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Polynomials with All Zeros Real and in a Prescribed Interval

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Abstract. We provide a characterization of the real-valued univariate polynomials that have only real zeros, all in a prescribed interval $[a, b]$. The conditions are stated in terms of positive semidefiniteness of related Hankel matrices.

Keywords: algebraic combinatorics, real algebraic geometry, the $\mathbb{K}$-moment problem

1. Introduction

From a fundamental result of Aissen et al. [1], a real-valued univariate polynomial has all its zeros real and nonpositive, if and only if a certain infinite Toeplitz matrix is totally nonnegative (see also [9, Theorem 1, p. 21]). However, despite its theoretical significance, this result involves checking *infinitely many* conditions, and therefore, cannot be applied directly for practical purposes (see Stanley [9] on some open problems in Algebraic Combinatorics). Using a modified Routh array, Šiljak has provided a finite algebraic procedure to count the number of positive (or negative) zeros, with their multiplicity (see also the more recent paper [8, Theorem 3.9, p. 140]).

In this paper we provide a characterization of such polynomials $\theta : \mathbb{R} \to \mathbb{R}$ different from that of Šiljak. Our conditions are stated in terms of two Hankel matrices $M(n, s)$, $B(n, s)$ formed with some functions s of the coefficients of the polynomial θ (the normalized Newton's sums). The conditions state that $M(n, s)$ and $-B(n, s)$ must be positive semidefinite ($M(n, s) \succeq 0$, $B(n, s) \preceq 0$) and the rank of $M(n, s)$ gives the number of *distinct* zeros. This condition is of the same flavour as Gantmacher's conditions for the number of real zeros of θ (see Gantmacher [4]). If we drop the nonpositivity condition on the zeros, then the condition reduces to $M(n, s) \succeq 0$, that is, a necessary and sufficient condition for θ to have only real zeros (as before, the rank of $M(n, s)$ also giving the number of distinct zeros). The basic idea is to consider conditions for a probability measure to have its support on the real zeros of θ. Then, we use a deep result in algebraic geometry of Curto and Fialkow [3] on the $\mathbb{K}$-moment problem.

In addition, this methodology allows us to also provide a similar necessary and sufficient condition on the coefficients for θ to have all its zeros real and in a prescribed interval $[a, b]$ of the real line.

2. Notation and definitions

Let $\mathbb{R}[x]$ be the ring of real-valued univariate polynomials $u : \mathbb{R} \to \mathbb{R}$. In a standard fashion, we identify u with its vector of coefficients $\{u_i\}$ when we write

$$u(x) = \sum_{i=0}^{n} u_i x^i, \tag{2.1}$$

in the canonical basis

$$1, x, x^2, \ldots \tag{2.2}$$

The problem under investigation is thus to characaterize the polynomials u with all its zeros real and nonpositive.

2.1. Moment matrix

Given an infinite vector $y \in \mathbb{R}^\infty$, let $M(n, y), B(n, y) \in \mathbb{R}^{(n+1)\times(n+1)}$ be the Hankel matrices

$$M(n, y) = \begin{bmatrix} 1 & y_1 & y_2 & \cdots & y_n \\ y_1 & y_2 & y_3 & \cdots & y_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_n & y_n & y_{n+1} & \cdots & y_{2n} \end{bmatrix},$$

and

$$B(n, y) = \begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_{n+1} \\ y_2 & y_3 & y_4 & \cdots & y_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{n+1} & y_{n+2} & y_{n+3} & \cdots & y_{2n+1} \end{bmatrix},$$

respectively. $M(0, y)$ is just the $(1, 1)$-matrix [1]. $M(n, y)$ is called a *moment matrix*. Whenever y is the vector of moments of some measure μ, then for every vector $q \in \mathbb{R}[x]$ of degree less than n, with vector of coefficients $q \in \mathbb{R}^{n+1}$, we have

$$\langle q, M(n, y)q \rangle = \int q(x)^2 \mu(dx) \geq 0, \tag{2.3}$$

and therefore, as (2.3) is true for every $q \in \mathbb{R}^{n+1}$, we must have $M(n, y) \succeq 0$, that is, $M(n, y)$ is positive semidefinite.

