = b_0$ vertices and valency a_1. Let $\mathcal{C}$ be a graph (or a class of graphs). The graph Γ is said to be **locally** (resp. **μ-locally**) $\mathcal{C}$, when each local graph (resp. each μ-graph) of Γ is isomorphic to (or a member of) $\mathcal{C}$.

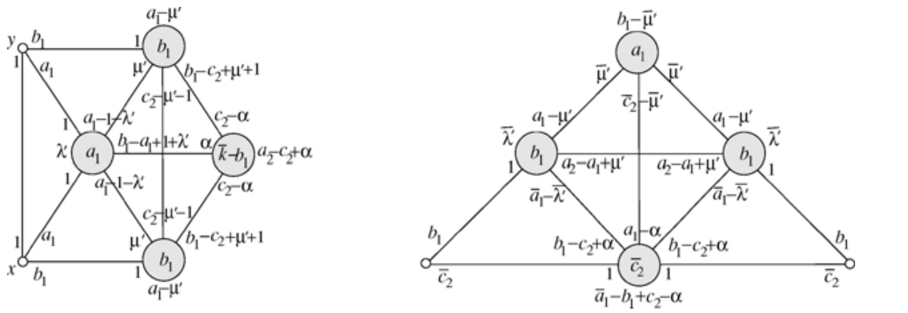

Figure 1.1. Let Γ be a strongly regular graph (v, k, a_1, c_2) with $a_2 = k - c_2 \neq 0$. Then Γ is 1-homogeneous, i.e., it is a locally strongly regular graph (v', k', λ', μ'), and for which α exists, if and only if its complement is 2-homogeneous. For the second subconstituent of the complement of Γ is isomorphic to the complement of a local graph of Γ and for vertices x and y of Γ at distance 2 a vertex in $D_1^1(x, y)$ has $a_1 - \alpha$ neighbours in the set $D_2^2(x, y)$. (a) the distance partition of 1-homogeneous graph Γ corresponding to two adjacent vertices; (b) the distance partition of the complement of 1-homogeneous graph Γ, corresponding to two vertices at distance 2, where $\overline{a_1} = v - 2k + c_2 - 2$, $\overline{c_2} = v - 2k + a_1$, $\overline{\lambda'} = k - 2a_1 + \mu' - 2$, $\overline{\mu'} = k - 2a_1 + \lambda'$, $\bar{k} = kb_1/c_2$ and $\bar{k} - b_1 = a_2 b_1/c_2$.

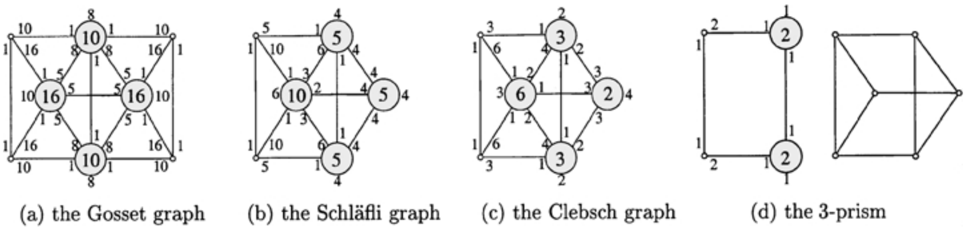

Figure 1.2. A tower of graphs with their distance partitions corresponding to two adjacent vertices (all but the last one are 1-homogeneous graphs): (a) the Gosset graph is a unique distance-regular graph with intersection array $\{27, 10, 1; 1, 10, 27\}$, an antipodal 2-cover of the complete graph K_{28}, and it is locally Schläfli graph see [3, pp. 103, 313]; (b) the Schläfli graph is a unique strongly regular graph (27, 16, 10, 8) and it is locally halved 5-cube, see [3, p. 103]; (c) the halved 5-cube, also known as the Clebsch graph, is a unique strongly regular graph (16, 10, 6, 6) and it is locally $J(5, 2)$, i.e., the complement of the Petersen graph, see [3, p. 264] (so the local graph is not 1-homogeneous), the Johnson graph $J(5, 2)$ is a unique strongly regular graph (10, 6, 3, 4) and is locally the 3-prism; (d) the 3-prism has two different distance partitions corresponding to an edge.

Note that the distance partition of the Gosset graph corresponding to two adjacent vertices is at the same time also its distance partition corresponding to two vertices at distance 2 (actually, in general, a 1-homogeneous 2-cover with diameter D is also $(D-1)$-homogeneous).

We are now ready to state the main result of this paper.

Theorem 1.1 *Let Γ be a 1-homogeneous graph with diameter $d \geq 2$. Recall that for each vertex x of Γ the local graph $\Delta(x)$ is strongly regular with parameters independent of x; we denote these parameters by (v', k', λ', μ'). If $c_2 = \mu' + 2 \geq 3$, then either $\lambda' = 0$, $\mu' = 2$, $d \geq 3$ and $\alpha = 1$, or Γ is one of the following graphs*:

(i) *a Johnson graph $J(2m, m)$ with $m \geq 2$,*

(ii) *a folded Johnson graph* $\bar{J}(4m, 2m)$ *with* $m \geq 2$,
(iii) *a halved m-cube with* $m \geq 4$,
(iv) *a folded halved* $(4m)$*-cube with* $m \geq 2$,
(v) *a Cocktail Party graph* $K_{m\times 2}$ *with* $m \geq 3$,
(vi) *the Schläfli graph with intersection array* $\{16, 5; 1, 8\}$,
(vii) *the Gosset graph with intersection array* $\{27, 10, 1; 1, 10, 27\}$.

Remark The Gosset graph is locally the Schläfli graph, see figure 1.2.

The graph Γ in the above statement is μ-locally the Cocktail Party graph. Our study is part of a larger project to classify 1-homogeneous graphs that are μ-locally complete multipartite. There are also very interesting examples of 1-homogeneous graphs that are μ-locally several copies of complete multipartite graphs or even the 2-extension of the halved 5-cube.

The multipartite graph $K_{t\times n}$ is the complement of t cliques of size n, i.e., the multipartite graph $K_{n1,n2,\dots,nt}$ with $n_1 = n_2 = \cdots = n_t = n$. In particular, $K_{t\times 2}$ is the Cocktail Party graph. Let Γ be a distance-regular graph with diameter $d \geq 2$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. An easy eigenvalue interlacing argument guarantees $\theta_1 \geq 0$ and $\theta_d \leq -\sqrt{2}$. We say that Γ is **tight** whenever it is not bipartite and

$$k(a_1 + b^+ b^-) = (a_1 - b^+)(a_1 - b^-),$$

where

$$b^+ = -1 - \frac{b_1}{1+\theta_d} \quad \text{and} \quad b^- = -1 - \frac{b_1}{1+\theta_1}.$$

For $d = 2$ we have $b_1 = -(1+\theta_1)(1+\theta_2)$, $b^+ = \theta_1$, $b^- = \theta_2$, and therefore Γ is tight (i.e., $\theta_1 = 0$) if and only if it is a complete multipartite graph $K_{t\times n}$ with $t > 2$ (i.e., $a_1 \neq 0$ and $\mu = k$). Tight graphs of diameter $d \geq 3$ were characterized in a number of ways in [10]. For example, if Γ is a distance-regular graph with diameter $d \geq 2$, then Γ is tight if and only if Γ is 1-homogeneous with $a_1 \neq 0$ and $a_d = 0$. Some examples of tight graphs are the Patterson graph and 10 tight antipodal distance-regular graphs with diameter four.

