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Character Formulas for q-Rook Monoid Algebras

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Abstract. The q-rook monoid $R_n(q)$ is a semisimple $\mathbb{C}(q)$-algebra that specializes when $q \to 1$ to $\mathbb{C}[R_n]$, where R_n is the monoid of $n \times n$ matrices with entries from $\{0, 1\}$ and at most one nonzero entry in each row and column. We use a Schur-Weyl duality between $R_n(q)$ and the quantum general linear group $U_q\mathfrak{gl}(r)$ to compute a Frobenius formula, in the ring of symmetric functions, for the irreducible characters of $R_n(q)$. We then derive a recursive Murnaghan-Nakayama rule for these characters, and we use Robinson-Schensted-Knuth insertion to derive a Roichman rule for these characters. We also define a class of standard elements on which it is sufficient to compute characters. The results for $R_n(q)$ specialize when $q = 1$ to analogous results for R_n.

Keywords: rook monoid, character, Hecke algebra, symmetric functions

0. Introduction

The rook monoid R_n is the monoid of $n \times n$ matrices with entries from $\{0, 1\}$ and at most one nonzero entry in each row and column (these correspond with the possible placements of nonattacking rooks on an $n \times n$ chessboard). It contains an isomorphic copy of the symmetric group S_n as the rank n (permutation) matrices. The q-rook monoid $R_n(q)$ is an "Iwahori-Hecke algebra" of R_n. It is a semisimple $\mathbb{C}(q)$-algebra so that when $q \to 1$, $R_n(q)$ specializes to the complex monoid algebra $\mathbb{C}[R_n]$. Recently, the representation theory of $R_n(q)$ was analyzed. Solomon [20] found a faithful action of $R_n(q)$ on tensor space. Halverson [10] showed that $R_n(q)$ and the quantum general linear group are in Schur-Weyl duality and found explicit combinatorial constructions for the irreducible $R_n(q)$-representations.

In this paper we study the combinatorics of $R_n(q)$-characters. First, we use Schur-Weyl duality to prove the following identity in the ring of symmetric functions

$$q_\mu(1, x_1, \ldots, x_r; q) = \sum_{k=0}^{n} \sum_{\lambda \vdash k} \chi^{\lambda}_{R_n(q)}(T_\mu) s_\lambda(x_1, \ldots, x_r). \tag{0.1}$$

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Here q_μ is a q-analog of the power sum symmetric function p_μ, s_λ is the Schur function, and $\chi^\lambda_{R_n(q)}(T_\mu)$ is the irreducible character of $R_n(q)$ indexed by λ and evalauated at a certain element T_μ. This is a generalization of the Frobenius formula of Ram [15] for the Iwahori-Hecke algebra $H_n(q)$ of the symmetric group S_n, which in turn is a generalization of Frobenius' [5] original formula from 1900,

$$p_\mu(x_1, \ldots, x_r) = \sum_{\lambda \vdash n} \chi^\lambda_{S_n}(\mu) s_\lambda(x_1, \ldots, x_r). \tag{0.2}$$

Here $\chi^\lambda_{S_n}(\mu)$ is the irreducible character of S_n indexed by λ and evaluated on the conjugacy class with cycle type μ.

We use our Frobenius formula to derive two combinatorial methods for computing $\chi^\lambda_{R_n(q)}(T_\mu)$:

(1) We give a recursive rule for computing $\chi^\lambda_{R_n(q)}(T_\mu)$ by removing broken border strips from λ. This rule is an analog of the Murnaghan-Nakayama rule for S_n characters, which was generalized to $H_n(q)$-characters in [15].
(2) We give a rule for computing $\chi^\lambda_{R_n(q)}(T_\mu)$ as weighted sums of standard tableaux. This rule is a generalization of Roichman's rule [17] for the irreducible characters of $H_n(q)$.

We use our Frobenius formula to show that the character table of $R_n(q)$, denoted $\Xi_{R_n(q)}$, is of the form

$$\Xi_{R_n(q)} = \begin{array}{|c|c|} \hline \Xi_{R_{n-1}(q)} & 0 \\ \hline * & \Xi_{H_n(q)} \\ \hline \end{array}, \tag{0.3}$$

where $\Xi_{R_{n-1}(q)}$ is the character table of $R_{n-1}(q)$ and $\Xi_{H_n(q)}$ is the character table of $H_n(q)$. The elements in $*$ are explicitly determined by either our Murnaghan-Nakayama rule or our Roichman rule.

The characters of the rook monoid R_n $(q = 1)$ were originally studied in the 1950s by Munn [14], who writes R_n characters in terms of S_k characters with $0 \le k \le n$. As an example, Munn produces the character table of R_4. In the Appendix we produce the character table of $R_4(q)$. Setting $q = 1$ in our table gives Munn's table, exactly. Munn also determines a "cycle-link" type for the elements of R_n, and he shows that R_n-characters are constant on cycle-link classes. In Section 5, we show that the irreducible $R_n(q)$-characters are completely determined by their values on the set of standard elements $T_\mu, \mu \vdash k, 0 \le k \le n$. Our element T_μ specializes at $q = 1$ to a rook element with "cycle-link" type μ, and we show how to use "rook diagrams" determine cycle-link type.

Solomon [19] determined yet another way to compute $R_n(q = 1)$ characters. He writes the R_n character table as a product $AY = YB$ where Y is a block diagonal matrix whose blocks are the characters of the symmetric groups S_k, $0 \le k \le n$, and A and B are matrices that can be computed combinatorially.

The q-rook monoid was first introduced by Solomon [18] as an analog of the Iwahori-Hecke algebra for the finite algebraic monoid $\mathrm{M}_n(\mathbb{F}_q)$ of $n \times n$ matrices over a finite field with q elements with respect to its "Borel subgroup" of invertible upper triangular matrices. In [20], Solomon gives a presentation of $R_n(q)$ and defines a faithful action of $R_n(q)$ on tensor space. In [8], Halverson and A. Ram show that $R_n(q)$ is a quotient of the Iwahori-Hecke algebra of type B_n and prove that $R_n(q)$ is semisimple over $\mathbb{C}$ whenever $[n]! \neq 0$, where $[n]! = [n][n-1]\cdots[1]$ and $[k] = q^{k-1} + q^{k-2} + \cdots + 1$.

1. q-Rook monoid algebras

1.1. The rook monoid

Let S_n denote the group of permutations of the set $\{1, 2, \ldots, n\}$. Identify $\sigma \in S_n$ with the matrix having a 1 in the (i, j)-position if $\sigma(i) = j$. For $1 \le i \le n-1$, let $s_i \in S_n$ be the transposition of i and $i+1$.

The *rook monoid* R_n is the monoid of $n \times n$ matrices having entries from $\{0,1\}$ with *at most* one nonzero entry in each row and column. There are $\binom{n}{k}^2 k!$ matrices in R_n having rank k, and thus

$$|R_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k!. \tag{1.1}$$

We have $S_n \subseteq R_n$ as the rank n matrices.

Let $E_{i,j}$ be the $n \times n$ matrix unit with a 1 in the (i, j)-position and 0s everywhere else. In R_n, define

$$\begin{aligned} \nu &= E_{1,2} + E_{2,3} + \cdots + E_{n-1,n}, & & \\ \pi_j &= E_{j+1,j+1} + E_{j+2,j+2} + \cdots + E_{n,n}, & & 1 \le j \le n-1 \\ \varepsilon_j &= I_n - E_{j,j}, & & 1 \le j \le n, \end{aligned} \tag{1.2}$$

where I_n is the identity matrix. Let π_n be the zero matrix, and note that $\pi_1 = \varepsilon_1$. Munn [14] shows that the complex monoid algebra

$$\mathbb{C}[R_n] = \left\{ \sum_{x \in R_n} \alpha_x x \mid \alpha_x \in \mathbb{C} \right\}$$

is semisimple. Note that π_n is the zero matrix but it is not the zero element in $\mathbb{C}[R_n]$ (the zero element in $\mathbb{C}[R_n]$ is the linear combination with $\alpha_x = 0$ for all x).

1.2. The q-rook monoid

Let q be an indeterminate. For $n \geq 2$, define the *q-rook monoid* $R_n(q)$ to be the associative $\mathbb{C}(q)$-algebra with generators $1, T_1, \ldots, T_{n-1}, P_1, \ldots, P_n$ and defining relations

$$\begin{array}{lll}
(A1) & T_i^2 = q \cdot 1 + (q-1)T_i, & \text{for } 1 \leq i \leq n-1, \\
(A2) & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i \leq n-2, \\
(A3) & T_i T_j = T_j T_i, & \text{when } |i-j| \geq 2, \\
(A4) & T_i P_j = P_j T_i = q P_j, & \text{for } 1 \leq i < j \leq n, \\
(A5) & T_i P_j = P_j T_i, & \text{for } 1 \leq j < i \leq n-1, \\
(A6) & P_i^2 = P_i, & \text{for } 1 \leq i \leq n, \\
(A7) & P_{i+1} = q P_i T_i^{-1} P_i, & \text{for } 2 \leq i \leq n.
\end{array} \tag{1.3}$$

Define $R_0(q) = \mathbb{C}(q)$ and define $R_1(q)$ to be the associative $\mathbb{C}(q)$-algebra spanned by 1 and P_1 subject to $P_1^2 = P_1$. The subalgebra of $R_n(q)$ generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra of type $H_n(q)$ of type A_{n-1} (see [20] or [10] for a proof that they can be identified).