2.2. Localizing matrix

Similarly, given a polynomial $\theta \in \mathbb{R}[x]$ of degree s, and given an infinite vector $y \in \mathbb{R}^\infty$, define the *localizing* matrix $M_\theta(n, y)$ (with respect to θ) to be

$$M_\theta(n, y)(i, j) = \sum_{k=0}^{s} \theta_k y_{i+j+k}, \quad \forall i, j \leq n.$$

Observe that $B(n, y) = M_x(n, y)$, that is, $B(n, y)$ is a localizing matrix with respect to the polynomial $x \mapsto \theta(x) := x$. The term *localizing* is used in Curto and Fialkow [3] because if y is the vector of moments of some measure μ, $M_\theta(n, y) \succeq 0$ states a necessary condition for μ to have its support contained in the algebraic set $\{x \in \mathbb{R} : \theta(x) \geq 0\}$. Indeed if y is the vector of moments of some measure μ, then for every vector $q \in \mathbb{R}[x]$ of degree less than n, with vector of coefficients $q \in \mathbb{R}^{n+1}$, we have

$$\langle q, M_\theta(n, y)q \rangle = \int \theta(x) q(x)^2 \mu(dx), \tag{2.4}$$

and therefore, as (2.4) is true for every $q \in \mathbb{R}^{n+1}$, we must have $M_\theta(n, y) \succeq 0$, whenever the support of μ is contained in the set $\{x \in \mathbb{R} \mid \theta(x) \geq 0\}$.

Therefore, if y is the vector of moments of some measure μ, the condition $M_\theta(n, y) = 0$ will state a necessary condition for μ to have its support on the real zeros of $\theta(x)$. With the additional condition $B(n, y) \preceq 0$, we will state a necessary condition for μ to have its support on the nonpositive real zeros of θ.

Remark 2.1 In the sequel, we will use the following observation. Let $\theta \in \mathbb{R}[x]$ be a polynomial of degree $n+1$, and let $\{a_i\}$, $i = 1, \ldots, q$, be its distinct zeros (real or complex) with associated multiplicity n_i. Let $s \in \mathbb{R}^\infty$ be the infinite sequence defined by

$$s_k = \frac{1}{n+1} \sum_{i=1}^{q} n_i a_i^k, \quad k = 1, 2, \ldots \tag{2.5}$$

From the definition of $M_\theta(n, .)$, it then follows that $M_\theta(n, s) = 0$ for all $n = 1, 2, \ldots$

3. Main result

For notational convenience, we consider a polynomial $\theta \in \mathbb{R}[x]$ of degree $n+1$ and, with no loss of generality, we may and will assume that $\theta_{n+1} = 1$, that is, we will consider the polynomial $\theta \in \mathbb{R}[x]$:

$$x \mapsto \theta(x) := x^{n+1} + \sum_{i=0}^{n} \theta_i x^i, \quad x \in \mathbb{R}.$$

We first need to introduce some additional material. Given n fixed, let $e_k : \mathbb{R}^{n+1} \to \mathbb{R}$ be the elementary symmetric functions

$$e_k := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n+1} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad k = 1, 2, \ldots$$

It is well known that every symmetric polynomial $p \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ is also a member of $\mathbb{C}[e_1, \ldots, e_{n+1}]$.

In particular, denote by $\{q_\alpha^{(k)}\}$ the coefficients in $\mathbb{C}$ of the expansion of $(n+1)^{-1} \sum_{i=1}^{n+1} x_i^k$ in the basis $(e_1, \ldots, e_{n+1})$. That is,

$$\begin{aligned} (n+1)^{-1} \sum_{i=1}^{n+1} x_i^k &= q_k(e_1, \ldots, e_{n+1}) \\ &= \sum_{|\alpha| \le k} q_\alpha^{(k)} e_1^{\alpha_1} \cdots e_{n+1}^{\alpha_{n+1}}, \quad k = 1, \ldots \end{aligned} \tag{3.1}$$

with $q_\alpha^{(k)} \in \mathbb{C}$, for all α, and $|\alpha| := \sum_i \alpha_i$. In fact, the coefficients $\{q_\alpha^{(k)}\}$ are all in $\mathbb{Q}$ and have a well-known combinatorial interpretation (see e.g. Macdonald [6, Ch. I, Section 6, Example 8] and Beck et al. [2]).

