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Actions of Finite Hypergroups

V.S. SUNDER sunder@imsc.res.in
Institute of Mathematical Sciences, Madras 600113, India

N.J. WILDBERGER norman@maths.unsw.edu.au
School of Mathematics, UNSW, Sydney 2052, Australia

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Abstract. This paper is concerned with actions of finite hypergroups on sets. After introducing the definitions in the first section, we use the notion of 'maximal actions' to characterise those hypergroups which arise from association schemes, introduce the natural sub-class of *-actions of a hypergroup and introduce a geometric condition for the existence of *-actions of a Hermitian hypergroup. Following an insightful suggestion of Eiichi Bannai we obtain an example of the surprising phenomenon of a 3-element hypergroup with infinitely many pairwise inequivalent irreducible *-actions.

Keywords: hypergroups, actions, *-actions, association schemes

1. Introduction

We begin with a brief bird's eye overview of this paper.

Section 1 is devoted to a review of the definition of a hypergroup, some of its consequences, and some of the better known examples of hypergroups.

In Section 2, we introduce the central notion of the paper—that of an 'action of a hypergroup'. After a preliminary result, we focus on what we term 'maximal actions', and use these to obtain a characterisation—see Theorem 2.9—of those hypergroups which come from association schemes.

After a brief Section 3, in which we direct attention to '*-actions', we show in Section 4, that these '*-actions', at least in the case of Hermitian hypergroups (those hypergroups where the involution is trivial), admit a pleasant geometric reformulation—see Theorem 4.2.

In the final Section 5, we use Theorem 4.2 to exhibit an example of the phenemenon of a finite Hermitian hypergroup possessing an infinite number of actions which are pairwise inequivalent.

Definition 1.1 A **hypergroup**[1] is a distinguished linear basis $K = \{c_0, c_1, \ldots, c_n\}$ of a complex unital associative $*$-algebra $\mathbb{C}K$ satisfying the following conditions, for $0 \leq i, j \leq n$:

(i) $c_i c_j = \sum_{k=0}^{n} n_{ij}^k c_k$, where

$$n_{ij}^k \geq 0 \quad \forall\, k \tag{1.1}$$

$$\sum_{k=0}^{n} n_{ij}^k = 1; \tag{1.2}$$

(ii) c_0 is the multiplicative identity for $\mathbb{C}K$—i.e.,

$n_{0i}^j = n_{i0}^j = \delta_{i,j}$ (the 'Kronecker delta') and

(iii) K is a self-adjoint set—i.e., there exists an involutive mapping $i \mapsto i^*$ of $\{0, 1, \ldots, n\}$ such that $c_{i^*} = c_i^*$; and further,

$$n_{ij}^0 > 0 \Leftrightarrow i = j^*. \tag{1.3}$$

With the foregoing notation, the **weight** of the element c_i is defined by $w(c_i) = (n_{i^*i}^0)^{-1}$, and the weight of the hypergroup K is defined by $w(K) = \sum_{i=0}^{n} w(c_i)$.

Here are some simple consequences of these axioms. (Some of these facts have also been discussed in [5].)

Proposition 1.2 *Suppose* $K = \{c_0, c_1, \ldots, c_n\}$ *is as in Definition* 1.1. *Fix* $0 \leq i, j, k \leq n$; *then, we have*

(a) $$n_{ij}^k = n_{j^*i^*}^{k^*}$$

(b) $$\frac{n_{ij}^k}{w(c_k)} = \frac{n_{i^*k}^j}{w(c_j)}$$

(c) $$\frac{n_{ij}^k}{w(c_{k^*})} = \frac{n_{kj^*}^i}{w(c_{i^*})}.$$

Proof:

(a) This follows from the equation defining the structure constants n_{ij}^k (upon taking adjoints and using the fact that the structure constants are real).
(b) It follows from (a) that $c_{k^*} c_i = \sum_{l=0}^{n} n_{i^*k}^l c_{l^*}$; hence, the coefficient of c_0 in the product $(c_{k^*} c_i) c_j$ is seen to be $n_{i^*k}^j (w(c_j))^{-1}$. On the other hand, the coefficient of c_0 in the product $c_{k^*}(c_i c_j)$ is clearly $n_{ij}^k (w(c_k))^{-1}$.
(c) Exactly as in (b), compute the coefficient of c_0 in the product $(c_i c_j c_{k^*})$ in two ways.

□

Given a (finite) hypergroup K as above, its **Haar measure** is the element $e_0 \in \mathbb{C}K$ defined by

$$e_0 = w(K)^{-1} \sum_{i=0}^{n} w(c_i) c_i. \tag{1.4}$$

It is well-known (and is a consequence of the following lemma—for whose explicit statement and proof we thank the referee) that e_0 is a central projection in $\mathbb{C}K$; more precisely,

$$e_0 = e_0^* = e_0^2 = c_i e_0 = e_0 c_i \quad \forall\, i. \tag{1.5}$$

Lemma 1.3 *With the foregoing notation, we have*

$$w(c_i) = w(c_{i^*}) \quad \forall\, i.$$

Proof: Deduce from parts (a) and (b) of Proposition 1.2 that for all i, j, k,

$$\frac{n_{ij}^k}{w(c_k)} = \frac{n_{k^*i}^{j^*}}{w(c_j)} = \frac{n_{k^*i}^{j^*}}{w(c_{j^*})}\frac{w(c_{j^*})}{w(c_j)} = \frac{n_{kj^*}^i}{w(c_i)}\frac{w(c_{j^*})}{w(c_j)}. \tag{1.6}$$

Then,

$$\begin{aligned}
w(K)e_0c_j &= \sum_i w(c_i)c_ic_j \\
&= \sum_{i,k} w(c_i)n_{ij}^k c_k \\
&= \sum_{i,k} w(c_i)\frac{n_{ij}^k}{w(c_k)}w(c_k)c_k \\
&= \sum_{i,k} w(c_i)\frac{n_{kj^*}^i}{w(c_i)}\frac{w(c_{j^*})}{w(c_j)}w(c_k)c_k \quad \text{(by Eq. (1.6))} \\
&= \frac{w(c_{j^*})}{w(c_j)}\sum_k\left(\sum_i n_{kj^*}^i\right)w(c_k)c_k \\
&= \frac{w(c_{j^*})}{w(c_j)}w(K)e_0.
\end{aligned}$$