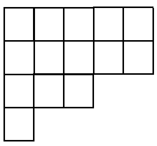
Solomon defined $R_n(q)$ in [18] and gave it a presentation in [20]. The presentation (1.3) is proved in [10]. Solomon [18, 20] shows that $R_n(q)$ is semisimple with dimension

$$\dim(R_n(q)) = \sum_{k=0}^{n} \binom{n}{k}^2 k!. \tag{1.4}$$

When $q \to 1$, $R_n(q)$ specializes to $\mathbb{C}[R_n]$. Under this specialization, we have $T_i \to s_i$ and $P_i \to \pi_i$.

1.3. Partitions and tableaux

We use the notation of [12] for partitions and compositions. A composition λ of the positive integer n, denoted $\lambda \models n$, is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ such that $|\lambda| = \lambda_1 + \cdots + \lambda_t = n$. The composition λ is a partition, denoted $\lambda \vdash n$, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$. The length $\ell(\lambda)$ is the number of nonzero parts of λ. The Young diagram of a partition λ is the left-justified array of boxes with λ_i boxes in the ith row. We let $m_i(\lambda)$ be the number of parts of λ equal to i, and we sometimes write $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots)$. For example,

if $\lambda = (5, 5, 3, 1) = (1, 3, 5^2) =$ , then $|\lambda| = 14$ and $\ell(\lambda) = 4$.

If λ is a partition with $0 \leq |\lambda| \leq n$, then we say that an *n-standard tableau of shape* λ is a filling of the diagram of λ with numbers from $\{1, 2, \ldots, n\}$ such that

(1) each number from $\{1, 2, \ldots, n\}$ appears in λ at most once,
(2) the rows of λ increase from left to right, and
(3) the columns of λ increase from top to bottom.

Similarly, an *n-column strict tableaux of shape* λ is the same as an n-standard tableau except that we allow the rows to weakly increase. Thus,

$$\begin{array}{|c|c|c|c|}\hline 2 & 3 & 9 & 10 \\ \hline 5 & 7 & 12 & 15 \\ \hline 6 & 11 & 13 \\ \cline{1-3} \end{array} \quad \text{is a 16-standard tableau of shape } (4, 4, 3)$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 3 & 3 & 8 & 8 \\ \hline 8 & 8 & 9 \\ \cline{1-3} \end{array} \quad \text{is a 16-column strict tableau of shape } (4, 4, 3).$$

1.4. Irreducible representations

The irreducible representations of $R_n(q)$ and $\mathbb{C}[R_n]$ are indexed by partitions in the set

$$\Lambda_n = \{\lambda \vdash k \mid 0 \leq k \leq n\}. \tag{1.5}$$

For $\lambda \in \Lambda_n$, we let M^λ be the irreducible $\mathbb{C}[R_n]$-module indexed by λ and let $\chi^\lambda_{R_n}$ be its character, and we let M^λ_q be the irreducible $\mathbb{C}[R_n]$-module indexed by λ and let $\chi^\lambda_{R_n(q)}$ be its character. The dimensions of M^λ and M^λ_q are given by

$$\dim(M^\lambda) = \dim\left(M^\lambda_q\right) = \#(n\text{-standard tableaux of shape } \lambda) = \binom{n}{|\lambda|} f_\lambda, \tag{1.6}$$

where f_λ is the number of $|\lambda|$-standard tableaux of shape λ given by the hook formula (see [21], Theorem 3.10.2).

The R_n-module M^λ is studied in [6, 14, 19]. In [6], C. Grood determines the analog of Young's natural basis for M^λ. In [10], analogs of Young's seminormal bases of both M^λ_q and M^λ are constructed, and the action of the generators of $R_n(q)$ and R_n on this basis are described explicitly.

1.5. Standard elements

Define $\gamma_1 = T_{\gamma_1} = 1$ and

$$\left.\begin{aligned} \gamma_t &= s_1 s_2 \cdots s_{t-1}, \\ T_{\gamma_t} &= T_1 T_2 \cdots T_{t-1}, \end{aligned}\right. \quad \text{for } 2 \leq t \leq n. \tag{1.7}$$

For a composition $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ with $|\mu| = k$ and $0 \le k \le n$, define

$$\begin{aligned} \gamma_\mu &= \gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_\ell}, \\ T_{\gamma_\mu} &= T_{\gamma_{\mu_1}} \otimes T_{\gamma_{\mu_2}} \otimes \cdots \otimes T_{\gamma_{\mu_\ell}}, \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} d_\mu &= \pi_{n-k} \otimes \gamma_\mu, \\ T_\mu &= P_{n-k} \otimes T_{\gamma_\mu}, \end{aligned} \tag{1.9}$$

where we view $T_\mu = P_k \otimes T_{\gamma_{\mu_1}} \otimes \cdots \otimes T_{\gamma_{\mu_\ell}} \in R_k(q) \otimes R_{\mu_1}(q) \otimes \cdots \otimes R_{\mu_\ell}(q) \subseteq R_n(q)$. For example, if $n = 15$ and $\mu = (5, 3, 2, 2)$, then

$$T_\mu = P_3(T_4T_5T_6T_7)(T_9T_{10})(T_{12})(T_{14}).$$

In [15] it is shown that $H_n(q)$-characters are completely determined by their value on T_{γ_μ}. In Section 5 we show that characters of $R_n(q)$ and $\mathbb{C}[R_n]$ are completely determined by their values on T_μ and d_μ. Since both the irreducible representations and the standard elements are indexed by Λ_n, we see that the character table is square with these labels.

When $q \to 1$, we have $R_n(q) \to \mathbb{C}[R_n]$ with $T_\mu \to d_\mu$. Furthermore, in [10], we construct M_q^λ so that $M_1^\lambda = M^\lambda$, and the action of T_μ specializes at $q = 1$ to the action of d_μ. It follows that the characters also specialize upon setting $q = 1$,

$$\chi^\lambda_{R_n(q)}(T_\mu)|_{q=1} = \chi^\lambda_{R_n}(d_\mu). \tag{1.10}$$

2. A Frobenius formula for the q-rook monoid

In this section, we use the Schur-Weyl duality between $R_n(q)$ and the quantum general linear group $U_q\mathfrak{gl}(r)$ to derive a Frobenius formula for the irreducible characters of $R_n(q)$.

We define $U_q\mathfrak{gl}(r)$ as in Jimbo [11], except with his parameter q replaced by $q^{1/2}$. Let $U_q\mathfrak{gl}(r)$ be the $\mathbb{C}(q^{1/4})$-algebra given by generators

$$e_i,\ f_i\ (1 \le i < r) \quad \text{and} \quad q^{\pm\varepsilon_i/2}\ (1 \le i \le n),$$

with relations

$$q^{\varepsilon_i/2}q^{\varepsilon_j/2} = q^{\varepsilon_j/2}q^{\varepsilon_i/2}, \quad q^{\varepsilon_i/2}q^{-\varepsilon_i/2} = q^{-\varepsilon_i/2}q^{\varepsilon_i/2} = 1,$$

$$q^{\varepsilon_i/2}e_jq^{-\varepsilon_i/2} = \begin{cases} q^{-\frac{1}{2}}e_j, & \text{if } j = i-1, \\ q^{\frac{1}{2}}e_j, & \text{if } j = i, \\ e_j, & \text{otherwise,} \end{cases} \qquad q^{\varepsilon_i/2}f_jq^{-\varepsilon_i/2} = \begin{cases} q^{\frac{1}{2}}f_j, & \text{if } j = i-1, \\ q^{-\frac{1}{2}}f_j, & \text{if } j = i, \\ f_j, & \text{otherwise,} \end{cases}$$

$$e_if_j - f_je_i = \delta_{ij}\frac{q^{\frac{1}{2}(\varepsilon_i - \varepsilon_{i+1})} - q^{-\frac{1}{2}(\varepsilon_i - \varepsilon_{i+1})}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

$$
\begin{aligned}
&e_{i\pm1}e_i^2 - \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)e_ie_{i\pm1}e_i + e_i^2e_{i\pm1} = 0,\\
&f_{i\pm1}f_i^2 - \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)f_if_{i\pm1}f_i + f_i^2f_{i\pm1} = 0,\\
&e_ie_j = e_je_i, \qquad f_if_j = f_jf_i, \quad \text{if } |i-j| > 1.
\end{aligned}
$$

Define

$$
t_i = q^{\frac{\varepsilon_i}{4}} \ (1 \le i \le r) \qquad k_i = t_it_{i+1}^{-1} \ (1 \le i \le r-1).
$$

There is a Hopf algebra structure (see [11], p. 248) on $U_q\mathfrak{gl}(r)$ with comultiplication Δ and counit u given by

$$
\begin{aligned}
\Delta(e_i) &= e_i \otimes k_i^{-1} + k_i \otimes e_i, \qquad & u(e_i) &= 0,\\
\Delta(f_i) &= f_i \otimes k_i^{-1} + k_i \otimes f_i, \qquad & u(f_i) &= 0,\\
\Delta(t_i) &= t_i \otimes t_i, & u(t_i) &= 1.
\end{aligned}
\tag{2.1}
$$

2.1. *Representations and characters of $U_q\mathfrak{gl}(r)$*

Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{gl}(r)$, and let $\varepsilon_1, \ldots, \varepsilon_r$ be an orthonormal basis for $\mathfrak{h}^*$ with respect to an inner product (,). The weight lattice is $L = \sum_{i=1}^r \mathbb{Z}\varepsilon_i$, and the dominant integral weights are of the form

$$
\lambda = m_1\varepsilon_1 + \cdots + m_r\varepsilon_r, \qquad m_i \in \mathbb{Z},\ m_1 \ge m_2 \ge \cdots \ge m_r.
$$

We identify the dominant weight λ with the sequence $(\lambda_1, \ldots, \lambda_r)$, and we let $V_q(\lambda)$ denote the irreducible $U_q\mathfrak{gl}(r)$-module with dominant weight λ (see [2], Section 10.1, for example).