Corollary 1.2 *Let* Γ *be a tight graph with diameter* $d \geq 2$. *Recall that for each vertex* x *of* Γ *the local graph* $\Delta(x)$ *is strongly regular with parameters independent of* x; *we denote these parameters by* (v', k', λ', μ'). *If* $c_2 = \mu' + 2$, *then either* $\lambda' = 0$, $\mu' = 2$, $d \geq 3$ *and* $\alpha = 1$, *or* Γ *is one of the following graphs*:
(i) *a Johnson graph* $J(2m, m)$, *with* $m \geq 2$,
(ii) *a halved* $(2m)$*-cube*, *with* $m \geq 2$,
(iii) *a Cocktail Party graph* $K_{m\times 2}$, *with* $m \geq 3$,
(iv) *the Gosset graph with intersection array* $\{27, 10, 1; 1, 10, 27\}$.

Proof: If $d = 2$ then Γ is a complete multipartite graph $K_{m\times n}$ with $m \geq 3$, and $c_2 - \mu' = (m-2)n - (m-3)n = n = 2$ implies that Γ is a Cocktail Party graph $K_{m\times 2}$ with $m \geq 3$. So let us now assume $d \geq 3$. Since Γ is a tight graph, it is locally connected and $a_d = 0$ by [10, Theorem 12.6 and Theorem 11.7]. Now $\mu' = 0$, implies that the local graph is complete, and hence Γ is complete as well, so we have $\mu' \geq 1$ and $c_2 = \mu' + 2 \geq 3$. Hence, Γ is one of the graphs in the list of Theorem 1.1 with $a_d = 0$ and we are done. $\square$

The paper is organized in the following way. In Section 2 we introduce some local conditions that are satisfied by a 1-homogeneous graph having all the μ-graphs equal to the complete multipartite graph $K_{t\times n}$. Then we establish some basic properties of graphs that satisfy these local conditions. The most important such property is that the intersection parameter α can only be t or $t-1$. Let Γ be a graph that satisfies these local conditions. In Section 3 we study the smallest eigenvalue of the local graphs of Γ. If $\alpha = t$, then $-n$ is the smallest eigenvalue of Γ. If $\alpha = t-1$, then either $n \neq 2$, or $\lambda' = 0$ and $\mu' = 2$. This sets the stage for our classification of 1-homogeneous graphs with $c_2 = \mu' + 2$. In Section 4 we determine all such graphs that are additionally locally grid graphs or locally triangular graphs. In Section 5 we prove the main theorem.

2. Local regularity conditions

We establish some basic properties of graphs that satisfy certain local regularity conditions.

If a graph Γ is regular with v vertices and valency k in which any two vertices at distance 2 have precisely $\mu = \mu(\Gamma)$ common neighbours, then it is called **co-edge-regular** with parameters (v, k, μ), see [3, p. 3]. Let Γ be a distance-regular graph with diameter d. For vertices x and y of Γ at distance i, $1 \leq i \leq d$, we define the sets $C_i(x, y) = \Gamma_{i-1}(x) \cap \Gamma(y)$, $A_i(x, y) = \Gamma_i(x) \cap \Gamma(y)$ and $B_i(x, y) = \Gamma_{i+1}(x) \cap \Gamma(y)$, and say that Γ has the **CAB$_j$ property**, $j \geq 1$, when the partition

$$\mathrm{CAB}_i(x, y) = \{C_i(x, y), A_i(x, y), B_i(x, y)\}$$

of the local graph of y is equitable for each pair of vertices x and y of Γ at distance $i \leq j$. Since the graph Γ with $a_1 \neq 0$ is 1-homogeneous graph if and only if it has CAB$_d$ property (see [11, Theorem 3.1]), we can now take a local approach to 1-homogeneous graphs.

We will start with a study of a distance-regular graph Γ with diameter at least 2 that is μ-locally the complete multipartite graph $K_{t\times n}$, $n, t \in \mathbb{N}$, and for which $a_2 \neq 0$ and the intersection number α exists with $\alpha \geq 1$. Since $\alpha \leq a_1$ we have also $a_1 \neq 0$. The intersection number α exists in a distance-regular graph with $a_2 \neq 0$ when it has additionally the CAB$_2$ property (α is equal to the number of neighbours that a vertex of A_2 has in C_2), therefore also for 1-homogeneous graphs. Certain examples of tight graphs are μ-locally the complete multipartite graph $K_{t\times n}$, $n, t \in \mathbb{N}$, cf. [10] and [12]. If Γ is 1-homogeneous, then $c_2 = \mu' + 1$ if and only if Γ is a Terwilliger graph, i.e., Γ is μ-locally K_t. Such graphs with $c_2 > 1$ have been classified in [11, Theorem 4.10].

It is quite natural to assume $a_2 \neq 0$ in Γ, since otherwise we have, by [3, Proposition 5.5.1] and [3, Proposition 1.1.5], either

(a) $a_1 = 0$, in which case any partition of a local graph of Γ is equitable, or
(b) $a_1 \neq 0$ and $d = 2$, in which case Γ is $K_{(t+1)\times n}$.

Similarly, as in the case of Terwilliger 1-homogeneous graphs [11, Lemma 4.1], there are also only two possibilities for α in the present situation.

Lemma 2.1 *Let Γ be a distance-regular graph with diameter at least 2 that is μ-locally the complete multipartite graph $K_{t\times n}$, and for which $a_2 \neq 0$ and the intersection number α exists with $\alpha \geq 1$. Then the following holds.*

(i) *$c_2 = nt$, each local graph of Γ is μ-locally $K_{(t-1)\times n}$ and co-edge-regular with parameters (v', k', μ'), where $v' = k, k' = a_1$, and $\mu' = n(t-1)$. Moreover, $\alpha a_2 = c_2(a_1 - \mu')$.*

(ii) *Let x and y be vertices of Γ at distance 2. Then for all $z \in D_2^1(x, y)$ the subgraph induced by $\Gamma(z) \cap D_1^1(x, y)$ is complete.*

(iii) *$\alpha \in \{t-1, t\}$, i.e., $t-1 \leq \alpha \leq t$,*

(iv) *Γ is a Terwilliger graph (i.e., $c_2 = \mu' + 1$) if and only if $n = 1$, and*

(v) *Γ is locally connected if and only if $t \neq 1$ (in which case every local graph has diameter 2).*

Proof:

(i) A μ-graph of Γ is $K_{t\times n}$, so it has $c_2 = nt$ vertices. Let x be a vertex of Γ. Since Γ has diameter at least 2, the local graph $\Delta(x)$ is not a complete graph. Any two nonadjacent vertices of $\Delta(x)$ have $\mu' = (t-1)n$ common neighbours in $\Delta(x)$ and these common neighbours induce $K_{(t-1)\times n}$. Hence $\Delta(x)$ is co-edge-regular with parameters (k, a_1, μ'). By a two way counting of the edges between $D_1^1(x, y)$ and $D_2^1(x, y)$, we find $\alpha a_2 = c_2(a_1 - \mu')$.

(ii) Suppose the opposite. Then $|D_1^1(x, y) \cap \Gamma(z)| \geq 2$ and there exist two nonadjacent vertices $u, v \in D_1^1(x, y) \cap \Gamma(z)$. Then $D_1^1(u, v) \supseteq \{x, y, z\}$ and the subgraph induced by $D_1^1(u, v)$ is not complete multipartite, since y and z are in the same coclique as x and are adjacent.

