Journal of Algebraic Combinatorics **16** (2002), 301–314

Lie Representations and an Algebra Containing Solomon's

FRÉDÉRIC PATRAS patras@math.unice.fr
CNRS UMR 6621, Université de Nice, Mathématiques, Parc Valrose, 06108 Nice cedex 2, France

CHRISTOPHE REUTENAUER christo@lacim.uqam.ca
Université du Québec à Montréal, Mathématiques, Montréal, CP 8888 succ A, Canada H3C3P8

Received February 16, 2001; Revised July 11, 2002

Abstract. We introduce and study a Hopf algebra containing the descent algebra as a sub-Hopf-algebra. It has the main algebraic properties of the descent algebra, and more: it is a sub-Hopf-algebra of the direct sum of the symmetric group algebras; it is closed under the corresponding inner product; it is cocommutative, so it is an enveloping algebra; it contains all Lie idempotents of the symmetric group algebras. Moreover, its primitive elements are exactly the Lie elements which lie in the symmetric group algebras.

Keywords: descent algebra, Hopf algebra, Lie idempotent, symmetric group algebras, quasi-symmetric functions, Lie elements

1. Introduction

Recall the definition of the descent algebra. Let S_n denote the symmetric group of order n and consider in the group algebra $\mathbf{Q}[S_n]$ the elements D_I, indexed by subsets I of $\{1, \dots, n\}$ with $D_I := \sum_{Desc(\sigma)=I} \sigma$, where the descent set of $\sigma \in S_n$ is defined by:

$$Desc(\sigma) := \{i,\ 1 \leq i \leq n-1,\ \sigma(i) > \sigma(i+1)\}.$$

Let Σ_n denote the linear span of the D_I. The descent algebra Σ is the direct sum of the Σ_n:

$$\Sigma := \bigoplus_{n\geq 0} \Sigma_n \subset \mathbf{S} := \bigoplus_{n\geq 0} \mathbf{Q}[S_n].$$

The graded vector space $\mathbf{S}$ becomes a ring by setting $\sigma\alpha := 0$ if σ and α are not in the same S_n. Note that $\mathbf{S}$ is a ring without a unit for this product (called the inner product). Solomon proved that Σ is a subring of $\mathbf{S}$ for this product [15]. Besides this inner product, $\mathbf{S}$ has another product (the outer product) and a coproduct which make it a Hopf algebra, which is not cocommutative (so that $\mathbf{S}$ is not generated by its primitive elements, and is not an enveloping algebra). The descent algebra is also a cocommutative sub-Hopf-algebra of $\mathbf{S}$, and thus is the enveloping algebra of the Lie algebra of its primitive elements [5, 7]. It is identical with the algebra of noncommutative symmetric functions [3]. It is naturally dual to the algebra of quasi-symmetric functions [4, 7].

A descent algebra can also be naturally associated to any graded connected commutative or cocommutative bialgebra. Its properties yield, for example, new combinatorial proofs of the Cartier-Milnor-Moore [9] or the Leray theorems [10].

Besides its rich algebraic structure, a striking feature of the descent algebra is that it is big enough so as to contain all classical Lie idempotents of the symmetric group algebras, such as the Dynkin idempotents, the canonical idempotents, the Klyachko idempotents. However, if the classical Lie idempotents belong to the descent algebra, there are other Lie idempotents, which do not lie in it. Among others, there is the mysterious Garsia idempotent, which projects onto the free Lie algebra parallel to the space of proper shuffles, see [2]. Thus one needs, algebraically speaking, a greater algebra in order to study free Lie algebras from the representation-theoretic point of view. We introduce here such an algebra, denoted $\mathcal{A}$.

It has the main algebraic properties of the descent algebra, and more: it is a sub-Hopf-algebra of **S**, which is cocommutative, so it is the enveloping algebra of the Lie algebra of its primitive elements; it is closed for the inner product; it contains *all* Lie idempotents of the symmetric group algebras. Moreover, its primitive elements are exactly the Lie elements which lie in the symmetric group algebras (they span the Lie representation of the symmetric group). From this fact, we can deduce the Hilbert series of $\mathcal{A}$.

This algebra $\mathcal{A}$ also appears to be the natural algebraic setting for studying the Lie properties of the symmetric group algebras, such as the properties of the Lie morphisms (that is, the morphisms from the free associative algebra to the free Lie algebra which belong to the symmetric group algebras). For example, the maps from the free associative algebra to higher components of the derived series of the free Lie algebra which belong to the symmetric group algebras are Lie morphisms and belong to this new algebra (cf. [12]), as do mappings related to subspaces of the free Lie algebra which are defined by the geometry of the bracketings, see Barcelo-Sundaram [1].

2. Tensor algebra and symmetric groups

The ground field is fixed once for all in the article: it is the field $\mathbf{Q}$ of rational numbers. Tensor products, vector spaces, ... have to be understood as tensor products over $\mathbf{Q}$, $\mathbf{Q}$-vector spaces, ...

Let $\mathcal{T}$ be the tensor algebra on an infinite alphabet X: $\mathcal{T} = \oplus_{n\in\mathbf{N}} \mathcal{T}_n = \oplus_{n\in\mathbf{N}}(\mathbf{Q}X)^{\otimes n}$, where $\mathbf{Q}X$ stands for the $\mathbf{Q}$-vector space spanned by X. The tensor algebra is naturally graded. We use the word notation for the elements of $\mathcal{T}$. For example, we write $y_1 \cdots y_n$ for $y_1 \otimes \cdots \otimes y_n \in \mathcal{T}_n$, where the y_is are elements of X. Such a tensor will be called a *word*; the words form a basis of $\mathcal{T}$. We shall assume that X contains enough elements: that is, it contains $\mathbf{N}$, and an element x_n for each $n \in \mathbf{N}$. We shall often represent permutations by words; so they are themselves elements of $\mathcal{T}$. For example, the permutation $\alpha \in S_n$ is represented by a word of lenght n: $\sigma = \sigma(1)\cdots\sigma(n)$; in general, we shall call *numerical words* the words whose letters belong to $\mathbf{N}$.

