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Irreducible Representations of Wreath Products of Association Schemes

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Abstract. The wreath product of finite association schemes is a natural generalization of the notion of the wreath product of finite permutation groups. We determine all irreducible representations (the Jacobson radical) of a wreath product of two finite association schemes over an algebraically closed field in terms of the irreducible representations (Jacobson radicals) of the two factors involved.

Keywords: association scheme, irreducible representation, wreath product

1. Introduction

In general, representation theory is a valuable tool for the study of association schemes. In this article, we consider the representations of the wreath product of association schemes. We consider the association scheme as defined in [1], but we do not assume the commutativity of it. Historically, this is also called a homogeneous coherent configuration. Here we will consider irreducible representations of wreath products of association schemes.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be association schemes. Then we can define the wreath product $\mathfrak{X} \wr \mathfrak{Y}$ of $\mathfrak{X}$ and $\mathfrak{Y}$. Let F be an algebraically closed field, and let $F\mathfrak{X}$, $F\mathfrak{Y}$, and $F(\mathfrak{X} \wr \mathfrak{Y})$ be the adjacency algebras of $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{X} \wr \mathfrak{Y}$ over F, respectively. We define representations of $F(\mathfrak{X} \wr \mathfrak{Y})$ in terms of irreducible representations of $F\mathfrak{X}$ and $F\mathfrak{Y}$. They are also irreducible with some exceptions. Next we determine the Jacobson radical of $F(\mathfrak{X} \wr \mathfrak{Y})$ and its dimension. Then we can conclude that every irreducible representation of $F(\mathfrak{X} \wr \mathfrak{Y})$ is defined from an irreducible representation of $F\mathfrak{X}$ or $F\mathfrak{Y}$. Also we will describe all irreducible characters of the wreath product. In K. See and S. Y. Song [5], they wrote that they can calculate the character table of the wreath product of association schemes. But they assume the commutativity of association schemes. In the non-commutative case, there are some difficulties.

2. Preliminaries

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be association schemes, in the sense of [1], with adjacency matrices $\{A_0, \dots, A_d\}$ and $\{B_0, \dots, B_h\}$, respectively. We suppose that A_0 and B_0 are the identity matrices.

We denote by n and n' the sizes of matrices A_i and B_j, respectively. We keep these notations throughout this paper. In [5], the wreath product $\mathfrak{X} \wr \mathfrak{Y}$ of $\mathfrak{X}$ and $\mathfrak{Y}$ is defined as follows. (Some notations differ from [5], but they are essentially the same.) We consider the set of matrices

$$\{A_0 \otimes B_0, \ldots, A_d \otimes B_0, J_n \otimes B_1, \ldots, J_n \otimes B_h\},$$

where J_n is the all one matrix of degree n. Then the wreath product $\mathfrak{X} \wr \mathfrak{Y}$ of $\mathfrak{X}$ and $\mathfrak{Y}$ is defined by the above matrices as adjacency matrices. It is easy to verify that it satisfies the definition of an association scheme. This can be considered as a generalization of the wreath product of transitive finite permutation groups. Let G and H be transitive finite permutation groups on the sets X and Y, respectively. Then we can define association schemes $\mathfrak{X}(G, X)$ and $\mathfrak{X}(H, Y)$ by [1, II, Example 2.1]. Also we can define the wreath product $G \wr H$ of G and H [4, Section 1.2]. The group $G \wr H$ is transitive on the set $X \times Y$, and the association scheme $\mathfrak{X}(G \wr H, X \times Y)$ is isomorphic to $\mathfrak{X}(G, X) \wr \mathfrak{X}(H, Y)$.

Let F be a field. We define the adjacency algebra $F\mathfrak{X}$ of $\mathfrak{X}$ over F by

$$F\mathfrak{X} = \bigoplus_{i=0}^{d} FA_i$$

as a matrix algebra over F. Since A_i is a 01-matrix, this definition has meaning. Clearly the dimension of $F\mathfrak{X}$ is $d + 1$. A representation of $F\mathfrak{X}$ is a matrix representation of $F\mathfrak{X}$, namely an algebra homomorphism from $F\mathfrak{X}$ to the full matrix ring of some degree over F. A representation of $F\mathfrak{X}$ is irreducible if the corresponding right $F\mathfrak{X}$-module has no proper submodule.

We state here some facts about finite dimensional algebras. From here, we always assume that the field F is algebraically closed. Let A be a finite dimensional algebra over F. The Jacobson radical $\mathrm{Rad}(A)$ of A is the intersection of all maximal right ideals of A.