(iii) We have $\alpha \leq t$ by (ii). Let x and y be vertices of Γ at distance 2. Let $z \in D_1^2(x, y)$ and $A = \Gamma(z) \cap D_1^1(x, y)$. Suppose $\alpha \leq t-2$. Then there are two adjacent vertices $u, v \in D_1^1(x, y)$ such that the subgraph induced by $\{u\} \cup \{v\} \cup A$ is complete and $\partial(u, z) = 2 = \partial(v, z)$. Then the set $\Gamma(u) \cap D_1^1(v, z)$ contains $A \cup \{x\}$, which means that $\alpha = |A| \geq |A \cup \{x\}| = \alpha + 1$. Contradiction! Hence $\alpha \geq t-1$.

(iv) and (v) Let $t \neq 1$ and w_1, w_2 be nonadjacent vertices of the local graph $\Delta(x)$. Then $x \in D_1^1(w_1, w_2)$ so $\partial_\Gamma(w_1, w_2) = 2$ and there is $(t-1)n$ neighbours of x in the μ-graph of w_1 and w_2 and hence also in the local graph $\Delta(x)$. Hence $\Delta(x)$ has diameter 2. The rest follows now directly from (i). $\square$

We could relax the distance-regularity assumption on the graph Γ in Lemma 2.1 to the requirement that the intersection numbers k, c_2, a_1 and a_2 exist (see figure 2.1(a)).

Since the 1-homogeneous graphs that are μ-locally $K_{t\times n}$, with $t = 1$ have been classified in [16], we assume from now on $t \geq 2$. Lemma 2.1 implies that we can calculate a_2 in terms

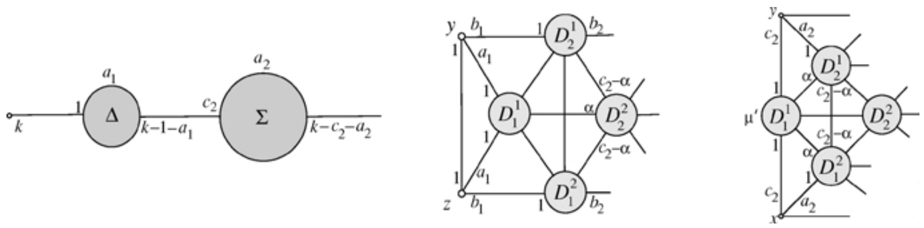

Figure 2.1. (a) The distance distribution corresponding to a vertex and the intersection numbers k, a_1, c_2 and a_2; (b) The distance distribution corresponding to y and z, $\partial(y, z) = 1$. Then we have $|D_1^1(y, z)| = a_1$ and $|D_1^2(y, z)| = |D_1^2(y, z)| = b_1$. (c) The distance distribution corresponding to x and y, $\partial(x, y) = 2$. Then we have $|D_1^1(x, y)| = c_2$ and $|D_1^2(x, y)| = |D_1^2(x, y)| = a_2$.

of a_1, n and t:

$$a_2 = \begin{cases} na_1 - (t-1)n^2 & \text{if } \alpha = t, \\ t\dfrac{a_1 n}{t-1} - n^2 t & \text{if } \alpha = t-1. \end{cases} \tag{1}$$

Let us now assume additionally that the local graphs of Γ are strongly regular with parameters (v', k', λ', μ'). We have already mentioned that $\alpha \le a_1$ implies $a_1 \ne 0$. In Lemma 2.1(i) we expressed v', k' and μ' in terms of a_1, n and t, therefore we can do the same for λ':

$$\lambda' = a_1 - 1 + \mu' - \frac{\mu'(k-1)}{a_1} = a_1 - 1 + n(t-1) - \frac{n(t-1)(k-1)}{a_1}. \tag{2}$$

If $d = 2$, then Γ is strongly regular and 1-homogeneous.

3. Eigenvalues of local graphs

Let Γ be a distance-regular graph with diameter at least 2, that is locally strongly regular and μ-locally the complete multipartite graph $K_{t\times n}$, $t \ge 2$, for which $a_2 \ne 0$, and the intersection number α exists with $\alpha \ge 1$. We study the smallest eigenvalue of a local graph of Γ.

Let $x_1, \ldots, x_n$ be vertices of a graph Γ. Then we denote the intersection $\Gamma(x_1) \cap \cdots \cap \Gamma(x_n)$ by $\Gamma(x_1, \ldots, x_n)$ and the corresponding induced subgraph by $\Delta(x_1, \ldots, x_n)$.

Lemma 3.1 *Let Γ be a distance-regular graph with diameter at least 2 that is locally strongly regular with parameters (v', k', λ', μ') and μ-locally the complete multipartite graph $K_{t\times n}$, $t \ge 2$, for which $a_2 \ne 0$, and the intersection number α exists with $\alpha \ge 1$. For an edge xy of Γ, the subgraph $\Delta(x, y)$ is co-edge-regular with parameters (v'', k'', μ''), where*

$$v'' = k', \quad k'' = \lambda', \quad \text{and} \quad \mu'' = n(t-2),$$

for $t \geq 3$ *the subgraph* $\Delta(x, y)$ *has diameter* 2, *and it contains an equitable partition* $\pi = \{P_1, P_2\}$ *with quotient matrix*

$$\begin{pmatrix} n(t-2) & \lambda' - n(t-2) \\ \alpha - 1 & \lambda' - \alpha + 1 \end{pmatrix}.$$

In particular, $|P_1| = n(t-1)$, $|P_2| = a_1 - n(t-1)$ *and*

$$(\alpha - 1)(a_1 - n(t-1)) = (\lambda' - (t-2)n)n(t-1). \tag{3}$$

Proof: The verification of co-edge-regularity is similar as in the case of Lemma 2.1(i). Let $z \in D_2^1(x, y)$. By Lemma 2.1(iii) and the fact that the valency of $\Delta(x, y)$ is λ', the partition

$$\{\Gamma(x) \cap \Gamma(y) \cap \Gamma(z), \Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)\},$$

is an equitable partition of the graph $\Delta(x, y)$ with the required quotient matrix, see Figure 2.1(b, c). The first set in the above partition has $\mu' = n(t-1)$ vertices, while the other one has $a_2' = k' - \mu' = a_1 - n(t-1)$ vertices. We obtain (3) by a two way counting of edges that are connecting vertices from different parts of the above partition. $\square$

The relation (3) gives us

$$\lambda' = 1 - n - \alpha + n(t-1) + \frac{a_1(\alpha - 1)}{n(t-1)}, \tag{4}$$

hence $n(t-1) \mid a_1(\alpha - 1)$. The above relation and (2) imply that one can express k in terms of n, α, t and a_1.

Theorem 3.2 *Let* Γ *be a distance-regular graph with diameter at least* 2, *that is locally strongly regular and* μ*-locally the complete multipartite graph* $K_{t\times n}$, *for which* $a_2 \neq 0$, *and the intersection number* α *exists with* $\alpha = t \geq 2$. *Then, for all vertices* x *of* Γ, *the smallest eigenvalue of* $\Delta(x)$ *equals* $-n$.