Any finite dimensional $U_q\mathfrak{gl}(r)$-module V has a basis B consisting of weight vectors, where, for each $b \in B$, there exists $\mathrm{wt}(b) \in L$ such that

$$
t_ib = q^{\frac{1}{4}(\varepsilon_i, \mathrm{wt}(b))}b, \quad 1 \le i \le r.
$$

Let $x_1, \ldots, x_r$ be indeterminates, and define the character of V to be

$$
\mathrm{ch}(V) = \sum_{b\in B} x^{\mathrm{wt}(b)},
\tag{2.2}
$$

where if $\mathrm{wt}(b) = a_1\varepsilon_1 + \cdots + a_r\varepsilon_r$, then $x^{\mathrm{wt}(b)} = x_1^{a_1}\cdots x_r^{a_r}$. It is known (see [2], Proposition 10.1.5) that $\mathrm{ch}(V_q(\lambda))$ is the same as the corresponding character of $\mathfrak{gl}(r)$, and so it is given by the Weyl denominator formula. Thus, when λ is a partition, the character of $V_q(\lambda)$ is given by the Schur function,

$$
\mathrm{ch}(V_q(\lambda)) = s_\lambda(x_1, \ldots, x_r) = \frac{\det\left(x_i^{\lambda_j+r-j}\right)}{\det\left(x_i^{r-j}\right)}.
\tag{2.3}
$$

2.2. The bitrace

If V is a finite-dimensional $U_q\mathfrak{gl}(r)$-module and $Z = End_{U_q\mathfrak{gl}(r)}(V)$ is its centralizer algebra, then define, for each $\phi \in Z$,

$$btr(\phi) = \sum_{b\in B} x^{\mathrm{wt}(b)}(\phi b|_b), \tag{2.4}$$

where B is a weight basis of V (a basis consisting of weight vectors) and $\phi b|_b$ is the coefficient of b in ϕb. For $\mu \in L$, let V_μ denote the μ-weight space of V. Then, since Z commutes with $U_q\mathfrak{gl}(r)$, we know that Z preserves weight spaces, and so by summing over weight spaces we get

$$btr(\phi) = \sum_{\mu\in L} \dim(V_\mu)x^\mu tr_{V_\mu}(\phi).$$

where $tr_{V_\mu}(\phi)$ is the trace of ϕ on V_μ. In particular $btr(\phi)$ is a weighted sum of usual traces, and it satisfies the trace property, $btr(\phi_1\phi_2) = btr(\phi_2\phi_1)$ for all $\phi_1, \phi_2 \in Z$.

Now, by double centralizer theory (see for example [3], Section 3D), we have a decomposition of the form

$$V \cong \bigoplus_\lambda V_q(\lambda) \otimes Z^\lambda$$

where Z^λ is an irreducible Z-module and the sum is over the highest weights λ for which $V_q(\lambda)$ is a constituent of V. For each module $V_q(\lambda) \otimes Z^\lambda$ we choose a basis $\{b_i^\lambda \otimes z_j^\lambda\}$, where $B_\lambda = \{b_i\}$ is a weight basis of $V_q(\lambda)$ and $\{z_j^\lambda\}$ is a basis of Z^λ. The bitrace becomes

$$btr(\phi) = \sum_\lambda \sum_{b\in B_\lambda} x^{\mathrm{wt}(b)} \sum_j \phi z_j^\lambda|_{z_j^\lambda} = \sum_\lambda \mathrm{ch}(V_q(\lambda))\chi_Z^\lambda(\phi). \tag{2.5}$$

Here $\phi z_j^\lambda|_{z_j^\lambda}$ is the coefficient of z_j^λ in ϕz_j^λ, and $\chi_Z^\lambda(\phi) = \sum_j \phi z_j^\lambda|_{z_j^\lambda}$ is the character of Z^λ evaluated at ϕ. We thank Arun Ram for suggesting this derivation of (2.5).

2.3. Schur-Weyl duality

The "fundamental" r-dimensional $U_q\mathfrak{gl}(r)$-module $V = V_q((1)) = V_q(\omega_1)$ is the vector space

$$V = \mathbb{C}(q^{1/4})\text{-span}\{v_1, \ldots, v_r\}$$

(so that the symbols v_i form a basis of V) with $U_q\mathfrak{gl}(r)$-action given by (see [11], Proposition 1)

$$e_i v_j = \begin{cases} v_{j+1}, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \qquad f_i v_j = \begin{cases} v_{j-1}, & \text{if } j = i+1, \\ 0, & \text{if } j \neq i+1, \end{cases}$$

and

$$t_i v_j = \begin{cases} q^{1/4} v_j, & \text{if } j = i, \\ v_j, & \text{if } j \neq i. \end{cases}$$

The "trivial" 1-dimensional $U_q\mathfrak{gl}(r)$-module $W = V_q(\emptyset)$ is the vector space

$$W = \mathbb{C}(q^{1/4})\text{-span}\{v_0\}$$

(so that the symbol v_0 is a basis of W) with $U_q\mathfrak{gl}(r)$-action given by the counit u

$$e_i v_0 = f_i v_0 = 0 \quad \text{and} \quad t_i v_0 = v_0.$$

Let $U = V \oplus W$ so that U has basis $v_0, v_1, \dots, v_r$. The coproduct on $U_q\mathfrak{gl}(r)$ is coassociative, so we can form the n-fold tensor product representation $U^{\otimes n}$. The simple tensors $v_{i_1} \otimes \cdots \otimes v_{i_n}$ form a basis for $U^{\otimes n}$, *i.e.*,

$$U^{\otimes n} = \mathbb{C}(q)\text{-span}\left\{v_{i_1} \otimes \cdots \otimes v_{i_n} \;\middle|\; 0 \leq i_j \leq n\right\}.$$

Define an action of $R_n(q)$ on $U^{\otimes n}$ as follows. The action of a generator T_k, $1 \leq k \leq n-1$, and P_j, $1 \leq j \leq n$, on a simple tensor $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_n}$ in $U^{\otimes n}$ is given by

$$\begin{aligned} T_k \mathbf{v} &= \begin{cases} q\mathbf{v}, & \text{if } i_k = i_{k+1}, \\ (q-1)\mathbf{v} + q^{1/2} s_k \mathbf{v}, & \text{if } i_k < i_{k+1}, \\ q^{1/2} s_k \mathbf{v}, & \text{if } i_k > i_{k+1}. \end{cases} \\ P_j \mathbf{v} &= \begin{cases} \mathbf{v}, & \text{if } i_1 = i_2 = \cdots = i_j = 0, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{2.6}$$

where s_k acts on $\mathbf{v}$ by place permutation,

$$s_k\big(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}} \otimes \cdots \otimes v_{i_n}\big) = v_{i_1} \otimes \cdots \otimes v_{i_{k+1}} \otimes v_{i_k} \otimes \cdots \otimes v_{i_n}.$$

Solomon [20] first proved that (2.6) extends to an action of $R_n(q)$ on tensor space, although he used a different generator N in place of the P_i, and he proved that the action is faithful when $r \geq n$.

Halverson [10] proved that $R_n(q)$ commutes with $U_q\mathfrak{gl}(r)$ on $U^{\otimes n}$, and so if $r \geq n$, we have $R_n(q) \cong End_{U_q\mathfrak{gl}(r)}(U^{\otimes n})$. Furthermore, [10] shows that $U^{\otimes n}$ decomposes into irreducibles as

$$U^{\otimes n} \cong \bigoplus_{k=0}^{n} \bigoplus_{\lambda \vdash k} V_q(\lambda) \otimes M_q^\lambda \tag{2.7}$$

as a bimodule for $U_q\mathfrak{gl}(r) \otimes R_n(q)$. Here, $V_q(\lambda)$ is the irreducible $U_q\mathfrak{gl}(r)$-module of highest weight λ, and M_q^λ is the irreducible $R_n(q)$-module corresponding to λ.

2.4. A Frobenius formula

Putting together (2.3), (2.5) and (2.7), proves

Proposition 2.1 *For all $h \in R_n(q)$, we have*

$$btr(h) = \sum_{k=0}^{n} \cdot \sum_{\lambda \vdash k} s_\lambda(x_1, \ldots, x_r)\chi^\lambda_{R_n(q)}(h),$$

where $\chi^\lambda_{R_n(q)}$ is the irreducible $R_n(q)$ character labeled by λ.