There is a right action of S_n on $\mathcal{T}$ defined by:

$$\forall \sigma \in S_n,\ y_1 \cdots y_n \cdot \sigma := y_{\sigma(1)} \cdots y_{\sigma(n)}$$

and $y_1 \cdots y_m \cdot \sigma = 0$ if $n \neq m$. Notice that this action is not a group action, since the unit of S_n does not act as the identity map on $\mathcal{T}$. It follows from this definition that the direct sum of the opposite algebras to the symmetric group algebras $\oplus_{n\in\mathbf{N}}\mathbf{Q}[S_n]^{op}$, with the algebra structure defined componentwise, embeds into the algebra of graded linear endomorphisms of $\mathcal{T}$, $End(\mathcal{T}) = \Pi_{n\in\mathbf{N}}End(\mathcal{T}_n)$. We write $\circ$ for the corresponding product on $\mathbf{S} = \oplus_{n\in\mathbf{N}}\mathbf{Q}[\mathbf{S_n}]$ and $\sigma(y_1 \cdots y_n)$ for $y_1 \cdots y_n \cdot \sigma$. Notice that, if we write $\cdot$ for the usual product on S_n (the composition of permutations), we have:

$$\forall(\sigma, \beta, \gamma) \in S_n \times S_n \times S_m, n \neq m : \sigma \circ \beta = \beta \cdot \sigma, \sigma \circ \gamma = \gamma \cdot \sigma = 0.$$

Recall also some general facts on Hopf algebras. A graded Hopf algebra over $\mathbf{Q}$ is a graded vector space $A = \oplus_{n\in\mathbf{N}}A_n$ together with morphisms of graded vector spaces $\pi: A\otimes A \to A$, $\eta: \mathbf{Q} \to A$, $\delta: A \to A\otimes A$, $\zeta: A \to \mathbf{Q}$ and $S: A \to A$. Here, the graduation on $A \otimes A$ is defined by: $(A \otimes A)_n := \oplus_{i+j=n}A_i \otimes A_j$, and $\mathbf{Q}$ is identified with the graded vector space with $\mathbf{Q}$ as a unique non trivial component, in degree 0. These morphisms, called respectively the product, the unit, the coproduct, the counit and the antipode of A, are subject to the following conditions:

1. The product and the unit provide A with the structure of an associative algebra with unit,
2. The coproduct and the counit provide A with the dual structure of a coassociative coalgebra with counit,
3. The coproduct is a morphism of algebras from A to $A \otimes A$ or, equivalently, the product is a morphism of coalgebras from $A \otimes A$ to A,
4. The antipode S satisfies the equation: $\pi \circ (S \otimes I) \circ \delta = \pi \circ (I \otimes S) \circ \delta = \eta \circ \zeta$, where we write I for the identity of A.

A graded vector space together with morphisms π, η, δ, ζ satisfying conditions 1–3 is called a graded bialgebra. A graded vector space V is connected if $V_0 = \mathbf{Q}$. A coalgebra C (resp. a bialgebra, a Hopf algebra) is cocommutative if the coproduct is cocommutative, that is if $T \circ \delta = \delta$, where T is the intertwining operator acting on $C \otimes C$: $T(x \otimes y) := y \otimes x$. An element a in a Hopf algebra is called primitive if and only if $\delta(a) = a \otimes 1 + 1 \otimes a$. A graded connected cocommutative bialgebra is naturally provided with the structure of a Hopf algebra: the antipode acts as –1 on the graded vector space of primitive elements of the bialgebra and is characterized by this property. See e.g. [8, 16] for further details on Hopf algebras.

The tensor algebra $\mathcal{T}$ is the free associative algebra on X and we write μ for its product:

$$\mu : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T},$$
$$\mu(y_1 \cdots y_n \otimes z_1 \cdots z_l) = y_1 \cdots y_n z_1 \cdots z_l.$$

We write ϵ for the empty word, which is the unit of $\mathcal{T}$. The tensor algebra $\mathcal{T}$ is a graded connected cocommutative Hopf algebra for the coproduct δ and antipode S defined on the generators y of $\mathcal{T}$ by:

$$\delta(y_i) := y_i \otimes \epsilon + \epsilon \otimes y_i$$
$$S(y_i) := -y_i.$$

The unit $\eta : \mathbf{Q} \to \mathcal{T}$ and counit $\zeta : \mathcal{T} \to \mathbf{Q}$ are defined using the isomorphism: $\mathbf{Q} \cong \mathcal{T}_0 \subset \oplus_{n\in\mathbf{N}}\mathcal{T}_n = \mathcal{T}$.

A closed formula for δ can be given as follows. Write $\mathcal{P}(n)$ for the set of couples (I, J) of disjoint complementary subsets of $[n] = \{1, \ldots, n\}$, with $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_{n-k}\}$. Then:

$$\delta(y_1 \cdots y_n) = \sum_{(I,J)\in\mathcal{P}(n)} y_{i_1} \cdots y_{i_k} \otimes y_{j_1} \cdots y_{j_{n-k}}.$$

For example:

$$\begin{aligned}\delta(x_2x_1x_3) = {} & x_2x_1x_3 \otimes \epsilon + x_2x_1 \otimes x_3 + x_1x_3 \otimes x_2 + x_2x_3 \otimes x_1 + x_2 \otimes x_1x_3 \\ & + x_1 \otimes x_2x_3 + x_3 \otimes x_2x_1 + \epsilon \otimes x_2x_1x_3.\end{aligned}$$

3. The coproduct

Definition 1 Let $s \in \mathbf{S}$. We say that s has a coproduct in $\mathbf{S}$ if there exists $\tilde{s}$ in $\mathbf{S} \otimes \mathbf{S} \subset End(\mathcal{T} \otimes \mathcal{T})$ such that:

$$\tilde{s} \circ \delta = \delta \circ s.$$

We will see below that such an $\tilde{s}$, if it exists, is unique.

Example: Let $s := 12 - 21 \in \mathbf{Q}[S_2]$. Then, for all (x, y) in $X \times X$, we have:

$$s(xy) = xy - yx$$

and $\delta \circ s(xy) = s(xy) \otimes \epsilon + \epsilon \otimes s(xy)$. Therefore, $\tilde{s} := s \otimes \zeta + \zeta \otimes s$ is a coproduct in $\mathbf{S}$ for s.

Counterexample: Let $s := 213 \in S_3$. If s had a coproduct in $\mathbf{S}$, the corresponding element $\tilde{s}$ of $\mathbf{S} \otimes \mathbf{S}$ should satisfy the equation $\tilde{s} \circ \delta(x_1x_2x_3) = \delta \circ s(x_1x_2x_3) = \delta(x_2x_1x_3)$. In particular, the equation:

$$\phi(x_1x_2 \otimes x_3 + x_1x_3 \otimes x_2 + x_2x_3 \otimes x_1) = x_1x_3 \otimes x_2 + x_2x_3 \otimes x_1 + x_2x_1 \otimes x_3$$

should have a solution ϕ in $\mathbf{Q}[S_2] \otimes \mathbf{Q}[S_1]$. An easy computation shows that this is not the case (see also the end of Section 4).