Proof: Suppose that the parameters of the local graphs that are strongly regular are (v', k', λ', μ'). It follows directly from the relation (3) and $\alpha = t$ that $a_1 - n(t-1) = (\lambda' - (t-2)n)n$. Now using that $n(t-1) = \mu'$ and $a_1 = k'$ it follows that $k' - \mu' = (\lambda' - \mu' + n)n$ and hence $-n$ is the negative eigenvalue of the local graph $\Delta(x)$ with parameters (v', k', λ', μ'). $\square$

Theorem 3.3 *Let* Γ *be a distance-regular graph with diameter at least* 2, *that is locally strongly regular with parameters* (v', k', λ', μ') *and* μ*-locally the complete multipartite* *graph $K_{t\times n}$, for which $a_2 \neq 0$, and the intersection number α exists with $\alpha = t-1 \geq 2$. Then there exists a positive integer a such that $-a-n$ is the smallest eigenvalue of every local graph and*

$$n(\lambda' - n(t-2)) = a(t-2)(\lambda' - n(t-3) + a). \tag{5}$$

In particular, $a(t-2) < n$.

Proof: Fix a vertex x of Γ. Let s be the smallest eigenvalue of the local graph $\Delta(x)$. Then

$$\mu' - k' = (\lambda' - \mu' - s)s.$$

On the other hand, $k' = a_1$ and $\mu' = n(t-1)$ by Lemma 2.1(i), so by (3) and $\alpha = t-1$, we have

$$(\lambda' - \mu' + n)n(t-1) = (t-2)(\lambda' - \mu' - s)(-s).$$

This means that $-s > n$, in particular s is negative. Set $a := -n-s$ in the above identity and we obtain (5). To show that a is integral, suppose the opposite. Then $\Delta(x)$ is a conference graph with parameters $(4\mu'+1, 2\mu', \mu'-1, \mu')$, so $k = 4\mu'+1$, $a_1 = 2\mu'$ and $\lambda' = \mu'-1$. Applying (4) we obtain $n = t-1$, so $k = 4(t-1)^2+1$, $a_1 = b_1 = 2(t-1)^2$, $c_2 = nt = (t-1)t$, $\lambda' = t(t-2)$ and $k_2 = kb_1/c_2 = 2(4t^2-8t+5)(t-1)/t$, which implies $t|10$, thus, by $t \geq 3$, we have $t = 5$ or $t = 10$. Let $y \in \Gamma(x)$. Then, by $|D_2^1(x,y)| = b_1$ and $|D_2^2(x,y)| = a_1(b_1 - b_1')/\alpha = 2(t-1)^3$, we obtain

$$k_2 - \left|D_2^2(x,y)\right| - \left|D_2^1(x,y)\right| = -2\frac{(t-1)(-5t^2+8t-5+t^3)}{t} < 0,$$

which is impossible. Therefore, $\Delta(x)$ is not a conference graph, and so a is integral.

Suppose $a(t-2) \geq n$. Then, by (5), we obtain

$$\lambda' - n(t-2) \geq \lambda' - n(t-3) + a, \quad \text{i.e.,} \quad -n \geq a,$$

which is not possible since a is a positive integer. Therefore, we have $a(t-2) < n$. □

Note that the assumptions $\alpha = t-1$ and $\alpha \geq 1$ imply $t = \alpha + 1 \geq 2$.

Corollary 3.4 *Let Γ be a distance-regular graph with diameter at least* 2, *that is locally strongly regular with parameters (v', k', λ', μ') and μ-locally the Cocktail Party graph $K_{t\times 2}$, for which $a_2 \neq 0$, and the intersection number α exists with $\alpha = t-1$. Then $\alpha = 1$, $\lambda' = 0$ and $\mu' = 2$.*

Proof: Suppose $\alpha > 1$, i.e., $t \geq 3$. By Theorem 3.3, we obtain $a(t-2) < n = 2$, and therefore $a = 1$ and $t = 3$. This implies that every local graph in Γ is strongly regular with parameters (57, 16, 5, 4). However, this is not possible, since for these parameters we do not have integral eigenvalue multiplicities. Therefore, $\alpha = 1$, and so $t = 2$. Hence $\mu' = 2$ and $\lambda' = 0$ by Lemma 2.1(i) and relation (4). □

Lemma 3.5 *Let Γ be a distance-regular graph with diameter at least 2 that is locally strongly regular with parameters $(k, a_1, 0, 2)$ and eigenvalues $a_1 > p > q$. Then p is a nonnegative integer, not congruent $3 \,(mod\ 4)$, $q = -p - 2$, $a_1 = (p+1)^2 + 1$, $k = 1 + a_1(a_1+1)/2$ and $b_1 = a_1(a_1 - 1)/2$.*

Proof: The graph Γ is not locally a conference graph, since $\lambda' = 0 \neq 1 = \mu' - 1$, so p is a nonnegative integer and we have

$$(p+1)^2 = \left(p + \frac{\lambda' - \mu'}{2}\right)^2 = \frac{(\lambda' - \mu')^2}{4} + (k' - \mu') = a_1 - 1.$$

The multiplicities of the nontrivial eigenvalues are integral if and only if p is not congruent $3 \,(\mathrm{mod}\ 4)$. The remaining relations are straightforward. □

Remark 3.6

(i) For $p = 0, 1, 2$, i.e., $a_1 = 2, 5, 10$, the local graphs of Γ are respectively the quadrangle, the folded 5-cube with intersection array $\{5, 4; 1, 2\}$ (also called the Clebsch graph), and the Gewirtz graph with intersection array $\{10, 9; 1, 2\}$. These are the only known strongly regular graphs with $\lambda = 0$ and $\mu = 2$.

(ii) We are interested in the case, when Γ is additionally μ-locally $K_{t\times 2}$, $t \geq 2$, $a_2 \neq 0$ and the intersection number $\alpha = 1$. Then, by Lemma 2.1(iii) and (i), Lemma 3.5 and the nonexistence of a strongly-regular graph with parameters (57, 16, 5, 4), we have

$$t = 2, \quad c_2 = 4, \quad a_1 > 5, \quad a_2 = 4(a_1 - 2),$$
$$b_2 = (a_1 - 5)(a_1 - 2)/2 \quad \text{and} \quad d \geq 3.$$

Finally, if we additionally assume that Γ is 1-homogeneous, then we can apply [11, Algorithm 4.7] in order to obtain that $d \neq 3$ and that Γ is not locally Gewirtz, i.e., $a_1 \neq 10$.

Conjecture 3.7 There is no 1-homogeneous distance-regular graph with diameter at least 2, that is locally strongly regular with parameters $(v', k', 0, 2)$, that has $a_2 \neq 0$, the intersection number $\alpha = 1$ and that Γ is μ-locally $K_{t\times 2}$, $t \geq 2$.

4. Locally grid and locally triangular graphs

Before we start to study locally grid and locally triangular graphs, we need to introduce some basic notions about codes. Let Γ be a graph with diameter d and the vertex set X. A

code C in Γ is a nonempty subset of X. Then the **distance of a vertex** $x \in X$ **to** C and the **covering radius** of C respectively are

$$\partial(x, C) := \min\{\partial(x, y) \mid y \in C\} \quad \text{and} \quad t(C) := \max\{\partial(x, C) \mid x \in X\}.$$

Let P_i be the set of vertices at distance i from C and $t = t(C)$. The code C is **completely regular** when the partition $\{P_i \mid i = 0, \dots, t\}$ is equitable. This definition is due to Neumaier [14], who showed that in the case of distance-regular graphs it is equivalent to the original Delsarte's definition, that the code C is completely regular when for each vertex x of Γ and for each $i \in \{0, 1, \dots, t\}$, the intersection number $|C \cap \Gamma_i(x)|$ depends only on $\partial(x, C)$, see [6] or [3, p. 351]. A partition π of a graph Γ gives rise to the **quotient graph** G/π with cells as vertices and two distinct cells P_i to P_j adjacent if there is an edge of Γ joining some vertex of P_i to some vertex in P_j. An equitable partition π is **uniformly regular** if there are constants e_{01} and e_{11} such that the parameters of the equitable partition are

$$c_{ij} = \begin{cases} e_{01} & \text{if } i = j, \\ e_{11} & \text{if } P_i \sim P_j \text{ in } \Gamma/\pi. \end{cases}$$

The line graph of $K_{m,n}$, i.e., the graph $K_m \times K_n$, will be called the $(m \times n)$-**grid**.

