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Strongly Regular Decompositions of the Complete Graph

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Abstract. We study several questions about amorphic association schemes and other strongly regular decompositions of the complete graph. We investigate how two commuting edge-disjoint strongly regular graphs interact. We show that any decomposition of the complete graph into three strongly regular graphs must be an amorphic association scheme. Likewise we show that any decomposition of the complete graph into strongly regular graphs of (negative) Latin square type is an amorphic association scheme. We study strongly regular decompositions of the complete graph consisting of four graphs, and find a primitive counterexample to A.V. Ivanov's conjecture which states that any association scheme consisting of strongly regular graphs only must be amorphic.

Keywords: association scheme, strongly regular graph

1. Introduction

In this paper we tackle several questions about so-called amorphic association schemes. Such an association scheme has the property that all of its graphs are strongly regular. Moreover, they are all of Latin square type, or all of negative Latin square type. For background on amorphic association schemes, we refer the reader to the expository paper [6].

To study the relevant questions on amorphic association schemes, we introduce more general strongly regular decompositions of a complete graph. These are decompositions of the edge set of a complete graph into spanning subgraphs that are all strongly regular. A trivial example is given by a strongly regular graph and its complement, which form a strongly regular decomposition of a complete graph consisting of two graphs. Such a decomposition is also an amorphic association scheme (trivially).

An important role in this paper is played by commutative decompositions. These are decompositions for which the adjacency matrices of all graphs in the decomposition commute. In Section 3 we therefore investigate how two commuting edge-disjoint strongly regular graphs G_1 and G_2 interact algebraically, and characterize the case when the commutative decomposition $\{G_1, G_2, \overline{G_1 \cup G_2}\}$ is an association scheme. We also give examples of all possible cases.

One of our main goals in this paper is to show that any decomposition of a complete graph into three strongly regular graphs is an amorphic association scheme. This surprising

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fact was claimed to be true by Gol'fand et al. [7], but no proof was given. Now two quite different proofs are available. One is contained in Section 4; for the other proof see [6]. The result generalizes a result by Rowlinson [13] and independently Michael [12] who showed that any decomposition of a complete graph into three isomorphic strongly regular graphs is an association scheme.

In Section 5 it is shown that any strongly regular decomposition consisting of Latin square type graphs only, or of negative Latin square type graphs only, is also an amorphic association scheme. This generalizes a result by Ito et al. [9] who showed that any association scheme consisting of (negative) Latin square type graphs only is amorphic.

Since a strongly regular decomposition of a complete graph into three graphs necessarily is an amorphic association scheme, the next case to investigate would be decompositions into four strongly regular graphs. In the general (not necessarily commutative) case it seems hard to characterize such decompositions. We have only one (!) non-commutative example (on 6 vertices), which we found by classifying the decompositions into four strongly regular graphs of which at least three are disconnected.

In the commutative case with four graphs Theorem 5 reduces the number of cases considerably. The commutative case is also of interest because of A.V. Ivanov's conjecture (cf. [10, Problem 1.3]). This conjecture states that any association scheme consisting of strongly regular graphs only must be amorphic. Already in [5] counterexamples to this conjecture were found, but these were all imprimitive. Theorem 5 helped us to find a first primitive counterexample to the conjecture, which was also one of the main goals of the research which led to this paper.

2. Preliminaries

In this paper we consider simple undirected graphs without loops, unless otherwise indicated. A *decomposition* of a graph is defined as an edge-decomposition into spanning subgraphs, i.e. a partition of the edge set of the original graph into graphs on the same vertex set. More formally, we say that $\{G_1, G_2, \ldots, G_d\}$ is a decomposition of a graph G if for any two adjacent vertices in G, there is exactly one i for which the two vertices are adjacent in the graph G_i, and the vertex set of G_i is the same as the one of G, for all i. For any graph G_1, the graph and its complement $G_2 = \overline{G_1}$ form a (trivial) decomposition of a complete graph. Association schemes are other examples of decompositions of the complete graph. A decomposition is called *strongly regular* if all graphs in the decomposition are strongly regular graphs.

2.1. Strongly regular graphs and restricted eigenvalues

A *strongly regular graph G* with parameters (v, k, λ, μ) is a non-complete graph on v vertices which is regular with valency k, and which has the property that any two adjacent vertices have λ common neighbours, and any two non-adjacent vertices have μ common neighbours. It is well known that the adjacency matrix of a strongly regular graph has two or three distinct eigenvalues, depending on whether the graph is disconnected or not. Since

G is regular with valency k, it has the all-ones vector $\mathbf{j}$ as an eigenvector with eigenvalue k. To unify the cases where G is connected and where it is not, we introduce the concepts of restricted eigenvalues and multiplicities. We say that a regular graph (and its adjacency matrix) has a *restricted eigenvalue* θ if it has an eigenvector for θ which is orthogonal to the all-ones vector $\mathbf{j}$ (i.e., these are all eigenvalues except the valency in case the graph is connected, and all eigenvalues if the graph is not connected). The *restricted multiplicity* for a restricted eigenvalue θ is the dimension of its eigenspace intersected with $\mathbf{j}^{\perp}$. It now follows that a regular graph is strongly regular if and only if it has two distinct restricted eigenvalues, say r and s. It is well known that for a strongly regular (v, k, λ, μ) graph the restricted eigenvalues r and s follow from the equations $r + s = \lambda - \mu$ and $rs = \mu - k$. The restricted multiplicities f for r and g for s are given by the equations $f = -\frac{k+(v-1)s}{r-s}$ and $g = \frac{k+(v-1)r}{r-s}$. If G is connected then these numbers are the usual multiplicities for a strongly regular graph.

A strongly regular graph is of *Latin square type* or of *negative Latin square type* if there are integers n and t (positive or negative, depending on the "type") such that the graph has n^2 vertices, valency $t(n-1)$, and restricted eigenvalues $n-t$ and $-t$.

We remark finally that the complement of a strongly regular graph is also strongly regular, hence a strongly regular graph and its complement form a (trivial) strongly regular decomposition of a complete graph. For more information on strongly regular graphs we refer the reader to [4].

2.2. *Commutative decompositions and association schemes*

We call a decomposition of a complete graph *commutative* if the adjacency matrices A_i, $i = 1, 2, \dots, d$ of all graphs in the decomposition commute. It follows that in a commutative decomposition of a complete graph all graphs are regular, since each adjacency matrix A_i, $i = 1, 2, \dots, d$ commutes with the all-ones matrix $J = I + A_1 + \cdots + A_d$.

A (d-class) *association scheme* is a decomposition $\{G_1, G_2, \dots, G_d\}$ of a complete graph such that there are p_{ij}^h, $h, i, j = 1, \dots, d$, called the intersection numbers, such that for any two vertices x and y adjacent in G_h, there are p_{ij}^h vertices z that are adjacent to x in G_i and adjacent to y in G_j. It follows that an association scheme is a commutative decomposition, and hence that all graphs in the decomposition are regular. Because of this regularity we can extend the definition of the p_{ij}^h to $h, i, j = 0, 1, \dots, d$, if we define G_0 as the graph consisting of a loop at each vertex (with adjacency matrix $A_0 = I$).

If A_i denotes the adjacency matrix of G_i, then the conditions of the association scheme translate into $A_i A_j = A_j A_i = \sum_{h=0}^{d} p_{ij}^h A_h$ for $i, j = 0, 1, \dots, d$. This implies that a decomposition $\{G_1, G_2, \dots, G_d\}$ of a complete graph is an association scheme if and only if the vector space $\langle A_0 = I, A_1, A_2, \dots, A_d \rangle$ forms an algebra (i.e. is closed under taking products) over the real number field. This algebra is called the Bose-Mesner algebra of the association scheme. We remark that in the literature more general definitions of association schemes are being used. In this respect we add that the association schemes in this paper will always be symmetric and hence commutative.

Association schemes are generalizations of strongly regular graphs in the sense that a decomposition of a complete graph into a graph and its complement is an association scheme if and only if the graph is strongly regular. For more information on association schemes we refer the reader to [1] or [3].