Consider the moment matrix $M(n, s) \in \mathbb{R}^{(n+1)\times(n+1)}$ defined as follows: For all $2 < i + j \le 2n + 2$,

$$M(n, s)(i, j) = s_{i+j-2} = q_{i+j-2}(-\theta_n, \theta_{n-1}, \ldots, (-1)^{n+1}\theta_0), \tag{3.2}$$

where the q_i's are defined in (3.1). Thus, the s_i's are the Newton's sums (here normalized) already considered in Gantmacher [4]. More precisely, if $\theta \in \mathbb{R}[x]$ has q distinct zeros $a_1, \ldots, a_q$ (real or complex) with associated multiplicity $n_1, \ldots, n_q$, then

$$s_k = \frac{1}{n+1} \sum_{i=1}^{q} a_i^k n_i, \quad k = 0, 1, \ldots \tag{3.3}$$

It is important to notice that the number q of all distinct zeros of θ (real or complex) is equal to the rank of the matrix associated with the quadratic form $Q_n : \mathbb{R}^n \to \mathbb{R}$,

$$x \mapsto Q_n(x, x) := \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k, \quad \forall n \ge q, \tag{3.4}$$

see Gantmacher [4, Theorem 6, p. 202]. Similarly, let $B(n, s) \in \mathbb{R}^{(n+1)\times(n+1)}$ be such that for all $1 \le i, j \le n + 1$,

$$B(n, s)(i, j) = s_{i+j-1} = q_{i+j-1}(-\theta_n, \theta_{n-1}, \ldots, (-1)^{n+1}\theta_0). \tag{3.5}$$

Theorem 3.1 *Let $\theta \in \mathbb{R}[x]$ be the polynomial $x \mapsto \theta(x) := x^{n+1} + \sum_{i=0}^{n} \theta_i x^i$, and let $s \in \mathbb{R}^\infty$ be the infinite vector of (normalized) Newton's sums defined in* (3.3). *Then the following two propositions are equivalent*:

(i) *All the zeros of* θ *are real*, *nonpositive*, *and* q *are distinct.*
(ii) $M(n,s) \succeq 0$, $B(n,s) \preceq 0$ *and rank*$(M(n,s)) = q$.

Proof: (i) $\Rightarrow$ (ii). Let $a_1; a_2, \ldots, a_q$ be the q real zeros of θ, all assumed to be nonpositive, and with associated multiplicity n_i, $i = 1, \ldots, q$. Let μ be the probability measure on $\mathbb{R}$, defined by

$$\mu := \frac{1}{n+1} \sum_{i=1}^{q} n_i \delta_{a_i},$$

(where δ_x stands for the Dirac measure at the point $x \in \mathbb{R}$), and let $s \in \mathbb{R}^\infty$ be its associated infinite vector of moments, that is,

$$s_k = \int_{\mathbb{R}} x^k \, d\mu = \frac{1}{n+1} \sum_{i=1}^{q} n_i a_i^k, \quad k = 1, 2, \ldots.$$

In other words, the moments of μ are the (normalized) Newton's sums defined in (3.3).

Therefore, $M(n,s) \succeq 0$ (as it is the moment matrix associated with μ) and moreover, since every zero of θ is real and nonpositive, then, necessarily, μ has its support contained in $(-\infty, 0]$. This clearly implies $B(n,s) \preceq 0$. Finally, observe that $M(n,s)$ is the matrix associated with the quadratic form $x \mapsto Q_{n+1}(x,x)$ (cf. (3.4)). Therefore, as the number of distinct zeros is q, from Gantmacher [4, Theorem 6, p. 202], we must have $q = \text{rank}(M(n,s))$.

(ii) $\Rightarrow$ (i). Remember that since $M(n,s)$ is the matrix associated with the quadratic form $x \mapsto Q_{n+1}(x,x)$ (cf. (3.4)), we know that $\text{rank}(M(n+k,s)) = q$ for all $k = 0, 1, \ldots$ as it is the number of distinct zeros (real or complex) of θ (and we will show that they all are real). Next, from $M(n,s) \succeq 0$ and $\text{rank}(M(n+k,s)) = \text{rank}(M(n,s)) = q$, it follows that $M(n+k,s) \succeq 0$ for all $k = 0, 1, \ldots$. In other words, and in the terminology of Curto and Fialkow [3], the matrices $M(n+k,s)$ are all *flat positive extensions* of $M(n,s)$, for all $k = 1, 2, \ldots$.

In addition, observe that from the definition of the s_k's, and as $\theta(a_i) = 0$ for all $i = 1, 2, \ldots, q$, we also have $M_\theta(n,s) = 0$ (cf. Remark 2.1). Therefore, s also satisfies

$$M(2n+1,s) \succeq 0; \quad B(n,s) \preceq 0; \quad M_\theta(n,s) = 0. \tag{3.6}$$

Equivalently,

$$M(2n+1,s) \succeq 0; \quad M_{-x}(n,s) \succeq 0; \quad M_\theta(n,s) = 0. \tag{3.7}$$

But then, from Theorem 1.6 in Curto and Fialkow [3, p. 6] (adapated here to the one-dimensional case), s is the vector of moments of a rank$(M(n,s))$-atomic (or, q-atomic) probability measure with support contained in $\{\theta(x) = 0\} \cap (-\infty, 0]$ (the constraint $M_\theta(n,s) = 0$ is equivalent to $M_\theta(n,s) \succeq 0$ and $M_{-\theta}(n,s) \succeq 0$).

