Journal of Algebraic Combinatorics **16** (2002), 195–207

Singular Polynomials of Generalized Kasteleyn Matrices

NICOLAU C. SALDANHA nicolau@mat.puc-rio.br
Depto. de Matemática, PUC-Rio, R. Mq. de S. Vicente 225, Rio de Janeiro RJ 22453-900, Brazil

Received March 16, 2001; Revised March 16, 2001

Abstract. Kasteleyn counted the number of domino tilings of a rectangle by considering a mutation of the adjacency matrix: a *Kasteleyn matrix K*. In this paper we present a generalization of Kasteleyn matrices and a combinatorial interpretation for the coefficients of the characteristic polynomial of KK^* (which we call the *singular polynomial*), where K is a generalized Kasteleyn matrix for a planar bipartite graph. We also present a q-version of these ideas and a few results concerning tilings of special regions such as rectangles.

Keywords: domino tilings, dimers, Kasteleyn matrix, singular values

Introduction

Kasteleyn [2] counted the number of domino tilings of a rectangle by considering a mutation of the adjacency matrix, since then known as a *Kasteleyn matrix* [3,5]. Given a planar bipartite graph $\mathcal{G}$ there are several Kasteleyn matrices K for $\mathcal{G}$ but, as has been shown independently by David Wilson and Horst Sachs, the singular values of K or, equivalently, the eigenvalues of KK^*, are independent of the choice of K. Following a question posed by James Propp [4], we search for a combinatorial interpretation for these numbers.

In Section 1 we introduce generalized Kasteleyn matrices for planar bipartite graphs and present a combinatorial interpretation for the determinant of such matrices A in terms of counting matchings. In Section 2 we address the main issue of understanding what the coefficients of the characteristic polynomial of AA^* represent, and then, in Section 3, we consider the special case of Kasteleyn matrices. In Section 4 we present the q-analogs of these ideas. In Section 5 we take a look at rectangles in the plane and present a few other small examples. We find the language of homology theory helpful and use it throughout the paper. We thank James Propp, Richard Kenyon and Horst Sachs for helpful conversations and emails.

1. Generalized Kasteleyn matrices and their determinants

Let $\mathcal{G}$ be a planar bipartite graph with n white vertices and n' black vertices. We number the white (black) vertices $1, 2, \ldots, n$ $(1, 2, \ldots, n')$. A *generalized Kasteleyn matrix* for $\mathcal{G}$

is an $n \times n'$ complex matrix A such that

$$|a_{ij}| = \begin{cases} 1, & \text{if the } i\text{-th white vertex and the } j\text{-th black vertex are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Such matrices are conveniently represented by labeling the edges of $\mathcal{G}$ with complex numbers of norm 1.

We may identify a generalized Kasteleyn matrix A with a cocomplex $\mathbf{A} \in C^1(\mathcal{G}, \mathbb{S}^1)$ by making the convention that a_{ij} indicates the value of $\mathbf{A}(e_{ij})$, e_{ij} being the oriented edge going from the j-th black to the i-th white vertex. A notational confusion must be avoided here: the complex numbers of norm 1 form a multiplicative group but the coefficients for homology or cohomology should be additive groups. Thus, from now on, the symbol $\mathbb{S}^1$ shall denote the additive group $\mathbb{R}/\mathbb{Z}$ and we denote the exponential $x \mapsto \exp(2\pi i x)$ by $\eta : \mathbb{S}^1 \to \mathbb{C}$. In particular, we write $a_{ij} = \eta(\mathbf{A}(e_{ij}))$. Since $C^2(\mathcal{G}, \mathbb{S}^1) = 0$, any cocomplex $\mathbf{A}$ is automatically closed and a generalized Kasteleyn matrix A defines an element $\mathbf{a} \in H^1(\mathcal{G}, \mathbb{S}^1)$.