Proposition 4.1 *Let Γ be a distance-regular graph with diameter at least 2. If Γ is locally the $(m \times n)$-grid and $c_2 = 4$, then Γ is the Johnson graph $J(n+m, n)$, or $m = n$ and Γ is the folded Johnson graph $\bar{J}(2m, m)$.*

Proof: By [3, Theorem 9.1.3], the graph Γ is the Johnson graph $J(n+m, n)$, or $m = n$ and Γ is a quotient of the Johnson graph $J(2m, m)$. More precisely, in the latter case we can partition the vertex set of $J(2m, m)$ into a uniform partition $\pi := \{P_i \mid i = 1, \dots, \binom{2m}{m}/2\}$, where $|P_i| = 2$. By [3, Theorem 11.1.6], we obtain that π is completely regular, i.e. the sets $P_i = \{x_i, y_i\}$ are completely regular with the same intersection numbers. Suppose $\partial(x_i, y_i) = h < d = m$. Then, by $b_h \neq 0$, there exists a neighbour v of x_i that is at distance $h+1$ from y_i. Therefore, each neighbour of a vertex in P_i is at distance $h+1$ from the other vertex of P_i. Hence $h = 1$ (since otherwise $c_h = 0$) and $a_1 = 0$. Since $a_1 = 2m-2$, this is not possible, thus $\partial(x_i, y_i) = d$ for every i. It follows that Γ is the folded Johnson graph $\bar{J}(2m, m)$. □

The last part of the above proof was motivated by the proof of [13, Theorem 2.3.3].

The line graph of the complete graph K_n is the **triangular graph** $T(n)$, i.e., the Johnson graph $J(n, 2)$. Note that $T(1)$ is an empty graph, $T(2)$ is K_1, $T(3)$ is K_3 and $T(4)$ is the complete multipartite graph $K_{2,2,2}$, i.e., the octahedron, and $T(5)$ is the complement of the Petersen graph.

Proposition 4.2 *Let Γ be a distance-regular graph with diameter $d \geq 3$ and let*
(i) *Γ be locally a triangular graph,*
(ii) *Γ have the CAB$_i$ property for some $i \in \{2, \dots, d-1\}$.*

Then for $1 \le j \le i$ and for all vertices x and y at distance j the induced subgraph on $C_j(x, y)$ is the triangular graph $T(2j)$. Furthermore, if the distance between vertices x and y is $i+1$, then the subgraph induced on $C_{i+1}(x, y)$ is a disjoint union of triangular graphs $T(2i+2)$.

Proof: By [3, Proposition 4.3.9 and Lemma 4.3.10] (cf. [15, 20] and [21]), the condition (i) implies

- there exists an integer n such that the graph Γ is locally $T(n)$,
- Γ is the halved graph of a bipartite graph Γ' with intersection numbers $c_i(\Gamma') = i$ for $i \le 3$, and
- the μ-graphs in Γ are isomorphic to the disjoint union of at most $\lfloor n/4 \rfloor$ copies of $K_{2,2,2}$.

So $k = n(n-1)/2$ and $a_1 = 2(n-2)$. Since $c_3(\Gamma') = 3$ and $a_2(\Gamma') = 0$, any 3-claw in Γ' determines a unique 3-cube by [3, Lemma 4.3.5(ii)]. Therefore, by [3, Proposition 4.3.6 and Corollary 4.3.7], the n-cube Q_n covers Γ'. More precisely, there exists a map $\pi' : V(Q_n) \to V(\Gamma')$ that preserves distances ≤ 3. It induces a map $\pi : V(\frac{1}{2}Q_n) \to V(\Gamma)$, that preserves adjacency (see figure 4.1). Let us denote by V' the set of vertices of Γ' corresponding to the vertices of Γ.

Let us define $c'_m(u', v') := c_m(\Gamma')(u', v')$ for vertices u' and v' at distance m in Γ' and $m = 1, 2, \ldots, \mathrm{diam}(\Gamma')$. Suppose we have shown for an integer t, where $1 \le t < i$, and for all $j \in \{1, \ldots, t\}$ that

(a) the subgraph of Γ induced by $C_j(x, y)$ is $T(2j)$ for all $x, y \in V(\Gamma)$ with $\partial(x, y) = j$,
(b) $c'_m(x', y') = m$ for all $m \in \{1, \ldots, 2t+1\}$, $x' \in V'$ and $y' \in \Gamma'_m(x')$.

Conditions (a) and (b) are certainly true for $t = 1$ by the observation made at the beginning of this proof and since $C_1(x, y)$ contains only one vertex, which means that it induces $T(2)$.

Before continuing with the induction, we need to introduce some new notations. Let m be a positive integer, $x' \in V'$ and $y' \in \Gamma'_m(x')$. We say that the number c'_m *semi-exists* if $c'_m(u', v') = c'_m(x', y')$ for all $u' \in V'$ and $v' \in \Gamma'_m(u')$ and $c'_m = c'_m(x', y')$. For vertices u and v at distance s in a graph X we denote by $I_X(u, v)$ the **interval graph**, that is the

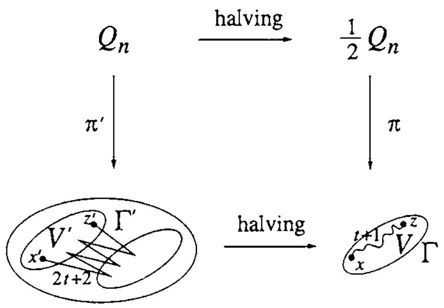

Figure 4.1. The halved graph of Q_n is denoted by $\frac{1}{2}Q_n$ and called the **halved** n**-cube**.

subgraph of X induced by the set $\{w \in V(X) \mid \partial_X(u,w) + \partial_X(v,w) = s\}$, i.e., the set of vertices that lie on a shortest path between u and v.

Let x be a vertex of Γ and x' the corresponding vertex of Γ'. Without loss of generality we may choose π' to map the vector $\mathbf{0}$ to x'. Since $c_m(Q_n) = m = c'_m$ for all $m \in \{1, 2, \ldots, 2t+1\}$, by the induction assumption and since both graphs Γ' and Q_n are bipartite, the map π' preserves distances $\leq 2t+1$ when at least one of the vertices is from V', so also the map π preserves distances $\leq t$, and the words of weight m in Q_n are in 1-1 correspondence with the vertices at distance m from x' in Γ'.

Let $z \in \Gamma_{t+1}(x)$ and let z' be the corresponding vertex of Γ'. Then $z' \in \Gamma_{2t+2}(x')$ and since $c'_{2t+1} = 2t+1$ and Γ' is bipartite, the words of weight $2t+2$ in the preimage $\pi'^{-1}(z')$ are mutually disjoint. Moreover, as $2t+1 \geq 3$, we have

$$c_{t+1} = c_{t+1}(x,z) = \frac{c'_{2t+2}(x',z')c'_{2t+1}}{c'_2} = \frac{c'_{2t+2}(x',z')(2t+1)}{2},$$

and therefore c'_{2t+2} semi-exists and it is equal to $2c_{t+1}/(2t+1)$. The interval graph $I_{\Gamma'}(x',z')$ consist of $p := c'_{2t+2}/(2t+2)$ copies of the $(2t+2)$-cubes sharing only the vertices x' and z' with each other.

