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The Peak Algebra of the Symmetric Group

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Abstract. The peak set of a permutation σ is the set $\{i : \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$. The group algebra of the symmetric group S_n admits a subalgebra in which elements are sums of permutations with a common descent set. In this paper we show the existence of a subalgebra of this descent algebra in which elements are sums of permutations sharing a common peak set. To prove the existence of this peak algebra we use the theory of enriched (P, γ)-partitions and the algebra of quasisymmetric peak functions studied by Stembridge (*Trans. Amer. Math. Soc.* 349 (1997) 763–788).

Keywords: peaks, Solomon's descent algebra, quasisymmetric functions

1. Introduction

In 1976 Louis Solomon [12] introduced a collection of algebras associated to finite Coxeter groups. In the case of the symmetric group the elements of the associated algebra are sums (in the group algebra of S_n over a field $\mathbf{k}$) of permutations sharing a common descent set. We refer to this algebra as the descent algebra and denote it by $Sol(A_{n-1})$ (see [3, 8], and [4]). In addition commutative subalgebras of the descent algebra have been studied in which permutations are grouped according to the number of descents ([1, 8, 10], and [6]) and according to the shapes of the permutations [7].

As an analogue to the descent algebra of the symmetric group we look at a subalgebra of the group algebra of S_n indexed by the position of peaks in the permutations. To prove the existence of this subalgebra we will utilize a set of maps on labeled posets, introduced by Stembridge [15], called enriched (P, γ)-partitions which are a variation of Stanley's notion of (ordinary) P-partitions [14]. A more in-depth discussion of enriched (P, γ)-partitions can be found in [13].

We begin with several definitions and notation. We will write permutations of $[n]$ in the form $\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n))$, and for $\sigma, \gamma \in S_n$ the product $\sigma\gamma$ will indicate first applying γ and then applying σ. The descent set of a permutation, denoted $D(\sigma)$, is the set $\{i : \sigma(i) > \sigma(i+1)\}$. Thus $D(\sigma) \subseteq [n-1]$. For example the descent set of $\sigma = (31524)$ is $\{1, 3\}$. Let $\mathcal{D}_T = \sum_{\sigma : D(\sigma)=T} \sigma$. Since there are 2^{n-1} possible descent sets, the symmetric group S_n is partitioned into 2^{n-1} disjoint descent classes. Solomon's result shows the product of two descent classes is a linear combination, with non-negative integer coefficients, of descent classes:

$$\mathcal{D}_T \cdot \mathcal{D}_Q = \sum_K a_K \mathcal{D}_K \quad a_K \in \mathbb{Z}_+.$$

A formal power series f of bounded degree in a countable number of indeterminates $x_1, x_2, \ldots$ with coefficients in a field $\mathbf{k}$ is a *quasisymmetric function* if for any $a_1, \ldots, a_k \in \mathbb{Z}_+$, the coefficient of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ in f is equal to the coefficient of $x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$ whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. Clearly symmetric functions are quasisymmetric functions. The series $\sum_{i<j} x_i x_j^2$ and $\sum_{i<j<k} x_i^3 x_j x_k^2$ are examples of quasisymmetric functions that are not symmetric.

Let $QSym = \bigoplus_{n \geq 0} \mathcal{Q}_n$ denote the algebra of quasisymmetric functions over a field $\mathbf{k}$. For $\alpha = (\alpha_1, \ldots, \alpha_k)$ a composition of n (denoted $\alpha \models n$) and

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \tag{1}$$

the set $\{M_\alpha : \alpha \models n\}$ forms a basis for $\mathcal{Q}_n$. Since there is a bijection between compositions of n and subsets of $[n-1]$, we will sometimes write M_E for $E \subseteq [n-1]$ to denote the basis element M_α where α is the composition of n corresponding to E. We will also use Gessel's fundamental basis [9],

$$F_D = \sum_{D \subseteq E} M_E.$$

For $f \in \mathcal{Q}_m$ and $g \in \mathcal{Q}_n$, the product $fg \in \mathcal{Q}_{m+n}$ making $QSym$ a graded $\mathbf{k}$-algebra.

2. The peak algebra of the symmetric group

We say that a permutation $\gamma \in S_n$ has a *peak* at position i if $\gamma(i-1) < \gamma(i) > \gamma(i+1)$. The *peak set* of γ is the set

$$\Lambda(\gamma) := \{i : \gamma(i-1) < \gamma(i) > \gamma(i+1)\}.$$

For example the permutation $\gamma = (325461) \in S_6$ has peaks at positions 3 and 5 since $\gamma(2) < \gamma(3) > \gamma(4)$ and $\gamma(4) < \gamma(5) > \gamma(6)$. We note that every subset S of $[n-1]$ is the descent set of at least one permutation of $[n]$, and every subset Γ of $[n-1]$ satisfying $1 \notin \Gamma$ and $k \in \Gamma$ implies $(k+1) \notin \Gamma$ is the peak set of at least one permutation of $[n]$. For this reason we refer to arbitrary subsets of $[n-1]$ as descent sets and to subsets Γ of $[n-1]$ with $1 \notin \Gamma$ and $k \in \Gamma$ implies $(k+1) \notin \Gamma$ as *peak sets*. We remark here that it is possible to allow peaks to occur at position 1 and this forms the basis of future work.