Let $n = n_1 + n_2$, $d_1 \in R_{n_1}(q)$, and $d_2 \in R_{n_2}(q)$. Then the bitrace of $d_1 \otimes d_2 \in R_n(q)$ on $U^{\otimes n}$ satisfies $btr(d_1 \otimes d_2) = btr(d_1)btr(d_2)$, where $btr(d_i)$ is the bitrace of d_i on $U^{\otimes n_i}$ (the proof is identical to that in [7] Section 5, since d_1 acts on the first n_1 tensor slots and d_2 acts on the last n_2 tensor slots). Thus if $\mu = (\mu_1, \ldots, \mu_\ell)$ is a composition with $0 \le |\mu| \le n$, and T_μ is defined as in (1.9), then

$$btr(T_\mu) = btr(P_{n-k})btr\big(T_{\mu_1}\big) \cdots btr\big(T_{\mu_\ell}\big). \tag{2.8}$$

As in [15], let $q_0(x_0, x_1, \ldots, x_r; q) = 1$, and for a positive integer k define

$$q_k(x_0, x_1, \ldots, x_r; q) = \sum_{I=(i_1,\ldots,i_k)} q^{e(I)}(q-1)^{\ell(I)} x_{i_1} \cdots x_{i_k}, \tag{2.9}$$

where the sum is over all weakly increasing sequences $I = (0 \le i_1 \le \cdots \le i_k \le r)$, $e(I)$ is the number of $i_j \in I$ such that $i_j = i_{j+1}$, and $\ell(I)$ is the number of $i_j \in I$ such that $i_j < i_{j+1}$. For a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, define

$$q_\mu = q_{\mu_1} q_{\mu_2} \cdots q_{\mu_\ell}. \tag{2.10}$$

Proposition 2.2
(a) *The bitrace of T_{γ_k} on $U^{\otimes k}$ is $btr(T_{\gamma_k}) = q_k(x_0, x_1, \ldots, x_r; q)$.*
(b) *The bitrace of P_k on $U^{\otimes k}$ is $btr(P_k) = 1$.*
(c) *For a composition μ with $0 \le |\mu| \le n$, the bitrace of T_μ on $U^{\otimes n}$ is*

$$btr(T_\mu) = q_\mu(x_0, \ldots, x_r; q).$$

Proof: Recall from Section 2.3, that $t_i v_0 = v_0$, and for $1 \le j \le r$, $t_j v_j = q^{\frac{1}{4}} v_j$ and $t_i v_j = v_j$ if $i \ne j$. Let $x_0 = 1$. Then

$$x^{\mathrm{wt}(v_0)} = 1 = x_0 \qquad \text{and} \qquad x^{\mathrm{wt}(v_j)} = x^{\varepsilon_j} = x_j, \quad 1 \le j \le r,$$

so the simple tensors $v_{i_1} \otimes \cdots \otimes v_{i_n}$ form a weight basis of $U^{\otimes n}$ satisfying

$$x^{\mathrm{wt}(v_{i_1} \otimes \cdots \otimes v_{i_n})} = x_{i_1} \cdots x_{i_n}.$$

Now, the proof of (a) is exactly as the proof of [15], Theorem 4.1. For (b), we have $P_k(v_{i_1}\cdots v_{i_k}) = 0$ unless $i_1 = \cdots = i_n = 0$, and $P_k(v_0\cdots v_0) = v_0\cdots v_0$. Part (c) follows from (a), (b), and (2.8). $\square$

Combining Propositions 2.1 and 2.2(c), we have the following Frobenius formula for $R_n(q)$.

Theorem 2.3 *Let μ be a composition with $0 \le |\mu| \le n$. Then*

$$q_\mu(1, x_1, \ldots, x_r; q) = \sum_{k=0}^{n}\sum_{\lambda\vdash k} \chi^\lambda_{R_n(q)}(T_\mu)s_\lambda(x_1,\ldots,x_r)$$

where T_μ is defined in (1.9) *and $\chi^\lambda_{R_n(q)}$ is the irreducible $R_n(q)$-character labeled by λ.*

We saw in (1.10) that upon setting $q = 1$ we have $\chi^\lambda_{R_n(q)}(T_\mu)|_{q=1} = \chi^\lambda_{R_n}(d_\mu)$. Furthermore, it is easy to see that $q_\mu(x_0, x_1, \ldots, x_r; 1) = p_\mu(x_0, x_1, \ldots, x_r)$, since when $q = 1$ in (2.9) we must have $i_1 = i_2 = \cdots = i_k$. Thus, setting $q = 1$ in Theorem 2.3, gives

Theorem 2.4 *Let μ be a composition with $0 \le |\mu| \le n$. Then*

$$p_\mu(1, x_1, \ldots, x_r) = \sum_{k=0}^{n}\sum_{\lambda\vdash k} \chi^\lambda_{R_n}(d_\mu)s_\lambda(x_1,\ldots,x_r),$$

where d_μ is defined in (1.9) *and $\chi^\lambda_{R_n}$ is the irreducible R_n-character indexed by λ.*

The next corollary (of Theorem 2.3) tells us that the character table of $R_n(q)$ has the form shown in (0.3).

Corollary 2.5 *Let $\lambda \in \Lambda_n$ and let μ be a composition with $0 \le |\mu| \le n$, then*
(a) *if $|\lambda| > |\mu|$, then $\chi^\lambda_{R_n(q)}(T_\mu) = 0$.*
(b) *if $|\lambda| \le |\mu|$, then $\chi^\lambda_{R_n(q)}(T_\mu) = \chi^\lambda_{R_{|\mu|}(q)}(T_{\gamma_\mu})$.*

Proof: From Theorem 2.3, we see that

$$\sum_{k=0}^{n}\sum_{\lambda\vdash k} \chi^\lambda_{R_n(q)}(T_\mu)s_\lambda(x_1,\ldots,x_r) = \sum_{k=0}^{|\mu|}\sum_{\lambda\vdash k} \chi^\lambda_{R_{|\mu|}(q)}\big(T_{\gamma_\mu}\big)s_\lambda(x_1,\ldots,x_r),$$

since each side of this equation equals $q_\mu(x_0, x_1, \ldots, x_r; q)$. This is an identity in the ring of symmetric functions, and the Schur functions are linearly independent, so the corollary follows from equating the coefficient of s_λ on both sides. In particular, when $|\lambda| > |\mu|$ the coefficient of s_λ on the right side is 0, proving part (a). $\square$

3. Murnaghan-Nakayama rules

If λ and μ are partitions, we say that $\mu \subseteq \lambda$ if $\mu_i \le \lambda_i$ for each i. The skew shape λ/ν consists of the boxes that are in λ and not in μ. Two boxes in λ/μ are adjacent if they share a common edge, and λ/ν is connected if you can travel from any box to any other via a path of adjacent boxes. A skew shape λ/ν is a *broken border strip* (bbs) if it does not contain any 2×2 blocks of boxes, and a broken border strip is a *border strip* if it is a single connected component. Each broken border strip λ/ν contains $cc(\lambda/\nu)$ connected components (border strips).

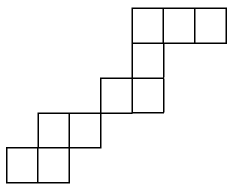
The width and height of a border strip b are defined, respectively, by

$$\begin{aligned} w(b) &= (\text{the number of columns that } b \text{ occupies}) - 1, \\ h(b) &= (\text{the number of rows that } b \text{ occupies}) - 1. \end{aligned} \tag{3.1}$$

For a skew shape λ/ν, we define

$$wt_{\lambda/\nu}(q) = \begin{cases} (q-1)^{cc(\lambda/\nu)-1} \prod_b q^{w(b)}(-1)^{h(b)}, & \text{if } \lambda/\nu \text{ is a bbs,} \\ 0, & \text{otherwise;} \end{cases} \tag{3.2}$$

where the product is over the connected components (border strips) b in λ/ν. For example

$(7, 5, 5, 3, 2)/(4, 4, 3, 1) =$ 

is a broken border strip consisting of two connected components b_1 and b_2 with $w(b_1) = 2, h(b_1) = 1$ and $w(b_2) = 3, h(b_2) = 2$. Thus its weight is $(q-1)q^2(-1)q^3(-1)^2 = -(q-1)q^5$.

A key step in proving the Murnaghan-Nakayama rule for $H_n(q)$ is the following proposition [15] (see also [7]), which is a q-analog of [12], Section 3, Example 11(2),

Proposition 3.1 (Ram [15]) *If* $\nu \vdash (n-k)$, *then*

$$q_k(x_1, \ldots, x_r; q) s_\nu(x_1, \ldots, x_r) = \sum_{\lambda \vdash n} wt_{\lambda/\nu}(q) s_\lambda(x_1, \ldots, x_r),$$

where q_k *is defined in* (2.9), s_ν *is the Schur function, and the sum is over all partitions* λ *such that* λ/ν *is a broken border strip of size* k.

To extend this result to our setting, we first expand $q_t(1, x_1, x_2, \ldots, x_r; q)$ in terms of $q_k(x_1, x_2, \ldots, x_r; q)$.