Now consider a commutative decomposition with adjacency matrices A_i, $i = 1, 2, \ldots, d$, and let $A_0 = I$. Since the matrices A_i commute, they can be diagonalized simultaneously, and hence they have a common basis of (mutually orthogonal) eigenvectors. It follows that the vector space $\mathbb{R}^v$ can be decomposed into maximal common eigenspaces V_i, $i = 0, 1, \ldots, t$ (for some t). Since all graphs are regular, one of these maximal common eigenspaces is $\langle \mathbf{j} \rangle$, which will be denoted by V_0. Let E_i be the idempotent matrix representing the orthogonal projection onto the eigenspace V_i, for $i = 0, 1, \ldots, t$, then $E_i E_j = \delta_{ij} E_i$. Moreover, we can express the matrices A_i in terms of the matrices E_j, i.e. $A_i = \sum_{j=0}^{t} P_{ji} E_j$ for $i = 0, 1, \ldots, d$, where P_{ji} is the eigenvalue of A_i on the eigenspace V_j. The matrix P is called the *eigenmatrix* of the decomposition. The following lemma characterizes the association schemes among the commutative decompositions.

Lemma 1 *Let $\{G_1, G_2, \ldots, G_d\}$ be a commutative decomposition of a complete graph with idempotents E_j, $j = 0, \ldots, t$, and eigenmatrix P as defined above. Then $t \geq d$ with equality if and only if the decomposition is an association scheme. If $t = d$, then P is nonsingular.*

Proof: Let $\mathbf{A} = \langle A_0 = I, A_1, A_2, \ldots, A_d \rangle$, and $\mathbf{E} = \langle E_0, E_1, \ldots, E_t \rangle$. Since $A_i = \sum_{j=0}^{t} P_{ji} E_j$ for $i = 0, \ldots, d$ it follows that $\mathbf{A} \subseteq \mathbf{E}$, hence we have that $d \leq t$ (since clearly the matrices A_i, $i = 0, 1, \ldots, d$ are linearly independent).

We claim that $\mathbf{E}$ is the smallest algebra containing $\mathbf{A}$. This shows that $\mathbf{A}$ is an algebra if and only if $t = d$, which proves the first part of the lemma. If $t = d$, then P is the matrix which transforms one basis of the algebra into another (since also the matrices E_j, $j = 0, 1, \ldots, t$ are linearly independent), and hence P is nonsingular.

To prove the claim, we first remark that clearly $\mathbf{E}$ is an algebra. The remainder of the proof is similar to the argument in [1, Theorem 3.1]. Fix i. Because of the maximality of the common eigenspaces V_j, there is a $k(j)$ such that $P_{jk(j)} \neq P_{ik(j)}$, for each $j \neq i$. Now the matrix $F = \prod_{j \neq i} (A_{k(j)} - P_{jk(j)} I)$ is contained in each algebra containing $\mathbf{A}$, and hence also in $\mathbf{E}$. Moreover, F vanishes on each V_j, $j = 0, 1, \ldots, t$ except on V_i. This means that E_i is a multiple of F, and hence that E_i is contained in each algebra containing $\mathbf{A}$. Since this holds for all i, the claim is proven. □

2.3. *Fusions and amorphic association schemes*

A decomposition $\{H_1, \ldots, H_e\}$ of some graph G is called a *fusion* of a decomposition $\{G_1, \ldots, G_d\}$ of G if for all i, the edge set of H_i is the union of the edge sets of some of the graphs G_j.

An association scheme is called *amorphic* if any of its fusions is also an association scheme. It follows that each of the graphs in an amorphic association scheme must be strongly regular. Thus amorphic association schemes are strongly regular decompositions of a complete graph. Moreover, it was shown in [7] that in an amorphic association scheme with at least three graphs all graphs are of Latin square type, or all graphs are of negative

Latin square type. For more information on amorphic association schemes we refer the reader to [6].

The following association schemes are in some cases amorphic, and they will play an important role in Section 6. Let $q = p^m$, where p is prime, and let d be a divisor of $q-1$. The d-class cyclotomic association scheme on vertex set $GF(q)$ is defined as follows. Let α be a primitive element of $GF(q)$. Then two vertices are adjacent in graph G_j if their difference equals α^{di+j} for some i, for $j = 1, \ldots, d$. Note that in some cases this association scheme is non-symmetric, but these association schemes will not be used in this paper. It was proven by Baumert et al. [2, Theorem 4] that, for $d > 2$, the cyclotomic scheme is amorphic if and only if -1 is a power of p modulo d.

3. Commuting strongly regular graphs and three-class association schemes

Suppose we have two edge-disjoint strongly regular graphs G_1 and G_2 on the same vertex set. Assume that the two graphs are not complementary, so that there is a remaining nonempty third graph $G_3 = \overline{G_1 \cup G_2}$ with adjacency matrix $A_3 = J - I - A_1 - A_2$, where A_1 and A_2 are the adjacency matrices of G_1 and G_2, respectively. We would like to know when these three graphs form a 3-class association scheme. It is clear that a necessary condition for this is that the matrices A_1 and A_2 commute. Examples of noncommuting A_1 and A_2 are not hard to find, for example, take the triangular graph $T(5)$ and the graph consisting of five vertex-disjoint edges which are not in the triangular graph (a matching in the Petersen graph, the complement of $T(5)$). These two graphs leave two vertex-disjoint 5-cycles as the third graph. This example is particularly interesting since it shows that although the third graph is part of some 3-class association scheme (the wreath product of K_2 and C_5), it does not form one with G_1 and G_2.

Is the property that G_1 and G_2 (i.e. A_1 and A_2) commute sufficient for G_1, G_2, and G_3 to form an association scheme? Before answering this question we first need the following lemma. This lemma, which in a sense tells us how G_1 and G_2 interact algebraically, will also play an important role in Section 6.

Lemma 2 *Let G_1 and G_2 be edge-disjoint strongly regular graphs on v vertices, with G_i having adjacency matrix A_i, valency k_i, and restricted eigenvalues r_i and s_i, for $i = 1, 2$. If A_1 and A_2 commute, then $A_1 + A_2$ has restricted eigenvalues $\vartheta_1 = s_1 + s_2$, $\vartheta_2 = s_1 + r_2$, $\vartheta_3 = r_1 + s_2$, and $\vartheta_4 = r_1 + r_2$, with respective restricted multiplicities*

$$m_1 = \frac{vr_1r_2 - (k_1 - r_1)(k_2 - r_2)}{(r_1 - s_1)(r_2 - s_2)}, \qquad m_2 = -\frac{vr_1s_2 - (k_1 - r_1)(k_2 - s_2)}{(r_1 - s_1)(r_2 - s_2)},$$

$$m_3 = -\frac{vs_1r_2 - (k_1 - s_1)(k_2 - r_2)}{(r_1 - s_1)(r_2 - s_2)}, \qquad m_4 = \frac{vs_1s_2 - (k_1 - s_1)(k_2 - s_2)}{(r_1 - s_1)(r_2 - s_2)}.$$

If moreover $r_i > s_i$ for $i = 1, 2$, then $m_2 > 0$ and $m_3 > 0$.

Proof: Since A_1 and A_2 commute, they have a common basis of eigenvectors. Clearly, this is also a basis of eigenvectors for $A_1 + A_2$. The all-ones vector $\mathbf{j}$ is a common eigenvector with eigenvalues k_1, k_2, and $k_1 + k_2$, respectively. It is also clear that $A_1 + A_2$ has restricted eigenvalues ϑ_i, $i = 1, \ldots, 4$, as stated. Now let m_i be the restricted multiplicity of ϑ_i, for $i = 1, \ldots, 4$ (more precisely, m_1 is the dimension of the intersection of the restricted eigenspaces of s_1 (of A_1) and s_2 (of A_2), etc.), and let g_i be the restricted multiplicity of the restricted eigenvalue s_i of the matrix A_i, for $i = 1, 2$. From the equations $m_1 + m_2 = g_1$, $m_1 + m_3 = g_2$, $m_1 + m_2 + m_3 + m_4 = v - 1$ and $(k_1 + k_2)^2 + m_1(s_1 + s_2)^2 + m_2(s_1 + r_2)^2 + m_3(r_1 + s_2)^2 + m_4(r_1 + r_2)^2 = v(k_1 + k_2)$ (which follows from the trace of $(A_1 + A_2)^2$), and the property that $g_i = \frac{vr_i + k_i - r_i}{r_i - s_i}$ for $i = 1, 2$, the multiplicities m_i, $i = 1, \ldots, 4$ follow.