Lemma 2 *Assume that $\tau \in \mathbf{S} \otimes \mathbf{S}$ is such that $\tau \circ \delta = 0$, then $\tau = 0$.*

To prove the lemma, notice first that, since the action of $\mathbf{S} \otimes \mathbf{S} = \oplus_{n,m}\mathbf{Q}[S_n] \otimes \mathbf{Q}[S_m]$ on $\mathcal{T} \otimes \mathcal{T}$ preserves the direct sum decomposition $\oplus_{n,m}\mathcal{T}_n \otimes \mathcal{T}_m$ of $\mathcal{T} \otimes \mathcal{T}$, we may assume that $\tau \in \mathbf{Q}[S_n] \otimes \mathbf{Q}[S_m]$.

Besides, $\mathcal{T}$ is naturally a multigraded algebra: $\mathcal{T} = \oplus_{\alpha\in\mathcal{I}}\mathcal{T}_\alpha$. Here, $\mathcal{I}$ is the set of functions α with finite support from X to $\mathbf{N}$, and $\mathcal{T}_\alpha$ is the span of the words having, for any letter $x \in X$, $\alpha(x)$ occurences of x.

The decomposition $\mathcal{T} = \oplus_{\alpha\in\mathcal{I}}\mathcal{T}_\alpha$ is invariant under the action of $\mathbf{S}$, and the same property holds for the decomposition $\oplus_{\alpha,\alpha'\in\mathcal{I}\times\mathcal{I}}\mathcal{T}_\alpha \otimes \mathcal{T}_{\alpha'}$ of $\mathcal{T}\otimes\mathcal{T}$ under the action of $\mathbf{S}\otimes\mathbf{S}$.

In particular, assume that $\tau \circ \delta = 0$, where $\tau \in \mathbf{Q}[S_n] \otimes \mathbf{Q}[S_m]$. Then, since $x_1 \cdots x_n \otimes x_{n+1} \cdots x_{n+m}$ is the projection of $\delta(x_1 \cdots x_{n+m})$ on one of the components of the decomposition $\oplus_{\alpha,\alpha'\in\mathcal{I}\times\mathcal{I}}\mathcal{T}_\alpha \otimes \mathcal{T}_{\alpha'}$ we must have:

$$\tau(x_1 \cdots x_n \otimes x_{n+1} \cdots x_{n+m}) = 0.$$

Write $\mathcal{T}_{(n,m)}$ for the component of $\oplus_{\alpha,\alpha'\in\mathcal{I}\times\mathcal{I}}\mathcal{T}_\alpha\otimes\mathcal{T}_{\alpha'}$ containing $x_1 \cdots x_n \otimes x_{n+1} \cdots x_{n+m}$. As a $(S_n \times S_m)^{op}$ -module, $\mathcal{T}_{(n,m)}$ is isomorphic to the regular (right) representation of $S_n \times S_m$ and is generated by $x_1 \cdots x_n \otimes x_{n+1} \cdots x_{n+m}$. This implies $\tau = 0$ and the lemma follows.

It follows from the lemma that, if $s \in \mathbf{S}$ has a coproduct $\tilde{s}$ in $\mathbf{S}$, $\tilde{s}$ is uniquely defined.

Proposition-Definition 3 Let $\mathcal{A} := \{s \in \mathbf{S} \mid \exists\tilde{s} \in \mathbf{S}\otimes\mathbf{S}, \tilde{s}\circ\delta = \delta\circ s\}$. Then, we also have:

$$\mathcal{A} = \{s \in \mathbf{S} \mid \exists!\tilde{s} \in \mathbf{S}\otimes\mathbf{S}, \tilde{s}\circ\delta = \delta\circ s\}.$$

For $s \in \mathcal{A}$, we call from now on the unique element $\tilde{s} \in \mathbf{S}\otimes\mathbf{S}$ the coproduct of s and denote it by $\Delta(s)$.

For the time being, $\mathcal{A}$ is just a subspace of $\mathbf{S}$ and Δ maps $\mathcal{A}$ to $\mathbf{S}\otimes\mathbf{S}$. We shall see that $\mathcal{A}$ has a rich algebraic structure. Before that, we characterize combinatorially the elements of $\mathcal{A}$, and deduce that Δ maps $\mathcal{A}$ into $\mathcal{A}\otimes\mathcal{A}$.

4. A combinatorial characterization of $\mathcal{A}$

Recall that, by Weyl duality, a linear endomorphism of $\mathcal{T}_n$ is in $\mathbf{Q}[S_n]$ if and only if it commutes with each homogeneous algebra endomorphism of $\mathcal{T}$. On the other hand, let $\mathcal{T}_{(n)}$ be the component of $\mathcal{T} = \oplus_{\alpha\in\mathcal{I}}\mathcal{T}_\alpha$ spanned by $x_1 \cdots x_n$. For any $\alpha \in \mathcal{I}$ such that $\sum_{x\in X}\alpha(x) = n$, there are homogeneous algebra endomorphisms of $\mathcal{T}$ mapping $\mathcal{T}_{(n)}$ onto $\mathcal{T}_\alpha$.

Since δ, elements of $\mathbf{S}$, and elements of $\mathbf{S}\otimes\mathbf{S}$ commute with homogeneous algebra endomorphisms of $\mathcal{T}$, it follows from the previous remark that $f \in \mathbf{Q}[S_n]$ is in $\mathcal{A}$ if and only if there exists $\tilde{f} \in \mathbf{S}\otimes\mathbf{S}$ such that:

$$\tilde{f}\circ\delta(x_1 \cdots x_n) = \delta\circ f(x_1 \cdots x_n)\ (*)$$

In other words, to check if an element f in $\mathbf{Q}[S_n]$ belongs to $\mathcal{A}_n$, it is enough to check that the equation $(*)$ has a solution $\tilde{f}$ in $\oplus_{m\leq n}\mathbf{Q}[S_m]\otimes\mathbf{Q}[S_{n-m}]$.

We need some notation and conventions. The *standard permutation* $st(w)$ associated to a numerical word $w = w_1 \cdots w_k$ of length $k \leq n$, which is *multilinear*, that is, without repetition of letters, is the unique permutation in S_k defined by:

$$st(w)(i) < st(w)(j) \Leftrightarrow w_i < w_j.$$

For example $st(3745) = 1423$. We also write st for the extension by linearity of st.