Given a descent set $S = \{s_1, \ldots, s_k\} \subseteq [n-1]$ we form the corresponding peak set $\Lambda'(S)$ by removing each s_i such that $s_i - s_{i-1} = 1$, where $s_0 = 0$. For example the descent set $S = \{1, 3, 4, 7, 8\}$ gives rise to the peak set $\Lambda'(S) = \{3, 7\}$.

Our main result shows the existence of a subalgebra of the descent algebra of S_n in which elements are sums of permutations sharing a peak class.

Theorem 1 *In the group algebra $\mathbb{Q}S_n$ of S_n, define for each peak set $\Gamma \subset [n-1]$*

$$P_\Gamma = \sum_{w : \Lambda(w) = \Gamma} w$$

where $\Lambda(w)$ is the peak set of $w \in S_n$. Then the subspace $\mathcal{P}_n$ spanned by the P_Γ's is a subalgebra of Sol(A_{n-1}). We call this the peak algebra.

Example 2 Consider the peak algebra of S_4. $\mathcal{P}_4$ consists of 3 elements: permutations with no peaks, permutations with a peak at position 2 and permutations with a peak at position 3. We will denote these three classes by $P_\emptyset$, P_2 and P_3 respectively.

$$\begin{aligned}
P_\emptyset &= (1234)+(2134)+(3124)+(4123)+(4312)+(3214)+(4213)+(4321)\\
P_2 &= (1432)+(2431)+(3421)+(1423)+(2413)+(3412)+(1324)+(2314)\\
P_3 &= (1243)+(1342)+(2341)+(4132)+(4231)+(3241)+(3142)+(2143)\\
P_3 \cdot P_2 &= (1342)+(1243)+(2143)+(4231)+(4132)+(3142)+(3241)\\
&\quad +(2341)+(2341)+(3241)+(3142)+(1234)+(2134)+(2143)\\
&\quad +(1243)+(1342)+(4321)+(4231)+(4132)+(3214)+(3124)\\
&\quad +(4123)+(4213)+(4312)+(1324)+(1234)+(2134)+(4213)\\
&\quad +(4123)+(3124)+(3214)+(2314)+(2314)+(3214)+(3124)\\
&\quad +(1243)+(2143)+(2134)+(1234)+(1324)+(4312)+(4213)\\
&\quad +(4123)+(3241)+(3142)+(4132)+(4231)+(4321)+(1423)\\
&\quad +(1432)+(2431)+(4312)+(4321)+(3421)+(3412)+(2413)\\
&\quad +(2413)+(3412)+(3421)+(1342)+(2341)+(2431)+(1432)\\
&\quad +(1423)\\
&= 3P_\emptyset + 2P_2 + 3P_3
\end{aligned}$$

The multiplication in $\mathcal{P}_4$ is summarized in Table 1 in which (x, y, z) refers to the sum $xP_\emptyset + yP_2 + zP_3$. We note here that a combinatorial multiplication rule which gives the coefficients a_k, where $P_i \cdot P_j = \sum_k a_k P_k$ is still an open question.

Denote by (P, γ) the *labeled poset* P with labels $\gamma(x)$ for $x \in P$, where γ is an injective map to a set of totally ordered elements (we will take this set to be $[n]$ where $n = |P|$; see figure 1). We represent $x < y$ in a poset if there is a path between x and y and if x is "under" y.

For a poset P with n elements, a *linear extension* $\overline{v} = \{v_1 < \cdots < v_n\}$ is a total ordering of the elements of P that preserves the partial order of P, i.e., $v_k < v_j$ in P implies $v_k < v_j$ in $\overline{v}$. We next consider the set of linear extensions of a labeled poset (P, γ) regarded as permutations of the labels of P.

Table 1. The multiplication of $\mathcal{P}_4$.

$\cdot$	$P_\emptyset$	P_2	P_3
$P_\emptyset$	(4, 2, 2)	(2, 3, 3)	(2, 3, 3)
P_2	(2, 4, 2)	(3, 2, 3)	(3, 2, 3)
P_3	(2, 2, 4)	(3, 3, 2)	(3, 3, 2)

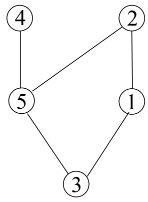

Figure 1. A labeled poset (P, γ).

Definition 3 Recall for a labeled poset (P, γ) the *Jordan-Hölder set* $\mathcal{L}(P, \gamma)$ is

$$\mathcal{L}(P, \gamma) = \{(\gamma(v_1) \dots \gamma(v_n)) : v_1 < \cdots < v_n \text{ is a linear extension of } P\}.$$

For example the Jordan-Hölder set of the labeled poset (P, γ) in figure 1 is $\mathcal{L}(P, \gamma) = \{(35142), (35124), (35412), (31524), (31542)\}$.