There is a natural inclusion $\mathbb{Z}/(2) \subseteq \mathbb{S}^1$; this defines $\eta : \mathbb{Z}/(2) \to \mathbb{C}$ with $\eta(m) = (-1)^m$. We also obtain induced inclusions $C^1(\mathcal{G}, \mathbb{Z}/(2)) \subseteq C^1(\mathcal{G}, \mathbb{S}^1)$ and $H^1(\mathcal{G}, \mathbb{Z}/(2)) \subseteq H^1(\mathcal{G}, \mathbb{S}^1)$. For a generalized Kasteleyn matrix A, $\mathbf{A} \in C^1(\mathcal{G}, \mathbb{Z}/(2))$ if and only if A is a real matrix.

Lemma 1.1 *There is a unique element* $\mathbf{k} \in H^1(\mathcal{G}, \mathbb{Z}/(2))$ *such that for any cycle* C,

$$\mathbf{k}(C) \equiv m + l + 1 \pmod 2 \tag{1}$$

where m is the number of vertices in the interior of C and $2l$ *is the length of C.*

In the statement above, the word 'cycle' is used in the sense of graph theory: C is a simple closed curve in the plane composed of edges and vertices of $\mathcal{G}$ and the interior of C is well defined by Jordan's theorem. Of course, graph theory cycles define homology cycles (i.e., closed elements of $C_1(\mathcal{G}, \mathbb{Z})$) but the converse is not always true; any homology cycle may nevertheless be written as a linear combination of graph theory cycles.

Proof: Uniqueness is obvious since the above equation gives the value of $\mathbf{k}$ computed against any cycle and thus, by linearity, against any element of $H_1(\mathcal{G}, \mathbb{Z})$ (recall that $H^1(\mathcal{G}, \mathbb{Z}/(2)) = \mathrm{Hom}(H_1(\mathcal{G}, \mathbb{Z}), \mathbb{Z}/(2))$).

In order to construct $\mathbf{k}$, we first notice that $H^1(\mathcal{G}, \mathbb{Z}/(2)) = (\mathbb{Z}/(2))^h$, where h is the number of *holes* (bounded connected components of the complement) of $\mathcal{G}$: in this identification, the coordinates of $\mathbf{a}$ corresponding to a given hole is $\mathbf{a}(C)$, C being the outer boundary of the said hole. It is then clear that there exists a unique element of $H^1(\mathcal{G}, \mathbb{Z}/(2))$, which we call $\mathbf{k}$, satisfying Eq. (1) for all such C.

Figure 1 illustrates a minor complication which has to be kept in mind: the boundary of a hole is not always a cycle in the sense of graph theory. There should be no confusion,

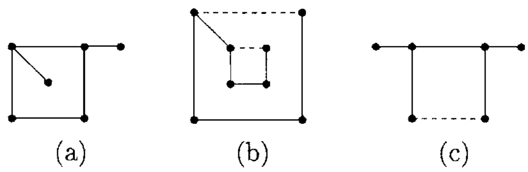


Figure 1. Examples of Kasteleyn classes.

however: in (a) we have $l = 2, m = 1$ for the only hole and in (b) we have $l = 2, m = 0$ and $l = 2, m = 4$ for the smaller and bigger hole, respectively.

Let C be an arbitrary cycle: we prove that Eq. (1) holds for C. The interior of C minus $\mathcal{G}$ is a union of holes. If we discard the holes which are completely surrounded by other holes in C and consider the outer boundaries $C_1, \dots, C_k$ of the remaining ones, we have $\mathbf{k}(C) = \sum_i \mathbf{k}(C_i)$. Since Eq. (1) holds for each C_i we have

$$\mathbf{k}(C) \equiv k + \sum_i l_i + \sum_i m_i \pmod 2,$$

where l_i and m_i correspond to C_i. Notice that $L = l + \sum_i l_i$, L denoting the number of edges of some C_i on C or in the interior of C, and $M = 2l + m - \sum_i m_i$, M denoting the number of vertices of some C_i on C or in the interior of C. Finally, by Euler characteristic, $k - L + M = 1$ and we have Eq. (1) for C, proving our lemma. □