If moreover $r_1 > s_1$ and $r_2 > s_2$, then $(k_1 - r_1)(k_2 - s_2) > vr_1s_2$ (since $s_2 < 0 \le r_1 \le k_1$), hence $m_2 > 0$. Similarly $m_3 > 0$. □

We remark furthermore that if $r_i > s_i$, then we also have that $\vartheta_1 < \vartheta_i < \vartheta_4$ for $i = 2, 3$, and that $\vartheta_2 = \vartheta_3$ if and only if $r_1 - s_1 = r_2 - s_2$. In this case the restricted multiplicity of $\vartheta_2 = \vartheta_3$ is of course $m_2 + m_3$.

An immediate consequence of Lemma 2 is the following corollary, which will be used in Theorem 3.

Corollary 1 *Let G_1 and G_2 be edge-disjoint strongly regular graphs, both of Latin square type or both of negative Latin square type. If the adjacency matrices of the two graphs commute, then their union $G_1 \cup G_2$ is also strongly regular of Latin square type, or negative Latin square type, respectively.*

Proof: There are integers n, t_1, and t_2 (positive or negative, depending on the "type" of the graphs) such that the number of vertices is n^2, and G_i has valency $t_i(n - 1)$ and restricted eigenvalues $r_i = n - t_i$ and $s_i = -t_i$, for $i = 1, 2$. It follows from Lemma 2 that the multiplicity m_4 equals zero. It thus follows that $G_1 \cup G_2$ has valency $(t_1 + t_2)(n - 1)$ and restricted eigenvalues $n - t_1 - t_2$ and $-t_1 - t_2$, hence it is also a strongly regular graph of Latin square type or negative Latin square type graph, respectively. □

Now we shall answer our earlier question.

Theorem 1 *Let G_1 and G_2 be commuting, edge-disjoint strongly regular graphs on v vertices, with valencies k_1 and $k_2 < v - 1 - k_1$, and restricted eigenvalues $r_1 > s_1$ and $r_2 > s_2$, respectively. Then $\{G_1, G_2, \overline{G_1 \cup G_2}\}$ is an association scheme if and only if $vr_1r_2 = (k_1 - r_1)(k_2 - r_2)$ or $vs_1s_2 = (k_1 - s_1)(k_2 - s_2)$.*

Proof: Let A_1, A_2, $A_3 = J - I - (A_1 + A_2)$ be the adjacency matrices of G_1, G_2, and $G_3 = \overline{G_1 \cup G_2}$, respectively. Since G_i is regular for $i = 1, 2, 3$, and A_1 and A_2 commute, it follows that the decomposition $\{G_1, G_2, G_3\}$ is commutative. Moreover, G_3 has valency $k_3 = v - 1 - k_1 - k_2$ and restricted eigenvalues $\theta_i = -1 - \vartheta_i$ with restricted multiplicities $m_i, i = 1, \ldots, 4$ (where the ϑ_i are the restricted eigenvalues of $A_1 + A_2$, as given in Lemma 2). Now let $V_0 = \langle \mathbf{j} \rangle$, and let V_i be the restricted eigenspace corresponding to the eigenvalue ϑ_i of $A_1 + A_2$, for $i = 1, \ldots, 4$ (more precisely, V_1 is the intersection of the restricted eigenspaces of s_1 (of A_1) and s_2 (of A_2), etc.). The eigenmatrix of the commutative decomposition (as defined in Section 2.2) is thus given by

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_3 = v - 1 - k_1 - k_2 \\ 1 & s_1 & s_2 & \theta_1 = -1 - s_1 - s_2 \\ 1 & s_1 & r_2 & \theta_2 = -1 - s_1 - r_2 \\ 1 & r_1 & s_2 & \theta_3 = -1 - r_1 - s_2 \\ 1 & r_1 & r_2 & \theta_4 = -1 - r_1 - r_2 \end{pmatrix}.$$

Consequently, since m_2 and m_3 are positive (by Lemma 2), we find from Lemma 1 that at most one of the multiplicities m_1 and m_4 can be zero, and if indeed one of them is, then $\{G_1, G_2, G_3\}$ is a 3-class association scheme. The result now follows from the expressions for m_1 and m_4 in Lemma 2. □

The theorem we just proved allows us to make a first step towards proving that any strongly regular decomposition of a complete graph consisting of three graphs is an amorphic association scheme.

Corollary 2 *A commutative strongly regular decomposition* $\{G_1, G_2, G_3\}$ *of a complete graph is an amorphic association scheme.*

Proof: From the proof of Theorem 1 it follows that $G_3 = \overline{G_1 \cup G_2}$ has two restricted eigenvalues only if $m_1 = 0$ or $m_4 = 0$. Hence the decomposition is an association scheme. Moreover, from the definition of amorphic schemes it follows easily that any 3-class association scheme in which all graphs are strongly regular is amorphic. □

In the next section we shall generalize this corollary, and prove that any strongly regular decomposition with three graphs is an amorphic association scheme.

The following examples of commuting strongly regular graphs G_1 and G_2 show that there are indeed cases where the decomposition $\{G_1, G_2, G_3 = \overline{G_1 \cup G_2}\}$ is not an association scheme. Moreover, they show that the number of distinct eigenvalues of G_3 is not a criterion for forming a scheme.

We just saw that the case where G_3 is strongly regular gives an amorphic scheme. However, in the following infinite family of examples G_3 has three distinct eigenvalues also, and the graphs do not form a scheme (G_3 will be disconnected). Let A_1 and A_2 be the following adjacency matrices of the Clebsch graph and the lattice graph $L_2(4)$, respectively:

$$A_1 = \begin{pmatrix} O & J-I & I & I \\ J-I & O & I & I \\ I & I & O & J-I \\ I & I & J-I & O \end{pmatrix} \quad \text{and}$$

$$A_2 = \begin{pmatrix} J-I & I & K & K \\ I & J-I & K & K \\ K & K & J-I & I \\ K & K & I & J-I \end{pmatrix},$$

where all submatrices are 4×4, and K is a symmetric permutation matrix with zero diagonal. These matrices commute, and the corresponding graphs have no edges in common. By taking $H_1 = J - 2(A_1 + I)$ and $H_2 = J - 2A_2$ we have two commuting regular symmetric Hadamard matrices with constant diagonal (cf. [4]), which have no entry -1 in common. Now define $H_1' = H^{\otimes t} \otimes H_1$ and $H_2' = H^{\otimes t} \otimes H_2$, for $t = 0, 1, \ldots,$ where

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix},$$

and $\otimes$ denotes the Kronecker product (and superscript $\otimes t$ the t-th Kronecker power), then H_1' and H_2' are also commuting regular symmetric Hadamard matrices with constant diagonal, which have no entry -1 in common (we leave the technical details to the reader). Now define $A_1' = \frac{1}{2}(J - H_1') - I$ and $A_2' = \frac{1}{2}(J - H_2')$, so that two vertices are adjacent if there is a -1 in the corresponding entry of the Hadamard matrix. Now also A_1' and A_2' commute, and the graphs defined by them are edge-disjoint, and both are strongly regular on $v = 4^{t+2}$ vertices (cf. [4]). Moreover, A_1' has eigenvalues $k_1 = 2^{2t+3} - 2^{t+1} - 1$, $r_1 = 2^{t+1} - 1$ and $s_1 = -2^{t+1} - 1$, while A_2' has eigenvalues $k_2 = 2^{2t+3} - 2^{t+1}$, $r_2 = 2^{t+1}$ and $s_2 = -2^{t+1}$. From this, it follows that the corresponding eigenmatrix is the following.

$$P = \begin{pmatrix} 1 & 2^{2t+3} - 2^{t+1} - 1 & 2^{2t+3} - 2^{t+1} & k_3 = 2^{t+2} \\ 1 & -2^{t+1} - 1 & -2^{t+1} & \theta_1 = 2^{t+2} \\ 1 & -2^{t+1} - 1 & 2^{t+1} & \theta_2 = 0 \\ 1 & 2^{t+1} - 1 & -2^{t+1} & \theta_3 = 0 \\ 1 & 2^{t+1} - 1 & 2^{t+1} & \theta_4 = -2^{t+2} \end{pmatrix}$$

with multiplicities $m_0 = 1$, $m_1 = 2^{t+1} - 1$, $m_2 = 2^{t+2}(2^{t+1} - 1)$, $m_3 = 2^{2t+3}$, $m_4 = 2^{t+1}$, hence the graphs do not form a scheme. Here G_3 has 3 distinct eigenvalues, in fact it is the disjoint union of (strongly regular) complete bipartite graphs. Note that in this example G_1 is of negative Latin square type, while G_2 is of Latin square type.

