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Type II Matrices and Their Bose-Mesner Algebras

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Abstract. Type II matrices were introduced in connection with spin models for link invariants. It is known that a pair of Bose-Mesner algebras (called a dual pair) of commutative association schemes are naturally associated with each type II matrix. In this paper, we show that type II matrices whose Bose-Mesner algebras are imprimitive are expressed as so-called generalized tensor products of some type II matrices of smaller sizes. As an application, we give a classification of type II matrices of size at most 10 except 9 by using the classification of commutative association schemes.

Keywords: type II matrix, spin model, Bose-Mesner algebra

1. Introduction

Throughout this paper $M[i, j]$ denotes the (i, j)-entry of a matrix M and $\boldsymbol{u}[h]$ denotes the h-th entry of a vector $\boldsymbol{u}$. Let M be an $m \times n$ matrix whose entries are all nonzero. We associate an $n \times m$ matrix M^- defined by the following.

$$M^-[i, j] = \frac{1}{M[j, i]}.$$

Let I denote the identity matrix and let J denote the all 1 square matrix of suitable size.

Let $\mathrm{Mat}_n(\boldsymbol{C})$ denote the set of $n \times n$ complex matrices. $W \in \mathrm{Mat}_n(\boldsymbol{C})$ is said to be a *type II matrix* if $WW^- = nI$. It is clear that if W is a type II matrix, then the transpose ${}^tW$ of the matrix and W^- are type II matrices as well. Hence for a matrix $W \in \mathrm{Mat}_n(\boldsymbol{C})$ whose entries are nonzero, we have the following.

$$\begin{aligned} WW^- = n \cdot I &\Leftrightarrow \sum_{h=1}^{n} \frac{W[i, h]}{W[j, h]} = \delta_{i,j} \cdot n \quad \text{for all } 1 \le i, j \le n \\ &\Leftrightarrow \sum_{h=1}^{n} \frac{W[h, i]}{W[h, j]} = \delta_{i,j} \cdot n \quad \text{for all } 1 \le i, j \le n \\ &\Leftrightarrow W^- W = n \cdot I. \end{aligned}$$

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The definition of type II matrices was first introduced explicitly in the study of *spin models*. See [1, 3, 4, 6–9, 13] for details.

Example 1.1

(1) Let ζ be a primitive n-th root of 1. Then the matrix $W = W(\boldsymbol{Z}_n) \in \mathrm{Mat}_n(\boldsymbol{C})$ defined by $W[i, j] = \zeta^{(i-1)(j-1)}$ is a type II matrix. $W(\boldsymbol{Z}_n)$ is called a *cyclic type II matrix* of size n.
(2) Let α be a root of the quadratic equation $t^2 + nt + n = 0$. Then the matrix $W \in \mathrm{Mat}_n(\boldsymbol{C})$ defined by $W[i, j] = 1 + \delta_{i,j}\alpha$ is a type II matrix. W is called a *Potts type II matrix* of size n.

Let $W \in \mathrm{Mat}_n(\boldsymbol{C})$ be a type II matrix. If $S, S' \in \mathrm{Mat}_n(\boldsymbol{C})$ are permutation matrices and $D, D' \in \mathrm{Mat}_n(\boldsymbol{C})$ are nonsingular diagonal matrices, then it is easy to see that $SDWD'S'$ is also a type II matrix (See Section 2). We say that two type II matrices W and W' are *type II equivalent* if $W' = SDWD'S'$ for suitable choices of permutation matrices S, S' and diagonal matrices D, D'. It is clear that this defines an equivalence relation on the set of type II matrices.

For a type II matrix $W \in \mathrm{Mat}_n(\boldsymbol{C})$ and for $1 \leq i, j \leq n$, we define an n-dimensional column vector $\boldsymbol{u}^W_{i,j}$ by the following.

$$\boldsymbol{u}^W_{i,j}[h] = \frac{W[h, i]}{W[h, j]}.$$

Let

$$\mathcal{N}(W) = \left\{ M \in \mathrm{Mat}_n(\boldsymbol{C}) \;\middle|\; \boldsymbol{u}^W_{i,j} \text{ is an eigenvector for } M \text{ for all } 1 \leq i, j \leq n \right\}.$$

It is known that $\mathcal{N}(W)$ is the Bose-Mesner algebra of a commutative association scheme. $\mathcal{N}(W)$ is called a *Nomura algebra*. Moreover, there exists a duality map from $\mathcal{N}(W)$ to $\mathcal{N}({}^tW)$. $\mathcal{N}({}^tW)$ is called the *dual* of $\mathcal{N}(W)$. We often say $\mathcal{N}(W)$ has a dual (See Section 2).

We are interested in determining type II matrices of small sizes. Type II matrices at most size 5 have been completely determined (See [7, 14]). We are also interested in the Bose-Mesner algebras which appear as the Nomura algebras of type II matrices. Type II matrices whose Nomura algebras are $\mathrm{Span}(I, J)$ are difficult to determine. On the other hand, in the classification of spin models, we do not need to determine type II matrices of this case [15]. In this paper we consider the case $\mathcal{N}(W) \neq \mathrm{Span}(I, J)$.

Let $U_1, U_2, \ldots, U_m$ be square matrices of size n, and let $V_1, V_2, \ldots, V_n$ be square matrices of size m. Let $\tilde{W} = (U_1, U_2, \ldots, U_m) \otimes (V_1, V_2, \ldots, V_n)$ be a square matrix of size mn such that the (i, j)-block $\tilde{W}[[i], [j]]$ is defined by the following.

$$\tilde{W}[[i], [j]] = \Delta_{i,j} V_j,$$

where $\Delta_{i,j}[h, k] = \delta_{h,k} U_h[i, j]$ ($i, j = 1, \ldots, n$ and $h, k = 1, \ldots, m$). We call $\tilde{W}$ the *generalized tensor product* of $U_1, U_2, \ldots, U_m$ and $V_1, V_2, \ldots, V_n$. In Lemma 4.1, we show that if $U_1, U_2, \ldots, U_m$ and $V_1, V_2, \ldots, V_n$ are type II matrices, then $\tilde{W}$ is a type II matrix. We are informed by K. Nomura that the special case of this result was already noticed by V.F.R. Jones. K. Nomura and U. Haagerup also considered some special cases of this result [5, 12].

Now we state our main result.

Theorem 1.1 *Let W be a type II matrix of size mn. Let J_n be the all 1 matrix of size $n \geq 2$, and let I_m be the identity matrix of size $m \geq 2$. Then the following are equivalent.*

(i) *There exists a permutation matrix S such that $J_n \otimes I_m \in \mathcal{N}(SW)$.*

(ii) *$\mathcal{N}(W)$ is an imprimitive Bose-Mesner algebra with a system of imprimitivity having blocks of size n.*

(iii) *W is type II equivalent to a generalized tensor product $(U_1, U_2, \ldots, U_m) \otimes (V_1, V_2, \ldots, V_n)$, where $U_1, U_2, \ldots, U_m$ and $V_1, V_2, \ldots, V_n$ are type II matrices of size n and m respectively.*

According to the classification of commutative association schemes [11] and considering the fact that $\mathcal{N}(W)$ has a dual, for type II matrices of size at most 10, one of the following holds.

(a) $\mathcal{N}(W)$ is a Bose-Mesner algebra of an imprimitive association scheme.

(b) $\dim \mathcal{N}(W) = 3$ or p for W of size $p = 5, 7, 9$.

(c) $\mathcal{N}(W) = \mathrm{Span}(I, J)$.