Hence

$$e_0c_j = \frac{w(c_{j^*})}{w(c_j)}e_0.$$

On the other hand,

$$\begin{aligned}
w(K)c_je_0 &= \sum_i w(c_i)c_jc_i \\
&= \sum_{i,k} w(c_i)n_{ji}^k c_k
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,k} w(c_i) \frac{n_{ji}^k}{w(c_k)} w(c_k) c_k \\
&= \sum_{i,k} w(c_i) \frac{n_{j^*k}^i}{w(c_i)} w(c_k) c_k \quad \text{(by Proposition 1.2(b))} \\
&= \sum_k \left(\sum_i n_{k*j}^{i^*} \right) w(c_k) c_k \quad \text{(by Proposition 1.2(a))} \\
&= w(K) e_0.
\end{aligned}
$$

Hence, $c_j e_0 = e_0$, and

$$
e_0^2 = e_0(c_j e_0) = (e_0 c_j) e_0 = \frac{w(c_{j^*})}{w(c_j)} e_0^2.
$$

Now,

$$
\begin{aligned}
e_0^2 &= w(K)^{-2} \sum_{i,j,k} w(c_i) w(c_j) n_{ij}^k c_k \\
&= w(K)^{-2} \sum_k \left(\sum_{i,j} w(c_i) w(c_j) n_{ij}^k \right) c_k \\
&\neq 0,
\end{aligned}
$$

since the coefficient of c_0 is non-negative and at least

$$
w(K)^{-2} = w(K)^{-2} w(c_0)^2 n_{00}^0.
$$

We may hence conclude that $w(c_{j^*}) = w(c_j)$ and that $e_0 c_j = e_0$. This proves the lemma and explicitly demonstrates that e_0 is a central idempotent. □

We turn next to some examples; in addition to three hypergroups which are naturally associated with any finite group, and those associated with (not necessarily commutative) association schemes, we mention examples stemming from the theory of *subfactors*—see [2]—which are of a much more general nature than those coming from groups (in a sense that can be made precise—see [3]).

Example 1.4

(a) Let G be a finite group.

  (i) Define $K = G$, with the involutive algebra structure on the complex group algebra being the natural one.

  (ii) Let $\{C_0 = \{e\}, C_1, \ldots, C_n\}$ be the set of conjugacy classes in G. (In the preceding sentence and throughout this example, we denote the identity element of G by e.)

Consider the elements $c_i \in \mathbb{C}G$ defined by $c_i = |C_i|^{-1} \sum_{g \in C_i} g$; then $K(G) = \{c_i : 0 \leq i \leq n\}$ is a basis for the algebra $\mathbb{C}K(G)$, which can be identified with the centre of the group algebra $\mathbb{C}G$. This set $K(G)$ is a hypergroup, and is called the **class hypergroup** of the group G. In this case, it is easy to see that $w(c_i) = |C_i|$.

(iii) Let $\{\chi_0 = 1, \chi_1, \ldots, \chi_n\}$ be an enumeration of the set of irreducible characters of G, and define $c_i = (\chi_i(e))^{-1}\chi_i$. Again, it is seen that $\hat{K}(G) = \{c_0, c_1, \ldots, c_n\}$ is a basis for the algebra $\mathbb{C}\hat{K}(G)$, which can be identified with the algebra of central functions on G. This is again a hypergroup, and is called the **character hypergroup** of G. It is seen that in this case, $w(c_i) = \chi_i(e)^2$.

(It is a fact—see [8], for instance—that the hypergroups $K(G)$ and $\hat{K}(G)$ are commutative and are 'duals' of one another.)

The preceding examples all had the feature that the weight of every element of the hypergroup is an integer. The next example also has this feature, but need have nothing to do with groups.

(b) Suppose $\{A_0, A_1, \ldots, A_n\}$ is (the set of 0,1-matrices corresponding to) an association scheme—see [1]—on a finite set of k elements. This means (essentially) that each A_i is a $k \times k$ matrix with all entries being 0 or 1, such that (i) A_0 is the $k \times k$ identity matrix I_k, (ii) there exist non-negative integers p^l_{ij} such that $A_i A_j = \sum_{l=0}^n p^l_{ij} A_l$, (iii) the collection $\{A_i : 0 \leq i \leq n\}$ is closed under formation of matrix-transpose, and (iv) $\sum_{i=0}^n A_i$ is the matrix J_k, all of whose entries are equal to 1. If A_{i^*} is the transpose of A_i, then the number $w_i = (p^0_{i^* i})$ is called the 'valency' of the 'i-th class' of the association scheme. It is a fact that if we define $c_i = w_i^{-1} A_i$, then $\{c_0, c_1, \ldots, c_n\}$ is a hypergroup with $w(c_i) = w_i$, and we shall call this the hypergroup of the given association scheme. (We will return later to the question of which hypergroups arise from association schemes in this fashion.)

(c) The next example has to do with tensor-products (or 'Connes' fusion') of bimodules (which are 'of finite type'), over von Neumann factors of type II_1. Given two such bimodules ${}_P X_Q$ and ${}_Q Y_R$, where P, Q and R are II_1 factors, this construction yields a bimodule ${}_P(X \otimes_Q Y)_R$, while the 'contragredient' of X is a bimodule ${}_Q \bar{X}_P$. It turns out that, analogous to example (a)(ii), but for an infinite compact group, the collection $\mathcal{G}(R)$ of isomorphism classes of $R - R$ bimodules satisfies all the requirements of a hypergroup with the exception of our finiteness requirement. However, it turns out—see [6], for instance—that $\mathcal{G}(R)$ has many interesting finite sub-hypergroups.