Furthermore, if $I \subset X$, define a linear endomorphism P_I of $\mathcal{T}$ as follows: for any word w, either each letter in I appears in w and then $P_I(w)$ is obtained by removing in w each letter not in I; or some letter in I is not in w and we put $P_I(w) = 0$. For example, if $I = \{1, 2\}$ and $w = 24351$, then $P_I(w) = 21$ and $P_I(352) = 0$. Note that if $P_I(w) \neq 0$, then the set of letters in $P_I(w)$ is I.

Theorem 4 *An element $f \in \mathbf{Q}[S_n]$ is in $\mathcal{A}_n$ if and only if it satisfies the following property: for any disjoint ordered subsets I and J of $[n]$ such that $I \cup J = [n]$,*

$$(st \otimes st) \circ (P_I \otimes P_J)(f)$$

depends only on $|I|$ and $|J|$. In this case, $\Delta(f) = \sum_{0 \leq i \leq n}(st \otimes st) \circ (P_{\{1,\ldots,i\}} \otimes P_{\{i+1,\ldots,n\}})(f)$.

Note that $P_I \otimes P_J$ denotes the natural linear mapping from $\mathcal{T}$ into $\mathcal{T} \otimes \mathcal{T}$ defined on the words w of $\mathcal{T}$ by $(P_I \otimes P_J)(w) = P_I(w) \otimes P_J(w)$; from the context there should be no confusion with the actual tensor product of the mappings P_I and P_J, also written $P_I \otimes P_J$.

To prove the theorem, suppose that $(*)$ holds. Replacing the letter x_i by the letter i, this equation may be rewritten equivalently as $\tilde{f} \circ \delta(1 \cdots n) = \delta \circ f(1 \cdots n)$. This becomes $\tilde{f} \circ \delta(1 \cdots n) = \delta(f)$, since f may be viewed as an element of $\mathcal{T}$.

Now, if g is a linear combination of permutations in S_n, then $\delta(g)$ is equal to $\sum_{(I,J)}(P_I \otimes P_J)(g)$, where the sum is over all $(I, J) \in \mathcal{P}(n)$. In particular $\delta(1 \cdots n) = \sum_{I,J} P_I(1 \cdots n) \otimes P_J(1 \cdots n) = \sum_{I,J} \sigma_I \otimes \sigma_J$, where we write σ_I for the product in increasing order of the elements in I. For example $\delta(123) = 123 \otimes \epsilon + 12 \otimes 3 + 13 \otimes 2 + 23 \otimes 1 + 1 \otimes 23 + 2 \otimes 13 + 3 \otimes 12 + \epsilon \otimes 123$.

Thus we obtain $\sum_{I,J} \tilde{f}(\sigma_I \otimes \sigma_J) = \sum_{I,J}(P_I \otimes P_J)(f)$. For reasons of multi-homogeneity, this equality splits into many equalities: $\forall (I, J)$, $\tilde{f}(\sigma_I \otimes \sigma_J) = (P_I \otimes P_J)(f)$. Note that for fixed I, J, the latter equality is equivalent to $(st \otimes st)(\tilde{f}(\sigma_I \otimes \sigma_J)) = (st \otimes st)((P_I \otimes P_J)(f))$. But the left-hand side is equal to $\tilde{f}(\sigma_{\{1,\ldots,|I|\}} \otimes \sigma_{\{1,\ldots,|J|\}}) = \tilde{f}_{|I|,|J|}$, if we write $\tilde{f} = \sum_{i+j=n} \tilde{f}_{i,j}$, with $\tilde{f}_{i,j} \in \mathbf{Q}[S_i] \otimes \mathbf{Q}[S_j]$. So the right-hand depends only on $|I|, |J|$, what was to be shown.

Conversely, suppose that the property of the theorem holds. Define $\tilde{f} = \Delta(f)$ as in the statement. Note that $(st \otimes st) \circ (P_I \otimes P_J)(f)$ is equal to $(st \otimes st) \circ (P_{\{1,\ldots,i\}} \otimes P_{\{i+1,\ldots,n\}})(f)$, with $i = |I|$. Hence to $\tilde{f}_{i,n-i}$. Thus the previous calculations taken backwards imply $(*)$, hence the theorem.

We give now another characterization of the elements of $\mathcal{A}$. Denote by ω the shuffle product and by $\langle\, , \rangle$ the scalar product on $\mathcal{T}$ for which the set of all words is an orthonormal basis; this scalar product extends naturally to $\mathcal{T} \otimes \mathcal{T}$, with the same notation. Then it is well-known that for any elements t, u, v in $\mathcal{T}$, one has $\langle \delta(t), u \otimes v \rangle = \langle t, u \omega v \rangle$, see [13] Proposition I.1.8.

Note that if u, v are multilinear numerical words, having both the same set of letters, then $\langle u, v\rangle = \langle st(u), st(v)\rangle$. This equality extends linearly and to the tensor product.

Corollary 5 *An element $f \in \mathbf{Q}[S_n]$ is in $\mathcal{A}_n$ if and only if, for any words u and v such that $uv \in S_n$, $\langle f, u\omega v\rangle$ depends only on $st(u)$ and $st(v)$. In this case,*

$$\langle \Delta(f), st(u) \otimes st(v)\rangle = \langle f, u\omega v\rangle.$$

Indeed, let A, B be the set of letters appearing in u, v respectively. Then $\langle f, u\omega v\rangle = \langle \delta(f), u \otimes v\rangle = \langle (P_A \otimes P_B)(\delta(f)), u \otimes v\rangle = \langle (P_A \otimes P_B)(\sum_{(I,J)}(P_I \otimes P_J)(f)), u \otimes v\rangle = \langle (P_A \otimes P_B)(f), u \otimes v\rangle = \langle (st \otimes st) \circ (P_A \otimes P_B)(f), st(u) \otimes st(v)\rangle$. The first part of the corollary follows therefore from Theorem 4. For the last assertion, we have $\langle \Delta(f), st(u) \otimes st(v)\rangle = \langle \sum_{0\le i\le n}(st \otimes st) \circ (P_{\{1,\ldots,i\}} \otimes P_{\{i+1,\ldots,n\}})(f), st(u) \otimes st(v)\rangle = \langle (st \otimes st)(P_{\{1,\ldots,a\}} \otimes P_{\{a+1,\ldots,n\}})(f), st(u) \otimes st(v)\rangle$, if we put $a = |A|$. This is equal by the theorem to $\langle (st \otimes st)(P_A \otimes P_B)(f), st(u) \otimes st(v)\rangle$ which is equal to $\langle f, u\omega v\rangle$ by our previous computation.