Proposition 2.1 ([2, Proposition 3.1.9]) *The Jacobson radical* $\mathrm{Rad}(A)$ *of* A *is a nilpotent* (*two-sided*) *ideal containing all nilpotent right and left ideals.*

The right socle $\mathrm{Soc}(A)$ is the sum of all irreducible right A-submodules of A. Then, for any $x \in \mathrm{Soc}(A)$ and $y \in \mathrm{Rad}(A)$, we have $xy = 0$.

It is well known that $A/\mathrm{Rad}(A)$ is semisimple. Since F is algebraically closed, we have

$$A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^{r} M_{d_i}(F),$$

where d_i's are the degrees of irreducible representations of A. So we have the following.

Proposition 2.2 *Let* $S_1, \ldots, S_r$ *be all non-equivalent irreducible representations of* A. *Then* $\dim_F A = \sum_{i=1}^{r} (\deg S_i)^2 + \dim_F \mathrm{Rad}(A)$.

3. Irreducible representations

The adjacency algebra $F(\mathfrak{X} \wr \mathfrak{Y})$ of the wreath product $\mathfrak{X} \wr \mathfrak{Y}$ has a basis

$$\{A_0 \otimes B_0, \ldots, A_d \otimes B_0, J_n \otimes B_1, \ldots, J_n \otimes B_h\}.$$

So $\dim_F F(\mathfrak{X} \wr \mathfrak{Y}) = d + h + 1$. In this section, we determine all irreducible representations of $F(\mathfrak{X} \wr \mathfrak{Y})$ in terms of irreducible representations of $F\mathfrak{X}$ and $F\mathfrak{Y}$. Let $S_1, \ldots, S_r$ be all non-equivalent irreducible representations of $F\mathfrak{X}$, and let $T_1, \ldots, T_s$ be all non-equivalent irreducible representations of $F\mathfrak{Y}$. We denote by k_i the valency of A_i, and denote by k'_j the valency of B_j. The map $A_i \mapsto k_i$ defines a representation of $F\mathfrak{X}$ of degree 1. We assume that S_1 is this representation, and also assume that $T_1 : B_j \mapsto k'_j$. Note that $J_n = \sum_{i=0}^{d} A_i$ and FJ_n is a one-dimensional $F\mathfrak{X}$-module affording the representation S_1. So $S_\mu(J_n) = 0$ for $\mu \neq 1$.

For $\mu \neq 1$, we put

$$\begin{cases} \tilde{S}_\mu(A_i \otimes B_0) = S_\mu(A_i) \\ \tilde{S}_\mu(J_n \otimes B_j) = 0, \end{cases}$$

and extend this linearly. Note that the definition of $\tilde{S}_\mu(J_n \otimes B_0) = \sum_{i=0}^{d} \tilde{S}_\mu(A_i \otimes B_0)$ is duplicated. But, since $\mu \neq 1$, we have $\sum_{i=0}^{d} \tilde{S}_\mu(A_i \otimes B_0) = S_\mu(J_n) = 0$. So $\tilde{S}_\mu$ is well-defined. Also we define

$$\begin{cases} \tilde{T}_\nu(A_i \otimes B_0) = k_i E \\ \tilde{T}_\nu(J_n \otimes B_j) = nT_\nu(B_j), \end{cases}$$

where E is the identity matrix of degree $\deg T_\nu$. In this case, $\sum_{i=0}^{d} \tilde{T}_\nu(A_i \otimes B_0) = nE = nT_\nu(B_0)$.

Lemma 3.1 *The maps $\tilde{S}_\mu$ $(\mu \neq 1)$ and $\tilde{T}_\nu$ defined above are representations of $F(\mathfrak{X} \wr \mathfrak{Y})$.*

Proof: By direct calculations, we have the result. □

Lemma 3.2 *The representations $\tilde{S}_\mu$ $(\mu \neq 1)$ and $\tilde{T}_1$ are irreducible. If* char $F \nmid n$ *or* char $F = 0$, *then $\tilde{T}_\nu$ is irreducible. Moreover they are non-equivalent to each other.* (*Note that, if* char $F \mid n$, *then $\tilde{T}_\nu$ $(\nu \neq 1)$ is reducible.*)

Proof: Since S_μ is irreducible over an algebraically closed field F, we have $\mathrm{Im} S_\mu = M_\ell(F)$, where $\ell = \deg S_\mu$. Now $\mathrm{Im}\tilde{S}_\mu \supseteq \mathrm{Im} S_\mu$, so we have $\mathrm{Im}\tilde{S}_\mu = M_\ell(F)$. This means that $\tilde{S}_\mu$ is irreducible. The representation $\tilde{T}_1$ has the degree one, so it is irreducible.