Thus, if we have an inclusion $N \subset M$ of II_1 factors, and let α denote the isomorphism class of $L^2(M, tr)$ regarded as an $N - N$ bimodule, then this bimodule is 'of finite type' precisely when the so-called Jones index $[M : N]$ is finite; and the smallest subclass K of $\mathcal{G}(N)$ which contains α and is 'closed under Connes' fusion' turns out to be finite in many interesting cases (the so-called finite depth case). Furthermore, most of the examples discussed in (a) above, are known to arise in this fashion. It is to be noted that these examples often exhibit dimension functions which assume non-integral values. For example, the hypergroup K_n of the 'so-called' A_n-subfactor has $[\frac{n+1}{2}]$ elements (where $[m]$ denotes the integral part of m) and the corresponding weights are given by $w(c_j) = (\frac{\sin(\frac{(2j+1)\pi}{n+1})}{\sin(\frac{\pi}{n+1})})^2$. (See [7] for details.)

2. Actions

In the sequel, given a finite set X, we shall write sX for the *simplex* based on X, by which we mean the subset of $\mathbb{R}^X$ defined by $sX = \{\alpha \in \mathbb{R}^X : \alpha_x \geq 0 \,\forall\, x \in X, \sum_{x \in X} \alpha_x = 1\}$. Let $M_X(\mathbb{C})$ denote the set of matrices with rows and columns indexed by X. We shall denote by $\mathit{Aff}(X)$ the set of affine (or convex) maps of sX, and make the natural identification—as in linear algebra—between $\mathit{Aff}(X)$ and those elements of $M_X(\mathbb{C})$ which are *column-stochastic*. Thus the map $T \in \mathit{Aff}(X)$ is identified with the column-stochastic matrix $(t_{x,y})_{x,y \in X}$ precisely when $(T\alpha)_x = \sum_y t_{x,y}\alpha_y$.

We come now to the central notion of this paper.

Definition 2.1 An **action** of a hypergroup K on a finite set X is a mapping $K \ni c_i \mapsto \pi(c_i) \in \mathit{Aff}(X)$ such that

$$\pi(c_0) = I \tag{2.7}$$

$$\pi(c_i)\pi(c_j) = \sum_{k=1}^{n} n_{ij}^k \pi(c_k) \tag{2.8}$$

where we think of elements of $\mathit{Aff}(X)$ as column-stochastic $X \times X$ matrices and I denotes the identity matrix.

The action $\pi : K \to \mathit{Aff}(X)$ is called a ***-action** if, in addition, the following condition is satisfied:

$$\pi(c_{i^*}) = \pi(c_i)^* \quad \forall\, i.$$

Observe that if π is a *-action, then each $\pi(c_i)$ is a doubly stochastic matrix.

Note that if $\pi : K \to \mathit{Aff}(X)$ is an action, we may, by linearity, extend π to a map from the convex hull, $co(K)$, of K in $\mathbb{C}K$, (which can be identified in a natural manner with sK) to $\mathit{Aff}(X)$. Thus, for instance, $\pi(e_0) = w(K)^{-1} \sum_{i=0}^{n} w(c_i)\pi(c_i)$.

Example 2.2

(i) Let K be any finite hypergroup and let $X = K$, and define the regular action by $\pi(c_i)_{c_j,c_k} = n_{ik}^j$. This is easily seen to be an action of K (because of associativity of multiplication in $\mathbb{C}K$). It is immediate from the definition that this **left-regular action** of K is a *-action if and only if the following condition is satisfied:

$$n_{ij}^k = n_{i^*k}^j \quad \forall\, i, j, k. \tag{2.9}$$

On the other hand, it is seen from Proposition 1.2(b) (and the fact that $w(c_0) = 1$) that a hypergroup will satisfy condition 2.9 precisely when $w(c_i) = 1 \;\forall\, i$, which, in turn, is seen to happen precisely when the hypergroup is a group in the sense of Example 1.4(a)(i).

(ii) If G is a finite group, then the class hypergroup $K(G)$ admits a natural action on the set G, which is inherited from the product in the group algebra $\mathbb{C}G$.

(iii) If we have an inclusion $N \subset M$ of II_1 factors—see Example 1.4(c)—such that the Jones index $[M : N]$ is finite, then it turns out that the 'infinite hypergroup' $\mathcal{G}(N)$ of isomorphism classes of 'irreducible $N - N$ bimodules acts, by Connes' fusion, on the set $\mathcal{G}(N, M)$ of isomorphism classes of irreducible $N - M$ bimodules. As before, if this subfactor turns out to be of 'finite depth', then we can extract a finite sub-hypergroup K of $\mathcal{G}(N)$ and a finite subset X of $\mathcal{G}(N, M)$ such that K acts on X.

Proposition 2.3 *The following conditions on an action $\pi : K \to \mathit{Aff}(X)$ are equivalent*:

(i) *there exists no non-empty proper subset $X_0 \subset X$ with the property that the sub-simplex sX_0 is stable under $\pi(c_i)$ for all i*;

(ii) *there exist strictly positive numbers α_x, $x \in X$ such that $\sum_{x \in X} \alpha_x = 1$ and $\pi(e_0)_{x,y} = \alpha_x \ \forall\, x, y \in X$*;

(iii) *for each $x, y \in X$, there exists $c_i \in K$ such that $\pi(c_i)_{x,y} > 0$.*

When these equivalent conditions are satisfied, the action is said to be **irreducible***.*

Proof:

(i) $\Rightarrow$ (ii) Suppose there are $x, y \in X$ with $\pi(e_0)_{x,y} = 0$. Let u be the $|X|$-tuple with y-coordinate 1 and all others 0. Then the vector defined by $\alpha = \pi(e_0)u$ has x-coordinate equal to 0. If $\alpha = 0$ then the y-column of $\pi(e_0)$ must be 0; but this dominates a (strictly) positive multiple of the y-column of $\pi(c_0)$, which contradicts $\pi(c_0)_{y,y} = 1$. So $\alpha \neq 0$. Let $X_0 = \{x \in X : \alpha_x > 0\}$. Since $\pi(c_i)\alpha = \alpha$ for all i, it follows that $\pi(c_i)_{x,y} = 0$ for all $x \notin X_0,\ y \in X_0$. Hence $\pi(c_i)(sX_0) \subset sX_0$ for all i, which contradicts assumption (i). Therefore $\pi(e_0)_{x,y} > 0$ for all $x, y \in X$. Now $\pi(e_0)$ is a positive idempotent matrix, and Perron's theorem implies that $\pi(e_0)$ has column rank 1. Since all column sums are 1, the columns must be equal and (ii) follows.