Another consequence is the following.

Corollary 6 *If f is in $\mathcal{A}_n$, then $\delta(f)$ and $\Delta(f)$ are related as follows*:

$$\Delta(f) = \sum_{0\le i\le n} (st \otimes st) \circ \big(P_{\{1,\ldots,i\}} \otimes P_{\{i+1,\ldots,n\}}\big)(\delta(f))$$

and

$$\delta(f) = \sum_{uv\in S_n} \langle \Delta(f), st(u) \otimes st(v)\rangle u \otimes v.$$

In order to illustrate the previous results, consider $f = 213 + 312$. With I equal in turn to $\{1\}, \{2\}, \{3\}$ and J to its complement in $\{1, 2, 3\}$, we have: $(P_I \otimes P_J)(f) = 1 \otimes 23 + 1 \otimes 32, 2 \otimes 13 + 2 \otimes 31, 3 \otimes 21 + 3 \otimes 12$. After standardization, the three become all equal to $1 \otimes 12 + 1 \otimes 21$. Moreover, $\delta(f) = f \otimes \epsilon + 1 \otimes 23 + 1 \otimes 32 + 2 \otimes 13 + 2 \otimes 31 + 3 \otimes 21 + 3 \otimes 12 + 21 \otimes 3 + 12 \otimes 3 + 13 \otimes 2 + 31 \otimes 2 + 23 \otimes 1 + 32 \otimes 1 + \epsilon \otimes f$. Finally, $f \in \mathcal{A}_3$ and $\Delta(f) = f \otimes \zeta + 1 \otimes 12 + 1 \otimes 21 + 21 \otimes 1 + 12 \otimes 1 + \zeta \otimes f$.

Consider now $g = 213$. Then, with the same sets I, J, we obtain in turn: $1 \otimes 23, 2 \otimes 13, 3 \otimes 21$. After standardization the first and the last become: $1 \otimes 12, 1 \otimes 21$, which are not equal. Hence g is not in $\mathcal{A}$.

5. The coalgebra structure

The goal of this section is to prove that Δ maps $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{A}$ and that it is cocommutative and coassociative.

Lemma 7 *The mapping Δ is a graded cocommutative and coassociative coproduct on $\mathcal{A}$.*

Assume that $f \in \mathcal{A}_n$. We want to show that $\Delta(f) \in \oplus_{0\le m\le n}\mathcal{A}_m \otimes \mathcal{A}_{n-m}$. By multigraduation and uniqueness arguments as in the previous sections, this amounts to prove that f has an iterated coproduct of order 3, that is that there exists $\bar{f} \in \mathbf{S}\otimes\mathbf{S}\otimes\mathbf{S}$ such that

$$\bar{f}\circ\delta^{[3]} = \delta^{[3]}\circ f,$$

where $\delta^{[3]} = (\delta\otimes I)\circ\delta = (I\otimes\delta)\circ\delta$, with I the identity of $\mathcal{T}$. Indeed, if this is shown, then write $\Delta(f) = \tilde{f} = \sum_i f_i\otimes g_i$; in order to show that $\tilde{f}$ is in $\mathcal{A}\otimes\mathcal{A}$, it is enough by symmetry to show, the f_i being chosen linearly independant, that the g_i are in $\mathcal{A}$. We may write $\bar{f} = \sum_i f_i\otimes\tilde{g}_i$, and we have by assumption $(\sum_i f_i\otimes\tilde{g}_i)\circ(I\otimes\delta)\circ\delta = (I\otimes\delta)\circ\delta\circ f$; this is rewritten $\sum_i(f_i\otimes(\tilde{g}_i\circ\delta))\circ\delta = (I\otimes\delta)\circ(\sum_i f_i\otimes g_i)\circ\delta = (\sum_i f_i\otimes(\delta\circ g_i))\circ\delta$. Now, as in Lemma 2, this implies that $\sum_i f_i\otimes(\tilde{g}_i\circ\delta) = \sum_i f_i\otimes(\delta\circ g_i)$, and finally that $\tilde{g}_i\circ\delta = \delta\circ g_i$, hence that g_i is in $\mathcal{A}$.

In order to prove that f has an iterated coproduct of order 3, the same arguments as in Section 4 show that this property is equivalent to: for all disjoint subsets I, J and K of $[n]$ such that $I\cup J\cup K = [n]$,

$$\begin{aligned}&(st\otimes st\otimes st)\circ(P_I\otimes P_J\otimes P_K)(f)\\ &= (st\otimes st\otimes st)\circ\left(P_{\{1,\ldots,|I|\}}\otimes P_{\{|I|+1,\ldots,|I|+|J|\}}\otimes P_{\{|I|+|J|+1,\ldots,n\}}\right)(f),\end{aligned}$$

or, equivalently, to: $(*)$ for any words u, v, w such that $uvw \in S_n$, $\langle f, u\,\omega\,v\,\omega\,w\rangle$ depends only on $st(u)$, $st(v)$ and $st(w)$.

Let us prove that this last property is satisfied. We write k (resp. l, m) for the degree of u (resp. of v and w). Notice that the requirements on u, v and w imply that $k+l+m = n$.

In the algebra spanned by numerical words, denote by $st^{(m)}$ the composition with st of the endomorphism acting on the generators by: $i\mapsto i+m$. For example, $st^{(2)}(31) = (43)$ since $st(31) = 21$. To prove $(*)$, it is enough to prove that $\langle f, u\,\omega\,v\,\omega\,w\rangle = \langle f, st(u)\omega\, st^{(k)}(v)\,\omega\, st^{(k+l)}(w)\rangle$.

We know from Corollary 5 the two words version of this: if x, y are words such that $xy \in S_n$, with x of length m, then:

$$\langle f, x\,\omega\,y\rangle = \left\langle f, st(x)\omega\, st^{(m)}(y)\right\rangle.(**)$$

This equality extends linearly.

Thus we have $\langle f, u\,\omega\,v\,\omega\,w\rangle = \langle f, st(u\ \omega\ v)\omega\, st^{(k+l)}(w)\rangle$.