Now let $\mathbb{P}$ denote the set of non-zero integers with the total ordering:

$$-1 \prec 1 \prec -2 \prec 2 \prec -3 \prec 3 \prec \dots.$$

We will say $k > 0$ to indicate positive integers and $|k|$ to refer to the absolute value of an integer as usual.

Definition 4 (Stembridge [15]) For (P, γ) a labeled poset, an *enriched* (P, γ)*-partition* is a map $f : P \longrightarrow \mathbb{P}$ such that for all $x < y$ in P, we have:

1. $f(x) \preceq f(y)$,
2. $f(x) = f(y) > 0$ implies $\gamma(x) < \gamma(y)$,
3. $f(x) = f(y) < 0$ implies $\gamma(x) > \gamma(y)$.

Let $\mathcal{E}(P, \gamma)$ denote the set of enriched (P, γ)-partitions. Figure 2 gives an example of an enriched (P, γ)-partition in which the γ labels appear in the vertices while $f(x)$ is shown below each element x.

We can think of $v \in \mathcal{L}(P, \gamma)$ as a labeled poset by considering γ as labels on the underlying chain formed by the linear extension $v_1 < \cdots < v_n$. In this way we can define enriched (v, γ)-partitions.

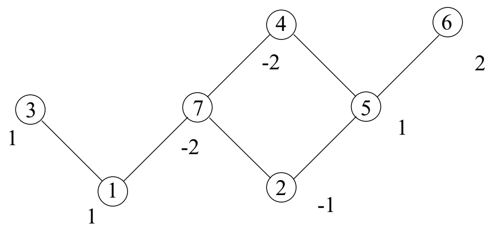

Figure 2. An enriched (P, γ)-partition f.

Lemma 5 (Stembridge [15]) *For any labeled poset* (P, γ), *we have*

$$\mathcal{E}(P, \gamma) = \bigcup_{v \in \mathcal{L}(P,\gamma)} \mathcal{E}(v, \gamma)$$

where $\mathcal{E}(v, \gamma)$ *is the set of enriched* (v, γ)*-partitions for the labeled chain* v.

For each enriched (P, γ)-partition Stembridge [15] defines a weight enumerator that assigns the weight z_k to both k and $-k$. Taking the product of weights for $f(x)$ over all $x \in P$, summed over all (P, γ)-partitions gives a homogeneous quasisymmetric function:

$$\Delta(P, \gamma) := \sum_{f \in \mathcal{E}(P,\gamma)} \prod_{x \in P} z_{|f(x)|}.$$

By Lemma 5 we have

$$\Delta(P, \gamma) = \sum_{v \in \mathcal{L}(P,\gamma)} \Delta(v, \gamma). \tag{2}$$

Stembridge demonstrates that $\Delta(v, \gamma)$ depends only on the peak set of $v \in \mathcal{L}(P, \gamma)$ [15, Proposition 2.2], that is, the set

$$\Lambda(v) := \{i : \gamma(v_{i-1}) < \gamma(v_i) > \gamma(v_{i+1}), 1 < i < n\}.$$

Following Stembridge define a family of quasisymmetric functions

$$\Theta_\Lambda := \Delta(v, \gamma) \tag{3}$$

for any labeled chain (v, γ) such that $\Lambda(v) = \Lambda$. These Θ_Λs form a basis of the algebra of quasisymmetric peak functions studied by Stembridge and discussed in Section 3 [15]. In this case we can rewrite (2) as

$$\Delta(P, \gamma) = \sum_{v \in \mathcal{L}(P,\gamma)} \Theta_{\Lambda(v)}. \tag{4}$$

Notice in calculating the weight enumerator Δ, only the absolute values of the mapping $f : P \to \mathbb{P}$ are of concern. As in [15] we will call two labelings γ and γ' of a poset P *weakly equivalent* if $\{|f| : f \in \mathcal{E}(P, \gamma)\} = \{|f| : f \in \mathcal{E}(P, \gamma')\}$ as multisets, where $|f|(x) := |f(x)|$. Thus $\Delta(P, \gamma) = \Delta(P, \gamma')$ for weakly equivalent labelings γ and γ'.

A criterion of Stembridge [15, Proposition 2.4] for testing weak equivalence of labelings involves the order ideals of a poset P, that is the subsets $I \subseteq P$ such that if $x \in I$ and $y < x$ then $y \in I$. For a labeled poset (P, γ), Stembridge defines binary relations $\to$ and $\leftarrow$ on the set, $J(P)$, of order ideals of P by

$$I \to J \quad \text{if } I \subset J, \text{ and } x, y \in J \backslash I, x < y \Rightarrow \gamma(x) > \gamma(y), \tag{5}$$

$$I \leftarrow J \quad \text{if } I \subset J, \text{ and } x, y \in J \backslash I, x < y \Rightarrow \gamma(x) < \gamma(y). \tag{6}$$

And finally, for any $I \subset J$,

$$N(P,\gamma)(I,J) := |\{K \in J(P) : I \to K \leftarrow J\}|.$$

A subposet Q of P is *convex* if for $x < y < z$ in P and $x, z \in Q$ we have $y \in Q$ [15].