Let Σ be the subgraph of Γ induced by the set $C_{t+1}(x,z)$. Then Σ consists of p disjoint graphs and each of them is the halved graph of the second neighbourhood of the $(2t+2)$-cube. The halved graph of the second neighbourhood of the s-cube is the Johnson graph $J(s,2)$, i.e., the triangular graph $T(s)$. It follows that the graph Σ is a disjoint union of p copies of the triangular graph $T(2t+2)$. Since Γ has the CAB_{t+1} property and $t < i \leq d-1$ (so also $t+1 \neq d$), the set $C_{t+1}(x,z)$ is a completely regular code with covering radius 2 in the triangular graph $T(n)$. The latter graph can be considered as the line graph of K_n, and the p copies of $T(2t+2)$ correspond to p distinguished disjoint $(2t+2)$-cliques of K_n, which do not cover all its vertices. Every edge of K_n corresponding to a vertex of $A_{t+1}(x,z)$ connects a vertex from one of the distinguished $(2t+2)$-cliques with one of the remaining vertices of K_n, or, if $p > 1$, it connects vertices from two distinct such $(2t+2)$-cliques. However, the latter is not possible by the CAB_{t+1} property, thus we conclude that $p = 1$ and so $c'_{2t+2} = 2t+2$.

Now we will show that $c'_{2t+2} = 2t+3$. By the fact that $C_{t+1}(x,z)$ is a completely regular code in $T(n)$, it follows that $b_{t+1} = \binom{n-2t-2}{2}$. We have chosen x and z to be vertices of Γ at distance $t+1$ and x', z' as their corresponding vertices of Γ' at distance $2t+2$, hence

$$\frac{(n-2t-2)(n-2t-3)}{2} = b_{t+1} = b_{t+1}(x,z) = \frac{1}{c'_2} \sum_{y' \in B_{2t+2}(x',z')} b'_{2t+3}(x',y'),$$

As Γ' is a bipartite graph with $c'_2 = c_2(\Gamma') = 2$, by [3, Proposition 1.9.1], it follows that $c'_i(u',v') \geq i$ for all vertices u' and v' of Γ' at distance i. So, by $|B_{2t+2}(x',z')| = b'_{2t+2} = n - c'_{2t+2} = n-(2t+2)$ and $b'_{2t+3}(x',y') = n - c'_{2t+3}(x',y') \leq n-(2t+3)$, we conclude $b'_{2t+3}(x',y') = n-2t-3$, which implies $c'_{2t+3} = 2t+3$. Now the proposition follows by induction. $\square$

For the convenience of the reader we give a proof of the following result.

Theorem 4.3 (A.E. Brouwer) *Let Γ be a bipartite distance-regular graph with diameter $d \geq 4$ and $c_i = i$ for $i \leq d-1$. Then Γ is a d-cube, a folded $(2d)$-cube or if $d = 4$, the coset graph of the extended binary Golay code.*

Proof: For $d = 4$ it follows from [4, Theorem 5.10 and Theorem 5.12] that either the valency equals 4, 8 or 24. For valency 24 it follows, by [2], that Γ is the coset graph of the binary Golay code. By [3, Theorem 11.1.6 and Corollary 4.3.7], there exists a completely regular code with the following distance partition (figure 4.2).

Figure 4.2. The distance partition of a certain completely regular code.

For $d \geq 5$, they were classified by van Tilborg, who showed $|C| \leq 2$. The result follows now. $\square$

Theorem 4.4 *Let Γ be a distance-regular graph with diameter $d \geq 2$. Then*
(i) *Γ is locally a triangular graph, and*
(ii) *Γ is 1-homogeneous*
if and only if Γ is the halved n-cube, $n \geq 4$, or Γ is the folded halved n-cube with $n = 4m$, $m \in \mathbb{N}$ and $m \geq 4$, or Γ is the halved coset graph of the extended binary Golay code.

Proof: Similarly as in the proof of Proposition 4.2, we start with the following:

(a) there exists an integer n such that the graph Γ is locally $T(n)$,
(b) Γ is the halved graph of a bipartite graph Γ' with intersection numbers $c_i(\Gamma') = i$ for $i \leq 3$, and
(c) the μ-graphs in Γ are isomorphic to the disjoint union of at most $\lfloor n/4 \rfloor$ copies of $K_{2,2,2}$.

If $d \geq 3$, then, by Proposition 4.2, for vertices x and y at distance $i \in \{1, \ldots, d-1\}$ the subgraph induced by $C_i(x, y)$ is the triangular graph $T(2i)$, and for vertices x and y at distance d the subgraph induced by $C_d(x, y)$ is a disjoint union of the triangular graphs $T(2d)$. Let us show that the same statement is true also when $d = 2$. We only need to check it for $i = 2$. Let x and y be vertices of Γ at distance 2. Then we want to show that the subgraph induced by $C_2(x, y)$, i.e., the μ-graph of x and y is a disjoint union of the triangular graphs $T(4)$. Since $K_{2,2,2}$ is isomorphic to $T(4)$, this statement coincides with the above property (c).

By the CAB$_d$ property, this means that the subgraph induced on $C_d(x, y)$ is either one copy of the triangular graph $T(2d)$, or the disjoint union of exactly $n/(2d)$ copies of $T(2d)$. Therefore, Γ is a distance-regular graph with intersection numbers

$$c_i = \binom{2i}{2}, \quad b_i = \binom{n-2i}{2} \quad \text{for } i \leq d-1, \quad \text{and} \quad c_d \in \left\{ \binom{2d}{2}, \frac{n}{2d}\binom{2d}{2} \right\}.$$

The first case can happen only when $2d \in \{n, n-1\}$. But then $|V(\Gamma)| = 2^{n-1}$ and, by [3, Corollary 4.3.8(ii)], the graph Γ is the halved n-cube, $n \geq 4$. So we may assume that we are in the second case. This can only happen when $d \geq 3$ and it is easy to see that $2d = d'$, where d' is the diameter of Γ'. We are going to show Γ' is a distance-regular graph with intersection numbers $c'_i = i$ for $i \leq d' - 1$ and $c_{d'} = n$. As in the proof of Proposition 4.2, let V' be the set of the vertices of Γ' of the corresponding to vertices of Γ and we have shown that $c'_i(x', y') = i$ for $i \leq d-1$ and $x' \in V'$, $y' \in V(\Gamma')$ at distance i. Furthermore, by assumptions we have $c_{d'} = n$.