Next, we shall construct infinite families of examples of commuting strongly regular graphs, where G_3 has 4 or 5 eigenvalues. In the latter case it is clear that the graphs cannot form an association scheme. In the first case we have examples forming schemes, and examples not forming schemes.

Consider a generalized quadrangle $GQ(q-1, q+1)$, with $q = 2^e$, as constructed by Hall [8], that is, with points those of the affine space $AG(3, q)$ and lines those of $q + 2$ parallel classes (spreads) of lines in $AG(3, q)$ corresponding to a hyperoval O in $PG(2, q)$ (the linear representation $T_2^*(O)$, cf. [14]). Take the line graph of this generalized quadrangle (vertices are lines, being adjacent if they intersect), which has $v = (q + 2)q^2$ vertices, valency $k_1 = q(q + 1)$, and restricted eigenvalues $r_1 = q$ and $s_1 = -q$. From the definition, it is clear that it has a partition of the vertices into $q + 2$ cocliques of size q^2. This partition is regular (equitable), meaning that the induced subgraph on the union of two cocliques is regular. This implies that when we take as second graph G_2 the corresponding disjoint union of q^2-cliques, then G_1 and G_2 commute. Since G_2 has valency $k_2 = q^2 - 1$ and restricted eigenvalues $r_2 = q^2 - 1, s_2 = -1$, it follows that the corresponding eigenmatrix is

$$P = \begin{pmatrix} 1 & q(q+1) & q^2-1 & k_3 = q^3 - q \\ 1 & -q & -1 & \theta_1 = q \\ 1 & -q & q^2-1 & \theta_2 = q - q^2 \\ 1 & q & -1 & \theta_3 = -q \end{pmatrix}$$

with multiplicities $m_1 = m_3 = \frac{1}{2}(q + 2)(q^2 - 1)$ and $m_2 = q + 1$. Since $m_4 = 0$ the three graphs form an association scheme, with G_3 having 4 distinct eigenvalues (unless $q = 2$, in which case it has 3 distinct eigenvalues).

Small adjustments of the previous construction will give us the other two cases. First, we shall refine the partition into $q + 2$ cocliques "plane-wise", that is, take a first parallel class (a coclique), and partition it into q parallel "planes". Since the set of all lines forms a generalized quadrangle, there is a unique other parallel class that can be partitioned into the "same planes", and we do so (again, we skip the technical details). Repeating this procedure with the remaining q parallel classes we find a partition of the vertex set into $(q + 2)q$ cocliques of size q, which is again regular. The partition gives us as the second graph G_2' a disjoint union of q-cliques, which commutes with $G_1' = G_1$, the line graph of the generalized quadrangle. Since G_2' now has valency $k_2' = q-1$ and restricted eigenvalues $r_2' = q - 1$ and $s_2' = -1$, we have eigenmatrix

$$P = \begin{pmatrix} 1 & q(q+1) & q-1 & k_3' = q^3 + q^2 - 2q \\ 1 & -q & -1 & \theta_1' = q \\ 1 & -q & q-1 & \theta_2' = 0 \\ 1 & q & -1 & \theta_3' = -q \\ 1 & q & q-1 & \theta_4' = -2q \end{pmatrix}$$

with multiplicities $m'_1 = m'_3 = \frac{1}{2}(q+2)q(q-1)$, $m'_2 = \frac{1}{2}(q^2+3q)$, and $m'_4 = \frac{1}{2}(q^2+q-2)$. Thus the three graphs do not form an association scheme; in this case G'_3 has 5 distinct eigenvalues.

The constructed partition into cocliques by "planes" also allows us to partition the lines regularly into $\frac{1}{2}(q+2)q$ cocliques of size $2q$ (simply reunite pairs of parallel "planes" consistently). This gives a graph G''_2 which also commutes with $G''_1 = G_1$, and which has valency $k''_2 = 2q-1$ and restricted eigenvalues $r''_2 = 2q-1$ and $s''_2 = -1$. Thus we obtain the eigenmatrix

$$P = \begin{pmatrix} 1 & q(q+1) & 2q-1 & k''_3 = q^3+q^2-3q \\ 1 & -q & -1 & \theta''_1 = q \\ 1 & -q & 2q-1 & \theta''_2 = -q \\ 1 & q & -1 & \theta''_3 = -q \\ 1 & q & 2q-1 & \theta''_4 = -3q \end{pmatrix}$$

with multiplicities $m''_1 = m''_3 = \frac{1}{4}(q+2)q(2q-1)$, $m''_2 = \frac{1}{4}q^2+q$, and $m''_4 = \frac{1}{4}q^2-1$, and hence also here the three graphs do not form an association scheme, unless $q=2$ (in which case $m_4 = 0$); in these examples G''_3 has 4 distinct eigenvalues.

4. Decompositions into three strongly regular graphs

In this section we shall consider decompositions of the complete graph into three strongly regular graphs. We shall prove that such decompositions must be (amorphic) association schemes. This generalizes the result of Rowlinson [13] and independently Michael [12] that any decomposition of a complete graph into three isomorphic strongly regular graphs forms an amorphic association scheme. The proof given here is based on techniques from linear algebra. An independent proof based on noncommutative algebra and representation theory is given in [6].

Theorem 2 *Let $\{G_1, G_2, G_3\}$ be a strongly regular decomposition of a complete graph. Then $\{G_1, G_2, G_3\}$ is an amorphic association scheme.*

Proof: Let G_i have parameters $(v, k_i, \lambda_i, \mu_i)$, restricted eigenvalues r_i and s_i, and adjacency matrix A_i, for $i = 1, 2, 3$. Denote by $E(r_i)$, $E(s_i)$ the restricted eigenspaces of r_i, s_i, respectively, that is, the spaces of corresponding eigenvectors orthogonal to the all-ones vector. Let $f_i = \dim E(r_i)$ and $g_i = \dim E(s_i)$ denote the restricted multiplicities of the eigenvalues, then $f_i = -\frac{k_i+(v-1)s_i}{r_i-s_i}$ and $g_i = \frac{k_i+(v-1)r_i}{r_i-s_i}$, for $i = 1, 2, 3$. Without loss of generality we rearrange the eigenvalues such that $f_i \geq \frac{1}{2}(v-1)$ (hence we do not necessarily have that $r_i > s_i$).

Our goal is to show that the graphs have a common basis of eigenvectors, hence that the decomposition is commutative, since then we have an association scheme by Corollary 2.

The all-ones vector $\mathbf{j}$ is a common eigenvector of the graphs with eigenvalues k_1, k_2, and k_3, respectively. Therefore, from now on we only have to consider restricted eigenvectors, i.e. eigenvectors in the $(v-1)$-dimensional subspace $\mathbf{j}^{\perp}$ of $\mathbb{R}^v$.

First we suppose that r_i and r_j, $i \neq j$ do not have a common eigenvector. Then necessarily $f_i + f_j = \dim E(r_i) + \dim E(r_j) = \dim(E(r_i) + E(r_j)) \leq v-1$, hence $f_i = f_j = \frac{1}{2}(v-1)$. But then $k_i = k_j = \frac{1}{2}(v-1)$ (if $f_i = \frac{1}{2}(v-1)$ then it follows that $k_i = -\frac{1}{2}(v-1)(r_i+s_i) = \frac{1}{2}(v-1)(\mu_i - \lambda_i)$, hence k_i is a multiple of, and hence equal to, $\frac{1}{2}(v-1)$). Thus the remaining graph is empty, which gives a contradiction. So in the remainder of the proof we may assume that r_i and r_j do have a common eigenvector.