Proposition 6 (Stembridge) *If γ and γ' are labelings of a poset P, the following are equivalent*:
1. *γ and γ' are weakly equivalent.*
2. *$\Delta(Q,\gamma) = \Delta(Q,\gamma')$ for every convex subposet Q of P.*
3. *$N(P,\gamma) = N(P,\gamma')$.*

We will omit the (P,γ) in the notation of $N(P,\gamma)$ when there is only one poset and labeling under consideration.

Definition 7 A *range poset* is a poset whose Hasse diagram is a path. As such each element belongs to ≤ 2 maximal chains with equality possible only for maximal or minimal elements of P.

A range poset has a natural left to right ordering on its elements as well as the usual partial ordering. Given a descent set $S \subseteq [n-1]$ we define the range poset M_S corresponding to S to be the poset such that if the elements are labeled $1, \ldots, n$ from left to right, then element $i > i+1$ if and only if $i \in S$. Figure 3 illustrates the range poset corresponding to the descent set $\{1, 4, 6, 7\}$. Given a permutation $\gamma \in S_n$ there is a natural labeling of an n element range poset P by assigning $\gamma(i)$ to the ith element from the left end of P.

Remark 8 For the remainder of this paper a peak of γ will refer to a peak of a permutation γ (i.e., $3 < 5 > 4$) and a trough of γ will refer to a trough of the permutation (i.e., $2 > 1 < 6$). A hill (respectively, a valley) of a range poset P will represent a maximal (minimal) element of P.

The next lemma can be found in [13, Exercise 7.95a]. The proof of Theorem 1 is an extension of the solution to [13, Exercise 7.95b] in which Gessel uses ordinary P-partitions to prove the existence of Solomon's descent algebra. When extending to enriched (P,γ)-partitions it is necessary to work with the union of range posets which share a common hill

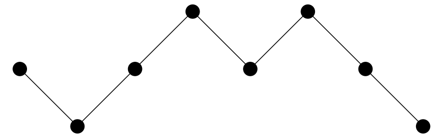

Figure 3. A range poset.

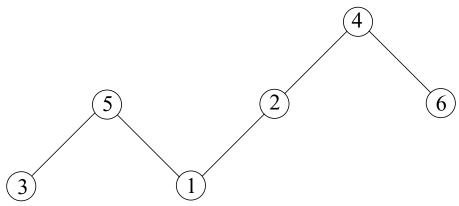

Figure 4. A range poset $M_{\{2,5\}}$ labeled by $\gamma = (351246)$.

set. In addition, to show the dependence of the weight enumerator of the enriched (P, γ)-partitions on the peak set of the labeling permutation we will require a result on the weak equivalence of labelings of range posets. We will give this result in Corollary 12.

Lemma 9 (Stanley) *Given $S \subseteq [n-1]$ and $\gamma \in S_n$, let (M_S, γ) be the labeled range poset corresponding to S. Then the Jordan-Hölder set $\mathcal{L}(M_S, \gamma)$ consists of all permutations $v \in S_n$ such that $D(v^{-1}\gamma) = S$.*

To illustrate this lemma consider the labeled range poset $(M_{\{2,5\}}, \gamma)$ in figure 4 where $\gamma = (351246)$. Let $v = (135264) \in \mathcal{L}(M_{\{2,5\}}, \gamma)$. Then $v^{-1}\gamma = (231465)$, and $D(v^{-1}\gamma) = \{2, 5\}$. We can see that this result is true by calling the ith element from the left end of M_S element i. Then $v^{-1}\gamma(i)$ gives the position of $\gamma(i)$ in v. Therefore a descent of $v^{-1}\gamma$ at position j indicates $\gamma(j+1)$ appears before $\gamma(j)$ in v and hence $j > (j+1)$ in M_S.

A similar result holds if we replace descent sets with peak sets. Here we must consider the set of all range posets with a particular peak set. For a peak set $\Gamma \subset [n-1]$ let $\mathcal{S}_\Gamma = \{S \subseteq [n-1] : \Lambda'(S) = \Gamma\}$ be the collection of descent sets having peak set Γ. And let $\mathcal{M}_\Gamma = \{M_S : S \in \mathcal{S}_\Gamma\}$ be the family of range posets with descents at $\{S : S \in \mathcal{S}_\Gamma\}$.