Define the words u', v', of respective length k, l, by $u'v' \in S_{k+l}$ and $st(uv) = u'v'$. Let also $w' = st^{(k+l)}(w)$. Then $st(u') = st(u), st(v') = st(v), st(u\omega v) = st(u'\ \omega\ v')$ and $w' = st^{(k+l)}(w')$. Observe that since v' has all its letters in the alphabet $\{1,\ldots,k+l\}$ and w' in $\{k+l+1,\ldots,n\}$, one has $st^{(k)}(v'\ \omega\ w') = st^{(k)}(v')\ \omega\ w'$. Note also that $u'v'w' \in S_n$. We deduce that

$$\langle f, u\,\omega\,v\,\omega\,w\rangle = \left\langle f, st(u'\omega v')\omega\, st^{(k+l)}(w')\right\rangle = \langle f, u'\ \omega\ v'\ \omega\ w'\rangle$$

by $(**)$. This is by $(**)$ equal to

$$\begin{aligned}\langle f, st(u')\omega\, st^{(k)}(v'\omega w')\rangle &= \langle f, st(u)\omega\, st^{(k)}(v')\omega w'\rangle \\ &= \langle f, st(u)\omega\, st^{(k)}(v)\omega\, st^{(k+l)}(w)\rangle.\end{aligned}$$

This proves what we wanted.

We show now that the coproduct Δ is cocommutative. Let $f \in \mathcal{A}$ and let T be the intertwining operator acting on $\mathcal{T} \otimes \mathcal{T}$. Since δ is cocommutative, we have: $T \circ \delta = \delta$. Therefore, $T \circ \Delta(f) \circ T \circ \delta = T \circ \Delta(f) \circ \delta = T \circ \delta \circ f = \delta \circ f$. Since we know that the equation $\tilde{f} \circ \delta = \delta \circ f$ has a unique solution $\tilde{f} = \Delta(f)$ in $\mathbf{S} \otimes \mathbf{S}$, we get: $\Delta(f) = T \circ \Delta(f) \circ T$.

On the other hand, let T' be the interwining operator acting on $\mathbf{S}\otimes\mathbf{S}$. We use the Sweedler notation [16, Section 1.2] and write $\sum_{(f)} f^{(1)} \otimes f^{(2)}$ for the coproduct of f. We then have: $\forall(u, v) \in \mathcal{T}^2$.

$$\begin{aligned}((T' \circ \Delta)(f))(u \otimes v) = (T'(\Delta(f)))(u \otimes v) &= \left(T'\left(\sum_{(f)} f^{(1)} \otimes f^{(2)}\right)\right)(u \otimes v) \\ &= \left(\sum_{(f)} f^{(2)} \otimes f^{(1)}\right)(u \otimes v) = \sum_{(f)} f^{(2)}(u) \otimes f^{(1)}(v) \\ &= (T \circ \Delta(f))(v \otimes u) = (T \circ \Delta(f) \circ T)(u \otimes v) = \Delta(f)(u \otimes v).\end{aligned}$$

Therefore, $T' \circ \Delta(f) = \Delta(f)$ and Δ is cocommutative.

Finally, we prove the coassociativity of Δ. Let $f \in \mathcal{A}_n$. We write $\mathcal{A}$ for the identity of $\mathcal{A}$. We then have:

$$\begin{aligned}((\Delta \otimes \mathcal{A}) \circ \Delta(f)) \circ \delta^{[3]} &= \sum_{(f)} \big((\Delta \otimes \mathcal{A})\big(f^{(1)} \otimes f^{(2)}\big)\big) \circ \delta^{[3]} \\ &= \sum_{(f)} \big(\Delta\big(f^{(1)}\big) \otimes f^{(2)}\big) \circ (\delta \otimes I) \circ \delta \\ &= \sum_{(f)} \big(\big(\Delta\big(f^{(1)}\big) \circ \delta\big) \otimes f^{(2)}\big) \circ \delta \\ &= \sum_{(f)} \big(\big(\delta \circ f^{(1)}\big) \otimes f^{(2)}\big) \circ \delta \\ &= \sum_{(f)} (\delta \otimes I) \circ \big(f^{(1)} \otimes f^{(2)}\big) \circ \delta = (\delta \otimes I) \circ \Delta(f) \circ \delta \\ &= (\delta \otimes I) \circ \delta \circ f = \delta^{[3]} \circ f.\end{aligned}$$

In the same way, we have:

$$((\mathcal{A} \otimes \Delta) \circ \Delta(f)) \circ \delta^{[3]} = \delta^{[3]} \circ f.$$

Besides, the same argument as in the proof of Lemma 2 shows that, if the equation $\tilde{f} \circ \delta^{[3]} = \delta^{[3]} \circ f$, where f is fixed, has a solution $\tilde{f}$ in $\mathbf{S} \otimes \mathbf{S} \otimes \mathbf{S}$, then $\tilde{f}$ is unique. Therefore,

$$((\mathcal{A} \otimes \Delta) \circ \Delta)(f) = ((\Delta \otimes \mathcal{A}) \circ \Delta)(f),$$

and the coproduct Δ is coassociative.

6. The Hopf algebra structure

The purpose of this section is to show that there is a graded connected cocommutative Hopf algebra structure on $\mathcal{A}$.

Recall that if V is a Hopf algebra, *End*(V) is endowed with the structure of an associative algebra, by the convolution product $*$ defined by:

$$\forall f, g \in \mathit{End}(V),\ f * g := \mu \circ (f \otimes g) \circ \delta.$$

In particular, *End*$(\mathcal{T})$ is an associative algebra for this product and a Weyl duality argument or a direct computation shows that $\mathbf{S}$ is a subalgebra of *End*$(\mathcal{T})$ for $*$ [13]. If V and V' are two Hopf algebras, the restriction of the convolution product on *End*$(V \otimes V')$ to *End*$(V) \otimes$ *End*(V') identifies with the product on *End*$(V) \otimes$ *End*(V') viewed as the tensor product of the two algebras *End*(V) and *End*(V'), equipped with the convolution product.

Lemma 8 *$\mathcal{A}$ is a subalgebra of $\mathbf{S}$ and of End$(\mathcal{T})$ for the convolution product.*

Indeed, let $f, g \in \mathcal{A}$. There is a Hopf algebra structure on $\mathcal{T} \otimes \mathcal{T}$, induced by the Hopf algebra structure on $\mathcal{T}$. For the corresponding convolution product, $\mathbf{S} \otimes \mathbf{S}$ is a subalgebra of *End*$(\mathcal{T} \otimes \mathcal{T})$ (this follows from the corresponding property for $\mathbf{S}$ and *End*$(\mathcal{T})$). We are going to prove that $\Delta(f) * \Delta(g)$, which belongs to $\mathbf{S} \otimes \mathbf{S}$, is a coproduct in $\mathbf{S}$ for $f * g$. According to the definition of $\mathcal{A}$, it will follow that $f * g \in \mathcal{A}$.