(ii) $\Rightarrow$ (iii) This follows from the strict positivity of the weights $w(c_i)$ and the fact that

$$\begin{aligned} 0 &< \alpha_x \\ &= \pi(e_0)_{x,y} \\ &= w(K)^{-1} \sum_{i=0}^{n} w(c_i)\pi(c_i)_{x,y}. \end{aligned}$$

(iii) $\Rightarrow$ (i) This is obvious. □

Remark 2.4

(a) Let G be a finite group and let $K = G$ as in Example 1.4(a)(i). It is then an easy matter to verify that the notion of an action of the hypergroup K on a set X is exactly the same as the notion of an action of the group G on the set X, and that further, irreducibility of the action of K is the same as transitivity of the action of the group G.

(b) There is a notion of a transitivity of an action of a hypergroup which is strictly stronger than the notion of irreducibility (at least for a general hypergroup), which has the pleasant feature that transitive actions of hypergroups are in bijective correspondence with sub-hypergroups. We shall not say more about this here.

Definition 2.5 Given an irreducible action of a hypergroup K on a set X as in Proposition 2.3, if the numbers $\alpha_x,\ x \in X$ are as in Proposition 2.3(b), we define weights on the set X, as well as the weight of the set X by the prescription

$$w(x) = \frac{\alpha_x}{\min_{y\in X}\alpha_y}, \tag{2.10}$$

$$w(X) = \sum_{x\in X} w(x)$$

$$= \frac{1}{\min_{y\in X}\alpha_y}. \tag{2.11}$$

The next theorem establishes two properties of the weights on a set underlying an irreducible action of a hypergroup.

Theorem 2.6 *Let $\pi : K \to Aff(X)$ be an irreducible action of a hypergroup K on a set X. Then,*
(i) $w(X) \leq w(K)$; *and*
(ii) *for any $\beta = ((\beta_x))_{x\in X}$, if we define $||\beta||_w = (\sum_x \frac{|\beta_x|^2}{w(x)})^{\frac{1}{2}}$, it follows that*

$$||\pi(c)\beta||_w \ \leq \ ||\beta||_w \quad \forall\, \beta \in \mathbb{C}^X, c \in co(K), \tag{2.12}$$

where we think of π as being extended by linearity to all of $co(K)$.

Proof:

(i) Note that if $\alpha_x, x \in X$ are as in Proposition 2.3(ii), and if $x_0 \in X$ is such that

$$\alpha_{x_0} = \min_{x\in X}\alpha_x = w(X)^{-1},$$

then,

$$\begin{aligned} w(X)^{-1} &= \alpha_{x_0} \\ &= \pi(e_0)_{x_0,x_0} \\ &= w(K)^{-1} \sum_{i=0}^{n} w(c_i)\,\pi(c_i)_{x_0,x_0} \\ &\geq w(K)^{-1}\, w(c_0)\,\pi(c_0)_{x_0,x_0} \\ &= w(K)^{-1}, \end{aligned}$$

as desired.

(ii) With x_0 as in the proof of (i) above, notice that $w(x) = \frac{\alpha_x}{\alpha_{x_0}}$. Let D denote the $X \times X$ matrix defined by $d_{x,y} = \delta_{x,y}\frac{\alpha_x}{\alpha_{x_0}}$. We shall think of X-tuples $\beta = ((\beta_x))_{x\in X}$ as column vectors, and write $||\beta|| = (\sum_x |\beta_x|^2)^{\frac{1}{2}}$. Then, by definition, we have

$$||\beta||_w = ||D^{-\frac{1}{2}}\beta||.$$

Thus, it is seen that we need to show that for arbitrary $c \in co(K)$, if we set $T = D^{-\frac{1}{2}}\pi(c) D^{\frac{1}{2}}$, then $||T|| \leq 1$, where $||\cdot||$ denotes the usual operator norm.

Observe that the matrix $P = D^{-\frac{1}{2}}\pi(e_0)D^{\frac{1}{2}}$ is given by $p_{x,y} = \sqrt{\alpha_x\alpha_y}$, and (since $\sum_x \alpha_x = 1$) represents the orthogonal projection onto the one-dimensional subspace spanned by the vector $v = ((\sqrt{\alpha_x}))$.

Notice next that $ce_0 = e_0$, so $\pi(c)((\alpha_x)) = ((\alpha_x))$, and consequently,

$$\begin{aligned} Tv &= (D^{-\frac{1}{2}}\pi(c)D^{\frac{1}{2}})((\sqrt{\alpha_x})) \\ &= D^{-\frac{1}{2}}\pi(c)\left(\left(\frac{\alpha_x}{\sqrt{\alpha_{x_0}}}\right)\right) \\ &= D^{-\frac{1}{2}}\left(\left(\frac{\alpha_x}{\sqrt{\alpha_{x_0}}}\right)\right) \\ &= v. \end{aligned}$$

On the other hand, $\pi(c)^*$ is a row-stochastic matrix (since $\pi(c)$ is column-stochastic) and hence,

$$\begin{aligned} T^*v &= D^{\frac{1}{2}}\pi(c)^*D^{-\frac{1}{2}}\left(\left(\sqrt{\alpha_x}\right)\right) \\ &= D^{\frac{1}{2}}\pi(c)^*\left(\left(\sqrt{\alpha_{x_0}}\right)\right) \\ &= D^{\frac{1}{2}}\left(\left(\sqrt{\alpha_{x_0}}\right)\right) \\ &= v. \end{aligned}$$

Thus $T^*Tv = v$. Since T^*T is a Hermitian matrix, it is unitarily diagonalisable, and hence its norm equals its spectral radius. Since T^*T has non-negative entries, and the positive vector v is fixed by T^*T, it follows from the Perron-Frobenius theorem that the spectral radius, and hence the norm, of T^*T must be 1. □

Remark 2.7 Theorem 2.6(ii), in the special case of the left-regular action of a commutative hypergroup, appears in [8], where it is interpreted as an 'entropy inequality'.