Corollary 10 *Given a peak set $\Gamma \subset [n-1]$ and a fixed permutation $\gamma \in S_n$ consider the family of labeled posets $\{(M_S, \gamma) : M_S \in \mathcal{M}_\Gamma\}$. The union of Jordan-Hölder sets $\bigcup_{\mathcal{M}_\Gamma} \mathcal{L}(M_S, \gamma)$ consists of all permutations $v \in S_n$ such that $\Lambda(v^{-1}\gamma) = \Gamma$.*

Proof: By Lemma 9, a permutation $v \in \mathcal{L}(M_S, \gamma)$, for $M_S \in \mathcal{M}_\Gamma$, if and only if the descent set $D(v^{-1}\gamma) = S$. Note if $S = D(u)$ then $\Lambda'(S) = \Lambda(u)$. Since $\Lambda'(S) = \Gamma$ for all $S \in \mathcal{S}_\Gamma$, any $v \in \bigcup_{\mathcal{M}_\Gamma} \mathcal{L}(M_S, \gamma)$ has $\Lambda(v^{-1}\gamma) = \Gamma$. Conversely, a permutation v such that $\Lambda(v^{-1}\gamma) = \Gamma$ has the property that $D(v^{-1}\gamma) = S$ for some $S \in \mathcal{S}_\Gamma$. Hence $v \in \mathcal{L}(M_S, \gamma)$ and thus is in $\bigcup_{\mathcal{M}_\Gamma} \mathcal{L}(M_S, \gamma)$. $\square$

Next we give a formula for $N(P, \gamma)(I, J)$, where P is a range poset, in terms of the positions of peaks of the labeling permutation γ. Notice that for a range poset P order ideals are unions of disjoint components, where a component consists of a set of consecutive elements of P (reading elements from left to right). These components are separated by hills of P. Similarly, for two order ideals $I \subseteq J$ the set $J \backslash I$ is composed of disjoint components separated by either hills or valleys of P (see figure 5).

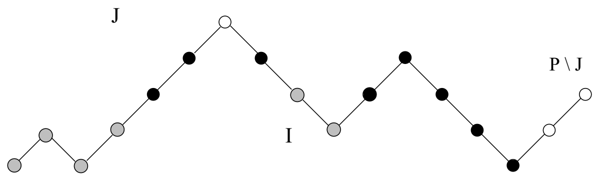

Figure 5. Order ideals $I \subset J$ of a range poset P. Ideal I is shaded, $J \backslash I$ is in black, and $P \backslash J$ is white.

We can think of a component M as a subposet of P. The interior of M will refer to elements which are not the first or last elements of M (reading from left to right). Interior hills or valleys of M will thus be maximal or minimal elements of P which are interior elements of M, and an interior slope of M is composed of interior elements which are neither maximal nor minimal elements of P. Given a component M, interior peaks or troughs of a labeling γ will refer to peaks or troughs of γ which occur on interior elements of M.

Proposition 11 *For P a range poset labeled by $\gamma \in S_n$ and $I \subset J$ order ideals*

$$N(P, \gamma)(I, J) = \prod_M k_M$$

where the product is taken over the components M of $J \backslash I$ and

$$k_M = \begin{cases} 0 & \text{if there is a peak of } \gamma \text{ on an interior slope of } M \\ 2^{\vartheta+1} & otherwise \end{cases}$$

for ϑ the number of peaks of γ on interior hills and valleys of M.

Corollary 12 *For a range poset M_S, two labelings $\gamma, \gamma' \in S_n$ are weakly equivalent if and only if $\Lambda(\gamma) = \Lambda(\gamma')$.*

Proof: For a range poset M_S, and arbitrary order ideals $I \subseteq J$, Proposition 11 gives $N(M_S, \gamma)(I, J)$ in terms of the positions of peaks of γ. Thus if $\Lambda(\gamma) = \Lambda(\gamma')$ we have $N(M_S, \gamma) = N(M_S, \gamma')$. So by Proposition 6, γ and γ' are weakly equivalent.

For the converse we show that if $\Lambda(\gamma) \neq \Lambda(\gamma')$ then $N(M_S, \gamma) \neq N(M_S, \gamma')$. Suppose $i \in \Lambda(\gamma)$, $i \notin \Lambda(\gamma')$. Take order ideals $I \subset J$ such that $J \backslash I$ consists of the elements $i - 1$, i, $i + 1$ of M_S (reading left to right). Since $i \notin \Lambda(\gamma')$, γ' has no peaks in the interior of $J \backslash I$. By Proposition 11, if element i is on a slope of M_S (and hence an interior slope of $J \backslash I$), $N(M_S, \gamma)(I, J) = 0$ while $N(M_S, \gamma')(I, J) = 2$. If i is on a hill or valley of M_S (and hence an interior hill or valley of $J \backslash I$), $N(M_S, \gamma)(I, J) = 2^2$ while $N(M_S, \gamma')(I, J) = 2$. Thus $N(M_S, \gamma) \neq N(M_S, \gamma')$. $\square$

We are now in a position to prove Theorem 1 and Proposition 11.