We write $\bar{\delta}$ (resp. $\bar{\mu}$) for the coproduct (resp. the product) on $\mathcal{T} \otimes \mathcal{T}$. Recall that we write T for the intertwining operator acting on $\mathcal{T} \otimes \mathcal{T}$. We then have:

$$\begin{aligned}(\Delta(f) * \Delta(g)) \circ \delta &= \bar{\mu} \circ (\Delta(f) \otimes \Delta(g)) \circ \bar{\delta} \circ \delta \\ &= \bar{\mu} \circ (\Delta(f) \otimes \Delta(g)) \circ (I \otimes T \otimes I) \circ (\delta \otimes \delta) \circ \delta\end{aligned}$$

(by definition of the Hopf algebra structure on the tensor product of two Hopf algebras)

$$= \bar{\mu} \circ (\Delta(f) \otimes \Delta(g)) \circ (\delta \otimes \delta) \circ \delta$$

(since δ is coassociative and cocommutative; the next identities follow from the definition of $\bar{\mu}$ and the fact that δ is a homomorphism)

$$\begin{aligned}&= \bar{\mu} \circ (\delta \otimes \delta) \circ (f \otimes g) \circ \delta \\ &= \delta \circ \mu \circ (f \otimes g) \circ \delta = \delta \circ (f * g).\end{aligned}$$

Hence $f * g$ has the coproduct $\Delta(f) * \Delta(g)$ and $\mathcal{A}$ is closed under convolution.

Theorem 9 *($\mathcal{A}$, $*$, Δ, u, e) is a graded connected cocommutative Hopf algebra.*

The unit of $\mathcal{A}$ is given by the inclusion: $u: \mathcal{A}_0 \cong \mathbf{Q} \to \mathcal{A}$. The counit is given by the projection $e : \mathcal{A} \to \mathcal{A}_0$. We have shown that $\mathcal{A}$ is a graded connected cocommutative coalgebra and an algebra. The equation $\Delta(f * g) = \Delta(f) * \Delta(g)$, which follows from our previous computatious, implies that Δ is an algebra map from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$. The existence of an antipode follows from the connectivity and the cocommutativity of $\mathcal{A}$: the antipode acts as -1 on the graded vector space of primitives and is characterized by this property. The theorem follows.

7. The inner product

Recall that the space $\mathbf{S}$, direct sum of all symmetric groups algebras, has a product, the *inner product* inherited from composition of permutations. The descent algebra has the striking property of being closed for this product, a result due originally to Solomon [15]. We show that $\mathcal{A}$ has the same property.

Theorem 10 *The vector spaces $\mathcal{A}_n \subset \mathbf{Q}[S_n] \subset End(\mathcal{T}_n)$, $n \in \mathbf{N}^*$ are closed under the composition of morphisms in $End(\mathcal{T}_n)$. Equivalently, $\mathcal{A}_n$ is closed under the products $\cdot$ and $\circ$ in $\mathbf{Q}[S_n]$. In particular, $\mathcal{A}_n$ is a subalgebra of the group algebra $\mathbf{Q}[S_n]$. Moreover Δ is a homomorphism for the inner product.*

The theorem follows immediately from the very definition of $\mathcal{A}$. Indeed, assume that f and g belong to $\mathcal{A}_n$, then we have:

$$\Delta(f) \circ \delta = \delta \circ f; \Delta(g) \circ \delta = \delta \circ g,$$

so that:

$$\Delta(f) \circ \Delta(g) \circ \delta = \Delta(f) \circ \delta \circ g = \delta \circ (f \circ g),$$

and $\Delta(f) \circ \Delta(g)$ is the coproduct of $f \circ g$. In particular, $f \circ g = g \cdot f \in \mathcal{A}_n$ and $\Delta(f \circ g) = \Delta(f) \circ \Delta(g)$.

8. Comparison with the descent algebra

First of all, let us show that $\mathcal{A}$ contains strictly the descent algebra. As a subalgebra of $\mathbf{S}$ for the convolution product, the descent algebra Σ is freely generated by the identity elements $1 \cdots n \in S_n, n \in \mathbf{N}^*$ [13]. Since these elements all have a coproduct in $\mathbf{S} \otimes \mathbf{S}$ (given explicitly by: $\Delta(1 \cdots n) = \sum_{i=0}^{n} 1 \cdots i \otimes 1 \cdots n-i$), Σ is certainly a subalgebra of $\mathcal{A}$. The fact that the inclusion is a strict one could be proved by general arguments involving the properties of the free Lie algebra or the Hilbert series computations below (Theorem 13). However, we think it useful and illuminating to give an example.

Let f be the linear combination of permutations in S_4 which maps $x_1 \dots x_4$ to the element $[[x_1, x_2], [x_3, x_4]]$ in the derived series of the free Lie algebra. A direct computation shows that:

$$f = 1234 - 2134 - 3412 + 3421 - 1243 + 2143 + 4312 - 4321.$$

Since f is a map from $\mathcal{T}$ to the free Lie algebra generated by X (that we view as a Lie subalgebra of $\mathcal{T}$), for all $t \in \mathcal{T}$, $f(t)$ is a primitive element and we therefore have:

$$\delta \circ f(t) = f(t) \otimes 1 + 1 \otimes f(t) = (f \otimes \zeta + \zeta \otimes f) \circ \delta(t),$$

so that f is in $\mathcal{A}$ (and is in fact a primitive element in $\mathcal{A}$). On the other hand, 2134 appears in the expansion of f but not 3124, which has the same descent set. It follows immediately that f does not belong to the Solomon algebra.

The same argument shows that, more generally, all the maps from the tensor algebra to higher components of the derived series of the free Lie algebra which belong to the symmetric group algebras also belong to the set of primitive elements in $\mathcal{A}$.

The last point that we want to investigate is the behavior of Δ with respect to the known Hopf algebra structure of $\mathbf{S}$ and on the descent algebra, which is a Hopf subalgebra of $\mathbf{S}$, cf. [3, 5–7, 11].

Theorem 11 *The Hopf algebra $\mathcal{A}$ is a Hopf subalgebra of* $\mathbf{S}$ *and Σ is a Hopf subalgebra of $\mathcal{A}$.*

Recall that the coproduct on $\mathbf{S}$ is defined for $f \in \mathbf{Q}[S_n]$ by [8]:

$$\Delta(f) = \sum_{0 \le i \le n} (st \otimes st)\big(P_{\{1,\dots,i\}} \otimes P_{\{i+1,\dots,n\}}\big)(f).$$

It follows from Theorem 4 that, if $f \in \mathcal{A}_n$, this coproduct is equal to the coproduct of f in $\mathbf{S}$ in the sense of Definition 1 and Proposition-Definition 3 above. Since Σ is a Hopf subalgebra of $\mathbf{S}$ and since $\mathcal{A}$ is a subalgebra of $\mathbf{S}$, the theorem follows.