Observe also that, in the notation of Theorem 2.6, the obvious inequality $|X| \leq w(X)$, together with Theorem 2.6(i), shows that $|X| \leq w(K)$. This is the justification for the terminology used in the next definition.

Definition 2.8 An irreducible action $\pi : K \to \mathit{Aff}(X)$ is said to be a **maximal action** if $|X| = w(K)$.

Thus, in order for a hypergroup to admit a maximal action, it is clearly necessary that $w(K)$ is an integer.

Theorem 2.9

(a) *Suppose a hypergroup K admits a maximal action $\pi : K \to \mathit{Aff}(X)$ which is also a *-action. Then the matrices $\{A_i = w(c_i)\pi(c_i), 0 \leq i \leq n\}$ define an association scheme—see Example* 1.4(*b*)*—and in particular, $w(c_i) \in \mathbb{N}\ \forall\, i$, and the hypergroup K comes from an association scheme* (*in the sense of Example* 1.4(*b*)).

(b) *Conversely, if a hypergroup K comes from an association scheme in the sense of Example* 1.4(*b*), *then K admits a maximal *-action.*

Proof:

(a) First observe that since $w(x) \geq 1 \; \forall \; x \in X$, we have

$$w(K) = |X| \leq \sum_{x \in X} w(x) = w(X) \leq w(K).$$

Hence we necessarily have $w(x) = 1 \; \forall \; x \in X$. In particular, $\pi(e_0)_{x,y} = \frac{1}{k}$, where $k = |X|$. Notice next that, for any $x \in X$, we have

$$\begin{aligned} \frac{1}{k} &= \pi(e_0)_{x,x} \\ &= w(K)^{-1} \sum_{i=0}^{k} w(c_i)\pi(c_i)_{x,x} \\ &\geq w(K)^{-1} w(c_0)\pi(c_0)_{x,x} \\ &= w(K)^{-1} \\ &= \frac{1}{k}, \end{aligned}$$

from which we may deduce that

$$\pi(c_i)_{x,x} = 0, \quad \forall \; x \in X, 0 < i \leq n. \tag{2.13}$$

Since π is a *-action, notice that if $0 \leq i, j \leq n$, then, since $n^0_{ij^*} = \delta_{i,j} w(c_i)^{-1}$, it follows from Eq. (2.13) that

$$\begin{aligned} (\pi(c_i)\pi(c_j)^*)_{x,x} &= \pi(c_i c_{j^*})_{x,x} \\ &= \sum_{l=0}^{n} n^l_{ij^*}\pi(c_l)_{x,x} \\ &= \delta_{i,j} w(c_i)^{-1} + \sum_{l=1}^{n} n^l_{ij^*}\pi(c_l)_{x,x} \\ &= \delta_{i,j} w(c_i)^{-1}. \end{aligned} \tag{2.14}$$

Since $\pi(c_i)$ has non-negative entries, this shows that

$$i \neq j, \pi(c_i)_{x,y} > 0 \Rightarrow \pi(c_j)_{x,y} = 0.$$

On the other hand,

$$\sum_{i=0}^{n} w(c_i)\pi(c_i)_{x,y} = w(K)\pi(e_0)_{x,y} = 1 \quad \forall \; x, y;$$

it follows from the last two equations that $A_i = w(c_i)\pi(c_i)$ is a matrix all of whose entries are 0 or 1. Further, $\sum_{i=0}^{n} A_i$ is the $k \times k$ matrix J_k (all of whose entries are equal to 1), and it follows from Lemma 1.3 that $A_{i^*} = A_i^*$. Deduce now from Eq. (2.14) that $(A_i A_{i^*})_{x,x} = w(c_i) \,\forall\, i$, and in particular, $w(c_i)$ is a positive integer for each i. It follows easily now that $\{A_0, A_1, \ldots, A_n\}$ is an association scheme as in Example 1.4(b); further, since the matrices $\{A_i : 0 \leq i \leq n\}$ are clearly linearly independent—they are actually orthogonal with respect to the natural inner-product on the set of matrices—it is seen that the mapping $c_i \mapsto w(c_i)^{-1} A_i$ induces a linear isomorphism of $\mathbb{C}K$ onto the algebra spanned by the A_i's and consequently, the hypergroup K does indeed come from the association scheme as asserted.

(b) This is easy: simply define $\pi(c_i) = w(c_i)^{-1} A_i$ and verify that this is an action with all the desired properties. □

3. *-Actions

In what follows, we shall classify all *-actions of some commutative hypergroups. (Actions $\pi_i : K \to \mathit{Aff}\, X_i$, $i = 1, 2$ are said to be *equivalent* if there exists a bijection $\sigma : X_1 \to X_2$ such that $\pi_1(c_j)_{x,y} = \pi_2(c_j)_{\sigma(x),\sigma(y)} \,\forall\, c_j \in K, x, y \in X_1$.) Since every hypergroup admits a unique (necessarily irreducible) *-action on a singleton set, we only consider non-trivial actions in what follows.

As a first step, we make the observation that any *-action breaks up naturally as a direct sum of irreducible *-actions. This is because any doubly stochastic matrix which is a self-adjoint projection of rank r is, up to conjugation by permutation matrices, nothing but a direct sum of matrices of the form P_k—where P_k denotes the $k \times k$ matrix all of whose entries are equal to $\frac{1}{k}$. It follows that in order to classify *-actions, we only need to classify irreducible *-actions. We shall use the following terminology and facts in the process.