Let $x' \in V'$ and $y' \in V(\Gamma')$. Then

$$|\Gamma_{2i}(x')| = \binom{n}{2i}, \quad |\Gamma_{2i}(y')| \leq \binom{n}{2i},$$
$$|\Gamma_{2d}(x')| = \binom{n-1}{2d-1} \quad \text{and} \quad |\Gamma_{2i}(y')| \leq \binom{n-1}{2d-1},$$

as $c'_i(u', v') \geq i$ for all $u', v' \in V(\Gamma')$ at distance i. But

$$\sum_{i=0}^{d} |\Gamma'_{2i}(x')| = \sum_{i=0}^{d} |\Gamma'_{2i}(y')|$$

and therefore $c'_i = i$ for $i \leq 2d-1$ and $c'_{2d} = n$. Hence Γ' is a distance-regular graph with intersection numbers $c'_i = i$ for $i \leq d'-1$ and $c_{d'} = n$, so Γ' is either the n-cube, the folded $2n$-cube or the coset graph of the extended binary Golay code by Theorem 4.3. As the halved folded $(4m+2)$-cube is not 1-homogeneous, the result follows now. □

5. Proof of the main result

By Gardiner [7], in an antipodal distance-regular graph Γ with diameter D a vertex x, which is at distance $i \leq \lfloor D/2 \rfloor$ from one vertex in an antipodal class, is at distance $D - i$ from all other vertices in this antipodal class, hence

$$\Gamma_{D-i}(x) = \bigcup \{\Gamma_D(y) \mid y \in \Gamma_i(x)\} \quad \text{for } i = 0, 1, \ldots, \lfloor D/2 \rfloor. \tag{6}$$

If Γ is 1-homogeneous and x, y are its adjacent vertices, then it is not hard to conclude by (6) that, by taking antipodal quotient of Γ, the cells $D^{d-j}_{d-i}(x, y)$ and $D^j_i(x, y)$ fold together for $0 \leq i, j \leq \lfloor d/2 \rfloor$. However, it is even more effective to follow antipodal folding through CAB$_i$ partitions of Γ and its antipodal quotient.

Theorem 5.1 *Let Γ be an antipodal graph with diameter $D \geq 4$ and let Σ be its antipodal quotient graph with diameter d. Then for $i \leq d-1$ the graph Σ has the CAB$_i$ property if and only if Γ has the CAB$_i$ property, and for $D = 2d$ the following are equivalent.*

(i) *The graph Γ is 1-homogeneous.*

(ii) *The graph Γ has the CAB$_d$ property.*

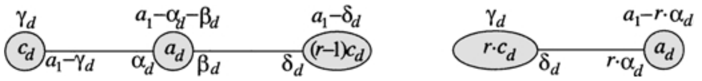

Figure 5.1. The CAB_d partition in Γ (left) and the CAB_d partition in the antipodal quotient graph Σ (right).

Moreover, if Γ *is* 1*-homogeneous and* $a_1 \neq 0$, *then* Σ *is* 1*-homogeneous if and only if* D *is even.*

Proof: The first part of the statement and (i) $\Leftrightarrow$ (ii) follow directly from the fact that a CAB_i partition of Σ, the corresponding CAB_{D-i} partitions of Γ and the corresponding CAB_i partition of Γ are isomorphic by the covering projection for $i = 1, \ldots, d-1$ (see figure 5.1).

Let $y_1, \ldots, y_r$ be the vertices of an antipodal class of Γ, let x be a vertex of Γ at distance d from y_1, and let $\hat{x}$ and $\hat{y}$ be the images of the covering projection corresponding to x and y_1 respectively. Consider the following partition

$$\Omega = D_1^{d-1}(x, y_1) \cup \cdots \cup D_1^{d-1}(x, y_r) \cup \left(\Gamma(x) \Big\backslash \bigcup_{i=1}^{r} D_1^{d-1}(x, y_i) \right)$$

of the local graph of x (see figure 5.2). If $D = 2d$, then the first r sets have size $c_d(\Gamma)$, the last one has size $a_d(\Gamma)$, and there are no edges between $D_1^{d-1}(x, y_i)$ and $D_1^{d-1}(x, y_j)$ when $i \neq j$, which means that the partitions $\mathrm{CAB}_d(x, y_i)$ for $i = 1, \ldots, r$ are equitable with the same parameters if and only if the partition Ω is equitable, in which case Ω has the

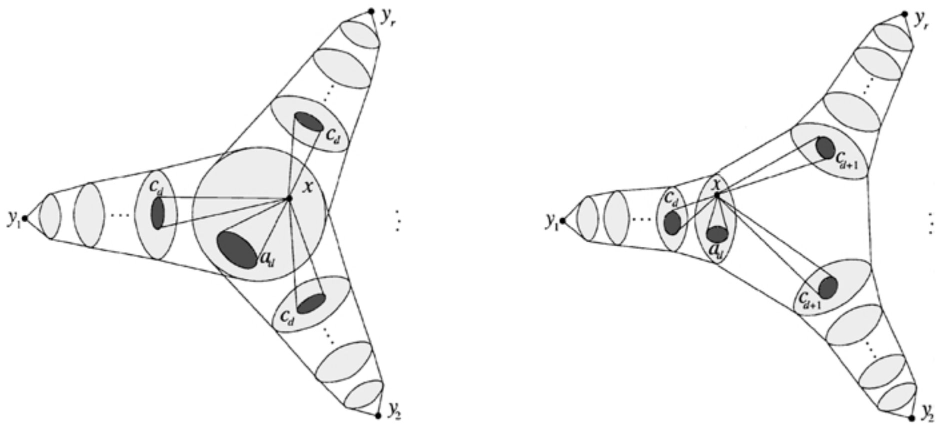

Figure 5.2. The partition corresponding to the distance distribution of the antipodal class $\{y_1, \ldots, y_r\}$ in the case when D is even (left) and the case when D is odd (right). We have chosen r to be three. Inside this partition there is a partition of the neighbourhood of the vertex x.

following quotient matrix

$$\begin{pmatrix} g & 0 & 0 & \cdots & 0 & a_1-g \\ 0 & g & 0 & \cdots & 0 & a_1-g \\ 0 & 0 & g & \cdots & 0 & a_1-g \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & g & a_1-g \\ h & h & h & \cdots & h & a_1-rh \end{pmatrix} \tag{7}$$

for some integers g and h. If (i)–(ii) holds, then $g = \gamma_d = a_1 - \delta_d$, $h = \alpha_d = \beta_d/(r-1)$ and the CAB_d partition of Σ is equitable with the quotient matrix $\left(\begin{smallmatrix} \gamma_d & \delta_d \\ r\alpha_d & a_1-r\alpha \end{smallmatrix}\right)$. Thus, we have shown that if Γ is 1-homogeneous and D is even, then Σ has the CAB_d property, i.e., Σ is 1-homogeneous.

Finally, let us suppose the antipodal quotient graph Σ is 1-homogeneous, i.e., Σ has the CAB_d property, and let $\left(\begin{smallmatrix} g, & a_1-g \\ a, & a_1-a \end{smallmatrix}\right)$ be the quotient matrix of the $\mathrm{CAB}_d(\hat{x}, \hat{y})$ partition in Σ. Suppose D is odd. Then the local graph of $\hat{x}$ is disconnected, (see figure 5.2 (right)), and we have $g = a_1$ and $a = 0$. By [11, Proposition 2.2], the set $C_d(\hat{y}, \hat{x})(\Sigma)$ is independent in Σ, which is not possible because $a_1 \neq 0$. Therefore, we have $D = 2d$ by [3, Proposition 4.2.2]. ☐

Remark 5.2 The Foster graph is an antipodal distance-regular graph with diameter 4 and intersection array $\{6, 4, 2, 1; 1, 1, 4, 6\}$. It is locally disconnected, therefore it is not a tight graph in the sense of [10], and hence not 1-homogeneous (see [10, Theorem 11.7]). However, its antipodal quotient is the complement of the triangular graph $T(6)$ and it is 1-homogeneous. Thus, under the assumptions of the above result Γ is not necessary 1-homogeneous when Σ is 1-homogeneous and $D = 2d$.