We call $\mathbf{k} \in H^1(\mathcal{G}, \mathbb{Z}/(2))$ as defined in the previous lemma the *Kasteleyn class*; when the graph $\mathcal{G}$ is not clear from the context, we write $\mathbf{k}_{\mathcal{G}}$. The definition of $\mathbf{k}$ involves m and thus appears to depend on the way $\mathcal{G}$ is drawn in the plane. Indeed, examples (a) and (c) in figure 1 represent equivalent graphs, but the Kasteleyn classes are different; solid lines stand for a label 1 and dashed lines stand for a label -1. A *Kasteleyn matrix* is a generalized Kasteleyn matrix corresponding to the Kasteleyn class.

We restrict ourselves for the rest of this section to the case $n = n'$ in order to explore the relationship between matchings and the determinant of square generalized Kasteleyn matrices A.

It is natural to interpret a matching of $\mathcal{G}$ as the sum of its edges oriented from black to white and thus as an element of $C_1(\mathcal{G}, \mathbb{Z})$. The boundary of any matching always equals the sum of all white vertices minus the sum of all black vertices; thus, the difference of two matchings of $\mathcal{G}$ is closed and may be identified with an element of $H_1(\mathcal{G}, \mathbb{Z})$. Notice furthermore that the difference of two matchings may be written in a unique way as a sum of disjoint graph theory cycles.

For $\mathbf{a} \in H^1(\mathcal{G}, \mathbb{S}^1)$ and two matchings μ_1 and μ_2, $\eta(\mathbf{a}(\mu_2 - \mu_1))$ is a complex number of absolute value 1. In particular, if $\mathbf{a} \in H^1(\mathcal{G}, \mathbb{Z}/(2))$ then $\eta(\mathbf{a}(\mu_2 - \mu_1))$ is 1 or -1. We then say that μ_1 and μ_2 have the same **a**-*parity* if $\eta(\mathbf{a}(\mu_2 - \mu_1)) = 1$; **a**-parity splits the set of matchings into two equivalence classes (occasionally one of these classes may turn out to be empty).

Lemma 1.2 *For any planar graph $\mathcal{G}$, for any $\mathbf{a} \in H^1(\mathcal{G}, \mathbb{S}^1)$ and for any given matching μ_0 we have*

$$\sum_{\mu_1,\mu_2} \eta(\mathbf{a}(\mu_2 - \mu_1)) = \left| \sum_{\mu} \eta(\mathbf{a}(\mu - \mu_0)) \right|^2,$$

where μ_1, μ_2 and μ range over all matchings.

Proof: We may write the right hand side as

$$\left(\sum_{\mu_2} \eta(\mathbf{a}(\mu_2 - \mu_0)) \right) \left(\sum_{\mu_1} \overline{\eta(\mathbf{a}(\mu_1 - \mu_0))} \right)$$
$$= \left(\sum_{\mu_2} \eta(\mathbf{a}(\mu_2 - \mu_0)) \right) \left(\sum_{\mu_1} \eta(\mathbf{a}(\mu_0 - \mu_1)) \right)$$

and distribute to get the left hand side. □

Define

$$\delta(\mathbf{a}, \mathcal{G}) = \sum_{\mu_1,\mu_2} \eta(\mathbf{a}(\mu_1 - \mu_2)),$$

where μ_1 and μ_2 range over all matchings; if $\mathcal{G}$ admits no matchings we define $\delta(\mathbf{a}, \mathcal{G}) = 0$. As an example, $\delta(0, \mathcal{G})$ is the square of the number of matchings of $\mathcal{G}$. Also, for $\mathbf{a} \in H^1(\mathcal{G}, \mathbb{Z}/(2))$, $\delta(\mathbf{a}, \mathcal{G})$ is the square of the difference between the number of matchings in each $\mathbf{a}$-parity equivalence class.

A matching may also be though of as a bijection from the set of white vertices to the set of black vertices. Thus, if μ_1 and μ_2 are matchings then $\mu_1^{-1} \circ \mu_2$ is a permutation of the set of white vertices: we say that these two matchings have the same *permutation parity* if and only if the this permutation is even.