Suppose $K = \{c_0, c_1, \ldots, c_n\}$ is a finite commutative hypergroup, and suppose $\hat{K} = \{\chi_0 = 1, \chi_1, \ldots, \chi_n\}$ is the set of characters of K. Thus, each χ_j is a multiplicative homomorphism from $\mathbb{C}K$ into $\mathbb{C}$ such that $\chi_j(c_0) = 1$. It is known that in general, $\hat{K}$ is a *signed hypergroup* (with respect to pointwise products and complex conjugation) meaning that there exist real, not necessarily non-negative, constants q_{ij}^k such that

$$\chi_i \chi_j = \sum_{k=0}^{n} q_{ij}^k \chi_k$$

$$\sum_{k=0}^{n} q_{ij}^k = 1$$

$$\exists \text{ unique } \hat{i} \text{ such that } \chi_{\hat{i}} = \overline{\chi_i}$$

$$q_{ij}^0 \neq 0 \Leftrightarrow j = \hat{i}$$

$$q_{ij}^0 > 0 \Leftrightarrow j = \hat{i}$$

(Most recently, and in the language used in this paper, these facts can be found in [8]. They may also be found in [1] (Theorems II.5.9 and II.5.10), where the result is credited to

Kawada [4]. Kawada worked with 'C-algebras' which are just signed hypergroups whose basis elements are multiplied by some positive scalars.)

In many cases, $\hat{K}$ might turn out to be a *bona fide* (positive) hypergroup. For instance, the dual of the class hypergroup $K(G)$ of a finite group G—see Example 1.4(a)(ii)—is precisely the character hypergroup $\hat{K}(G)$—see Example 1.4(a)(iii).

The positive number $(q^0_{\hat{i}i})^{-1}$ is called the weight of χ_i and denoted by $w(\chi_i)$. For convenience of reference, we list some facts concerning characters and 'duals' of finite commutative hypergroups. (These facts, in this language, may be found in [8]; in fact, parts (a), (c), (d) of the next proposition may also be found in Section II.5 of [1].)

Proposition 3.1 *Suppose $K = \{c_0, \dots, c_n\}$ is a finite commutative hypergroup and $\hat{K} = \{\chi_0, \dots, \chi_n\}$ is its dual signed hypergroup as above. Then,*

(a) $\chi_i(c_{j^*}) = \overline{\chi_i(c_j)}\ \forall\, i, j$;

(b) $|\chi_i(c_j)| \le 1,\ \forall\, i, j$;

(c) *if we let*

$$e_i = \frac{w(\chi_i)}{w(K)} \sum_{k=0}^{n} w(c_k)\chi_i(c_{k^*})c_k, \tag{3.15}$$

then $\{e_0, e_1, \dots, e_n\}$ is a basis of self-adjoint projections for $\mathbb{C}K$;

(d) $c_i = \sum_{j=0}^{n} \chi_j(c_i)e_j\ \forall\, i$.

Now suppose $\pi : K \to \mathit{Aff}(X)$ is a *-action of K. Then, π extends, by linearity, to a *-homomorphism from $\mathbb{C}K$ into $M_X(\mathbb{C})$, and consequently there exist well-defined non-negative integers $b_0, b_1, \dots, b_n$ which are the multiplicities with which the characters $\chi_0, \chi_1, \dots, \chi_n$ feature in the representation π. Alternatively, using Eq. (3.15), we see that $b_i = \chi(e_i)$, where we write $\chi(x) = Tr(\pi(x))$ for all $x \in \mathbb{C}K$.

In the sequel, we shall write $\mathbf{b} = [b_0, \dots, b_n]$ for the multiplicity vector for an action.

The following lists some facts concerning actions that we will use constantly in our subsequent discussion of *-actions.

Proposition 3.2 *Suppose $\pi : K \to \mathit{Aff}\,X$ is an irreducible *-action of a finite commutative hypergroup $K = \{c_0 \dots, c_n\}$ on a set X with k elements. With the foregoing notation, we have*:

(a) $\chi(c_0) = k = |X|$;

(b) $k \le w(K)$;

(c) $\chi(c_i) \ge 0\ \forall\, i$;

(d) $b_0 = \chi(e_0) = 1$;

(e) $b_i = \chi(e_i) \le w(\chi_i)\ \forall\, i$;

(f) *$\pi(e_0)$ is the matrix all of whose entries are equal to $\frac{1}{k}$; this matrix will henceforth be denoted by the symbol P_k.*

Proof: Assertions (a) and (c) are obvious while (b) and (d) are consequences of Theorem 2.6(i) and Proposition 2.3, respectively; also, (f) is a consequence of Proposition 2.3 and the obvious fact that P_k is the unique $k \times k$ doubly stochastic matrix which is a projection of rank one.

As for (e), since $\chi(c_k), w(c_k) \geq 0 \; \forall \, k$, it follows from Proposition 3.1(c) and (b) that

$$\begin{aligned} |b_i| &= |Tr\, \pi(e_i)| \\ &\leq \frac{w(\chi_i)}{w(K)} \sum_{k=0}^{n} w(c_k)\chi(c_k) \\ &= w(\chi_i)\, \chi(e_0) \\ &= w(\chi_i), \end{aligned}$$

and the proof is complete. □

4. *-Actions of Hermitian hypergroups

In this section, we first describe a reformulation of what it means to have an irreducible *-action of a Hermitian hypergroup—i.e., a hypergroup where $c_{i^*} = c_i$ for all i. We will need some terminology.

Remark 4.1 What is usually referred to as the standard $(k-1)$-simplex is the convex hull of the standard basis—call it $\{x_1, \ldots, x_k\}$—in $\mathbb{R}^k$; the centroid of this simplex is the vector with all co-ordinates equal to $\frac{1}{k}$. Hence, if we define $v_j = x_j - \frac{1}{k}\sum_{l=1}^{k} x_l$, then we see that $v_1, \ldots, v_k$ are k vectors of what might be called a regular simplex centered at the origin; all these vectors lie in the orthogonal complement of the vector $\sum_{l=1}^{k} x_l$, and are easily verified to satisfy the conditions:

$$\langle v_l, v_j \rangle = \delta_{lj} - \frac{1}{k} \tag{4.16}$$

$$\sum_{j=1}^{k} v_j = 0. \tag{4.17}$$

It should be observed—as was pointed out to us by the referee—that condition (4.17) is a consequence of condition (4.16), as is seen by computing the inner product of $\sum_{l=1}^{k} v_l$ with itself. It is clear, on the other hand that if $z_1, \ldots, z_k$ is any collection of k vectors in $\mathbb{R}^{k-1}$ satisfying condition 4.16, then there exists a unique orthogonal transformation mapping $\mathbb{R}^{k-1}$ onto the hyperplane spanned by $\{v_1, \ldots, v_k\}$ which maps z_j onto v_j for $1 \leq j \leq k$. For this reason, we shall say that a collection $\{z_1, \ldots, z_k\} \subset \mathbb{R}^{k-1}$ are the vertices of a *regular normalised* $(k-1)$*-simplex in* $\mathbb{R}^{k-1}$ *centered at* 0 precisely when they satisfy the condition 4.16. In this case the simplex they span is

$$\Delta_z = \left\{ v \in \mathbb{R}^{k-1} : \langle v, z_j \rangle + \frac{1}{k} \geq 0 \quad \forall\, j \right\}.$$