Now, suppose that some r_i and s_j $(i \neq j)$ have a common eigenvector. Then $-1-r_i-s_j$ is an eigenvalue of A_h (with the same eigenvector), where $h \neq i, j$. First suppose that this eigenvalue is r_h, i.e. $-1-r_i-s_j = r_h$. Since we know that r_i and r_j also have an eigenvector in common, it follows that $-1-r_i-r_j$ is also an eigenvalue of A_h, and this eigenvalue must then equal s_h. Also r_j and r_h have an eigenvector in common, and it now follows that $-1-r_j-r_h = s_i$. From these equations we find that $r_i - s_i = r_j - s_j = r_h - s_h$, and then

$$\begin{aligned} f_i &= -\frac{k_i + (v-1)s_i}{r_i - s_i} = -\frac{v-1-k_j-k_h+(v-1)(-1-r_j-r_h)}{r_i - s_i} \\ &= \frac{k_j + (v-1)r_j}{r_j - s_j} + \frac{k_h + (v-1)r_h}{r_h - s_h} = g_j + g_h, \end{aligned}$$

and so $g_i = f_j + f_h - (v-1)$. Since $s_i = -1-r_j-r_h$, we have that $E(r_j) \cap E(r_h) \subset E(s_i)$. However, the previous computation shows that $\dim E(s_i) = f_j + f_h - (v-1) \leq \dim E(r_j) + \dim E(r_h) - \dim(E(r_j) + E(r_h)) = \dim(E(r_j) \cap E(r_h))$, hence $E(s_i) = E(r_j) \cap E(r_h)$. Similarly we find that $E(s_j) = E(r_i) \cap E(r_h)$ and $E(s_h) = E(r_i) \cap E(r_j)$. Now (use that $f_i = g_j + g_h$) it is clear that each eigenvector of A_i is also an eigenvector of A_j, and consequently also of A_h. Hence A_1, A_2, and A_3 commute, so we have an association scheme.

Secondly, suppose that $-1-r_i-s_j = s_h$. Then $-1-r_i-r_j = r_h$, and so $r_h - s_h = s_j - r_j$. Now

$$\begin{aligned} g_h &= \frac{k_h + (v-1)r_h}{r_h - s_h} = \frac{v-1-k_i-k_j+(v-1)(-1-r_i-r_j)}{r_h - s_h} \\ &= \frac{k_i + (v-1)r_i}{r_j - s_j} + \frac{k_j + (v-1)r_j}{r_j - s_j} = \frac{r_i - s_i}{r_j - s_j} g_i + g_j. \end{aligned}$$

Because of the symmetry of j and h, we may assume without loss of generality that $g_h > g_j$ (equality cannot occur by the previous computation). If r_j and s_h do not have a common eigenvector, then $f_j + g_h \leq v-1$. However, $f_j + g_h > f_j + g_j = v-1$, which is a contradiction. So r_j and s_h do have a common eigenvector, and then $-1-r_j-s_h = s_i$, which together with the other equations gives that $r_i - s_i = r_j - s_j$, and $g_h = g_i + g_j$. Now we find similarly as before that $E(r_h) = E(r_i) \cap E(r_j)$, $E(s_i) = E(r_j) \cap E(s_h)$ and $E(s_j) = E(r_i) \cap E(s_h)$, and that the adjacency matrices commute, proving that we have an association scheme.

In the remaining case, which will need a different approach, r_i and r_j do have common eigenvectors, and r_i and s_j do not have common eigenvectors, for all i, j. Consequently $f_i + g_j \leq v - 1$, and also $f_j + g_i \leq v - 1$. But $f_i + g_j + f_j + g_i = 2(v - 1)$, hence we have equality, and so $f_i = f_j$. Thus $f_1 = f_2 = f_3$ and $g_1 = g_2 = g_3$. It is also clear from the assumptions that $r_1 + r_2 + r_3 = -1$, and since $(r_1 - s_1)g_1 = k_1 + (v - 1)r_1 = v - 1 - k_2 - k_3 + (v - 1)(-1 - r_2 - r_3) = -k_2 - (v - 1)r_2 - k_3 - (v - 1)r_3 = -(r_2 - s_2)g_2 - (r_3 - s_3)g_3 = -(r_2 - s_2 + r_3 - s_3)g_1$, so that $r_1 - s_1 + r_2 - s_2 + r_3 - s_3 = 0$, also $s_1 + s_2 + s_3 = -1$. Moreover, we see that $r_1 - s_1 + r_2 - s_2 \neq 0$, a property which we shall use later on. Now, finally, we need some combinatorics.

Let π_{ij}^h be the average number of vertices that are adjacent in G_i to vertex x, and in G_j to vertex y, over all pairs (x, y) that are adjacent in G_h, for $i, j, h \in \{1, 2, 3\}$. The parameters π_{ij}^h naturally resemble the intersection numbers p_{ij}^h of an association scheme. Obviously we have the equations $\pi_{ij}^h = \pi_{ji}^h$, $\pi_{i1}^h + \pi_{i2}^h + \pi_{i3}^h = k_i - \delta_{hi}$, $\pi_{ii}^h = \mu_i$ if $h \neq i$, and $\pi_{ii}^i = \lambda_i$. By counting ordered "(i, j, h)-triangles", we find that $k_h \pi_{ij}^h = k_i \pi_{hj}^i$.

Using these equations we derive that $\pi_{23}^1 \pi_{31}^3 = \frac{k_2}{k_1} \pi_{13}^2 \frac{k_1}{k_3} \mu_3 = \pi_{13}^2 \frac{k_2}{k_3} \pi_{33}^2 = \pi_{13}^2 \pi_{23}^3$. From the equations $\pi_{12}^3 + \pi_{13}^3 = k_1 - \mu_1$, $\pi_{12}^3 + \pi_{23}^3 = k_2 - \mu_2$, $\pi_{31}^3 + \pi_{32}^3 = k_3 - 1 - \lambda_3$ we derive that $2\pi_{12}^3 = k_1 - \mu_1 + k_2 - \mu_2 - (k_3 - 1 - \lambda_3)$, $2\pi_{31}^3 = k_1 - \mu_1 - (k_2 - \mu_2) + k_3 - 1 - \lambda_3$, and $2\pi_{32}^3 = -(k_1 - \mu_1) + k_2 - \mu_2 + k_3 - 1 - \lambda_3$. Similarly, we find that $2\pi_{23}^1 = -(k_1 - 1 - \lambda_1) + k_2 - \mu_2 + k_3 - \mu_3$, and $2\pi_{13}^2 = k_1 - \mu_1 - (k_2 - 1 - \lambda_2) + k_3 - \mu_3$. After plugging these into the equation $\pi_{23}^1 \pi_{31}^3 = \pi_{13}^2 \pi_{23}^3$, replacing the parameters by the eigenvalues ($k_i - \mu_i = -r_i s_i$ and $\lambda_i - \mu_i = r_i + s_i$), and using that $r_3 = -1 - r_1 - r_2$ and $s_3 = -1 - s_1 - s_2$, we find an equation which is equivalent to the equation $(r_1 s_2 - s_1 r_2)(r_1 - s_1 + r_2 - s_2) = 0$ (this argument is similar to that of [7, Lemma 4.5]). We mentioned before that the second factor in nonzero, hence we have that $r_1 s_2 = s_1 r_2$. But then also $k_1 r_2 = -f_1 r_1 r_2 - g_1 s_1 r_2 = -f_2 r_1 r_2 - g_2 r_1 s_2 = k_2 r_1$, and so r_1 and r_2 have the same sign. Similarly r_1 and r_3 have the same sign, which gives a contradiction to the fact that $r_1 + r_2 + r_3 = -1$. Thus this final case cannot occur, and so in all possible cases we have an association scheme. □

In Section 5 we shall prove that also any decomposition into (possibly more than three) strongly regular graphs of Latin square type, and any decomposition into strongly regular graphs of negative Latin square type forms an amorphic association scheme.

5. Decompositions of (negative) Latin square type

It was shown in [7] that in an amorphic association scheme (with at least three graphs) all graphs are of Latin square type, or all graphs are of negative Latin square type. We will show that there are no other strongly regular decompositions of a complete graph in which all graphs are of Latin square type, or in which all graphs are of negative Latin square type. This extends the result by Ito et al. [9] who showed that any association scheme in which all graphs are strongly regular of Latin square type, or in which all graphs are strongly regular of negative Latin square type, is amorphic.

Theorem 3 *Let $\{G_1, G_2, \ldots, G_d\}$ be a strongly regular decomposition of the complete graph on v vertices, such that the strongly regular graphs G_i, $i = 1, 2, \ldots, d$ are all of Latin square type or all of negative Latin square type. Then the decomposition is an amorphic association scheme.*

Proof: Let A_i be the adjacency matrix of G_i, which has valency k_i and restricted eigenvalues $r_i > s_i$, for $i = 1, 2, \ldots, d$. We shall give a proof for the case where all graphs are of Latin square type. The case of negative Latin square type is similar; only the roles of r_i and s_i have to be interchanged.

First we note that the all-ones vector $\mathbf{j}$ is a common eigenvector of the strongly regular graphs G_i with respective eigenvalues k_i, for $i = 1, 2, \ldots, d$.