Lemma 1.3 *Two matchings have the same permutation parity if and only if they have the same* $\mathbf{k}$*-parity*, $\mathbf{k}$ *being the Kasteleyn class.*

Proof: Let μ_1 and μ_2 be two matchings and write $\mu_1 - \mu_2$ as a sum of disjoint cycles $C_1, \ldots, C_N$ of lengths $2l_1, \ldots, 2l_N$. The interior and exterior of any of these cycles is matchable, thus $m_1, \ldots, m_N$ as in Lemma 1.1 are all even. From Eq. (1), $\mathbf{k}(C_i) \equiv l_i + 1 \pmod 2$ and thus $\mathbf{k}(\mu_0 - \mu_1) \equiv \sum(l_i + 1) \pmod 2$.

The permutation $\mu_0^{-1} \circ \mu_1$ can be written as a product of N cycles (in the permutation sense) corresponding to $C_1, \ldots, C_N$ with lengths $l_1, \ldots, l_N$ and the parity of the permutation $\mu_1^{-1} \circ \mu_2$ is thus $\sum(l_i + 1)$. This proves our claim. □

Notice that permutation parity, unlike the Kasteleyn class, does not depend on how $\mathcal{G}$ is drawn in the plane. A corollary of the previous lemma is thus that if differences of matchings generate $H_1(\mathcal{G}, \mathbb{Z})$ then the Kasteleyn class of $\mathcal{G}$ is the same for all planar embeddings.

Lemma 1.4 *For any generalized Kasteleyn matrix A we have* $|\det(A)|^2 = \delta(\mathbf{a} + \mathbf{k}, \mathcal{G})$.

Proof: Each non-zero monomial in the expansion of $\det(A)$ corresponds to a matching. Thus, each matching μ contributes with a complex number of absolute value 1 to $\det(A)$. The expression $\eta(\mathbf{a}(\mu - \mu_0))$ obtains, up to a fixed multiplicative constant of absolute value 1, the product of the corresponding elements of A. From Lemma 1.2, **k**-parity is permutation parity, i.e., gives the sign of the monomial in the definition of the determinant. Thus, the contribution of μ to $\det(A)$ is, again up to a fixed multiplicative constant of absolute value 1, $\eta((\mathbf{a} + \mathbf{k})(\mu - \mu_0))$, proving our lemma. □

As a special case, if K is a Kasteleyn matrix, $|\det(K)|$ is the number of matchings of $\mathcal{G}$: this is Kasteleyn's original motivation.

2. Singular polynomials of generalized Kasteleyn matrices

Having provided an interpretation for $|\det(A)|$ when A is square, it is natural to ask about other functions of A, specially if A is not square. We should not expect natural interpretations for the argument of $\det(A)$ since it depends on the way we assign labels to vertices. Also, a few simple experiments will show that the spectrum of A (even if A is square) is not a function of **a**. The following lemma tells us what functions of A are determined by **a**.

Lemma 2.1 *Let A be a generalized Kasteleyn matrix for $\mathcal{G}$ and let* **a** *be the corresponding element of $H^1(\mathcal{G}, \mathbb{S}^1)$. Then the generalized Kasteleyn matrices for $\mathcal{G}$ also corresponding to* **a** *are precisely the matrices of the form $D_1 A D_2$ where D_1 and D_2 are unitary diagonal matrices. Furthermore, if $\mathcal{G}$ is connected, $D_1 A D_2 = D_1' A D_2'$ if and only if there exists a complex number z of absolute value* 1 *with $D_1 = z D_1'$, $D_2 = z^{-1} D_2'$.*

It is possible to give a more elementary proof, but following the spirit of the rest of this paper we phrase the proof in homological language.