Applying Theorem 1.1 to the case of (a), W is expressed as a generalized tensor product of type II matrices of size at most 5. Moreover, the following hold.

Theorem 1.2 *Let W be a type II matrix of size at most 8 or size 10. Then one of the following holds.*

(i) *$\mathcal{N}(W) = \mathrm{Span}(I, J)$.*

(ii) *W is type II equivalent to a cyclic type II matrix.*

(iii) *W is type II equivalent to a generalized tensor product of type II matrices of smaller sizes.*

Recently, T. Matsumura [10] showed that there is no type II matrix W such that $\dim \mathcal{N}(W) = 3$. According to his result, the above theorem is true for the case of size 9. As for the results concerning small four-weight spin models, the reader is referred to [4, 15].

2. Preliminary results

Let W be a type II matrix. Then we can define a mapping $\Psi = \Psi_W$ from $\mathcal{N}(W)$ to $\mathrm{Mat}_n(\boldsymbol{C})$ by the following.

$$M\boldsymbol{u}^W_{i,j} = \Psi(M)[i,j]\boldsymbol{u}^W_{i,j} \quad \text{for } M \in \mathcal{N}(W),$$

i.e., the (i, j)-entry of $\Psi(M)$ is the eigenvalue of M associated with the eigenvector $\boldsymbol{u}_{i,j}^{W}$. The map Ψ is called the *duality map*.

Proposition 2.1 *Let W be a type II matrix in* $\mathrm{Mat}_n(\boldsymbol{C})$. *Then the following hold.*
(1) *For all $1 \le i \le n$, the set of vectors $\{\boldsymbol{u}_{i,j}^{W} \mid 1 \le j \le n\}$ is linearly independent.*
(2) *$\mathcal{N}(W)$ is the Bose-Mesner algebra of a commutative association scheme.*
(3) *The duality map $\Psi = \Psi_W$ is a linear isomorphism from $\mathcal{N}(W)$ to $\mathcal{N}({}^{t}W) = \mathcal{N}(W^{-})$ satisfying the following conditions.*
 (a) $\Psi(I) = J$ *and* $\Psi(J) = nI$.
 (b) $\Psi(MN) = \Psi(M) \circ \Psi(N)$ *for all* $M, N \in \mathcal{N}(W)$.
 (c) $\Psi(M \circ N) = (1/n)\Psi(M)\Psi(N)$ *for all* $M, N \in \mathcal{N}(W)$.
(4) *Let $\Psi' = \Psi_{{}^{t}W}$. Then for every $M \in \mathcal{N}(W)$, we have $\Psi'(\Psi(M)) = n\,{}^{t}M$.*

Proof: All assertions can be found in [7, Theorem 1]. (1) is the statement (23) in its proof. □

Let $\mathcal{B}$ denote the Bose-Mesner algebra of a commutative association scheme $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$. Then there are two canonical bases. One of them is the set of *adjacency* (*or associate*) *matrices* $\{A_0 = I, A_1, \ldots, A_d\}$ satisfying $A_i \circ A_j = \delta_{i,j} A_i$, and the other is the set of *primitive idempotents* $\{E_0 = (1/|X|)J, E_1, \ldots, E_d\}$ satisfying $E_i E_j = \delta_{i,j} E_i$. Let

$$A_i = \sum_{j=0}^{d} p_i(j) E_j, \quad E_i = \frac{1}{|X|} \sum_{j=0}^{d} q_i(j) A_j.$$

The base change matrices P with $P[i, j] = p_j(i)$ and Q with $Q[i, j] = q_j(i)$ are called *the first eigenmatrix* and *the second eigenmatrix* respectively. For the general theory of commutative association schemes and that of Bose-Mesner algebras, the reader is referred to the excellent monograph [2].

Let $W \in \mathrm{Mat}_n(\boldsymbol{C})$ be a type II matrix. We use the following notation. Let $\mathcal{X}$ (resp. $\mathcal{X}'$) be a commutative association scheme with the Bose-Mesner algebra $\mathcal{N}(W)$ (resp. $\mathcal{N}({}^{t}W)$). Let $A_0, A_1, \ldots, A_d$ be the adjacency matrices in $\mathcal{N}(W)$, and let $E_0, E_1, \ldots, E_d$ be the primitive idempotents. By the previous proposition, the dimensions of $\mathcal{N}(W)$ and $\mathcal{N}({}^{t}W)$ are equal. Let $A_0', A_1', \ldots, A_d'$ be the adjacency matrices in $\mathcal{N}({}^{t}W)$, and let $E_0', E_1', \ldots, E_d'$ be the primitive idempotents. Let $\Psi = \Psi_W$ and $\Psi' = \Psi_{{}^{t}W}$.

Corollary 2.2 *Let $W \in$* $\mathrm{Mat}_n(\boldsymbol{C})$ *be a type II matrix. Then, by a suitable arrangement of indices, the following hold.*
(1) $\Psi(A_i) = nE_i'$ *and* $\Psi(E_i) = A_i'$.
(2) $\Psi'(E_i') = {}^{t}A_i$ *and* $\Psi'(A_i') = n\,{}^{t}E_i$.
(3) *The first eigenmatrix P of $\mathcal{X}$ is the second eigenmatrix Q' of $\mathcal{X}'$ and the second eigenmatrix Q of $\mathcal{X}$ is the first eigenmatrix P' of $\mathcal{X}'$.*

Remarks In this paper, we use the above ordering of the idempotents so that $P = Q'$, although it is customary to use the standard ordering of them so that $P = \bar{Q}'$.

Proposition 2.3 *Let W be a type II matrix in* $\mathrm{Mat}_n(\boldsymbol{C})$, *let* $\Delta \in \mathrm{Mat}_n(\boldsymbol{C})$ *be a nonsingular diagonal matrix, and let S be a permutation matrix such that* $S[x, y] = \delta_{\sigma(x),y}$, *where σ is a permutation on* $X = \{1, 2, \ldots, n\}$. *Then the following hold.*
(1) $\boldsymbol{u}^W_{x,y} = \boldsymbol{u}^{\Delta W}_{x,y} = \frac{\Delta[y,y]}{\Delta[x,x]}\boldsymbol{u}^{W\Delta}_{x,y}$.
(2) $\boldsymbol{u}^{WS}_{x,y} = \boldsymbol{u}^W_{\sigma^{-1}(x),\sigma^{-1}(y)}$, *and* $\boldsymbol{u}^{SW}_{x,y} = S\boldsymbol{u}^W_{x,y}$.
(3) ΔW, $W\Delta$, SW *and* WS *are type II matrices.*
(4) $\mathcal{N}(W) = \mathcal{N}(\Delta W) = \mathcal{N}(W\Delta) = \mathcal{N}(WS)$ *and* $S\mathcal{N}(W)^t S = \mathcal{N}(SW)$.
(5) $\Psi_{WS}(M) = {}^t S\Psi_W(M)S$ *for* $M \in \mathcal{N}(WS) = \mathcal{N}(W)$.

Proof: Straightforward. See also [7, Section 3.2]. □

Remarks Two Bose-Mesner algebras $\mathcal{B}$ and $\mathcal{B}'$ in $\mathrm{Mat}_n(\boldsymbol{C})$ are *combinatorially isomorphic* if there exists a permutation matrix S in $\mathrm{Mat}_n(\boldsymbol{C})$ such that $\mathcal{B} = S\mathcal{B}'{}^tS$. Hence by (4) in the above proposition the Bose-Mesner algebras $\mathcal{N}(W)$, $\mathcal{N}(\Delta W)$, $\mathcal{N}(W\Delta)$, $\mathcal{N}(WS)$ and $\mathcal{N}(SW)$ are all combinatorially isomorphic.