The Latin square type graph G_i has the property that the restricted multiplicity of the positive resticted eigenvalue r_i is equal to the valency k_i of the graph; the negative restricted eigenvalue s_i has multiplicity $l_i = v - 1 - k_i$.

Now we fix j. Then $v - 1 - k_j = \sum_{i \neq j} k_i = (d-1)(v-1) - \sum_{i \neq j} l_i$, so $\sum_{i \neq j} l_i = (d-2)(v-1) + k_j$. From repeatedly using the observation that $\dim(A \cap B) = \dim(A) + \dim(B) - \dim(A + B) \geq \dim(A) + \dim(B) - (v-1)$ when A and B are subspaces of $\mathbf{j}^{\perp}$, it thus follows that $\dim(\bigcap_{i \neq j} E(s_i)) \geq k_j$, where $E(s_i)$ denotes the restricted eigenspace of s_i as eigenvalue of A_i, for $i = 1, 2, \ldots, d$. In other words, the matrices $A_i, i \neq j$ have a common eigenspace of dimension at least k_j with respective eigenvalues s_i. On this common eigenspace the matrix $A_j = J - I - \sum_{i \neq j} A_i$ has eigenvalue $-1 - \sum_{i \neq j} s_i$, which must be equal to r_j (since $s_i \leq -1$ for all i), and which has restricted multiplicity k_j. Hence the dimension of the common eigenspace is exactly k_j. Since the above holds for all $j = 1, 2, \ldots, d$, it follows that $\mathbb{R}^v$ can be decomposed into $d + 1$ common eigenspaces, and that the decomposition $\{G_1, G_2, \ldots, G_d\}$ is commutative. It then follows from Lemma 1 that the decomposition is an association scheme.

Since, by Corollary 1, the union of any two commuting edge-disjoint Latin square type (or negative Latin square type) graphs is again a Latin square type (negative Latin square type, respectively) graph, it follows from the above that any fusion of the association scheme is again an association scheme, hence the original association scheme is amorphic. $\square$

It would be natural to ask if a mixture of Latin square type graphs and negative Latin square type graphs is possible in a decomposition of a complete graph, and if so, if these (can) form an association scheme. In the next section we shall see examples of (commutative) decompositions which indeed contain such a mixture. These decompositions are not association schemes.

6. Decompositions into four strongly regular graphs

A decomposition of a complete graph into three strongly regular graphs necessarily is an association scheme. Next, we consider the case of decompositions of a complete graph into four strongly regular graphs.

6.1. Three disconnected graphs

First, we shall classify the decompositions in which at least three of the four strongly regular graphs are disconnected. First of all, we have the Latin square schemes $L_{1,1,1}(n)$, for $n > 2$. Such an amorphic association scheme is constructed from a Latin square of side n as follows. The vertices are the n^2 cells of the Latin square. In the first graph, two cells are adjacent if they are in the same row, in the second graph if they are in the same column, and in the third graph if they contain the same entry. The fourth graph is the remainder.

Secondly, there is a family of examples consisting of the following: one graph is the complete multipartite graph $K_{4,4,\ldots,4}$, and the other three graphs are matchings. These four graphs form an association scheme, the wreath product of a complete graph and $L_{1,1,1}(2)$. We note that this association scheme is not amorphic.

Finally, there is one example on 6 vertices, in which all four graphs are disconnected: three of the four are matchings, and the fourth is the disjoint union of two triangles. In this example the matchings do not commute, hence it does not give rise to an association scheme.

Theorem 4 *Let $\{G_1, G_2, G_3, G_4\}$ be a strongly regular decomposition of the complete graph on v vertices into four strongly regular graphs, of which at least three are disconnected. Then the decomposition is the above example on* 6 *vertices, or the wreath product* (*association scheme*) *of a complete graph and* $L_{1,1,1}(2)$, *or a Latin square association scheme* $L_{1,1,1}(n)$, $n > 2$.

Proof: Without loss of generality we assume that G_1, G_2, and G_3 are disconnected, say G_i is the disjoint union of t_i complete graphs on n_i vertices, for $i = 1, 2, 3$, where $n_1 \geq n_2 \geq n_3 \geq 2$. The union of G_1, G_2, and G_3, i.e. the complement of G_4, is also strongly regular, say with parameters ($v = t_i n_i$, $k = n_1 + n_2 + n_3 - 3$, λ, μ), and restricted eigenvalues r and s. By considering two adjacent vertices in G_i it follows that λ equals n_i, $n_i - 1$, or $n_i - 2$, for $i = 1, 2, 3$, hence it follows that $n_3 \geq \lambda \geq n_1 - 2$.

The cases with $n_3 = 2$ are easily checked: using the fact that $k\lambda$ is even and the condition that $\mu = \frac{k(k-1-\lambda)}{v-1-k}$ is an integer (at most k) reduces the number of cases substantially. Since there are no strongly regular graphs with parameters (8, 5, 2, 5), (16, 5, 2, 1), (12, 7, 2, 7), or (36, 7, 2, 1), and since the unique strongly regular graph with parameters (10, 3, 0, 1), the Petersen graph, does not decompose into three matchings, it subsequently follows that the only possibilities are the cases $n_1 = n_2 = n_3 = 2$, $\lambda = 0$, $v = 6$ and $n_1 = 3$, $n_2 = n_3 = 2$, $\lambda = 2$, $v = 6$, which both give rise to the stated example on 6 vertices; and the cases $n_1 = n_2 = n_3 = 2$, $\lambda = 2$, $v = 4m$, $m > 1$, which give rise to the wreath product of a complete graph and $L_{1,1,1}(2)$.

Next, we assume that $n_3 \geq 3$. The graph G_i has eigenvalue -1 with multiplicity $v - t_i$, for $i = 1, 2, 3$. Let V_i be the corresponding eigenspace of G_i for this eigenvalue -1. Since $\dim(V_1 \cap V_2 \cap V_3) \geq \dim(V_1) + \dim(V_2) + \dim(V_3) - 2\dim(\mathbf{j}^{\perp}) \geq v + 2 - 3t_3 = (n_3 - 3)t_3 + 2 > 0$, it follows that G_1, G_2, and G_3 have a common eigenvector with eigenvalue -1, hence their union has an eigenvalue $s = -3$. From the equations $\lambda - \mu = r + s$ and $\mu - k = rs$ we now derive that $\mu = \frac{3\lambda - k + 9}{2}$, and hence that $k + \lambda$ is odd.

The case $n_1 = n_2 = n_3 = n$ ($k = 3n - 3$), $\lambda = n$ gives $\mu = 6$. From the equation $\mu(v - 1 - k) = k(k - 1 - \lambda)$ we now find that $v = n^2$. It is clear now that in this case

G_1, G_2, and G_3 commute, and that the four graphs form a Latin square association scheme $L_{1,1,1}(n)$, $n > 2$.

The case $n_1 = n_2 = n_3 = n$ ($k = 3n - 3$), $\lambda = n - 2$ gives $\mu = 3$, and $v = 2n^2 - n$. This case leads to a contradiction as follows. Here the concepts adjacency and neighbour refer to the union of G_1, G_2, and G_3, unless otherwise specified. Let x, y, z be mutually adjacent in G_1. Let x_2 be adjacent to x in G_2. Since y and x_2 are not adjacent (otherwise x and y have more than $n - 2$ common neighbours), they have $\mu = 3$ common neighbours. One of these is x, the other two are, say, y_2 and y_3 which are adjacent to y in G_2 and G_3, respectively. It follows that y_3 and x_2 must be adjacent in G_1, and y_2 and x_2 must be adjacent in G_3. Similarly x_2 and z have three common neighbours, one of them being x, and the other two being, say, z_2 and z_3, which are adjacent to z in G_2 and G_3, respectively. Also here it follows that z_3 and x_2 are adjacent in G_1 (and hence y_3 and z_3 are adjacent in G_1), and z_2 and x_2 are adjacent in G_3. Since y_3 and z are not adjacent (otherwise y and z have too many common neighbours), they should have three common neighbours. This gives a contradiction, since y_3 and z can have at most two common neighbours, z_3 and y: any other common neighbour should be adjacent to both y_3 and z in G_2, and hence y_3 and z should themselves be adjacent in G_2, a contradiction.