Lemma 3.2 *For $t \geq 0$, we have*

$$q_t(1, x_1, \dots, x_r; q) = \sum_{k=0}^{t} f_{k,t}(q) q_k(x_1, \dots, x_r; q),$$

where

$$f_{k,t}(q) = \begin{cases} q^{t-1}, & \text{if } k = 0, \\ (q-1)q^{t-k-1}, & \text{if } 0 < k < t, \\ 1, & \text{if } k = t. \end{cases} \tag{3.3}$$

Proof: By definition $q_t(1, x_1, \dots, x_r; q) = \sum_I q^{e(I)}(q-1)^{\ell(I)} x_{i_1}, \dots, x_{i_k}$, where the sum is over all sequences $I = (i_1, \dots, i_t)$ of the form $0 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq r$. We let K represent the subsequence of I containing all the strictly positive terms in I, and let $k = |K|$.

Now we sum the terms in $q_t(1, x_1, \dots, x_r; q)$ according to k. The terms with $k = t$ contribute

$$\sum_{K=(i_1,\dots,i_t)} q^{e(K)}(q-1)^{\ell(K)} x_{i_1}, \dots, x_{i_t} = q_t(x_1, \dots, x_r; q),$$

since $1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq r$. The terms with $0 < k < t$ each have $t-k-1$ equalities between 0s and one jump from a 0 subscript to a nonzero subscript. Thus, they contribute

$$\begin{aligned} &(q-1)\sum_{k=1}^{t-1} q^{t-k-1} \sum_{K=(i_{t-k+1},\dots,i_t)} q^{e(K)}(q-1)^{\ell(K)} x_{i_{t-k+1}}, \dots, x_{i_t} \\ &\quad = (q-1)\sum_{k=1}^{t-1} q^{t-k-1} q_k(x_1, \dots, x_r; q). \end{aligned}$$

Finally, there is one term with $k = 0$. It has the form

$$q^{t-1} x_0, \dots, x_0 = q^{t-1} q_0(x_1, \dots, x_r; q).$$

Summing these three cases gives the desired result. □

Proposition 3.3 *If $\nu \in \Lambda_{n-t}$, then*

$$q_t(1, x_1, \dots, x_r; q) s_\nu(x_1, \dots, x_r) = \sum_{\lambda \in \Lambda_n} f_{|\lambda/\nu|,t}(q) wt_{\lambda/\nu}(q) s_\lambda(x_1, \dots, x_r),$$

where the nonzero terms in this sum are over the partitions $\lambda \in \Lambda_n$ such that λ/ν is a broken border strip with $0 \leq |\lambda/\nu| \leq t$.

Proof: By Proposition 3.1 and Lemma 3.2, if $\nu \in \Lambda_{n-t}$, we have

$$\begin{aligned} q_t(1, x_1, \ldots, x_r; q)s_\nu(x_1, \ldots, x_r) &= \sum_{k=0}^{t} f_{k,t}(q)q_k(x_1, \ldots, x_r; q)s_\nu(x_1, \ldots, x_r) \\ &= \sum_{k=0}^{n} f_{k,t}(q) \sum_{\lambda \vdash (|\nu|+k)} wt_{\lambda/\nu}(q)s_\lambda(x_1, \ldots, x_r). \end{aligned}$$

□

We now are ready to derive a Murnaghan-Nakayama rule for computing the irreducible characters of $R_n(q)$.

Theorem 3.4 *Let $\lambda \in \Lambda_n$ and let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a composition with $0 \leq |\mu| \leq n$. Let $\mu_\ell = t$ and $\bar{\mu} = (\mu_1, \ldots, \mu_{\ell-1})$. Then*

$$\chi^\lambda_{R_n(q)}(T_\mu) = \sum_{\nu \in \Lambda_{n-t}} f_{|\lambda/\nu|,t}(q)wt_{\lambda/\nu}(q)\chi^\nu_{R_{n-t}(q)}(T_{\bar{\mu}}),$$

where $wt_{\lambda/\nu}(q)$ is defined in (3.2) and $f_{k,t}(q)$ is defined in (3.3). The nonzero terms in this sum correspond to partitions $\nu \in \Lambda_{n-t}$ such that λ/ν is a broken border strip with $0 \leq |\lambda/\mu| \leq t$.

Proof: From Theorem 2.3 and Proposition 3.3, we have

$$\begin{aligned} &\sum_{\lambda \in \Lambda_n} \chi^\lambda_{R_n(q)}(T_\mu)s_\lambda(x_1, \ldots, x_r) \\ &\quad = q_\mu(1, x_1, \ldots, x_r; q) \\ &\quad = q_{\bar{\mu}}(1, x_1, \ldots, x_r; q)q_t(1, x_1, \ldots, x_r; q) \\ &\quad = \sum_{\nu \in \Lambda_{n-t}} \chi^\nu_{R_{n-t}(q)}(T_{\bar{\mu}})s_\nu(x_1, \ldots, x_r)q_t(1, x_1, \ldots, x_r; q) \\ &\quad = \sum_{\nu \in \Lambda_{n-t}} \chi^\nu_{R_{n-t}(q)}(T_{\bar{\mu}}) \sum_{\lambda \in \Lambda_n} f_{|\lambda/\nu|,t}(q)wt_{\lambda/\nu}(q)s_\lambda(x_1, \ldots, x_r) \\ &\quad = \sum_{\lambda \in \Lambda_n} \left(\sum_{\nu \in \Lambda_{n-t}} \chi^\nu_{R_{n-t}(q)}(T_{\bar{\mu}})f_{|\lambda/\nu|,t}(q)wt_{\lambda/\nu}(q) \right) s_\lambda(x_1, \ldots, x_r). \end{aligned}$$

Now compare coefficients of the s_λ, which are a basis in the ring of symmetric functions. □

When $q = 1$, definitions (3.3) and (3.2) become

$$f_{k,t}(1) = \begin{cases} 1, & \text{if } k = 0 \text{ or } k = t, \\ 0, & \text{otherwise,} \end{cases} \tag{3.4}$$

$$wt_{\lambda/\nu}(1) = \begin{cases} (-1)^{h(\lambda/\nu)}, & \text{if } \lambda/\nu \text{ is a border strip,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

It follows that the Murnaghan-Nakayama rule for the rook monoid is

Theorem 3.5 *Let $\lambda \in \Lambda_n$ and let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a composition with $0 \leq |\mu| \leq n$. Let $\mu_\ell = t$ and $\bar{\mu} = (\mu_1, \ldots, \mu_{\ell-1})$. Then*

$$\chi^{\lambda}_{R_n}(d_\mu) = \sum_{\nu \in \Lambda_{n-t}} (-1)^{h(\lambda/\nu)} \chi^{\nu}_{R_{n-t}}(d_{\bar{\mu}}),$$

where the sum is over partitions $\nu \in \Lambda_{n-t}$ such that either $\nu = \lambda$ or λ/ν is a border strip of size t.

4. Robinson-Schensted-Knuth insertion and Roichman weights

Fix $r \geq n$. For a partition $\mu = (\mu_1, \ldots, \mu_\ell) \in \Lambda_n$ define $B(\mu)$ to be the set of partial sums of μ so that

$$B(\mu) = \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_\ell\}. \tag{4.1}$$

For $\mu \vdash k$, define the μ-weight of $x_{i_1}, \ldots, x_{i_k}$, with $0 \leq i_j \leq r$, to be

$$wt_\mu\big(x_{i_1}, \ldots, x_{i_k}\big) = \prod_{\substack{j=1 \\ j \notin B(\mu)}}^{k} \phi_\mu\big(j, x_{i_1}, \ldots, x_{i_n}\big), \tag{4.2}$$

where

$$\phi_\mu\big(j, x_{i_1}, \ldots, x_{i_k}\big) = \begin{cases} -1 & \text{if } i_j < i_{j+1}, \\ 0, & \text{if } i_j \geq i_{j+1} \text{ and } i_{j+1} < i_{j+2} \text{ and } i_{j+1} \notin B(\mu), \\ q, & \text{otherwise.} \end{cases}$$

Proposition 4.1 ([16]) *We have $q_\emptyset = 1$, and for $\mu \vdash k$ with $1 \leq k \leq n$, we have*

$$q_\mu(x_0, x_1, \ldots, x_r; q) = \sum_{x_{i_1}, \ldots, x_{i_k}} wt_\mu\big(x_{i_1}, \ldots, x_{i_k}\big) x_{i_1}, \ldots, x_{i_k},$$

where the sum is over all words $x_{i_1}, \ldots, x_{i_k}$ with $0 \leq i_j \leq r$.

Let $\lambda \in \Lambda_n$ and recall our definition, in Section 1.3, of an n-standard tableau Q_λ of shape λ. In this section we will place the numbers that are missing from Q_λ in a standard tableau of shape $(n - |\lambda|)$ to the right of λ. Thus, our example from Section 1.3 becomes,

$$Q_\lambda = \left(\begin{array}{|c|c|c|c|}\hline 2 & 3 & 9 & 10 \\ \hline 5 & 7 & 12 & 15 \\ \hline 6 & 11 & 13 & \\ \cline{1-3} \end{array}, \begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 8 & 14 & 16 \\ \hline \end{array} \right) \text{ is a 16-standard tableau of shape (4, 4, 3).}$$

In this way we identify n-standard tableaux with ordered pairs of standard tableaux, such that the second tableau is a single row, and there is a total of n boxes. We write $Q_\lambda = (Q_\lambda^{(1)}, Q_\lambda^{(2)})$ where $Q_\lambda^{(1)}$ is the original tableau and $Q_\lambda^{(2)}$ is the single row of "missing" entries.