As q was the number of distinct (real or complex) zeros of θ, this shows that in fact θ has only real zeros, all nonpositive and q distinct. □

If in Theorem 3.1 we drop the condition $B(n, s) \preceq 0$, then $M(n, s) \succeq 0$ becomes a necessary and sufficient condition for θ to have only real zeros.

We next consider the case where all the zeros are real and in a prescribed interval $[a, b] \subseteq \mathbb{R}$.

Theorem 3.2 *Let $[a, b] \subseteq \mathbb{R}$, $\theta \in \mathbb{R}[x]$ be the polynomial $x \mapsto \theta(x) := x^{n+1} + \sum_{i=0}^{n} \theta_i x^i$, and let $s \in \mathbb{R}^\infty$ be the infinite vector of (normalized) Newton's sums defined in* (3.3). *Then the following two propositions are equivalent*:

(i) *All the zeros of θ are in $[a, b]$, and q are distinct.*

(ii) *$M(n, s) \succeq 0$, $bM(n, s) \succeq B(n, s) \succeq aM(n, s)$ and rank$(M(n, s)) = q$.*

Proof: The proof mimics that of Theorem 3.1. It is immediate to check that $bM(n, s) - B(n, s)$ is the localizing matrix $M_{b-x}(n, s)$ whereas $B(n, s) - aM(n, s)$ is the localizing matrix $M_{x-a}(n, s)$. Therefore, exactly as in the proof of Theorem 3.1, invoking Theorem 1.6 in Curto and Fialkow [3], the conditions in (ii) are necessary and sufficient for the vector s to be the vector of moments of a probability measure with support in the set

$$\{x \in \mathbb{R} \mid \theta(x) = 0; b - x \geq 0; x - a \geq 0\}. \qquad \square$$

When $a > -\infty$ and $b < \infty$, the condition $M(n, s) \succeq 0$ is implied by the other one. However, as it stands, Theorem 3.2 includes Theorem 3.1 as a particular case with $a = -\infty$ and $b = 0$.

Example Consider the 3rd degree polynomial

$$x \mapsto \theta(x) := x^3 + \theta_2 x^2 + \theta_1 x + \theta_0, \quad x \in \mathbb{R}.$$

$M(2, s) \in \mathbb{R}^{3\times 3}$ is the Hankel matrix

$$\begin{bmatrix} 1 & -\theta_2/3 & (\theta_2^2 - 2\theta_1)/3 \\ -\theta_2/3 & (\theta_2^2 - 2\theta_1)/3 & -\theta_2^3/3 + \theta_1\theta_2 - \theta_0 \\ (\theta_2^2 - 2\theta_1)/3 & -\theta_2^3/3 + \theta_1\theta_2 - \theta_0 & \theta_2^4/3 - 4\theta_2^2\theta_1/3 + 2\theta_1^2/3 + 4\theta_2\theta_0/3 \end{bmatrix},$$

whereas $B(2, s) \in \mathbb{R}^{3\times 3}$ is the Hankel matrix

$$\begin{bmatrix} -\theta_2/3 & (\theta_2^2 - 2\theta_1)/3 & -\theta_2^3/3 + \theta_1\theta_2 - \theta_0 \\ * & -\theta_2^3/3 + \theta_1\theta_2 - \theta_0 & \theta_2^4/3 - 4\theta_2^2\theta_1/3 + 2\theta_1^2/3 + 4\theta_2\theta_0/3 \\ * & * & -\theta_2^5/3 + 5(\theta_2^3\theta_1 - \theta_2^2\theta_0 - \theta_2\theta_1^2 + \theta_1\theta_0)/3 \end{bmatrix},$$

where we have displayed only the upper triangle.

4. Conclusion

In this paper we have provided finitely many necessary and sufficient conditions on the coefficients of a polynomial $\theta \in \mathbb{R}[x]$, for θ to have only real zeros, all in a prescribed interval $[a, b]$ of the real line. Those conditions are diffferent from those of Šiljak stated for $a, b = \pm\infty$.

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