9. Primitive elements and Hilbert series

According to Theorem 9 and to the Cartier-Milnor-Moore theorem [8, 9], $\mathcal{A}$ is the enveloping algebra of its primitive part. We write $Lie(X)$ for the free Lie algebra on X, identified with the Lie algebra of primitive elements in $\mathcal{T}$ (see [13], also for the general properties of the free Lie algebras and Lie representations that we are using hereafter).

Theorem 12 *The vector space $Prim_n\mathcal{A}$ of primitive elements of degree n in $\mathcal{A}$ is the vector space*:

$$Prim_n\mathcal{A} = \{f \in \mathbf{Q}[S_n] \subset End(\mathcal{T}_n) \mid Im(f) \subset Lie_n(X)\}$$

Moreover, $Prim_n\mathcal{A}$ *is canonically isomorphic* (*as a vector space*) *to the multilinear part of degree n of the free Lie algebra on n generators* $1, \dots, n$, *that is to the Lie representation* Lie_n *of* S_n.

Assume that f is a map whose image is contained in $Lie_n(X)$. Then, we have:

$$\forall t \in \mathcal{T}_n, \delta \circ f(t) = f(t) \otimes \epsilon + \epsilon \otimes f(t) = (f \otimes \zeta + \zeta \otimes f) \circ \delta,$$

and $f \in Prim_n\mathcal{A}$. Conversely, the same computation shows that if f is primitive, it is a map to the set of primitive elements in $\mathcal{T}$.

The last part of the theorem follows from the fact that elements f in $\mathcal{A}_n$ and their coproducts are characterized by the action of f on the numerical word $1 \dots n$ (see Section 4).

Note that this result implies that the Lie elements of **S** are primitive elements, for the coproduct defined in [7]; this fact, that was known to Daniel Krob (personal communication), may of course be proved directly.

The linear generators of Lie_n (the bracketings of $1, \dots, n$) may be represented graphically by binary trees with n leaves labelled by $1, \dots, n$ (see [1]). In particular, to each labelled tree is associated a primitive element of $\mathcal{A}_n$. It follows, for example, that the submodules of the action of the symmetric group S_n on Lie_n studied in [1] can be embedded naturally in $Prim_n\mathcal{A}$ and $\mathcal{A}_n$.

Since the dimension of the Lie representation of S_n is well-known (it is $(n-1)!$), the Hilbert series $Hilb(\mathcal{A})$ of $\mathcal{A}$ can be computed easily.

Theorem 13 *The Hilbert series of* $\mathcal{A}$ *is*:

$$Hilb(\mathcal{A}) = \prod_{n \geq 1} \left(\frac{1}{1 - t^n} \right)^{(n-1)!}.$$

The theorem follows from the Poincaré-Birkhoff-Witt theorem which implies that the Hilbert series of the enveloping algebra of a graded Lie algebra whose component of degree n has dimension α_n is $\prod_{n\geq 1}(\frac{1}{1-t^n})^{\alpha_n}$.

The first terms of the Hilbert series of $\mathcal{A}$ are:

$$Hilb(\mathcal{A}) = 1 + t + 2t^2 + 4t^3 + 11t^4 + 37t^5 + 167t^6 + 925t^7 + 6164t^8 + \cdots.$$

Note that $\mathcal{A}$ is not a subalgebra of the *Rahmenalgebra* of Armin Jöllenbeck [5]; the latter is not closed under the inner product, but has interesting applications to character theory of the symmetric group (it is actually a noncommutative character theory), and to enumeration of permutations. The algebra recently introduced by Manfred Schocker [14] is generated by the Lie idempotents of the symmetric group and is a proper subalgebra of $\mathcal{A}$; he has however theorems on the inner structure of his algebra and a mapping onto the character ring of the symmetric groups that has no analogue, for the time being, in $\mathcal{A}$.

References

1. H. Barcelo and S. Sundaram, "On some submodules of the action of the symmetric group on the free Lie algebra," *J. Algebra*. **154**(1) (1993), 12–26.
2. G. Duchamp, "Orthogonal projection onto the free Lie algebra," *Theor. Comput. Sci.* **79**(1) (1991), 227–239.
3. I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon, "Noncommutative symmetric functions," *Adv. Math.* **112**(2) (1995), 218–348.
4. I.M. Gessel, "Multipartite P-partitions and inner products of skew Schur functions," in *Combinatorics and algebra*, Boulder, CO., 1983, Contemp. Math., Vol. 34, Amer. Math. Soc., Providence, R.I., 1984, pp. 289–317.
5. A. Jöllenbeck, "Nichtkommutative Charaktertheorie der symmetrischen Gruppen," *Bayreuth. Math. Schr.* **56** (1999), 1–41.
6. D. Krob, B. Leclerc, and J.-Y. Thibon, "Noncommutative symmetric functions. II: Transformations of alphabets," *Int. J. Algebra Comput*. **7**(2) (1997), 181–264.
7. C. Malvenuto and C. Reutenauer, "Duality between quasi-symmetric functions and the Solomon descent algebra," *J. Algebra* **177**(3) (1995), 967–982.
8. J.W. Milnor and J.C. Moore, "On the structure of Hopf algebras," *Ann. Math.* Ser. II. **81** (1965), 211–264.
9. F. Patras, "L'algèbre des descentes d'une bigèbre graduée," *J. Algebra.* **170**(2) (1994), 547–566.
10. F. Patras and C. Reutenauer, "Higher Lie idempotents," *J. Algebra* **222**(1) (1999), 51–64.
11. S. Poirier and C. Reutenauer, "Algèbres de Hopf de tableaux," *Ann. Sci. Math. Québec* **19**(1) (1995), 79–90.
12. C. Reutenauer, "Dimensions and characters of the derived series of the free Lie algebra," in *Mots*, Mélanges offerts à Schützenberger, Hermès, 1990, pp. 171–184.
13. C. Reutenauer, *Free Lie algebras*, Oxford University Press, 1993.
14. M. Schocker, "Über die höheren Lie-Darstellungen der symmetrischen Gruppen," Ph.D. Thesis, Kiel, 2001.
15. L. Solomon, "A Mackey formula in the group algebra of a finite Coxeter group," *J. Algebra* **41** (1976), 255–268.
16. M. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.