Proof of Theorem 1: We show that the product of two peak classes is again a sum of peak classes; that is

$$P_\Upsilon \cdot P_\Gamma = \sum_i P_{\Theta_i}. \tag{7}$$

If γ and γ' have the same peak set they must appear the same number of times on the right of (7), so we must show for $\gamma, \gamma' \in S_n$ with $\Lambda(\gamma) = \Lambda(\gamma')$

$$\begin{aligned}&|\{(u, v) \in S_n \times S_n : vu = \gamma, \Lambda(u) = \Gamma, \Lambda(v) = \Upsilon\}|\\ &\quad = |\{(u, v) \in S_n \times S_n : vu = \gamma', \Lambda(u) = \Gamma, \Lambda(v) = \Upsilon\}|.\end{aligned}$$

And so, for each γ, the number of such pairs (u, v) in $S_n \times S_n$ as above, should depend only on the peak sets Γ, Υ, and $\Lambda(\gamma)$.

For $\gamma \in S_n$ and $S \subseteq [n-1]$ with $\Lambda'(S) = \Gamma$, consider the set $\mathcal{E}(M_S, \gamma)$ of enriched (M_S, γ)-partitions. By (4) the weight enumerator $\Delta(M_S, \gamma) = \sum_{v\in\mathcal{L}(M_s,\gamma)} \Theta_{\Lambda(v)}$ is a sum of basis elements of the algebra of quasisymmetric peak functions, and this representation is unique. Thus the multiset $\{\Lambda(v) : v \in \mathcal{L}(M_S, \gamma)\}$ is determined by $\Delta(M_S, \gamma)$.

If we consider the sum of weight enumerators taken over the union of enriched (M_S, γ)-partitions for all $M_S \in \mathcal{M}_\Gamma$, we again have a unique representation in terms of the basis elements Θ_Λ, and so the multiset

$$\Phi = \bigcup_{M_S\in\mathcal{M}_\Gamma} \{\Lambda(v) : v \in \mathcal{L}(M_S, \gamma)\} = \left\{\Lambda(v) : v \in \bigcup_{M_S\in\mathcal{M}_\Gamma} \mathcal{L}(M_S, \gamma)\right\}$$

depends only on

$$\sum_{M_S\in\mathcal{M}_\Gamma} \Delta(M_S, \gamma).$$

This sum in turn depends only upon the weak isomorphism classes of the pairs (M_S, γ). By Corollary 12 these weak equivalence classes depend only on the posets M_S and the peak set $\Lambda(\gamma)$, while the posets $\{M_S : M_S \in \mathcal{M}_\Gamma\}$ are completely determined by Γ.

By Corollary 10 we can conclude that as a multiset,

$$\Phi = \{\Lambda(v) : \Lambda(v^{-1}\gamma) = \Gamma\},$$

which by the above argument depends only upon Γ and $\Lambda(\gamma)$. Thus letting $u = v^{-1}\gamma$ we have the number of u, v for which $\Lambda(u) = \Gamma$, $\Lambda(v) = \Upsilon$, and $vu = \gamma$ depends only on Γ, Υ and $\Lambda(\gamma)$.

The subspace $\mathcal{P}_n$ forms a subalgebra of the descent algebra of S_n since elements of $\mathcal{P}_n$ are sums of descent classes. Specifically $P_\Gamma = \sum_{\Lambda'(S)=\Gamma} \mathcal{D}_S$ where $\mathcal{D}_S = \sum_{D(\gamma)=S} \gamma$. □

Proof of Proposition 11: It is sufficient to consider one component of $J\backslash I$ as $N(I, J)$ will be the product of the number of K satisfying $I \to K \leftarrow J$ in each component. For the remainder of this proof K will refer to an order ideal satisfying $I \to K \leftarrow J$.

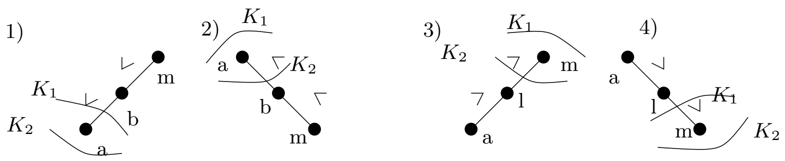

Figure 6. Choices for K in conditions 1 to 4.

Consider an arbitrary segment M of $J\backslash I$. We first note several conditions which any K must satisfy. These conditions are illustrated in figure 6 in which $<$ and $>$ on edges indicate the relative values of γ.

1. If γ is increasing on an increasing slope of M, that is if we have $a < b < \cdots < m$ and $\gamma(a) < \gamma(b) < \cdots < \gamma(m)$ then b can not be in K. Otherwise we have $a, b \in K\backslash I$ with $a < b$ and $\gamma(a) < \gamma(b)$, violating (5).
2. If γ is increasing on an decreasing slope of M, that is $a > b > \cdots > m$ and $\gamma(a) < \gamma(b) < \cdots < \gamma(m)$ then b must be in K. Otherwise we have $a, b \in J\backslash K$ with $a > b$ and $\gamma(a) < \gamma(b)$, violating (6).
3. Similarly, if γ is decreasing on an increasing slope of M, namely $a < \cdots < l < m$ and $\gamma(a) > \cdots > \gamma(l) > \gamma(m)$, then l must be in K.
4. Finally, if γ is decreasing on an decreasing slope of M, so $a > \cdots > l > m$ and $\gamma(a) > \cdots > \gamma(l) > \gamma(m)$, then l cannot be in K.