Theorem 4.2 *Let K be a Hermitian hypergroup.*

(a) *Suppose π is an irreducible *-action of K on a set X with $|X| = k$. Suppose the 'multiplicity vector' associated with this action is given by* $\mathbf{b} = [b_0 = 1, b_1, \ldots, b_n]$. *Consider the sequence*

$$\chi_1, \ldots, \chi_1; \chi_2, \ldots, \chi_2; \ldots; \chi_n, \ldots, \chi_n \tag{4.18}$$

where χ_i is repeated b_i times, for $1 \leq i \leq n$. Let us re-write the sequence displayed in (4.18) *as*: $\phi_1, \phi_2, \ldots, \phi_{k-1}$.

For $0 \leq j \leq n$, consider the $(k-1) \times (k-1)$ diagonal matrix defined by

$$T_j = diag(\phi_1(c_j), \ldots, \phi_{k-1}(c_j)).$$

Then there exist $z_1, \ldots, z_k \in \mathbb{R}^{k-1}$ such that

(i) *$z_1, \ldots, z_k$ are the vertices of a regular normalised $(k-1)$-simplex in $\mathbb{R}^{k-1}$ centered at the origin*; *and*

(ii) *each T_j, $0 \leq j \leq n$ maps the convex hull of $\{z_i : 1 \leq i \leq k\}$ into itself.*

(b) *Conversely, if there exists $\{z_i : 1 \leq i \leq k\} \subset \mathbb{R}^{k-1}$ satisfying* (i) *and* (ii) *above, then there exists an irreducible *-action of K on a set of k elements with multiplicity vector* $\mathbf{b}$.

(c) *Finally, if $\{z_i^{(\epsilon)} : 1 \leq i \leq k\}$, $\epsilon = 1, 2$ are two sets of points satisfying* (i) *and* (ii) *above, then the associated *-actions are equivalent if and only if there exist an orthogonal transformation $S : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ which commutes with each T_j, and a permutation $\sigma \in S_k$ such that $Sz_i^{(1)} = z_{\sigma(i)}^{(2)}\ \forall\, i$.*

Proof:

(a) We adopt the convention that the indices i, j, l always satisfy $0 \leq i \leq n$, $1 \leq j, l \leq k$.

Let us write $\phi_k(c_i) = 1\ \forall\, i$. Since the hypergroup is Hermitian (and hence commutative), it is clear that if we are given an irreducible *-action $\pi : K \to \mathit{Aff}\, X$ on a set $X = \{x_j\}_j$ with multiplicity vector $\mathbf{b}$, then $\{\pi(c_i)\}_i$ is a collection of commuting $k \times k$ Hermitian matrices with non-negative entries. We regard the $\pi(c_i)$'s as the matrices of linear operators on the real Hilbert space $\ell^2_{\mathbb{R}}(X)$ with respect to the standard basis $\{x_j\}_j$. Since these are pairwise commuting Hermitian operators, we can—by definition of $\mathbf{b}$—find an orthonormal basis $\{y_j\}_j$ of $\ell^2_{\mathbb{R}}(X)$ such that

$$\pi(c_i)y_j = \phi_j(c_i)y_j.$$

Note that $\pi(e_0) = P_k$, by the assumed irreducibility of the *-action. (See Proposition 3.2 for the definition of P_k.) So we may assume that $y_k = \frac{1}{\sqrt{k}} \sum_{j=1}^k x_j$. It follows from the discussion preceding the statement of this theorem that if we define $v_j = x_j - \frac{1}{\sqrt{k}} y_k$, then $\{v_1, \ldots, v_k\}$ are the vertices of a regular normalised $(k-1)$-simplex in the subspace $\{y_k\}^\perp = span\{y_1, \ldots, y_{k-1}\}$, which is centered at the origin; further, the simplex they span is given by $\Delta_v = \Delta_x - \frac{1}{\sqrt{k}} y_k = \Delta_x - \pi(e_0)(\Delta_x)$, and is consequently mapped into itself by each $\pi(c_i)$. If we now set $z_l = [z_{l1}, \cdots, z_{l,k-1}]$, where $z_{lj} = \langle v_l, y_j \rangle$, the construction implies that the z_l's are the vertices of a regular normalised simplex in $\mathbb{R}^{k-1}$ which is mapped into itself by each of the matrices T_i, $0 \leq i \leq n$.

(b) Conversely, if we are given a regular normalised simplex in $\mathbb{R}^{k-1}$ with vertices $\{z_l = [z_{l,1}, \ldots, z_{l,k-1}] : 1 \le l \le k\}$ which is centered at 0, and is left invariant by the matrices T_i for each i, define the vectors v_l in $\mathbb{R}^k$ by

$$v_l = \left[z_{l,1}, \ldots, z_{l,k-1}, \frac{1}{\sqrt{k}} \right]$$

and note that $\{v_l : 1 \le l \le k\}$ is an orthonormal basis for $\mathbb{R}^k$.