3. Type II matrices

In this section, we prove several results which will be useful to determine type II matrices W when a Bose-Mesner algebra contained in $\mathcal{N}(W)$ is given.

Proposition 3.1 *Let* $W \in \mathrm{Mat}_n(\boldsymbol{C})$ *be a type II matrix. Let $\mathcal{B}$ be the Bose-Mesner algebra of a commutative association scheme contained in $\mathcal{N}(W)$, and let $A_0, A_1, \ldots, A_d$ be its adjacency matrices and $E_0, E_1, \ldots, E_d$ be its primitive idempotents, which satisfy the conditions* (1)–(3) *in Corollary* 2.2. *Let* $V = \boldsymbol{C}^n$ *and* $V_i = E_i V$. *Suppose*

$$A_i = \sum_{h=0}^{d} p_i(h)E_h, \quad E_i = \frac{1}{n}\sum_{h=0}^{d} q_i(h)A_h.$$

Then the following hold.
(1) *Let* $\Pi = \Pi_i^{(j)} = \{h \mid \boldsymbol{u}^W_{h,j} \in V_i\}$. *If* $A_l[s,t] = 1$, *then*

$$\frac{W[t,j]}{W[s,j]}\sum_{h\in\Pi}\frac{W[s,h]}{W[t,h]} = n \cdot E_i[s,t] = q_i(l).$$

(2) *Let* $\Lambda = \Lambda_i^{(j)} = \{h \mid A_i[h,j] = 1\}$ *and let* $\Psi = \Psi_W$. *If* $\boldsymbol{u}^W_{s,t} \in V_l$, *then*

$$\frac{W[j,s]}{W[j,t]}\sum_{h\in\Lambda}\frac{W[h,t]}{W[h,s]} = \Psi(A_i)[s,t] = p_i(l).$$

(3) *Let* ${}^tE_i = E_{\hat{i}}$. *Then* $\boldsymbol{u}^W_{s,t} \in V_i$ *if and only if* $\boldsymbol{u}^W_{t,s} \in V_{\hat{i}}$.

Lemma 3.2 *Let* $\Psi = \Psi_W$ *be the duality map from* $\mathcal{N}(W)$ *to* $\mathcal{N}({}^tW)$. *Then the following hold.*

(1) $\boldsymbol{u}^{{}^{\mathrm{t}}W}_{s,t} = \boldsymbol{u}^{W^-}_{t,s}$ *for every* $1 \leq s, t \leq n$.
(2) $E_i \boldsymbol{u}^W_{s,t} = \boldsymbol{u}^W_{s,t}$ *if and only if* $\Psi(E_i)[s,t] = 1$.
(3) $\Psi(A_i)\boldsymbol{u}^{{}^{\mathrm{t}}W}_{s,t} = n\boldsymbol{u}^{{}^{\mathrm{t}}W}_{s,t}$ *if and only if* $A_i[t,s] = 1$.
(4) *For* $M \in \mathcal{N}(W)$, $\Psi({}^{\mathrm{t}}M) = {}^{\mathrm{t}}\Psi(M)$.

Proof: All of the assertions are clear from the definitions and Corollary 2.2. The last assertion is a consequence of the following.

$$\Psi({}^{\mathrm{t}}M) = \frac{1}{n}\Psi(\Psi'(\Psi(M)) = {}^{\mathrm{t}}\Psi(M),$$

by Corollary 2.2. □

Proof of Proposition 3.1: Let $\Psi = \Psi_W$ and $\Psi' = \Psi_{{}^{\mathrm{t}}W}$.

(1) Since $\Psi'(\Psi({}^{\mathrm{t}}E_i)) = n \cdot E_i$, we compute $\Psi({}^{\mathrm{t}}E_i)\boldsymbol{u}^{{}^{\mathrm{t}}W}_{s,t}$. Note that $\Psi({}^{\mathrm{t}}E_i)$ is a (0, 1) matrix as it is an idempotent with respect to a $\circ$-product. $\Psi({}^{\mathrm{t}}E_i)[j,h] = {}^{\mathrm{t}}\Psi(E_i)[j,h] = 1$ if and only if $\Psi(E_i)[h,j] = 1$. On the other hand, $\Psi(E_i)[h,j] = 1$ if and only if $E_i\boldsymbol{u}^W_{h,j} = \boldsymbol{u}^W_{h,j}$, i.e., $\Psi(E_i)[h,j] = 1$ if and only if $h \in \Pi^{(j)}_i = \Pi$. Hence, we have the following.

$$\begin{aligned} n \cdot E_i[s,t]\frac{W[s,j]}{W[t,j]} &= \left(\Psi\left({}^{\mathrm{t}}E_i\right)\boldsymbol{u}^{{}^{\mathrm{t}}W}_{s,t}\right)[j] \\ &= \sum_{h=1}^{n} \Psi\left({}^{\mathrm{t}}E_i\right)[j,h]\boldsymbol{u}^{{}^{\mathrm{t}}W}_{s,t}[h] \\ &= \sum_{h=1}^{n} \Psi\left({}^{\mathrm{t}}E_i\right)[j,h]\frac{W[s,h]}{W[t,h]} \\ &= \sum_{h\in\Pi} \frac{W[s,h]}{W[t,h]}. \end{aligned}$$

This proves (1).

(2) Since $\Psi'(\Psi({}^{\mathrm{t}}A_i)) = n \cdot A_i$ and $A_i[h,j] = 1$ if and only if $h \in \Lambda$, we have the following.

$$\begin{aligned} \sum_{h\in\Lambda} \frac{W[h,t]}{W[h,s]} &= \sum_{h=1}^{n} A_i[h,j]\frac{W[h,t]}{W[h,s]} \\ &= \sum_{h=1}^{n} {}^{\mathrm{t}}A_i[j,h]\frac{W[h,t]}{W[h,s]} \\ &= \frac{1}{n}\sum_{h=1}^{n} \Psi'(\Psi(A_i))[j,h]\frac{W[h,t]}{W[h,s]} \\ &= \frac{1}{n}\left(\Psi'(\Psi(A_i))\boldsymbol{u}^W_{t,s}\right)[j] \\ &= \frac{1}{n}\Psi(\Psi'(\Psi(A_i)))[t,s]\frac{W[j,t]}{W[j,s]} \end{aligned}$$

$$= {}^{t}\Psi(A_i)[t,s]\frac{W[j,t]}{W[j,s]}$$
$$= \Psi(A_i)[s,t]\frac{W[j,t]}{W[j,s]}.$$

Now it remains to show that $\Psi(A_i)[s,t] = p_i(l)$. This follows from the following.

$$\Psi(A_i)[s,t]\,\boldsymbol{u}^W_{s,t} = A_i\boldsymbol{u}^W_{s,t} = \sum_{h=0}^{d} p_i(h)E_h\boldsymbol{u}^W_{s,t} = p_i(l)\,\boldsymbol{u}^W_{s,t},$$

as $\boldsymbol{u}^W_{s,t} \in V_l = E_l V$. This proves (2).