The case $n_1 = n_2 = n, n_3 = n - 1$ ($k = 3n - 4$), $\lambda = n - 1$ gives $\mu = 5$ and $v = \frac{6}{5}n^2 - n + \frac{1}{5}$. On the other hand, since $v = t_1 n = t_3(n - 1)$, it follows that v is a multiple of $n(n - 1)$. This gives a contradiction, as is easily verified. The remaining three cases go similarly. □

Thus in any other example of a strongly regular decomposition with four graphs at least two of these must be connected. We conjecture that except for amorphic association schemes there are no strongly regular decompositions of a complete graph into four graphs of which exactly two are connected.

6.2. Commuting strongly regular graphs

For a commutative strongly regular decomposition of a complete graph into four graphs we have the following.

Theorem 5 *Let $\{G_1, G_2, G_3, G_4\}$ be a commutative strongly regular decomposition of the complete graph on v vertices. Let G_i have valency k_i and restricted eigenvalues r_i and s_i (where we do not assume that $r_i > s_i$), for $i = 1, \ldots, 4$. Then $\{G_1, G_2, G_3, G_4\}$ is* (i) *an amorphic association scheme*; *or* (ii) *an association scheme in which three of the graphs, say G_2, G_3, G_4, have the same parameters and which has eigenmatrix given by*

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_2 & k_2 \\ 1 & s_1 & r_2 & r_2 & r_2 \\ 1 & r_1 & s_2 & s_2 & r_2 \\ 1 & r_1 & s_2 & r_2 & s_2 \\ 1 & r_1 & r_2 & s_2 & s_2 \end{pmatrix}; \qquad (1)$$

or (iii) *it is not an association scheme, in which case the eigenmatrix is given by*

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_3 & k_4 \\ 1 & s_1 & s_2 & r_3 & r_4 \\ 1 & s_1 & r_2 & s_3 & r_4 \\ 1 & s_1 & r_2 & r_3 & s_4 \\ 1 & r_1 & s_2 & s_3 & r_4 \\ 1 & r_1 & s_2 & r_3 & s_4 \\ 1 & r_1 & r_2 & s_3 & s_4 \end{pmatrix}, \tag{2}$$

where possibly one row is removed.

Proof: Let G_i have adjacency matrix A_i, for $i = 1, \dots, 4$. First we assume that the restricted eigenvalues satisfy $r_i > s_i$ for $i = 1, \dots, 4$. During the proof we shall see that this assumption is not necessary, i.e. that we can interchange the role of the r_i and s_i (for all i simultaneously).

By using only the possible eigenvalues for A_1, A_2, and A_3 we obtain the following eigenmatrix for the decomposition:

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_3 & v-1-k_1-k_2-k_3 = k_4 \\ 1 & s_1 & s_2 & s_3 & -1-s_1-s_2-s_3 = \theta_1 \\ 1 & s_1 & s_2 & r_3 & -1-s_1-s_2-r_3 = \theta_2 \\ 1 & s_1 & r_2 & s_3 & -1-s_1-r_2-s_3 = \theta_3 \\ 1 & s_1 & r_2 & r_3 & -1-s_1-r_2-r_3 = \theta_4 \\ 1 & r_1 & s_2 & s_3 & -1-r_1-s_2-s_3 = \theta_5 \\ 1 & r_1 & s_2 & r_3 & -1-r_1-s_2-r_3 = \theta_6 \\ 1 & r_1 & r_2 & s_3 & -1-r_1-r_2-s_3 = \theta_7 \\ 1 & r_1 & r_2 & r_3 & -1-r_1-r_2-r_3 = \theta_8 \end{pmatrix}.$$

Since G_4 is strongly regular, many of the eigenvalues θ_j should coincide, and/or some of the multiplicities $m_j = \dim V_j$ ($j = 0, 1, \dots, 8$) should be zero (in which case some of the rows in P are deleted), where V_j is the eigenspace corresponding to the eigenvalue θ_j of G_4 (more precisely, V_1 is the intersection of the restricted eigenspaces of s_1 (of A_1), s_2 (of A_2), and s_3 (of A_3), etc.). On the other hand, by Lemma 1 at least five of the multiplicities m_i must be positive (one of them being $m_0 = 1$); and if exactly five multiplicities are positive, then the four graphs form a 4-class association scheme.

Now we assume without loss of generality that $r_1 - s_1 \geq r_2 - s_2 \geq r_3 - s_3 \geq r_4 - s_4 > 0$. Assume first that $r_1 - s_1 = r_2 - s_2 = r_3 - s_3 \geq r_4 - s_4 > 0$. Then $\theta_1 > \theta_2 = \theta_3 = \theta_5 > \theta_4 = \theta_6 = \theta_7 > \theta_8$. Since we must have two distinct values (r_4 and s_4) among the θ_j with positive multiplicity m_j (of which there are at least four), there are a few possibilities.

- The positive multiplicities are m_1, m_2, m_3, m_5. After removing the rows of P corresponding to the zero eigenspaces (V_4, V_6, V_7, V_8), this gives eigenmatrix

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_3 & k_4 \\ 1 & s_1 & s_2 & s_3 & \theta_1 = r_4 \\ 1 & s_1 & s_2 & r_3 & \theta_2 = s_4 \\ 1 & s_1 & r_2 & s_3 & \theta_3 = s_4 \\ 1 & r_1 & s_2 & s_3 & \theta_5 = s_4 \end{pmatrix}.$$

  From the eigenmatrix of this association scheme it follows that fusing any two of the strongly regular graphs gives another strongly regular graph. From this it easily follows that the association scheme is amorphic (case (i)). Similarly the case where the positive multiplicities are m_4, m_6, m_7, m_8 leads to an amorphic association scheme (with the roles of the r_i and s_i interchanged).
- The positive multiplicities are m_1, m_4, m_6, m_7. But now $r_4 - s_4 = \theta_1 - \theta_4 = r_2 - s_2 + r_3 - s_3 > r_4 - s_4$, which is a contradiction. Similarly the case where the positive multiplicities are m_2, m_3, m_5, m_8 leads to a contradiction.
- The positive multiplicities are among $m_2, m_3, m_5, m_4, m_6, m_7$. After removing the rows of P corresponding to the zero eigenspaces V_1 and V_8, we obtain eigenmatrix

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_3 & k_4 \\ 1 & s_1 & s_2 & r_3 & \theta_2 = r_4 \\ 1 & s_1 & r_2 & s_3 & \theta_3 = r_4 \\ 1 & s_1 & r_2 & r_3 & \theta_4 = s_4 \\ 1 & r_1 & s_2 & s_3 & \theta_5 = r_4 \\ 1 & r_1 & s_2 & r_3 & \theta_6 = s_4 \\ 1 & r_1 & r_2 & s_3 & \theta_7 = s_4 \end{pmatrix},$$

  which is of the form (2) (case (iii)). From this we derive that $r_4 - s_4 = r_1 - s_1 = r_2 - s_2 = r_3 - s_3$. Now the multiplicities $m_2, \ldots, m_7$ can be expressed easily in terms of the other parameters. For example, m_7 is the dimension of the intersection of the restricted eigenspaces of r_1 and r_2, hence $m_7 = \frac{vs_1s_2-(k_1-s_1)(k_2-s_2)}{(r_1-s_1)(r_2-s_2)}$, according to Lemma 2.

  Now suppose that in this case we have an association scheme. Then two of the multiplicities $m_2, \ldots, m_7$ must be zero. Without loss of generality we may take one of these to be m_7. Since, again by Lemma 2, any eigenvalue r_i has a common eigenvector with s_j, $j \neq i$, it follows that the other zero multiplicity must be m_2. But in this case the eigenmatrix of the association scheme would be singular, which is a contradiction.

  Hence, in this case at most one of the multiplicities $m_2, \ldots, m_7$ can be zero, and we do not have an association scheme. We remark that this case is symmetric with respect to the roles of the r_i and s_i.

Next, assume that $r_1 - s_1 = r_2 - s_2 > r_3 - s_3 \geq r_4 - s_4 > 0$. Then $\theta_1 > \theta_2 > \theta_3 = \theta_5 > \theta_4 = \theta_6 > \theta_7 > \theta_8$. Since at least four multiplicities must be positive, these must be m_3, m_5, m_4, m_6. Hence this case would give an association scheme. However, the corresponding eigenmatrix is singular, which is a contradiction.

Finally, we assume that $r_1 - s_1 > r_2 - s_2 \geq r_3 - s_3 \geq r_4 - s_4 > 0$. Then $\theta_1 > \theta_2 \geq \theta_3 > \theta_5, \theta_4 > \theta_6 \geq \theta_7 > \theta_8$. Also here at least 4 multiplicities must be positive, which gives the following possibilities.