Next consider the (real) diagonal $k \times k$ matrices $\tilde{T}_i$, $0 \le i \le n$, defined by

$$\tilde{T}_i = T_i \oplus 1_1,$$

where 1_1 denotes the 1×1 'identity matrix'; then we see that $\{\tilde{T}_i : 0 \le i \le n\}$ is a collection of Hermitian matrices which satisfy

$$\tilde{T}_i \tilde{T}_m = \sum_{p=0}^{n} n_{im}^{p} \tilde{T}_p \quad \forall\, 0 \le i, m \le n.$$

Further, it should be clear that for each i, $\tilde{T}_i$ maps the convex hull Δ_v of $\{v_j : 1 \le j \le k\}$ into itself.

It follows that if we write the matrices of the $\tilde{T}_i$'s with respect to the orthonormal basis $\{v_j : 1 \le j \le k\}$, i.e., if we define the matrices $\{\pi(c_i) : 0 \le i \le n\}$ by

$$(\pi(c_i))_{jl} = \langle \tilde{T}_i v_l, v_j \rangle,$$

then π will define a *-action of K with multiplicity vector **b**.

(c) is an easy exercise in linear algebra. □

5. An infinity of actions

We would like to acknowledge the suggestion made by E. Bannai that Theorem 4.2 was likely to produce an example of a finite hypergroup with infinitely many pairwise inequivalent irreducible *-actions.

Lemma 5.1 *Suppose $K = \{c_0, c_1, c_2\}$ is a Hermitian 3-element hypergroup with character table*

	c_0	c_1	c_2
χ_0	*1*	*1*	*1*
χ_1	1	x_1	x_2
χ_2	1	y_1	y_2

,

*where $x_1, x_2, y_1, y_2 \in (-\frac{1}{2}, \frac{1}{2})$. Then K admits infinitely many pairwise inequivalent irreducible *-actions on a three element set, all with the multiplicity vector $\mathbf{b} = [1, 1, 1]$.*

Proof: Fix $\theta \in [0, 2\pi]$, and define

$$z_j^{(\theta)} = \sqrt{\frac{2}{3}}\left[\cos\left(\theta + \frac{2j\pi}{3}\right), \sin\left(\theta + \frac{2j\pi}{3}\right)\right];$$

it is clear that $z_1^{(\theta)}, z_2^{(\theta)}, z_3^{(\theta)}$ are the vertices of a regular normalised 2-simplex in $\mathbb{R}^2$ centered at the origin. In the notation of Theorem 4.2, and with $x_0 = y_0 = 1$, let

$$T_0 = \begin{bmatrix} x_0 & 0 \\ 0 & y_0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}.$$

Let $\Delta^{(\theta)}$ denote the regular normalised simplex spanned by the $z_i^{(\theta)}$'s; then (see Remark 4.1), we have:

$$\Delta^{(\theta)} = \left\{ v \in \mathbb{R}^2 : \langle v, z_i^{(\theta)} \rangle \geq -\frac{1}{3} \right\}.$$

Notice then, that for $1 \leq k, l \leq 3$ and $j = 1, 2$, we have

$$\begin{aligned} |\langle T_j z_k^{(\theta)}, z_l^{(\theta)} \rangle| &\leq ||T_j z_k^{(\theta)}|| \, ||z_l^{(\theta)}|| \\ &< \frac{1}{2} \, ||z_k^{(\theta)}|| \, ||z_l^{(\theta)}|| \\ &= \frac{1}{2}\sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}} \\ &= \frac{1}{3}, \end{aligned}$$

thereby establishing that $\Delta^{(\theta)}$ is mapped into its interior by T_1 and T_2. Hence $\Delta^{(\theta)}$ is mapped into itself by each T_i.

Therefore, according to Theorem 4.2, each $\Delta^{(\theta)}$ accounts for one irreducible *-action of K on a 3-element set. On the other hand, since $\{\chi_j : 0 \leq j \leq 2\}$ are the distinct characters of K, it follows that the T_i's linearly span the set of all real diagonal matrices; consequently only diagonal orthogonal matrices can commute with all the T_i's. We may finally conclude from Theorem 4.2(c) that if $0 < |\theta - \theta'| < \frac{2\pi}{3}$, then the *-actions corresponding to $\Delta^{(\theta)}$ and $\Delta^{(\theta')}$ are inequivalent; and the lemma is proved. □

On the other hand, the existence of hypergroups satisfying the conditions of the above lemma has been demonstrated in [9]; in fact, in that example, we have (see the Table (7.11) on p. 30) $x_1 = \frac{2}{15}$, $y_1 = \frac{-1}{30}$, $x_2 = \frac{-1}{25}$, $y_1 = \frac{1}{125}$. We thus see that it can happen that a finite hypergroup admits a continuum of pairwise inequivalent irreducible *-actions. It should be remarked that since the above numbers are all bounded, in absolute value, by $\frac{1}{7}$, the reasoning in the above lemma can be imitated to construct a continuum of actions of this hypergroup on an 8-element set.

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Note

1. This definition actually only yields *finite* hypergroups, but we shall never consider any other kind here.

References

1. E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, Menlo Park, 1984.
2. V.F.R. Jones, "Index for subfactors," *Invent. Math.* **72** (1983), 1–15.
3. V.F.R. Jones and V.S. Sunder, *Introduction to Subfactors*, Cambridge University Press, 1997.
4. Y. Kawada, "Über den Dualitatssatz der Charaktere nichtcommutative Gruppen," *Proc. Phys. Math. Soc. Japan* **24**(3) (1942), 97–109.
5. N. Obata and N.J. Wildberger, "Generalized hypergroups and orthogonal polynomials," *Nagoya Math. J.* **142** (1996), 67–93.
6. V.S. Sunder, "II_1 factors, their bimodules and hypergroups," *Trans. Amer. Math. Soc.* **330** (1992), 227–256.
7. V.S. Sunder, "On the relation between subfactors and hypergroups," Applications of hypergroups and related measure algebras, *Contemp. Math.* **183** (1995), 331–340.
8. N.J. Wildberger, "Duality and entropy for finite abelian hypergroups and fusion rule algebras," *J. London Math. Soc.* **56**(2) (1997), 275–291.
9. N.J. Wildberger, "Lagrange's theorem and integrality for finite commutative hypergroups with applications to strongly regular graphs," *J. Algebra* **182** (1996) 1–37.