Theorem 5.3 *A graph Γ is an 1-homogeneous graph with diameter at least 2, that is μ-locally the complete multipartite graph $K_{t\times 2}$, $t \geq 2$, and for which $a_2 \neq 0$ and the intersection number α exists with $\alpha = t$ if and only if Γ is one of the following*:

(i) *a Johnson graph $J(2m, m)$ with $m \geq 3$,*
(ii) *a folded Johnson graph $\bar{J}(4m, 2m)$ with $m \geq 3$,*
(iii) *a halved m-cube with $m \geq 5$,*
(iv) *a folded halved $(4m)$-cube with $m \geq 3$,*
(v) *the Schläfli graph with intersection array* $\{16, 5; 1, 8\}$,
(vi) *the Gosset graph with intersection array* $\{27, 10, 1; 1, 10, 27\}$.

Proof: Let Γ be an 1-homogeneous graph with diameter at least 2 that is μ-locally the complete multipartite graph $K_{t\times 2}$, $t \geq 2$, and for which $a_2 \neq 0$ and the intersection number α exists with $\alpha = t$. Let x be a vertex of Γ. Then the subgraph $\Delta(x)$ is a connected strongly regular graph by [11, Theorem 3.1 and Proposition 2.1] and the smallest eigenvalue of $\Delta(x)$ is -2 by Theorem 3.2. By Seidel's classification [17], [3, Theorem 3.12.4], the local graph $\Delta(x)$ is either

- a triangular graph $T(m)$ with $m \geq 5$,
- a $(m \times m)$-grid with $m \geq 3$,
- a Cocktail Party graph $K_{m\times 2}$ with $m \geq 2$,
- the Petersen graph,
- the Clebsch graph (i.e., the folded 5-cube),
- the Schläfli graph,
- the Shrikhande graph, or
- one of the three Chang graphs.

The μ-graphs of the Shrikhande graph, the Petersen graph and all the three Chang graphs are not all isomorphic to $K_{(t-1)\times 2}$. If Γ is locally $K_{m\times 2}$ with $m \geq 2$, then Γ is the Cocktail Party graph $K_{(m+1)\times 2}$ by [3, Proposition 1.1.5], so $a_2 = 0$.

If Γ is locally Clebsch graph, see [3, p. 104], then $\mu' = 6$, Γ is the Schläfli graph, see [3, p. 312], which has $a_1 = 10$, $a_2 = 8$, $c_2 = 8 = \mu' + 2$ (so it is μ-locally $K_{4\times 2}$), and $\alpha = 4$ (so it is really 1-homogeneous graph, cf. [11, Theorem 3.9]).

If Γ is locally Schläfli graph, then Γ is the Gosset graph, see [3, p. 313], which is 1-homogeneous graph, see [10, Theorem 11.7 and 12.6], and has $a_1 = 16$, $c_2 = 10 = \mu' + 2$ (so it is μ-locally $K_{5\times 2}$), and $\alpha = 5$.

Suppose Γ is locally $(m \times m)$-grid with $m \geq 3$. Then the eigenvalues of the local graphs are $a_1 = 2m - 2, m - 2$ and -2. Furthermore, $\mu' = 2 = n(t-1)$, $c_2 = nt = 4$ by Lemma 2.1, and Γ is either a Johnson graph $J(2m, m)$ or a folded Johnson graph $\bar{J}(2m, m)$ by Proposition 4.1. The first graph is 1-homogeneous by [10, Theorem 11.7 and 12.6], with $\alpha = 2$ and has $c_2 = \mu' + 2$ (so it is μ-locally $K_{2\times 2}$). Suppose Γ is the second graph and θ_1, θ_d are respectively its second largest and its smallest eigenvalue. Then the 1-homogeneous property of Γ implies that $-1 - b_1/(1+\theta)$ for some $\theta \in \{\theta_1, \theta_d\}$ is an eigenvalue of all local graphs of Γ by [11, Theorem 3.9]. The latter can only happen when m is even by [3, Proposition 9.1.5]. The folded Johnson graph $\bar{J}(8, 4)$ is 1-homogeneous with $\alpha = 4$ by Theorem 5.1 and [12, Corollary 5.8], so $\alpha \neq 2 = t$. Thus $m = 2s$ with $s \geq 3$. By Theorem 5.1, the folded Johnson graph $\bar{J}(4s, , 2s)$ (obtained by folding 1-homogeneous antipodal graph of even diameter) is 1-homogeneous with $\alpha = 2$ and it has $c_2 = \mu' + 2$ (so it is μ-locally $K_{2\times 2}$).

Finally, we suppose Γ is locally triangular graph $T(m)$ with $m \geq 5$. Then $\mu' = 4$, $t = 3$ and Γ is either

- the halved m-cube, or
- the folded halved m-cube with $4|m$ and $m \geq 8$, or
- the halved coset graph of the extended binary Golay code,

by [11, Theorem 3.1] and Theorem 4.4. The first graph and the second graph in the case $m \neq 8$, are by [10, Theorem 12.6 and 11.7] and by Theorem 5.1 respectively, 1-homogeneous with $\alpha = 3$ and have $c_2 = 6 = \mu' + 2$ (so it is μ-locally $K_{3\times 2}$). The folded halved 8-cube is 1-homogeneous with $\alpha = 6$ by Theorem 5.1 and [12, Corollary 4.8], so $\alpha \neq t$. By a direct counting argument, we verify that also the halved $(2s+1)$-cubes with $s \geq 2$ and the folded halved $(4s+2)$-cubes with $s \geq 2$ are 1-homogeneous. The third graph is the halved graph of the distance-regular graph with intersection array $\{24, 23, 22, 21; 1, 2, 3, 24\}$ by

[3, Theorem 11.3.2] and is therefore a strongly regular graph with parameters (2048, 276, 44, 36) by [3, Proposition 4.2.2(i)], so $c_2 = 36 \neq 6 = \mu' + 2$.

The converse is straightforward, since we have already verified that the graphs in the above list have the required properties. $\square$

Proof of Theorem 1.1: Let Γ be a 1-homogeneous graph with diameter $d \geq 2$. Let for all vertices x of Γ the local graph $\Delta(x)$ be a strongly regular graph with parameters (v', k', λ', μ') and $c_2 = \mu' + 2 \geq 3$. Since μ' is the valency of the μ-graphs, each μ-graph is the Cocktail Party graph $K_{(c_2/2)\times 2}$, c_2 is even and $\mu' \geq 2$. Therefore, Γ is μ-locally $K_{(c_2/2)\times 2}$. If $a_2 = 0$, then $b_2 = 0$ by $\mu' \neq 0$ and connectivity of Γ, thus $d = 2$ and Γ is the Cocktail party graph $K_{m\times 2}$ with $m = t + 1 \geq 3$. Now we assume $a_2 \neq 0$. Since Γ is 1-homogeneous and $\mu' \neq 0$, the intersection parameter α exists, $\mu' = n(t-1)$ by Lemma 2.1(i), and $\alpha \in \{t-1, t\}$ by Lemma 2.1(iii). If $\alpha = t$, then by Theorem 5.3 we obtain that Γ has to be one of the listed examples except (v). If $\alpha = t - 1$, then, by Corollary 3.4, we have $\alpha = 1, t = 2, \mu' = 2, \lambda' = 0$ and $c_2 = 4$, in which case $d = 2$ implies, by (1) and (2), $k - 4 = 4a_1 - 8$ and $0 = a_1 + 1 - 2(k-1)/a_1$, hence $a_1 = 2$ or $a_1 = 5$. In the first case we obtain the octahedron that implies $a_2 = 0$, and the second one has already been considered in Remark 3.6 and is not possible.

The converse is straightforward. $\square$

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