(3) Since $\Psi({}^{t}E_i) = {}^{t}\Psi(E_i)$, we have the following.

$$\boldsymbol{u}^W_{s,t} \in V_i \Leftrightarrow 1 = \Psi(E_i)[s,t] = {}^{t}\Psi\left({}^{t}E_i\right)[s,t] = \Psi\left({}^{t}E_i\right)[t,s]$$
$$\Leftrightarrow \boldsymbol{u}^W_{t,s} \in V_{\hat{i}}.$$

□

Lemma 3.3 ([15]) *Let W be a type II matrix in* $\mathrm{Mat}_n(\boldsymbol{C})$, *and* $E_0 = \frac{1}{n}J, E_1, \ldots, E_d$ *be orthogonal idempotents of* $\mathcal{N}(W)$ *expressing I as a sum, i.e.,*

$$E_iE_j = \delta_{i,j}E_i, \quad \textit{for } 0 \le i, j \le d, \quad \textit{and} \quad I = E_0 + E_1 + \cdots + E_d.$$

Then W is type II equivalent to a matrix $U = [U_0, U_1, \ldots, U_d]$ with the following properties.

(1) $U_0 = \boldsymbol{j}$, *where $\boldsymbol{j}$ denotes the all ones column vector.*
(2) *U_i is an $n \times m_i$ matrix with ones in the first row and no zero entries, where* $m_i = \mathrm{rank}E_i$.
(3) *The column space of U_i equals the column space of E_i. In particular, the columns of each U_i are linearly independent.*
(4) $\mathcal{N}(W) = \mathcal{N}(U)$.

Proof: Since all entries of W are nonzero, there exist nonsingular diagonal matrices D and D' such that DWD' has $\boldsymbol{j}$ as the first column and the entries of the first row are all ones. Let $W' = DWD' = [\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n]$. Since $\boldsymbol{w}_1 = \boldsymbol{j}$, $\boldsymbol{u}^{W'}_{i,1} = \boldsymbol{w}_i$ for $i = 1, 2, \ldots, n$. Since $\mathcal{N}(W) = \mathcal{N}(DWD') = \mathcal{N}(W')$, the set of column vectors of W' forms a basis of common eigenvectors of $\mathrm{Span}(E_0, E_1, \ldots, E_d)$. Since E_i's are idempotents, the eigenvalues are 1 or 0. Hence $E_i\boldsymbol{w}_j = \boldsymbol{w}_j$ or 0. Since $\boldsymbol{w}_j = I\boldsymbol{w}_j = E_0\boldsymbol{w}_j + E_1\boldsymbol{w}_j + \cdots + E_d\boldsymbol{w}_j$, each $\boldsymbol{w}_j$ is contained in exactly one column space of E_i's. Hence by a suitable rearrangement of the order of the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$, we have a matrix U with the properties (1)–(3). Since U is obtained by multiplying a permutation matrix S to W' from the right, $U = W'S = DWD'S$ and it is type II equivalent to W and $\mathcal{N}(W) = \mathcal{N}(U)$ as desired. □

In the light of the previous lemma and Corollary 2.2, we consider the following situation. Let $W \in \mathrm{Mat}_n(\boldsymbol{C})$ be a type II matrix. Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme with the Bose-Mesner algebra $\mathcal{B} \subset \mathcal{N}(W)$. Let $A_0, A_1, \ldots, A_d$ be the adjacency matrices in $\mathcal{B}$, and let $E_0, E_1, \ldots, E_d$ be the primitive idempotents in $\mathcal{B}$. Let $\mathcal{B}' = \Psi(\mathcal{B})$.

Then $\mathcal{B}'$ is a Bose-Mesner algebra of a commutative association scheme $\mathcal{X}'$ which is dual to $\mathcal{X}$. Let $A_i' = \Psi(E_i)$ and $E_i' = \frac{1}{n}\Psi(A_i)$. Then $A_0', A_1', \ldots, A_d'$ are the adjacency matrices in $\mathcal{B}'$, and $E_0', E_1', \ldots, E_d'$ are the primitive idempotents in $\mathcal{B}'$.

For a matrix $M \in \mathrm{Mat}_n(\boldsymbol{C})$ and the set of indices $\Lambda = \{i_1, i_2, \ldots, i_k\}$ and $\Pi = \{j_1, j_2, \ldots, j_m\}$, let $M[\Lambda, \Pi]$ denote the submatrix of M consisting of the rows $i_1, i_2, \ldots, i_k$ and the columns $j_1, j_2, \ldots, j_m$. Let $W = [\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n]$. Assume $\boldsymbol{w}_1 = \boldsymbol{j}$ and the entries in the first row of W are 1. Assume the following.

1. $\Lambda_i = \{h \mid A_i[h, 1] = 1\}$.
2. $\Pi_j = \{h \mid E_j \boldsymbol{w}_h = \boldsymbol{w}_h\}$.

As an application of Proposition 3.1, the following hold.

Proposition 3.4 *Let W be a type II matrix satisfying the condition above. Let $W_{s,t} = W[\Lambda_s, \Pi_t]$. Let $\boldsymbol{j}$ denote the all one vector of appropriate size. Then the following hold.*
(1) $W_{i,h}\boldsymbol{j} = q_h(i)\boldsymbol{j}$.
(2) ${}^t\boldsymbol{j}W_{i,h} = \overline{p_i(h)}\,{}^t\boldsymbol{j}$.
(3) $(W_{j,h})^-\boldsymbol{j} = p_j(h)\boldsymbol{j}$.
(4) ${}^t\boldsymbol{j}(W_{j,h})^- = \overline{q_h(j)}\,{}^t\boldsymbol{j}$.
(5) $W_{i,h}(W_{j,h})^- = n \cdot E_h[\Lambda_i, \Lambda_j]$.

We define two matrices S and T of size $n \times (d+1)$ and $(d+1) \times n$.

$$S[h, j] = \begin{cases} 1 & \text{if } h \in \Pi_j \\ 0 & \text{otherwise} \end{cases}, \quad T[i, h] = \begin{cases} 1 & \text{if } h \in \Lambda_i \\ 0 & \text{otherwise} \end{cases}.$$

Corollary 3.5 *Under the hypothesis of Proposition* 3.4, *the following hold.*
(1) $WS = SQ$.
(2) $TW = {}^t\bar{P}T$.
(3) $W^-\,{}^tT = {}^tTP$.
(4) ${}^tSW^- = {}^t\bar{Q}\,{}^tS$.

Proof: The matrix equations are direct consequences of the assertions (1)–(4) in Proposition 3.4. □

4. Generalized tensor products

In this section we give some properties of generalized tensor products and the proof of Theorem 1.1.

Lemma 4.1 *Let $U_1, U_2, \ldots, U_m$ be square matrices of size n, and $V_1, V_2, \ldots, V_n$ be square matrices of size m. Let W be a generalized tensor product of $U_1, U_2, \ldots, U_m$ and $V_1, V_2, \ldots, V_n$. Then the following are equivalent.*
(1) *W is a type II matrix.*
(2) *$U_1, U_2, \ldots, U_m, V_1, V_2, \ldots, V_n$ are type II matrices.*

Proof: (1) ⇒ (2). Since W is a type II matrix,

$$\sum_{j=1}^{n}\sum_{y=1}^{m}\frac{W[m(h-1)+x, m(j-1)+y]}{W[m(i-1)+x, m(j-1)+y]} = mn\delta_{h,i},$$

for $1 \le h, i \le n$ and $1 \le x \le m$.

$$\begin{aligned} LHS &= \sum_{j=1}^{n}\sum_{y=1}^{m}\frac{\Delta_{h,j}V_j[x,y]}{\Delta_{i,j}V_j[x,y]} \\ &= \sum_{j=1}^{n}\sum_{y=1}^{m}\frac{U_x[h,j]V_j[x,y]}{U_x[i,j]V_j[x,y]} \\ &= \sum_{j=1}^{n}\frac{U_x[h,j]}{U_x[i,j]}\left(\sum_{y=1}^{m}\frac{V_j[x,y]}{V_j[x,y]}\right) \\ &= m\sum_{j=1}^{n}\frac{U_x[h,j]}{U_x[i,j]}. \end{aligned}$$

Hence

$$\sum_{j=1}^{n}\frac{U_x[h,j]}{U_x[i,j]} = n\delta_{h,i},$$

i.e., U_x is a type II matrix of size n for $1 \le x \le m$.