First consider the case in which there exists a peak of γ on an interior slope of M; that is $a < b < c$ in M with $\gamma(a) < \gamma(b) > \gamma(c)$. It cannot be the case that $b \in K$, since $b \in K\backslash I$ implies $a < b \in K\backslash I$ and $\gamma(a) < \gamma(b)$ contrary to (5). But it also cannot be the case that $b \notin K$, since that implies $b < c \in J\backslash K$, with $\gamma(b) > \gamma(c)$ contradicting (6). Thus there is no K satisfying $I \to K \leftarrow J$, and consequently $N(I, J) = 0$.

For the remainder of this proof we assume any peaks of γ in the interior of M occur on hills or valleys of M. Between each consecutive pair of such peaks there exists a trough of γ and we claim regardless of where such a trough is located there are exactly two choices for K satisfying $I \to K \leftarrow J$. Figure 7 illustrates the choices for K in Cases A, B, and C.

Case A: There exists a trough of γ on a slope of M. So $a < b < c$ with $\gamma(a) > \gamma(b) < \gamma(c)$. First note that we cannot have both a and b in $J\backslash K$ since $a < b$ with $\gamma(a) > \gamma(b)$. Thus element a must belong to K, but we have a choice to include b or not include b in K

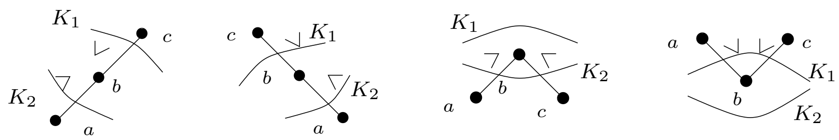

Figure 7. Choices for K in Cases A, B, and C.

as follows. If b is in K we have $\gamma(x_1) > \gamma(x_2)$ for all $x_1 < x_2 \leq b$ in M since there are no peaks of γ on internal slopes of M. Thus $b \in K$ is consistent with (5). If b is not in K, and since γ only peaks at hills or valleys of M we have $\gamma(y_1) < \gamma(y_2)$ for all $b \leq y_1 < y_2$ in M, which is consistent with (6). We note that c can not belong to K else we have $b < c$ in $K \backslash I$ with $\gamma(b) < \gamma(c)$ a contradiction to (5).

Case B: There exists a trough of γ on an interior peak of M, i.e., $a < b > c$ with $\gamma(a) > \gamma(b) < \gamma(c)$. Either a or $c \notin K$ violates (6) and again since interior peaks of γ only occur at hills or valleys of M we know that $\gamma(x_1) > \gamma(x_2)$ for all $x_1 < x_2 \leq b$ in M. Thus $b \in K$ or $b \notin K$, $a, c \in K$ both form valid choices for K.

Case C: There exists a trough of γ at an interior valley of M, i.e., $a > b < c$ with $\gamma(a) > \gamma(b) < \gamma(c)$. We know $\gamma(y_1) > \gamma(y_2)$ for all $y_1 > y_2 \geq b$ since interior peaks of γ are restricted to hills and valleys of M. Thus $b \notin K$ or $b \in K$, $a, c \notin K$ are valid choices for K. Furthermore a or $c \in K$ violate (5).

Hence there are two valid choices for K between each pair of consecutive γ peaks. These choices are independent because each choice is separated by either a valley or hill of M. We have left to consider the portion of M before the first γ peak and the portion of M after the last peak. (If there does not exist an interior γ peak on M then M can be treated as one of an initial segment or a final segment.)

If γ begins with a descent in the initial segment of M then there exists a trough of γ before the first γ peak and thus as in Cases A, B, and C there are exactly two choices for K. Otherwise γ ascends monotonically to the first peak. Then by condition 2 if M contains any decreasing slopes K must include all elements of the slope except (possibly) the top element. And by condition 1, K cannot include any element, except (possibly) the smallest, of any increasing slope of M. These requirements determine membership in K for all but the initial element i of M (see figure 8). If M begins with an ascent $i < j$ then i may belong to K, since j is not in K (by condition 1). But both i and j are eligible as members of $J \backslash K$ so there are two choices for K. If M begins with a descent $i > j$ then i may belong to $J \backslash K$ since j must be in K (by condition 2), or both i and j may belong to K since $i > j$ and $\gamma(i) < \gamma(j)$. So again there are 2 possible choices for K.

In the final segment of M, γ either has a final trough in which case we have two choices for K as before, or γ decreases monotonically to the end of M. In this situation by reading the elements of M from right to left, and consequently reading γ in reverse, this case reduces to the case in which γ increases on the initial segment of M. And so we have two choices for K by either including or not including the final element (from left to right) of M.