If char $F \nmid n$ or char $F = 0$, then $n \neq 0$ in F. So $\tilde{T}_\nu$ is irreducible by the similar argument as above. □

In the rest of this section, we will show that irreducible representations in Lemma 3.2 are all irreducible representations.

Lemma 3.3 *Assume that* char $F \nmid n$ *or* char $F = 0$, *and put*

$$I = \mathrm{Rad}(F\mathfrak{X}) \otimes B_0 + J_n \otimes \mathrm{Rad}(F\mathfrak{Y}).$$

Then the set I is a nilpotent ideal of $F(\mathfrak{X} \wr \mathfrak{Y})$ and $\dim_F I = \dim_F \mathrm{Rad}(F\mathfrak{X}) + \dim_F \mathrm{Rad}(F\mathfrak{Y})$. (*In fact, I is the Jacobson radical of $F(\mathfrak{X} \wr \mathfrak{Y})$. This will be shown in the Proof of Theorem* 3.4.)

Proof: Firstly, we note that $A_i \otimes B_0$ commutes with $J_n \otimes B_j$ for any i and j.

For $\alpha \in \mathrm{Rad}(F\mathfrak{X})$, we have $(\alpha \otimes B_0)(J_n \otimes B_j) = 0$, since J_n is in the socle of $F\mathfrak{X}$. Also, for $\beta \in \mathrm{Rad}(F\mathfrak{Y})$, we have $(A_i \otimes B_0)(J_n \otimes \beta) = k_i J_n \otimes \beta \in I$. Thus I is an ideal of $F(\mathfrak{X} \wr \mathfrak{Y})$.

If $\mathrm{Rad}(F\mathfrak{X})^\ell = 0$ and $\mathrm{Rad}(F\mathfrak{Y})^m = 0$, then

$$I^{\ell+m} = \sum_{i=0}^{\ell+m} (\mathrm{Rad}(F\mathfrak{X})^i \otimes B_0)(J_n \otimes \mathrm{Rad}(F\mathfrak{Y})^{\ell+m-i}) = 0.$$

So I is nilpotent.

Since $\dim_F \mathrm{Rad}(F\mathfrak{X}) = \dim_F \mathrm{Rad}(F\mathfrak{X}) \otimes B_0$, $\dim_F \mathrm{Rad}(F\mathfrak{Y}) = \dim_F J_n \otimes \mathrm{Rad}(F\mathfrak{Y})$, and $\mathrm{Rad}(F\mathfrak{X}) \otimes B_0 \cap J_n \otimes \mathrm{Rad}(F\mathfrak{Y}) = 0$, we have $\dim_F I = \dim_F \mathrm{Rad}(F\mathfrak{X}) + \dim_F \mathrm{Rad}(F\mathfrak{Y})$. □

Theorem 3.4 *Suppose that* char $F \nmid n$ *or* char $F = 0$. *Then $\tilde{S}_2, \ldots, \tilde{S}_r, \tilde{T}_1, \ldots, \tilde{T}_s$ are all non-equivalent irreducible representations of $F(\mathfrak{X} \wr \mathfrak{Y})$.*

Proof: We use Propositions 2.1 and 2.2. By Lemma 3.3, we have

$$\begin{aligned}
\dim_F F(\mathfrak{X} \wr \mathfrak{Y}) &\geq \sum_{\mu=2}^{r} (\deg \tilde{S}_\mu)^2 + \sum_{\nu=1}^{s} (\deg \tilde{T}_\nu)^2 + \dim_F \mathrm{Rad}(F(\mathfrak{X} \wr \mathfrak{Y}))\\
&\geq \sum_{\mu=2}^{r} (\deg \tilde{S}_\mu)^2 + \sum_{\nu=1}^{s} (\deg \tilde{T}_\nu)^2 + \dim_F I\\
&= (\dim_F F\mathfrak{X} - \dim_F \mathrm{Rad}(F\mathfrak{X}) - 1) + (\dim_F F\mathfrak{Y} - \dim_F \mathrm{Rad}(F\mathfrak{Y}))\\
&\quad + (\dim_F \mathrm{Rad}(F\mathfrak{X}) + \dim_F \mathrm{Rad}(F\mathfrak{Y}))\\
&= \dim_F F\mathfrak{X} + \dim_F F\mathfrak{Y} - 1 = \dim_F F(\mathfrak{X} \wr \mathfrak{Y}).
\end{aligned}$$