Similarly, since W is a type II matrix,

$$\sum_{h=1}^{n}\sum_{x=1}^{m}\frac{W[m(h-1)+x, m(i-1)+y]}{W[m(h-1)+x, m(i-1)+z]} = mn\delta_{y,z},$$

for $1 \le i \le n$ and $1 \le y, z \le m$. By computing the left hand side of the above equation, we have the following.

$$\sum_{x=1}^{m}\frac{V_i[x,y]}{V_i[x,z]} = m\delta_{y,z},$$

i.e., V_i is a type II matrix of size m for $1 \le i \le n$.

(2) ⇒ (1).

$$\begin{aligned} \sum_{h=1}^{n}\sum_{x=1}^{m}\frac{W[m(h-1)+x, m(i-1)+y]}{W[m(h-1)+x, m(j-1)+z]} &= \sum_{h=1}^{n}\sum_{x=1}^{m}\frac{\Delta_{h,i}V_i[x,y]}{\Delta_{h,j}V_j[x,z]} \\ &= \sum_{h=1}^{n}\sum_{x=1}^{m}\frac{U_x[h,i]V_i[x,y]}{U_x[h,j]V_j[x,z]} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x=1}^{m} \left(\sum_{h=1}^{n} \frac{U_x[h,i]}{U_x[h,j]} \right) \frac{V_i[x,y]}{V_j[x,z]} \\
&= n\delta_{i,j} \sum_{x=1}^{m} \frac{V_i[x,y]}{V_j[x,z]} \\
&= n\delta_{i,j} \sum_{x=1}^{m} \frac{V_i[x,y]}{V_i[x,z]} \\
&= mn\delta_{i,j}\delta_{y,z}. \qquad \square
\end{aligned}
$$

The following lemma is well known. See also [2].

Lemma 4.2 *Let $M \in \mathrm{Mat}_{mn}(\boldsymbol{C})$. Suppose M satisfies the following.*
(1) $M \circ I = I$.
(2) $M = M \circ M$.
(3) $mM = MM$.
(4) ${}^tM = M$.
Then there exists a permutation matrix S such that ${}^tSMS = I_n \otimes J_m$.

Lemma 4.3 *Let $W \in \mathrm{Mat}_{mn}(\boldsymbol{C})$ with nonzero entries. Then the following are equivalent.*
(i) $W = (U_1, \ldots, U_m) \otimes (V_1, \ldots, V_n)$ *for some matrices $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ of sizes n and m respectively.*
(ii) *For $1 \le i, h \le n$ and $1 \le x, y, z \le m$,*

$$\frac{W[x, m(i-1)+y]}{W[x, m(i-1)+z]} = \frac{W[m(h-1)+x, m(i-1)+y]}{W[m(h-1)+x, m(i-1)+z]}.$$

Proof: (i) $\Rightarrow$ (ii) is obtained by the direct computation. Assume (ii). We have the following.

$$\frac{W[m(h-1)+x, m(i-1)+y]}{W[x, m(i-1)+y]} = \frac{W[m(h-1)+x, m(i-1)+z]}{W[x, m(i-1)+z]}$$

for $1 \le i, h \le n$ and $1 \le x, y, z \le m$.

The above equation implies that the ratio of $W[m(h-1)+x, m(i-1)+y]$ to $W[x, m(i-1)+y]$ does not depend on the choice of y for $1 \le y \le m$.

Fix $y = 1$. Set

$$t_{h,i}^{x} = \frac{W[m(h-1)+x, m(i-1)+1]}{W[x, m(i-1)+1]},$$

where $1 \le h, i \le n$ and $1 \le x \le m$.

Define square matrices V_j of size m by $V_j = W[[1],[j]]$ for $1 \le j \le n$, and U_x of size n by $U_x[i,j] = t_{i,j}^{x}$ for $1 \le x \le m$ and $1 \le i, j \le n$.

Then we can verify $W = (U_1, \ldots, U_m) \otimes (V_1, \ldots, V_n)$. $\square$

Lemma 4.4 *Let $U_1, U_2, \ldots, U_m$ be type II matrices of size n and let $V_1, V_2, \ldots, V_n$ be type II matrices of size m. Let W be a generalized tensor product of $U_1, U_2, \ldots, U_m$ and $V_1, V_2, \ldots, V_n$. If $M \in \mathcal{N}(V_i)$ for $1 \le i \le n$, then $J_n \otimes M \in \mathcal{N}(W)$.*

Proof: Let $\mathbf{v}^{i,j}_{y,z}$ be a column vector of W defined as follows.

$$\mathbf{v}^{i,j}_{y,z}[m(h-1)+x] = \frac{\Delta_{h,i} V_i[x,y]}{\Delta_{h,j} V_j[x,z]}.$$

where $1 \le h, i, j \le n$ and $1 \le x, y, z \le m$.

The following hold.

$$\begin{aligned}
&\big((J_n \otimes M)\mathbf{v}^{i,j}_{y,z}\big)[m(h-1)+x] \\
&= \sum_{h'}\sum_{x'} (J_n \otimes M)[m(h-1)+x, m(h'-1)+x']\mathbf{v}^{i,j}_{y,z}[m(h'-1)+x'] \\
&= \sum_{h'}\sum_{x'} M[x,x'] \frac{\Delta_{h',i} V_i[x',y]}{\Delta_{h',j} V_j[x',z]} \\
&= \sum_{h'}\sum_{x'} M[x,x'] \frac{U_{x'}[h',i] V_i[x',y]}{U_{x'}[h',j] V_j[x',z]} \\
&= \sum_{x'} M[x,x'] \left(\sum_{h'} \frac{U_{x'}[h',i]}{U_{x'}[h',j]}\right) \frac{V_i[x',y]}{V_j[x',z]} \\
&= n\delta_{i,j} \sum_{x'} M[x,x'] \frac{V_i[x',y]}{V_j[x',z]} \\
&= n\delta_{i,j}\big(M\boldsymbol{u}^{V_i}_{y,z}\big)[x]
\end{aligned}$$

Since M belongs to $\mathcal{N}(V_i)$, the following hold.

$$(J_n \otimes M)\mathbf{v}^{i,j}_{y,z} = \alpha\delta_{i,j}\mathbf{v}^{i,j}_{y,z}$$

where $\alpha \in \mathbf{C}$. Hence $J_n \otimes M \in \mathcal{N}(W)$. □

Proof of Theorem 1.1: (i) ⇒ (ii). Suppose that there exists a permutation matrix S such that $J_n \otimes I_m \in \mathcal{N}(SW)$. Let $A_0, \ldots, A_d$ be the basis of Hadamard idempotents of $\mathcal{N}(SW)$, where $A_0 = I$ and $A_0 + \cdots + A_d = J$. By a suitable arrangement of indices, there exists a permutation matrix S' such that

$$A_0 + \cdots + A_s = S'(I_m \otimes J_n)^{\mathrm{t}}S' \in \mathcal{N}(SW)$$

for some s with $1 \le s \le d-1$. Hence $\mathcal{N}(W)$ is an imprimitive Bose-Mesner algebra whose imprimitive equivalence class is of size n. By Proposition 2.3, $\mathcal{N}(W)$ and $\mathcal{N}(SW)$ are combinatorially isomorphic. Therefore we obtain (i) ⇒ (ii). For the detail see [2].