In a similar fashion we identify n-column strict tableau P_λ with an ordered pair of column-strict tableau, but such that the second tableau is a single row of length $n - |\lambda|$ containing all 0s. Thus, our example from Section 1.3 becomes

$$P_\lambda = \left(\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 3 & 3 & 8 & 8 \\ \hline 8 & 8 & 9 & \\ \cline{1-3} \end{array}, \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \right) \text{ is a 16-column strict tableau of shape (4,4,3).}$$

We write $P_\lambda = (P_\lambda^{(1)}, P_\lambda^{(2)})$ where $P_\lambda^{(1)}$ is the original tableau and $P_\lambda^{(2)}$ is the single row of 0s. The Schur function can be written as

$$s_\lambda(x_1, \ldots, x_r) = \sum_{P_\lambda} x_1^{m_1(P_\lambda)} x_2^{m_2(P_\lambda)}, \ldots, x_r^{m_r(P_\lambda)}, \tag{4.3}$$

where the sum is over all r-column strict tableaux of shape λ and $m_i(P_\lambda)$ is the number of times that i appears in P_λ.

For an n-standard tableau Q_λ of shape λ define

$$wt_\mu(Q_\lambda) = \prod_{\substack{j=1 \\ j \notin B(\mu)}}^{k} \psi_\mu(j, Q_\lambda), \tag{4.4}$$

where $B(\mu)$ is as defined in (4.1) and

$$\psi_\mu(j, Q_\lambda) = \begin{cases} -1, & \text{if } j+1 \text{ is southwest of } j \text{ in } Q_\lambda, \\ 0, & \text{if } j+1 \text{ is northeast of } j \text{ in } Q_\lambda \text{ and } j+2 \text{ is} \\ & \quad \text{southwest of } j+1 \text{ in } Q_\lambda \text{ and } j+1 \notin B(\mu), \\ q, & \text{otherwise.} \end{cases}$$

Here, by "southwest" we mean south (below) and/or west (left), by "northeast" we mean north (above) and/or east (right) or both. Furthermore, we consider the entries of $Q_\lambda^{(1)}$ to be southwest of those in $Q_\lambda^{(2)}$. Notice that $j+1$ cannot be south and east of j in a standard tableau. For example, in the standard tableau Q_λ above, 2 is southwest of 1, 3 is northeast of 2, 4 is northeast of 3, 5 is southwest of 4, etc.

The Robinson-Schensted-Knuth (RSK) correspondence (see [21], Section 3.2) is an algorithm which gives a bijection between sequences $x_{i_1}, \ldots, x_{i_n}$, with $1 \le i_j \le m$, and pairs (P, Q) where P is a column-strict tableaux, Q is a standard tableau, and P and Q have shape λ for some partition λ with n boxes. The RSK algorithm constructs the pair of tableaux (P, Q) iteratively,

$$(\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), \ldots, (P_n, Q_n) = (P, Q),$$

in such a way that

(1) P_j is a column strict tableau that contains j boxes, and Q_j is a standard tableau that has the same shape as P_j,
(2) P_j is obtained from P_{j-1} by *column* inserting i_j into P_{j-1}, denoted $P_j = P_{j-1} \leftarrow i_j$, as follows
   (a) Insert i_j into the first column of P_{j-1} by displacing the smallest number $\ge i_j$; if every number is $< i_j$, add i_j to the bottom of the first column,
   (b) If i_j displaces x from the first column, insert x into the second column using the rules of (a),
   (c) Repeat for each subsequent column until a number is added to the bottom of some (possibly empty) column,
(3) Q_j is obtained from Q_{j-1} by putting j in the newly added box (*i.e.*, the box created in going from P_{j-1} to P_j).

The standard tableau Q is called the recording tableau.

We modify RSK insertion to work for n-standard and n-column strict tableaux. Given a sequence $x_{i_1}, \ldots, x_{i_n}$ with $0 \le i_j \le r$, our insertion scheme constructs a sequence $(\emptyset, \emptyset) = (P_0, Q_0), \ldots, (P_n, Q_n) = (P, Q)$, where $P_i = (P_i^{(1)}, P_i^{(2)})$ is an i-semistandard tableaux, $Q_i = (Q_i^{(0)}, Q_i^{(1)})$ is an i-standard tableaux, and P_i and Q_i have the same shape. Our insertion rule uses usual column insertion in one of the components according to the following rule:

$$\left(P_{j-1}^{(1)}, P_{j-1}^{(2)}\right) \leftarrow i_j = \begin{cases} \left(P_{j-1}^{(1)} \leftarrow i_j, P_{j-1}^{(2)}\right), & \text{if } i_j > 0, \\ \left(P_{j-1}^{(1)}, P_{j-1}^{(2)} \leftarrow i_j\right), & \text{if } i_j = 0. \end{cases}$$

That is, insert 0s into the second tableau and insert nonzero numbers into the first tableau, using usual column insertion in both cases. Again, we construct Q_i from Q_{i-1} by putting i into the new box added in the ith step. For example, the result of inserting x_2, x_1, x_0, x_2, x_0 is

$$P_i : (\emptyset,\emptyset),\ \left(\boxed{2}, \emptyset\right),\ \left(\boxed{1\,|\,2}, \emptyset\right),\ \left(\boxed{1\,|\,2}, \boxed{0}\right),\ \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \boxed{0}\right),\ \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \boxed{0\,|\,0}\right)$$

$$Q_i : (\emptyset,\emptyset),\ \left(\boxed{1}, \emptyset\right),\ \left(\boxed{1\,|\,2}, \emptyset\right),\ \left(\boxed{1\,|\,2}, \boxed{3}\right),\ \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \boxed{3}\right),\ \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \boxed{3\,|\,5}\right).$$

Two well-known properties of RSK insertion are

(4) If $i_j < i_{j+1}$ and $P_{j+1} = (P_{j-1} \leftarrow i_j) \leftarrow i_{j+1}$, then $j+1$ is southwest of j in Q_{j+1},
(5) If $i_j \geq i_{j+1}$ and $P_{j+1} = (P_{j-1} \leftarrow i_j) \leftarrow i_{j+1}$, then $j+1$ is southwest of j in Q_{j+1},

In our modified insertion, we always keep the 0s to the right (northeast) of the nonzero numbers, so properties (1) and (2) still hold. By property (2), the second tableau in our insertion will always be a single row. Thus, our insertion establishes a bijection between sequences $x_{i_1}, \ldots, x_{i_n}$, with $0 \leq i_j \leq r$, and pairs (P, Q), where P is an n-column strict tableau, and Q is an n-standard tableau, each having shape $\lambda \vdash k$ with $0 \leq k \leq n$ (note that k is the number of nonzero subscripts in $x_{i_1}, \ldots, x_{i_n}$). Furthermore, it follows from (4.2), (4.4), and properties (4) and (5) of RSK insertion that this bijection is weight preserving, i.e., for all $\mu \in \Lambda_n$,

$$\begin{aligned}&\text{if } (P, Q) \text{ results from inserting } x_{i_1}, \ldots, x_{i_n}, \text{ then}\\ &wt_\mu\big(x_{i_1}, \ldots, x_{i_n}\big) = wt_\mu(Q) \text{ and } x^P = x_{i_1}, \ldots, x_{i_n}.\end{aligned} \tag{4.5}$$

Proposition 4.2 *For each $\mu \in \Lambda_n$, we have*

$$q_\mu(x_0, \ldots, x_r; q) = \sum_{\lambda \in \Lambda_n} \left(\sum_{Q_\lambda} wt_\mu(Q_\lambda) \right) s_\lambda(x_1, \ldots, x_r)$$

where the inner sum is over all n-standard tableaux of shape λ.

Proof: Using (4.5) and Proposition 4.1, we have

$$\begin{aligned} q_\mu(x_0, \ldots, x_r; q) &= \sum_{x_{i_1}, \ldots, x_{i_n}} wt_\mu\big(x_{i_1}, \ldots, x_{i_n}\big) x_{i_1}, \ldots, x_{i_n} \\ &= \sum_{\lambda \in \Lambda_n} \sum_{P_\lambda, Q_\lambda} wt_\mu(Q_\lambda) x^{P_\lambda} \\ &= \sum_{\lambda \in \Lambda_n} \left(\sum_{Q_\lambda} wt_\mu(Q_\lambda) \right) \left(\sum_{P_\lambda} x^{P_\lambda} \right) \\ &= \sum_{\lambda \in \Lambda_n} \left(\sum_{Q_\lambda} wt_\mu(Q_\lambda) \right) s_\lambda(x_1, \ldots, x_r), \end{aligned}$$

where the sums are over all n-column strict tableaux P_λ of shape λ and all n-standard tableaux Q_λ of shape λ. □

By comparing coefficients of the s_λ, which are linearly independent in the ring of symmetric functions, in Theorem 2.3 and Proposition 4.2, we get the following theorem. It is a generalization of Roichman's rule [17] for $H_n(q)$.

Theorem 4.3 *For $\mu, \lambda \in \Lambda_n$, we have*

$$\chi^{\lambda}_{R_n(q)}(T_\mu) = \sum_{Q_\lambda} wt_\mu(Q_\lambda),$$

where the sum is over all n-standard tableaux of shape λ.