This completes the proof. (Also we can conclude that $I = \mathrm{Rad}(F(\mathfrak{X} \wr \mathfrak{Y}))$.) □

Lemma 3.5 *Assume that* char $F \mid n$, *and put* $I = \mathrm{Rad}(F\mathfrak{X}) \otimes B_0 + J_n \otimes F\mathfrak{Y}$. *Then the set I is a nilpotent ideal of $F(\mathfrak{X} \wr \mathfrak{Y})$ and* $\dim_F I = \dim_F \mathrm{Rad}(F\mathfrak{X}) + \dim_F F\mathfrak{Y} - 1$. (*In fact, I is the Jacobson radical of $F(\mathfrak{X} \wr \mathfrak{Y})$.*)

Proof: We note that $(J_n)^2 = 0$, in this case. It is easy to verify that I is a nilpotent ideal of $F(\mathfrak{X} \wr \mathfrak{Y})$. Since $\mathrm{Rad}(F\mathfrak{X}) \otimes B_0 \cap J_n \otimes F\mathfrak{Y} = F(J_n \otimes B_0)$, we have the result. □

Theorem 3.6 *Suppose that* char $F \mid n$. *Then* $\tilde{S}_2, \ldots, \tilde{S}_r, \tilde{T}_1$ *are all non-equivalent irreducible representations of* $F(\mathfrak{X} \wr \mathfrak{Y})$.

Proof: The proof is similar to the proof of Theorem 3.4. □

As a consequence of Theorem 3.4 and 3.6, we have the following corollary. (We note that a general criterion of the semisimplicity of an adjacency algebra is discussed in [3, Theorem 4.2].)

Corollary 3.7 *The algebra* $F(\mathfrak{X} \wr \mathfrak{Y})$ *is semisimple if and only if both* $F\mathfrak{X}$ *and* $F\mathfrak{Y}$ *are semisimple.*

Proof: If char $F \mid n$, then both $F\mathfrak{X}$ and $F(\mathfrak{X} \wr \mathfrak{Y})$ are not semisimple. If char $F \nmid n$, then the assertion holds by Theorem 3.4 and its proof. □

4. Irreducible characters

In this section, we describe all irreducible characters of $\mathfrak{X} \wr \mathfrak{Y}$ over the complex number field $\mathbb{C}$. The character means the trace function of a representation. Since the adjacency algebra of an association scheme over $\mathbb{C}$ is always semisimple, this is easy by Theorem 3.4.

Let $\chi_1, \ldots, \chi_r$ be all irreducible characters of $\mathbb{C}\mathfrak{X}$, and let $\varphi_1, \ldots, \varphi_s$ be all irreducible characters of $\mathbb{C}\mathfrak{Y}$. Suppose $\chi_1(A_i) = k_i$ and $\varphi_1(B_j) = k'_j$. We define

$$\begin{cases} \tilde{\chi}_\mu(A_i \otimes B_0) = \chi_\mu(A_i) \\ \tilde{\chi}_\mu(J_n \otimes B_j) = 0, \end{cases}$$

for $\mu \neq 1$ and

$$\begin{cases} \tilde{\varphi}_\nu(A_i \otimes B_0) = k_i \varphi_\nu(B_0) \\ \tilde{\varphi}_\nu(J_n \otimes B_j) = n\varphi_\nu(B_j). \end{cases}$$

Then we have the following.

Theorem 4.1 *In the above notations*, $\tilde{\chi}_2, \ldots, \tilde{\chi}_r, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_s$ *are all irreducible characters of* $\mathbb{C}(\mathfrak{X} \wr \mathfrak{Y})$.

Appendix

Professor A. Munemasa pointed out that the result in this article holds for more general situations. Let F be an algebraically closed field, and let A and B be finite dimensional F-algebras. Suppose A has a central element e such that Fe is a two-sided ideal of A, and that $e^2 = e$ or $e^2 = 0$. Put $C = A \otimes 1 + e \otimes B \subset A \otimes_F B$. If $e^2 = e$, then A is a direct sum of two-sided ideals $A(1-e)$ and $Ae = Fe$. In this case, we have $C \cong A(1-e) \oplus B$. If $e^2 = 0$, then $e \otimes B$ is contained in the Jacobson radical of C, so the irreducible representations of C are the same as those of A. Our main results Theorem 3.4 and 3.6 are easy consequences of these facts.

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