(ii) $\Rightarrow$ (i). Suppose $\mathcal{N}(W)$ is an imprimitive Bose-Mesner algebra, whose imprimitive equivalence class is of size n. There exists a permutation matrix S such that

$$A_0 + \cdots + A_s = {}^{t}S(I_m \otimes J_n)\,S$$

for adjacency matrices $A_0, \ldots, A_s$, where $1 \le s \le d-1$ by [2, Theorem 9.3]. Hence

$$I_m \otimes J_n \in S\mathcal{N}(W)\,{}^{t}S = \mathcal{N}(SW).$$

Since W and SW are type II equivalent, We have (ii).

(i) $\Rightarrow$ (iii). Let W be a type II matrix of size mn satisfying the condition (i). We may assume $J_n \otimes I_m \in \mathcal{N}(W)$. Then $\Psi_W(J_n \otimes I_m) \in \mathcal{N}({}^{t}W)$. Let $M = \Psi_W(\frac{1}{n} J_n \otimes I_m)$ and let $M' = \frac{1}{n} J_n \otimes I_m$. Then the following hold.

(1′) $M'J = J$.
(2′) $M' = M'M'$.
(3′) $\frac{1}{n}M' = M' \circ M'$.
(4′) ${}^{t}M' = M'$.

Next, we consider the duality. By Proposition 2.1(3) and Lemma 3.2(4), the following hold.

(1) $\Psi_W(M') \circ \Psi_W(J) = \Psi_W(J)$.
(2) $\Psi_W(M') = \Psi_W(M') \circ \Psi_W(M')$.
(3) $\frac{1}{n}\Psi_W(M') = \frac{1}{mn}\Psi_W(M')\Psi_W(M')$.
(4) ${}^{t}\Psi_W(M') = \Psi_W(M')$.

It is clear that $M = \Psi_W(M') = \Psi_W(\frac{1}{n} J_n \otimes I_m)$ satisfies the conditions (1)–(4) in Lemma 4.2. Hence there exists a permutation matrix S of size mn such that $\Psi_{WS}(J_n \otimes I_m) = {}^{t}S\Psi_W(J_n \otimes I_m)S = nI_n \otimes J_m$. Hence $nI_n \otimes J_m \in \mathcal{N}({}^{t}(WS))$ (See Proposition 2.3(3)). Since $\mathcal{N}(WS) = \mathcal{N}(W)$, we replace W by WS if necessary. Therefore we assume that W satisfies

$$\Psi_W(J_n \otimes I_m) = nI_n \otimes J_m.$$

Let $a = m(i-1) + y$ and $b = m(i-1) + z$ for $1 \le i \le n$, $1 \le y, z \le m$. Compare the $(m(h-1)+x)$ entry of the both sides of

$$(J_n \otimes I_m)\,\boldsymbol{u}^W_{a,b} = \Psi_W(J_n \otimes I_m)[a, b]\,\boldsymbol{u}^W_{a,b},$$

for $1 \le h \le n$, $1 \le x \le m$. The left hand side is

$$\sum_{k=1}^{mn} (J_n \otimes I_m)[m(h-1)+x, k]\,\boldsymbol{u}^W_{a,b}[k] = \sum_{j=1}^{n} \frac{W[m(j-1)+x, a]}{W[m(j-1)+x, b]}.$$

The right hand side is

$$n(I_n \otimes J_m)[a, b]\,\boldsymbol{u}^W_{a,b}[m(h-1)+x] = n\frac{W[m(h-1)+x, a]}{W[m(h-1)+x, b]}.$$

Hence we get

$$\sum_{j=1}^{n} \frac{W[m(j-1)+x,a]}{W[m(j-1)+x,b]} = n\frac{W[m(h-1)+x,a]}{W[m(h-1)+x,b]}.$$

Observe that the left hand side does not depend on h. Hence the right hand side is also independent of the choice of h. Therefore we have (ii) of Lemma 4.3, and thus $W = (U_1, \ldots, U_m) \otimes (V_1, \ldots, V_n)$ for some matrices $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ of sizes n and m respectively. By Lemma 4.1, $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ are type II matrices. This shows that W satisfies (iii) in Theorem 1.1.

(iii) $\Rightarrow$ (i). By Lemma 4.4, it is clear. □

Remarks K. Nomura illustrated generalized tensor products by functions as follows.

Fix nonempty finite sets X, Y. For functions:

$$f : X \times Y \times X \longrightarrow \mathbf{C},$$
$$g : Y \times X \times Y \longrightarrow \mathbf{C},$$

we define their generalized tensor product

$$f \otimes g : X \times Y \times X \times Y \longrightarrow \mathbf{C}$$

by

$$(f \otimes g)(x_1, y_1, x_2, y_2) = f(x_1, y_1, x_2)g(y_1, x_2, y_2).$$

5. Examples of generalized tensor products

5.1. Generalized tensor products of size 2m

In this section, we describe the method to express type II matrices as generalized tensor products. We use the same notation for $\Delta_{i,j}$ as in the previous section.

If $U_1 = U_2 = \cdots = U_m = U$ we write $U \otimes (V_1, V_2, \ldots, V_n)$ instead of $(U_1, U_2, \ldots, U_m) \otimes (V_1, V_2, \ldots, V_n)$.

Proposition 5.1 *Let $U_1, U_2, \ldots, U_m$ be type II matrices of size 2 and let V_1, V_2 be type II matrices of size m. Then the generalized tensor product $(U_1, U_2, \ldots, U_m) \otimes (V_1, V_2)$ is type II equivalent to the generalized tensor product $U \otimes (V_1, V_2')$ where $U = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, and $V_2' = \Delta_{11}^{-1}\Delta_{12}V_2$.*

Proof: Straightforward. □

For type II matrices W, W' of the same size, we say W is *right type II equivalent* to W' if there exist a nonsingular diagonal matrix D and a permutation matrix S such that $W' = WDS$.

Proposition 5.2 *Let* $U = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, *and let* V_1, V_2, V_1' *and* V_2' *be type II matrices of size* m. *Let* $W = U \otimes (V_1, V_2)$.*Then the following hold.*

(i) *If* V_1' *is type II equivalent to* V_1, *then* W *is type II equivalent to* $U \otimes (V_1', L)$ *for a type II matrix* L *of size* m.

(ii) *If* V_2' *is right type II equivalent to* V_2, *then* W *is type II equivalent to* $U \otimes (V_1, V_2')$.

Proof: Straightforward. □

5.2. Examples

Let W be a type II matrix such that $\mathcal{N}(W) \neq \mathrm{Span}(I, J)$. Recall $\mathcal{N}(W)$ has a dual.

5.2.1. The case of size 6. Let W be a type II matrix of size 6. According to the classification of association schemes of size 6 [11], it is easy to see that $\mathcal{N}(W)$ is imprimitive. Hence by Theorem 1.1, W is type II equivalent to a generalized tensor product of type II matrices of size 2 and those of size 3.

Let

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 & 1 & 1 \\ a & aw & aw^2 \\ b & bw^2 & bw \end{bmatrix},$$

where $w^3 = 1, w \neq 1$ and $a, b \in \mathbf{C} - \{0\}$. By Proposition 5.1 and Proposition 5.2, W is type II equivalent to $U \otimes (V_1, V_2)$.