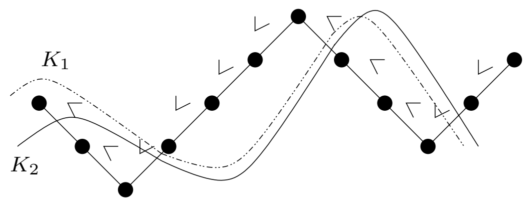

Figure 8. The two possible Ks satisfying $I \to K \leftarrow J$ for γ increasing on a segment M of $J \backslash I$.

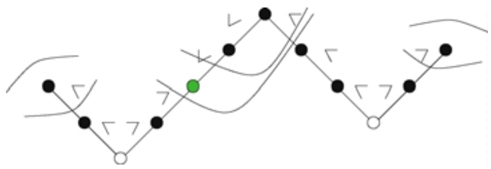

Figure 9. A range poset with γ peaks unfilled, and troughs shaded along with choices for where K "splits the trough" and end choices for K.

Thus there are two independent choices for K between each pair of consecutive γ peaks which occur at interior hills or valleys of M plus two choices for K at both the beginning and end of M for a total of $2^{\vartheta+1}$ such K, where ϑ is the number of peaks of γ occurring at interior hills or valleys of M (see figure 9). $\square$

3. Connections to Stembridge's peak algebra

Let Π_n be the $\mathbb{Z}$-module generated by the $\{\Theta_\Lambda\}$ (3) where Λ ranges over peak sets in $[n-1]$, and let $\Pi := \bigoplus_{n\geq 0} \Pi_n$. This defines a graded Hopf subalgebra of $QSym$ studied by Stembridge [15] and known as the algebra of peaks. Let $\mathcal{P} = \bigoplus_{n\geq 0} \mathcal{P}_n$. Although the peak algebra studied in this paper deals with a multiplication that is defined within each $\mathcal{P}_n$ (referred to as the inner multiplication), there are ties to Stembridge's peak algebra which we now address.

We denote by Sym the graded ring of symmetric functions and by Ω the subring generated by Schur Q-functions (see [13]). Stembridge [15] introduces a map $\theta : QSym \to \Pi$ defined linearly on each graded piece by

$$\theta(F_D) = \Theta_{\Lambda'(D)}$$

for each $D \subset [n-1]$. Furthermore there is map $\hat{\theta} : Sym \to \Omega$, and for $\lambda = (\lambda_1, \lambda_2, \ldots)$ a partition of n, $\hat{\theta}$ is defined by

$$\hat{\theta}(p_\lambda) = \begin{cases} 2^{|\lambda|} p_\lambda & \text{if all parts of } \lambda \text{ are odd} \\ 0 & \text{if any part of } \lambda \text{ is even} \end{cases}$$

where $|\lambda|$ is the length of λ, $p_\lambda = p_{\lambda_1} p_{\lambda_2}, \ldots$, and p_n is the power sum symmetric function, namely $p_n = \sum_i x_i^n$. The diagram of figure 10, in which the horizontal maps are inclusions and the vertical maps surjections, commutes [15, Remark 3.2].

$$\begin{tikzcd} Sym \arrow[r, hook] \arrow[d, "\hat{\theta}"'] & QSym \arrow[d, "\theta"] \\ \Omega \arrow[r, hook] & \Pi \end{tikzcd}$$

Figure 10. The algebra of peaks and the Schur Q algebra.

$$\begin{tikzcd}
\mathcal{P} \arrow[r, hook] \arrow[dd] & \mathcal{N}Sym \arrow[d, "ab"] & \\
 & Sym \arrow[r, hook] \arrow[dl, shift left, "\hat{\theta}"] & QSym \arrow[d, "\theta"] \\
\Omega \arrow[ur, hook, shift left] \arrow[rr, hook] & & \Pi
\end{tikzcd}$$

Figure 11. The dualization of figure 10.

Recall that the Hopf algebra $\mathcal{N}Sym$ is the graded dual of $QSym$ via $\mathcal{D}_K = F_K^*$ [11]. And the dual of the inclusion map $Sym \hookrightarrow QSym$ is the abelianization map from $\mathcal{N}Sym$ to Sym (recall Sym is a self dual Hopf algebra). Now we identify $\mathcal{P}$ as the dual vector space to Π by letting $P_S = \Theta_S^*$. Then as $\mathcal{D}_K$ is dual to F_K, the map θ is dual to the inclusion map $i(P_S) = \sum_{\Lambda'(J)=S} \mathcal{D}_J$. It is known that θ is a map of Hopf algebras [5, 15] and thus it follows that $\mathcal{P}$ is a Hopf subalgebra of $\mathcal{N}Sym$.

It is known that Ω is self dual and the dual of $\hat{\theta}$ is the inclusion map $\Omega \hookrightarrow Sym$. Hence the diagram of figure 10 dualizes producing the diagram of figure 11. Thus the image of $\mathcal{P}$ under the abelianization map is Ω. We note here that in Stembridge's work there is no consideration to the inner multiplication (namely the algebra structure on each $\mathcal{P}_n$) which is the main object of this work.

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