All of our calculations work when $q = 1$, and so we have

Corollary 4.4 *Let $\mu \in \Lambda_n$. Then*
(1) $p_\mu(1, x_1, \dots, x_r) = \sum_{\lambda\in\Lambda_n}(\sum_{Q_\lambda} wt_\mu(Q_\lambda))s_\lambda(x_1, \dots, x_r)$,
(2) *For each $\lambda \in \Lambda_n$, $\chi^{\lambda}_{R_n}(d_\mu) = \sum_{Q_\lambda} wt_\mu(Q_\lambda)$,*
where in each case Q_λ varies over all n-standard tableaux of shape λ and wt_μ is computed as in (4.4) *with $q = 1$.*

5. Standard elements

We now show that $R_n(q)$-characters are completely determined by their values on the standard elements T_μ. In doing so, we show that $R_n(q)$ satisfies a basic construction similar to the partition algebra [9]. In Section 5.2, we define rook diagrams and use them to show how to explicitly "conjugate" elements of R_n to get standard elements d_μ.

5.1. Standard elements in $R_n(q)$

The elements $s_1, \dots, s_{n-1}$ generate S_n, and a reduced word for $w \in S_n$ is a product $w = s_{i_1}, \dots, s_{i_k}$ with k minimal. Given a reduced word $w = s_{i_1}s_{i_2}, \dots, s_{i_k} \in S_n$, let $T_w = T_{i_1}T_{i_2}, \dots, T_{i_k} \in H_n(q)$. The element T_w is well-defined (independent of choice of the reduced word for w), and the elements T_w, $w \in S_n$, form a basis of $H_n(q)$. Furthermore,

Theorem 5.1 (Ram [15]) *The characters of $H_n(q)$ are completely determined by their values on the set $\{T_{\gamma_\mu} \mid \mu \vdash n\}$.*

The proof in [15] of Theorem 5.1 shows that for any $w \in S_n$ there exists $a_{w\mu} \in \mathbb{Z}[q]$ such that $T_w = \sum_{\mu\vdash n} a_{w\mu}T_{\gamma_\mu}$ In [7] we give a new basis $\{L_w \mid w \in S_n\}$ for $H_n(q)$ such that for any character χ, we have $\chi(L_w) = \chi(T_{\gamma_\mu})$, where μ is the cycle type of the permutation w.

For $1 \le i \le n$, define $T_{i,i} = 1$, and define

$$T_{i,j} = T_{j-1}T_{j-2}, \dots, T_i, \quad \text{for } 1 \le i < j \le n.$$

Let $A = \{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n\}$, and assume that $a_1 < a_2 < \cdots < a_k$. Define

$$T_A = T_{1,a_1}T_{2,a_2}, \dots, T_{k,a_k}.$$

Now for $0 \leq k \leq n$, define,

$$\Omega_k = \left\{ (A, B, w) \;\middle|\; \begin{array}{l} A, B \subseteq \{1, 2, \ldots, n\}, |A| = |B| = k, \\ w \in S_{\{k+1,\ldots,n\}}, \end{array} \right\}$$

where $S_{\{k+1,\ldots,n\}}$ is the symmetric group of permutations of $\{k+1, \ldots, n\}$. Define

$$T_{(A,B,w)} = T_A T_w P_k T_B^{-1}, \quad (A, B, w) \in \Omega_k.$$

Then let $\Omega = \bigcup_{k=0}^{n} \Omega_k$, and we have

Theorem 5.2 ([10]) *The set* $\{T_{(A,B,w)} \mid (A, B, w) \in \Omega\}$ *is a* $\mathbb{C}(q)$*-basis of* $R_n(q)$.

The following relations are easy to verify in $R_n(q)$

$$T_i^{-1} = q^{-1} T_i + (q^{-1} - 1) \cdot 1, \tag{5.1}$$

$$(T_1 T_2, \ldots, T_\ell) T_j = T_{j+1}(T_1 T_2, \ldots, T_\ell), \; 1 \leq j < \ell. \tag{5.2}$$

$$P_i P_j = P_{\max(i,j)}, \tag{5.3}$$

$$P_{i+1} = P_i T_i P_i - (q-1) P_i. \tag{5.4}$$

Lemma 5.3 *Let* A_{n-1} *be the subalgebra of* $R_n(q)$ *generated by* $T_2, \ldots, T_{n-1}, P_2, \ldots, P_n$. *Then for each* $b \in R_n(q)$ *there exists* $a \in A_{n-1}$ *such that* $P_1 b P_1 = a P_1 = P_1 a$.

Proof: First note that P_1 commutes with $a \in A_{n-1}$ by (A4) in (1.3) and by (5.3). It is sufficient to prove the statement for $b = g_1 g_2, \ldots, g_k$ where for each i, we have $g_i \in \{T_1, \ldots, T_{n-1}, P_1, \ldots, P_n\}$. We show that $P_1 g_1 g_2, \ldots, g_k P_1 = a P_1$, $a \in A_{n-1}$ by induction on k. When $k = 1$, we have $P_1 g_1 P_1 = g_1 P_1$ if $g_1 \neq T_1, P_1$. Furthermore, $P_1 P_1 P_1 = P_1 \cdot 1$, and by (5.4),

$$P_1 T_1 P_1 = P_2 + (q-1) P_1 = (P_2 + (q-1) \cdot 1) P_1.$$

When $k > 1$, we can assume $g_1 \notin A_{n-1}$, otherwise $P_1(g_1, \ldots, g_k) P_1 = g_1 P_1(g_2, \ldots, g_k) P_1$, and we can apply induction to $P_1(g_2, \ldots, g_k) P_1$. If $g_1 = P_1$ we have $P_1(g_1 g_2, \ldots, g_k) P_1 = P_1(g_2, \ldots, g_k) P_1$, and we can again apply induction. Thus, we assume that $g_1 = T_1$. First we see that,

$$P_1(T_1 T_2, \ldots, T_k) P_1 = P_1 T_1 P_1 (T_2, \ldots, T_k) = (P_2 + (q-1) \cdot 1)(T_2, \ldots, T_k) P_1.$$

Now assume that for some $\ell \geq 1$, $b = T_1, \ldots, T_\ell g_{\ell+1}, \ldots, g_k$, $g_{\ell+1} \neq T_{\ell+1}$, and consider the possibilities for $g_{\ell+1}$. If $g_{\ell+1} = T_\ell$, then we can use relation (A1) in (1.3) to write $P_1 b P_1 = q P_1 b' P_1 + (q-1) P_1 b'' P_1$, where b' and b'' are both shorter words than b, so we can apply induction to each term. If $g_{\ell+1} = T_j$ with $j > \ell + 1$, then T_j commutes with all the elements to its left, and so it can be factored out making the word shorter. If $g_{\ell+1} = T_j$ with $j < \ell$, then by (5.2), $P_1(T_1, \ldots, T_\ell) T_j = T_{j+1} P_1 (T_1, \ldots, T_\ell)$ and again induction can be applied. If $g_{\ell+1} = P_j$, then by (5.4) and induction, $P_1(T_1, \ldots, T_\ell)P_j(g_{\ell+2}, \ldots, g_k)P_1 = P_1(T_1, \ldots, T_\ell)P_1P_jP_1(g_{\ell+2}, \ldots, g_k)P_1 = a_1P_1P_ja_2P_1 = a_1P_ja_2P_1$ for some $a_1, a_2 \in A_{n-1}$. □

Proposition 5.4 *The map $\rho : R_{n-1}(q) \to P_1R_n(q)P_1$ defined by*

$$\begin{aligned} \rho(T_i) &= P_1T_{i+1} = T_{i+1}P_1, && 1 \le i \le n-2, \\ \rho(P_i) &= P_{i+1}, && 1 \le i \le n-1. \end{aligned}$$

is an isomorphism.

Proof: Lemma 5.3 tells us that $P_1R_n(q)P_1 = P_1A_{n-1} = A_{n-1}P_1$, and since A_{n-1} is generated by T_i, $2 \le i \le n-1$, and P_i, $2 \le i \le n$, we see that ρ maps $R_{n-1}(q)$ onto $P_1R_n(q)P_1$. Since P_1 commutes with A_{n-1}, it is easy to check that $\rho(T_i)$ and $\rho(P_i)$ satisfy the same relations (1.3) as T_i and P_i, and thus ρ is a homomorphism. To see that it is one-to-one, we compare dimensions. When we specialize $q = 1$ in $P_1R_n(q)P_1$, the specialized algebra is the $\mathbb{C}$-span of the rook matrices of the form $\pi_1 d\pi_1$, where $d \in R_n$. These are all the matrices in R_n having their first row and first column equal to 0. There are $|R_{n-1}| = \dim(R_{n-1}(q))$ such matrices. Furthermore, under such a specialization, the dimension cannot go up. This is because $P_1R_n(q)P_1$ is generated by elements $P_1T_iP_1$, $P_1P_kP_1$ whose structure constants are well-defined (no poles) at $q = 1$ (see [4], Section 68.A). □

Theorem 5.5 *If χ is an irreducible character of $R_n(q)$, then χ is completely determined by its values on T_μ, $\mu \vdash k$, $0 \le k \le n$.*

Proof: The proof is by induction on n with the cases $n = 0, 1$ being trivial. Let $n > 1$ and let χ be a character of $R_n(q)$. It is sufficient to compute χ on a basis element $T_{(A,B,w)}$, $(A, B, w) \in \Omega$. If $|A| = |B| = 0$, then $T_{(A,B,w)} = T_w \in H_n(q)$ and by Theorem 5.1, $\chi(T_w)$ can be written in terms of the values $\chi(T_{\gamma_\mu})$, $\mu \vdash n$. If $|A| = |B| > 0$, then we use the trace property of χ and (5.3) to get