Remarks Let $C_n \in \mathrm{Mat}_n(\boldsymbol{C})$ denote a permutation matrix defined by $C_n[i, j] = \delta_{i+1,j}$, where indices are regarded as elements in $\mathbf{Z}_n$. For type II matrices W of size 6 one of the following holds.

(i) $\mathcal{N}(W) = \mathrm{Span}(I, J)$.

(ii) W is type II equivalent to a cyclic type II matrix $W(\mathbf{Z}_6)$ and there is a permutation matrix $S \in \mathrm{Mat}_n(\boldsymbol{C})$ such that

$$S\mathcal{N}(W)^{\mathrm{t}}S = \mathcal{N}(SW) = \mathrm{Span}(I, C, C^2, \ldots, C^5),$$

where $C = C_6$.

(iii) W or ${}^{\mathrm{t}}W$ is type II equivalent to $U \otimes (V_1, V_2)$, where (a, b) is not a member of $\{(\pm 1, \pm 1), (\pm w, \pm w^2), (\pm w^2, \pm w)\}$. Moreover, there are permutation matrices $S, T \in \mathrm{Mat}_n(\boldsymbol{C})$ such that

$$S\mathcal{N}(W)^{\mathrm{t}}S = \mathcal{N}(SW) = \mathrm{Span}(I, C + C^3 + C^5, C^2, C^4),$$
$$T\mathcal{N}({}^{\mathrm{t}}W)^{\mathrm{t}}T = \mathcal{N}(T\,{}^{\mathrm{t}}W) = \mathrm{Span}(I, C + C^4, C^2 + C^5, C^3),$$

where $C = C_6$.

The statement (iii) implies the existence of the generalized tensor product which is essentially different from an ordinary tensor product since $\mathcal{N}(U \otimes (V_1, V_2)) \neq \mathcal{N}(U) \otimes \mathcal{N}(V_i) = \mathcal{N}(U \otimes V_i) = \mathcal{N}(W(\mathbf{Z}_6))$ for $i = 1, 2$, where U, V_1, V_2 are the matrices defined above and (a, b) satisfies the condition in the statement (iii).

5.2.2. The case of size 8. Let W be a type II matrix of size 8. According to the classification of association schemes of size 8 [11], it is clear that $\mathcal{N}(W)$ is imprimitive. Hence by Theorem 1.1, W is type II equivalent to a generalized tensor product of type II matrices of size 2 and those of size 4.

Let

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\lambda & \lambda \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \mu & -\mu \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\mu & \mu \end{bmatrix},$$

where $\lambda, \mu \in \mathbf{C} - \{0\}$. By Proposition 5.1 and Proposition 5.2, W is type II equivalent to $U \otimes (V_1, SDV_2)$ where S is a permutation matrix of size 4 and D is a diagonal matrix of size 4.

5.2.3. The case of size 10. Let W be a type II matrix of size 10. According to the classification of association schemes of size 10 [11], it is clear that $\mathcal{N}(W)$ is imprimitive. Hence by Theorem 1.1, W is type II equivalent to a generalized tensor product of type II matrices of size 2 and those of size 5.

Let

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \eta & \eta^2 & \eta^3 & \eta^4 \\ 1 & \eta^2 & \eta^4 & \eta & \eta^3 \\ 1 & \eta^3 & \eta & \eta^4 & \eta^2 \\ 1 & \eta^4 & \eta^3 & \eta^2 & \eta \end{bmatrix}, \quad V_2 = \begin{bmatrix} \alpha & 1 & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 & 1 \\ 1 & 1 & \alpha & 1 & 1 \\ 1 & 1 & 1 & \alpha & 1 \\ 1 & 1 & 1 & 1 & \alpha \end{bmatrix},$$

where $\eta^5 = 1$, $\eta \neq 1$, and α satisfies the equation $\alpha + \alpha^{-1} + 3 = 0$. By Proposition 5.1 and Proposition 5.2, W is type II equivalent to one of the following.

(i) $U \otimes (V_1, SDV_1)$,
(ii) $U \otimes (V_1, SDV_2)$,
(iii) $U \otimes (V_2, SDV_1)$,
(iv) $U \otimes (V_2, SDV_2)$,

where S is a permutation matrix of size 5 and D is a nonsingular diagonal matrix of size 5. See also [14].

6. Type II matrices of size 7

For the determination of type II matrices of size 7, we need to consider the case $\mathcal{N}(W)$ is primitive. In the first subsection, we prove a lemma which will be helpful to determine them.

6.1. Submatrices of type II matrices

Lemma 6.1 *Let U be a square matrix of size* 3, *whose entries are all nonzero. Let α be a complex number satisfying $\alpha\bar{\alpha} \neq 1$.*

(1) *Suppose $UJ = JU = \alpha J$ and $U^{-}J = JU^{-} = \bar{\alpha}J$. Then there are complex numbers u, v and w and a permutation matrix S of size* 3 *such that*

$$US = \begin{bmatrix} u & v & w \\ w & u & v \\ v & w & u \end{bmatrix}.$$

(2) *Moreover, suppose*

$$UU^{-} = \begin{bmatrix} 3 & \alpha & \bar{\alpha} \\ \bar{\alpha} & 3 & \alpha \\ \alpha & \bar{\alpha} & 3 \end{bmatrix}.$$

Then there is a complex number γ such that γu, γv, and γw are the roots of $x^3 - \alpha x^2 + \bar{\alpha}x - 1 = 0$.

Proof: Let $U[i, j] = u_{i,j}$, $s_i = u_{i,1}u_{i,2}u_{i,3}$ and $t_j = u_{1,j}u_{2,j}u_{3,j}$, where $1 \leq i, j \leq 3$.

(1) By our assumption, we have the following.

$$\alpha = u_{i,1} + u_{i,2} + u_{i,3} = u_{1,j} + u_{2,j} + u_{3,j} \quad \text{for every } 1 \leq i, j \leq 3.$$

$$\bar{\alpha} = \frac{1}{u_{i,1}} + \frac{1}{u_{i,2}} + \frac{1}{u_{i,3}} = \frac{1}{u_{1,j}} + \frac{1}{u_{2,j}} + \frac{1}{u_{3,j}}$$

$$= \frac{u_{i,2}u_{i,3} + u_{i,1}u_{i,3} + u_{i,1}u_{i,2}}{s_i} = \frac{u_{2,j}u_{3,j} + u_{1,j}u_{3,j} + u_{1,j}u_{2,j}}{t_j},$$

for every $1 \leq i, j \leq 3$.

Hence $u_{i,j}$ is a common root of the following equations.

$$x^3 - \alpha x^2 + s_i\bar{\alpha}x - s_i = 0,$$
$$x^3 - \alpha x^2 + t_j\bar{\alpha}x - t_j = 0.$$

Hence it is a root of the difference $(s_i - t_j)(\bar{\alpha}x - 1) = 0$. If $\bar{\alpha}x - 1 = 0$, then $u_{i,j}$ is a root of $x^3 - \alpha x^2 = 0$. Since $u_{i,j} \neq 0$, this implies $u_{i,j} = \alpha$ and $\alpha\bar{\alpha} = 1$. This is not the case. Thus $s_i = t_j$ for all i, j. Let $s = s_i = t_j$. We conclude that the both sets $\{u_{i,1}, u_{i,2}, u_{i,3}\}$ and $\{u_{1,j}, u_{2,j}, u_{3,j}\}$ coincide the set of roots of the equation,

$$x^3 - \alpha x^2 + s\bar{\alpha}x - s = 0.$$

Let $\{u, v, w\}$ be the set of the roots of the equation above. Then we have the assertion.