- The positive multiplicities are m_2, m_3, m_6, m_7, and $\theta_2 = \theta_3$ and $\theta_6 = \theta_7$. But then $r_4 - s_4 = \theta_2 - \theta_6 = r_1 - s_1 > r_4 - s_4$, which is a contradiction.
- The positive multiplicities are m_4, m_5, m_6, m_7, and $\theta_4 = \theta_5$ and $\theta_6 = \theta_7$. From this it follows that $r_1 - s_1 = r_2 - s_2 + r_3 - s_3$ and $r_2 - s_2 = r_3 - s_3 = r_4 - s_4$.

  Since s_1 and s_j have no common eigenvector, for $j = 2, 3, 4$, we find that $vr_1r_j = (k_1 - r_1)(k_j - r_j)$, for $j = 2, 3, 4$ (see Lemma 2). From this we can derive that

  $$\frac{r_2}{k_2} = \frac{r_3}{k_3} = \frac{r_4}{k_4}.$$

  Moreover, it follows that

  $$m_4 = \frac{vr_1s_j - (k_1 - r_1)(k_j - s_j)}{(r_1 - s_1)(r_j - s_j)} = \frac{vs_hs_j - (k_h - s_h)(k_j - s_j)}{(r_h - s_h)(r_j - s_j)} \quad \text{for}$$
  $$j, h = 2, 3, 4,\ j \neq h.$$

  From this it follows that

  $$vr_1(s_i - s_j) = (k_1 - r_1)(k_i - s_i - (k_j - s_j)) \quad \text{and}$$
  $$vs_h(s_i - s_j) = (k_h - s_h)(k_i - s_i - (k_j - s_j))$$

  for $\{i, j, h\} = \{2, 3, 4\}$. Since the signs of the right hand sides of these two equations are the same, while the signs of the left hand sides are opposite, it follows that in one of the equations both sides must be zero. If $k_i - s_i \neq k_j - s_j$, then $k_1 - r_1 = 0$, and then $s_i = s_j$, but then $k_h = s_h$, a contradiction. Thus $k_i - s_i = k_j - s_j$, and then also $s_i = s_j$, and $k_i = k_j$. Hence $k_2 = k_3 = k_3, s_2 = s_3 = s_4$, and also $r_2 = r_3 = r_4$. Now the eigenmatrix is of the form (1) (case (ii)).
- The last case, the one with positive multiplicities m_2, m_3, m_4, m_5 is similar to the previous one, and leads to the eigenmatrix (1) (case (ii)), where the r_i and s_i are interchanged. □

Examples of commutative strongly regular decompositions with four graphs which are not association schemes can be constructed as follows. Let $GF(3^{2m})$ be the vertex set, where m is even, and let α be a primitive element in this field. Consider the cyclotomic amorphic 4-class association scheme on $GF(3^{2m})$. Two distinct vertices are adjacent in G_j if their difference is of the form α^{4i+j} for some i ($j = 1, \dots, 4$). The graphs $G_1, \dots, G_4$ are (isomorphic) strongly regular graphs of negative Latin square type with valency $\frac{3^{2m}-1}{4}$, and restricted eigenvalues $\frac{3^m-1}{4}$ and $\frac{3^m-1}{4} - 3^m$. The union of G_2 and G_4 is the Paley graph on 3^{2m} vertices: two vertices are adjacent if their difference is a square.

Next, let $d = 3^m + 1$, and consider the cyclotomic amorphic d-class association scheme on $GF(3^{2m})$. Two distinct vertices are adjacent in H_j if their difference is of the form α^{di+j} for some i ($j = 1, \ldots, d$). The graphs $H_1, \ldots, H_d$ are (disconnected) strongly regular graphs of Latin square type with valency $3^m - 1$, and restricted eigenvalues $3^m - 1$ and -1. The union of $H_2, H_4, \ldots, H_d$ is the same as the union of G_2 and G_4 (the Paley graph on 3^{2m} vertices).

It is easy to show that all graphs G_i and H_j (i.e., their adjacency matrices) commute, hence $\{G_1, G_3, H_2, H_4, \ldots, H_d\}$ is a commutative strongly regular decomposition of a complete graph. Moreover, so is any of its fusions of the form $\{G_1, G_3, K_1, K_2\}$. Note that such a fusion consists of two strongly regular graphs of negative Latin square type (G_1 and G_3), and two of Latin square type (K_1 and K_2).

For $m = 2$ there are two possible fusions of $\{H_2, H_4, \ldots, H_{10}\}$ consisting of two graphs. For the corresponding decompositions $\{G_1, G_3, K_1, K_2\}$, the respective P-matrices (of the form (2) with one row deleted since $m_2 = 0$) are the following.

$$P = \begin{pmatrix} 1 & 20 & 20 & 8 & 32 \\ 1 & -7 & 2 & -1 & 5 \\ 1 & -7 & 2 & 8 & -4 \\ 1 & 2 & -7 & -1 & 5 \\ 1 & 2 & -7 & 8 & -4 \\ 1 & 2 & 2 & -1 & -4 \end{pmatrix} \quad \text{and} \quad P' = \begin{pmatrix} 1 & 20 & 20 & 16 & 24 \\ 1 & -7 & 2 & -2 & 6 \\ 1 & -7 & 2 & 7 & -3 \\ 1 & 2 & -7 & -2 & 6 \\ 1 & 2 & -7 & 7 & -3 \\ 1 & 2 & 2 & -2 & -3 \end{pmatrix}.$$

An example of a non-amorphic association scheme with four strongly regular graphs was already given by the wreath product of a complete graph and $L_{1,1,1}(2)$. Indeed, its eigenmatrix

$$P = \begin{pmatrix} 1 & v-4 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \end{pmatrix}$$

is of the form (1). We remark that this association scheme is a counterexample of A.V. Ivanov's conjecture [10, Problem 1.3] that any association scheme in which all graphs are strongly regular must be amorphic. Already in [5] counterexamples of this conjecture were found, and it was suggested that maybe Ivanov had intended to conjecture the above for primitive association schemes. We shall show next that also this weaker conjecture is false.

If a counterexample for the conjecture has four graphs, then the eigenmatrix must be of the form (1). We found by computer that there are only 7 "feasible" parameter sets for a primitive association scheme of the form (1) on at most 2048 vertices, the smallest one having 288 vertices. Besides these we found an infinite series of "feasible" parameter sets:

it has 8^t vertices and eigenmatrix

$$P = \begin{pmatrix} 1 & 2^{3t} - 4 - 3 \cdot 2^{2t} - 3 \cdot 2^t & 2^{2t} + 2^t + 1 & 2^{2t} + 2^t + 1 & 2^{2t} + 2^t + 1 \\ 1 & -4 - 3 \cdot 2^t & 1 + 2^t & 1 + 2^t & 1 + 2^t \\ 1 & -4 + 2^t & 1 - 2^t & 1 - 2^t & 1 + 2^t \\ 1 & -4 + 2^t & 1 - 2^t & 1 + 2^t & 1 - 2^t \\ 1 & -4 + 2^t & 1 + 2^t & 1 - 2^t & 1 - 2^t \end{pmatrix}.$$

We shall construct an association scheme which has this eigenmatrix for $t = 4$. It is obtained as a fusion scheme of the 45-class cyclotomic scheme on $GF(4096)$. Consider in this field a primitive element α satisfying $\alpha^{12} = \alpha^6 + \alpha^4 + \alpha + 1$. Two distinct vertices are adjacent in H_j if their difference is of the form α^{45i+j} for some i ($j = 1, \dots, 45$). De Lange [11] found that $G_2 = H_{45} \cup H_5 \cup H_{10}$ is a strongly regular graph with valency $k_2 = 273$ and restricted eigenvalues $r_2 = 17$ and $s_2 = -15$. Clearly $G_3 = H_{15} \cup H_{20} \cup H_{25}$ and $G_4 = H_{30} \cup H_{35} \cup H_{40}$ are isomorphic to G_2. Moreover, the union of these three graphs is one of the graphs in the 5-class cyclotomic amorphic association scheme on $GF(4096)$. Hence the complement G_1 of this union is strongly regular, and it has valency $k_1 = 3276$ and restricted eigenvalues $r_1 = 12$ and $s_1 = -52$. Since $r_1 - s_1 \neq r_2 - s_2$ it now follows from Theorem 5 that the four strongly regular graphs $G_1, \dots, G_4$ form a primitive 4-class association scheme which is not amorphic.

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