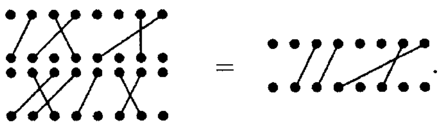
$$\chi\left(T_AT_wP_kT_B^{-1}\right) = \chi\left(T_B^{-1}T_AT_wP_k\right) = \chi\left(T_B^{-1}T_AT_wP_kP_1^2\right) = \chi\left(P_1T_B^{-1}T_AT_wP_kP_1\right).$$

Since $|A| = |B| > 0$, we have $P_1T_B^{-1}T_AT_wP_kP_1 \in P_1A_{n-1}P_1 \cong R_{n-1}(q)$. By induction, $\chi(P_1T_B^{-1}T_AT_wP_kP_1)$ can be written in terms of $\chi(\rho(T_\mu))$ where T_μ is a standard element in $R_{n-1}(q)$. Since ρ increases the subscripts of T_i and P_k by one and then multiplies by P_1, $\rho(T_\mu)$ is a standard element of $R_n(q)$. □

5.2. Rook diagrams

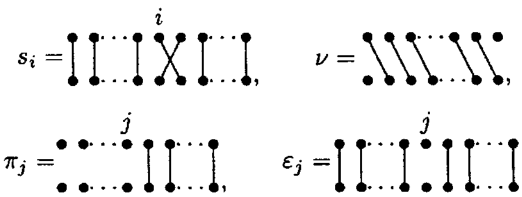
We say that a *rook diagram* is a graph on two rows of n vertices, having k edges with $0 \le k \le n$ such that each edge is adjacent to one vertex in each row, and each vertex is incident to at most one edge. We multiply two rook diagrams d_1 and d_2 by placing d_1 above

d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. For example,

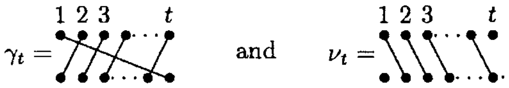


(5.5)

If we assign to each rook diagram d the $n \times n$, 0-1 matrix having a 1 in row i and column j if and only if the ith vertex in the top row of d is connected to the jth vertex in the bottom row, then this identification is an isomorphism with the rook monoid. Under this identification, we have, for $1 \leq i \leq n-1$ and $1 \leq j \leq n$,



Let $\gamma_1 = 1$ and let ν_1 be the diagram consisting of a single column of vertices with no edges. For $t > 1$, let



For a rook diagram d we compute the cycle and link type of d as follows: connect each vertex in the top row with the vertex directly below it by a dotted line. We call the connected components of this new diagram *blocks*. Each block is one of the following

(a) A *t-cycle* $(i_1, i_2, \ldots, i_t, i_1)$, with $1 \leq t \leq n$, where i_1 maps to i_2, i_2 maps to i_3, and so on until i_{t-1} maps to i_t and i_t maps to i_1.
(b) A *t-link* $[i_1, i_2, \ldots, i_t]$, with $1 \leq t \leq n$, where i_1 maps to i_2, i_2 maps to i_3, and so on until i_{t-1} maps to i_t. By definition, we let ν_1 be the 1-link (a single column of vertices with no edges).

The sizes of the cycles in d form a partition μ called the *cycle type* of d, and the sizes of the links form a partition τ called the *link type* of d. They satisfy $|\tau| + |\rho| = n$. For example, the diagram d_2 from (5.5) has cycle type (2) and link type (5, 1).

Two elements d_1 and d_2 of R_n are conjugate, written $d_1 \sim d_2$ if there exists $\pi \in S_n$ so that $\pi d_1 \pi^{-1} = d_2$. Notice that if d is a rook diagram, then $\pi d \pi^{-1}$ is the rook diagram given by rearranging the vertices of d, in both the top and bottom row, according to π. If d_1 and d_2 are rook diagrams in R_{n_1} and R_{n_2}, respectively, then $d_1 \otimes d_2$ is the rook diagram in $R_{n_1+n_2}$ obtained by placing d_2 to the right of d_1. It is easy to check that

(1) $\pi d \pi^{-1}$ has the same cycle-link type as d.
(2) $d \sim b_1 \otimes b_2 \otimes \cdots \otimes b_k$ where each b_i is a single block.
(3) $b_1 \otimes b_2 \sim b_2 \otimes b_1$
(4) Each t-cycle is conjugate to γ_t and each t-link is conjugate to ν_t.

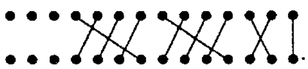
It follows that if d has cycle type $\mu = (\mu_1, \ldots, \mu_\ell)$ and link type $\tau = (\tau_1, \ldots, \tau_m)$, then

$$d \sim \nu_{\tau_1} \otimes \nu_{\tau_2} \otimes \cdots \otimes \nu_{\tau_m} \otimes \gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_\ell}. \tag{5.6}$$

For a composition $\mu = (\mu_1, \ldots, \mu_\ell)$ with $0 \leq |\mu| \leq n$, define

$$\begin{aligned} \gamma_\mu &= \gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_\ell}, \text{ and} \\ d_\mu &= \pi_{n-|\mu|} \otimes \gamma_\mu. \end{aligned} \tag{5.7}$$

For example, if $\mu = (4, 4, 2, 1) \vdash 11$ and $n = 14$, then

$d_\mu =$ 

This is the $q \to 1$ specialization of the element T_μ defined in (3.3). Using rook diagrams, we now provide a more streamlined proof to a result found in Munn [14].

Proposition 5.6 ([14]) *If χ is any character of $R_n(q)$, and d is a rook diagram with cycle type μ, then*

$$\chi(d) = \chi(d_\mu).$$

Proof: From (5.6) we know that $\chi(d) = \chi(d')$ where $d' = \nu_{\tau_1} \otimes \cdots \otimes \nu_{\tau_m} \otimes \gamma_\mu$. If d' has an isolated vertex (a vertex adjacent to no edges) in the kth position of the top row that is part of a link ν_t with $t > 1$, then

$$d' = \varepsilon_k d' \quad \text{and} \quad d\varepsilon_k = d''.$$

where in d'' the link ν_t is replaced by $\nu_{t-1} \otimes \nu_1$. Furthermore, by the trace property χ, we have $\chi(d') = \chi(\varepsilon_k d') = \chi(d'\varepsilon_k) = \chi(d'')$. Recursive application of this process, replaces all the t-links with the t-fold tensor product of ν_1, and the result is proved. $\square$

Appendix: Character table of $R_4(q)$

We index the rows of the character table for $R_n(q)$ by the irreducible representations and the columns by the standard elements. Thus, for $\lambda, \mu \in \Lambda_n$, we let the entry in the row indexed by λ and the column indexed by μ be $\chi^\lambda_{R_n(q)}(T_\mu)$. Below, we give the character table of $R_4(q)$. It contains the character tables of $R_3(q)$, $R_2(q)$, $R_1(q)$, and $R_0(q)$ as well as $H_4(q)$, $H_3(q)$, $H_2(q)$, $H_1(q)$, and $H_0(q)$ on the diagonal (see (0.3)). Upon setting $q = 1$ we obtain the character table of R_n first given in [14].

Character table of $R_4(q)$

	(1^4)	(21^3)	(2^2)	(31)	(4)
(1^4)	1	-1	1	1	-1
(21^3)	3	$q-2$	$1-2q$	$1-q$	q
(2^2)	2	$q-1$	q^2+1	$-q$	0
(31)	3	$2q-1$	$q(q-2)$	$q(q-1)$	$-q^2$
(4)	1	q	q^2	q^2	q^3
(1^3)	4	$q-3$	$2(1-q)$	$2-q$	$q-1$
(21)	8	$4(q-1)$	$2(q-1)^2$	$1-3q+q^2$	$q(1-q)$
(3)	4	$3q-1$	$2q(q-1)$	$q(2q-1)$	$q^2(q-1)$
(1^2)	6	$3(q-1)$	$1-4q+q^2$	$(q-1)^2$	$q(1-q)$
(2)	6	$2(2q-1)$	$1-2q+3q^2$	$2q(q-1)$	$q^2(q-1)$
(1)	4	$3q-1$	$2q(q-1)$	$q(2q-1)$	$q^2(q-1)$
$\emptyset$	1	q	q^2	q^2	q^3

	(1^3)	(21)	(3)	(1^2)	(2)	(1)	$\emptyset$
(1^4)	0	0	0	0	0	0	0
(21^3)	0	0	0	0	0	0	0
(2^2)	0	0	0	0	0	0	0
(31)	0	0	0	0	0	0	0
(4)	0	0	0	0	0	0	0
(1^3)	1	-1	1	0	0	0	0
(21)	2	$q-1$	$-q$	0	0	0	0
(3)	1	q	q^2	0	0	0	0
(1^2)	3	$q-2$	$1-q$	1	-1	0	0
(2)	3	$2q-1$	$q(q-1)$	1	q	0	0
(1)	3	$2q-1$	$q(q-1)$	2	$q-1$	1	0
$\emptyset$	1	q	q^2	1	q	1	1

Acknowledgments

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