(2) Since $UU^-[1, 2] = \alpha$ and $UU^-[2, 1] = \bar{\alpha}$, we have

$$\frac{u}{w} + \frac{v}{u} + \frac{w}{v} = \alpha,$$
$$\frac{1}{u/w} + \frac{1}{v/u} + \frac{1}{w/v} = \bar{\alpha},$$
$$\frac{u}{w} \cdot \frac{v}{u} \cdot \frac{w}{v} = 1.$$

Hence the set $\{u/w, v/u, w/v\}$ coincides with the set of the roots of the following equation.

$$x^3 - \alpha x^2 + \bar{\alpha}x - 1 = 0.$$

Now the assertions follow. □

6.2. *Type II matrices of size 7*

A commutative association scheme which properly contains the association scheme of class 1 has the relation matrix defined as follows.

$$R = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 & 2 \\ 2 & 0 & 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}, \quad P = \bar{Q} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & \alpha & \bar{\alpha} \\ 1 & \bar{\alpha} & \alpha \end{bmatrix},$$

where $\xi = e^{(2\pi\sqrt{-1})/7}$ and $\alpha = \xi^1 + \xi^2 + \xi^4$, $\bar{\alpha} = \xi^3 + \xi^5 + \xi^6$.

We now apply results in previous sections to determine type II matrices W of size 7 such that $\mathcal{N}(W)$ contains the Bose-Mesner algebra of the commutative association scheme determined by the data above.

First assume that W is normalized in the sense of Lemma 3.3. In particular, the entries in the first row and the first column are 1, and the second to the fourth column vectors span the column space of E_1 and the fifth to the seventh column vectors span the column space of E_2. Using the relation matrix R above, we have the following. Here $\boldsymbol{j}$ denotes the all one column vector of length 3.

1. $\Pi_0 = \{1\}$, $\Pi_1 = \{2, 3, 4\}$, and $\Pi_2 = \{5, 6, 7\}$.
2. $\Lambda_0 = \{1\}$, $\Lambda_1 = \{4, 6, 7\}$, and $\Lambda_2 = \{2, 3, 5\}$.
3. $W_{0,0} = [1]$, $W_{0,1} = W_{0,2} = {}^t\boldsymbol{j}$, $W_{1,0} = W_{2,0} = \boldsymbol{j}$.
4. $W_{1,1} = W[\{4, 6, 7\}, \{2, 3, 4\}]$, $W_{1,2} = W[\{4, 6, 7\}, \{5, 6, 7\}]$, $W_{2,1} = W[\{2, 3, 5\}, \{2, 3, 4\}]$, and $W_{2,2} = W[\{2, 3, 5\}, \{5, 6, 7\}]$.

By Proposition 3.4, we have the following lemma.

Lemma 6.2 *The following hold.*
(1) $W_{2,1}\boldsymbol{j} = W_{1,2}\boldsymbol{j} = (W_{2,2})^-\boldsymbol{j} = (W_{1,1})^-\boldsymbol{j} = \alpha\boldsymbol{j}$.
(2) $W_{2,2}\boldsymbol{j} = W_{1,1}\boldsymbol{j} = (W_{2,1})^-\boldsymbol{j} = (W_{1,2})^-\boldsymbol{j} = \bar{\alpha}\boldsymbol{j}$.
(3) ${}^t\boldsymbol{j}W_{2,1} = {}^t\boldsymbol{j}W_{1,2} = {}^t\boldsymbol{j}(W_{2,2})^- = {}^t\boldsymbol{j}(W_{1,1})^- = \alpha{}^t\boldsymbol{j}$.
(4) ${}^t\boldsymbol{j}W_{2,2} = {}^t\boldsymbol{j}W_{1,1} = {}^t\boldsymbol{j}(W_{2,1})^- = {}^t\boldsymbol{j}W_{1,2} = \bar{\alpha}{}^t\boldsymbol{j}$.
(5) $W_{2,1}(W_{2,1})^- = W_{1,1}(W_{1,1})^- = M$.
(6) $W_{2,1}(W_{1,1})^- = W_{1,2}(W_{2,2})^- = T$.
(7) $W_{2,2}(W_{2,2})^- = W_{1,2}(W_{1,2})^- = \bar{M}$.
(8) $W_{2,2}(W_{1,2})^- = W_{1,1}(W_{2,1})^- = \bar{T}$.
Here bars denote the complex conjugates and

$$M = \begin{bmatrix} 3 & \alpha & \bar{\alpha} \\ \bar{\alpha} & 3 & \alpha \\ \alpha & \bar{\alpha} & 3 \end{bmatrix}, \quad \textit{and} \quad T = \begin{bmatrix} \bar{\alpha} & \bar{\alpha} & \alpha \\ \bar{\alpha} & \alpha & \bar{\alpha} \\ \alpha & \bar{\alpha} & \bar{\alpha} \end{bmatrix}.$$

Proposition 6.3 *Let W be a type II matrix of size 7. If $\mathcal{N}(W)$ contains a Bose-Mesner algebra of an association scheme isomorphic to the association scheme defined by the relation matrix R above, then $\mathcal{N}(W)$ has dimension 7 and it is isomorphic to the Bose-Mesner algebra of a regular group scheme of $\mathbf{Z}_7$.*

Proof: We use Lemma 6.1 to determine the possibilities of $W_{i,j}$ defined above. Let $\zeta = e^{2\pi\sqrt{-1}/7}$. Then the roots of the following equation $x^3 - \alpha x^2 + \bar{\alpha}x - 1 = 0$ are ζ, ζ^2 and ζ^4. Hence there is a permutation matrix S of size 3 and we have one of the following.

$$W_{2,1}S = U, \quad \text{or} \quad \frac{\alpha}{\bar{\alpha}} \cdot U^-, \quad \text{where } U = \begin{bmatrix} \zeta & \zeta^4 & \zeta^2 \\ \zeta^2 & \zeta & \zeta^4 \\ \zeta^4 & \zeta^2 & \zeta \end{bmatrix}.$$

Moreover, $(W_{1,1})^- = (W_{2,1})^{-1}T$ and $W_{1,1} = M((W_{1,1})^-)^{-1}$. It is easily checked by the calculation that $W_{2,1}S = U$ is the only possibility. Since the complex conjugates of $W_{2,2}$ and $W_{1,2}$ satisfy the same equation, the following is the only solution if we take suitable permutation matrices S_1 and S_2 of size 3.

$$W_{2,1}S_1 = U, \quad \text{and} \quad W_{1,2}S_2 = \bar{U}.$$

Hence we have the following.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^4 & \zeta^2 & \zeta^6 & \zeta^3 & \zeta^5 \\ 1 & \zeta^2 & \zeta^1 & \zeta^4 & \zeta^5 & \zeta^6 & \zeta^3 \\ 1 & \zeta^3 & \zeta^5 & \zeta^6 & \zeta^4 & \zeta^2 & \zeta \\ 1 & \zeta^4 & \zeta^2 & \zeta & \zeta^3 & \zeta^5 & \zeta^6 \\ 1 & \zeta^5 & \zeta^6 & \zeta^3 & \zeta^2 & \zeta & \zeta^4 \\ 1 & \zeta^6 & \zeta^3 & \zeta^5 & \zeta & \zeta^4 & \zeta^2 \end{bmatrix}.$$

Now the assertion is obvious. □

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