Proof: As we saw in Section 1, generalized Kasteleyn matrices correspond to 1-cocomplexes in $C^1(\mathcal{G}, \mathbb{S}^1)$; two such 1-cocomplexes $\mathbf{A}$ and $\mathbf{A}'$ induce the same element of $H^1(\mathcal{G}, \mathbb{S}^1)$ if and only if their difference is the coboundary of a 0-cocomplex. A 0-cocomplex $\mathbf{D}$ is a function assigning an element of $\mathbb{S}^1$ to each vertex; the η's of these elements may conveniently be arranged in a pair of unitary diagonal matrices, D_w for the white and D_b for the black vertices. It is a simple translating process to verify that the cocomplex $\mathbf{A} + d(\mathbf{D})$ corresponds to the generalized Kasteleyn matrix $D_w A D_b^{-1}$, thus proving our first claim. The uniqueness of D_1 and D_2 up to a constant multiplicative factor corresponds to the fact that the only closed 0-cocomplexes are the constants, i.e., that $H^0(\mathcal{G}, \mathbb{S}^1) = \mathbb{S}^1$ if $\mathcal{G}$ is connected. □

We recall that for any complex $n \times n'$ matrix B, there are unitary matrices U_1 and U_2 such that $S = U_1 B U_2$ is a real diagonal matrix with non-increasing non-negative diagonal entries; S is well-defined given B and its diagonal entries (i.e., the s_{ii} entries of S, even if S is not square) are called the *singular values* of B. The rows of U_1 (resp., columns of U_2) are called the *left* (resp., *right*) *singular vectors* of B. It is easy to see that the singular values and left (resp. right) singular vectors of B are the non-negative square roots of the eigenvalues and the eignevectors of BB^* (resp., B^*B). Inspired in these classical notions, we call the characteristic polynomial of BB^* the *singular polynomial* of B: its roots are the squares of the singular values of B. Also, the singular polynomials of B and U_1BU_2 are equal and the singular polynomials of B and B^* differ by a factor of $t^{n-n'}$.

It follows from Lemma 2.1 and the remarks in the previous paragraph that the singular polynomial of A is determined by $\mathbf{a}$: we call it $P_{\mathbf{a}}$. The singular values of A and, if the singular values are simple, the absolute values of the coordinates of the singular vectors (up to a constant factor) are also determined by $\mathbf{a}$. We shall now present what we find to be a reasonably natural interpretation for the coefficients of $P_{\mathbf{a}}$. While these numbers determine the singular values the question remains whether a nice interpretation exists for the actual singular values and vectors.

Let $\mathcal{H} \subseteq \mathcal{G}$ be a balanced subgraph of $\mathcal{G}$: the inclusion induces a map $\pi_{\mathcal{G},\mathcal{H}} : H^1(\mathcal{G}, \mathbb{S}^1) \to H^1(\mathcal{H}, \mathbb{S}^1)$. More concretely, if $\mathbf{a}$ corresponds to a generalized Kasteleyn matrix A then $\pi_{\mathcal{G},\mathcal{H}}(\mathbf{a})$ corresponds to the submatrix of A obtained by picking only the elements for which both row and column correspond to elements of $\mathcal{H}$. The simplest interpretation is probably in terms of labels for edges: just keep the old labels. When this causes no confusion, we write $\mathbf{a}$ instead of $\pi_{\mathcal{G},\mathcal{H}}(\mathbf{a})$: for instance, we write $\delta(\mathbf{a}, \mathcal{H})$ instead of the more correct but cumbersome $\delta(\pi_{\mathcal{G},\mathcal{H}}(\mathbf{a}), \mathcal{H})$.

Theorem 2.2 *Let A be a generalized Kasteleyn matrix and let $P_{\mathbf{a}}(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n$ be the singular polynomial of A. Then*

$$a_m = (-1)^m \sum_{|\mathcal{H}|=2m} \delta(\mathbf{a} + \mathbf{k}_{\mathcal{H}}, \mathcal{H}) \tag{2}$$

where $\mathcal{H}$ ranges over all balanced subgraphs with $2m$ vertices.

Notice that for $m = n$ Eq. (2) is equivalent to Lemma 1.4. For $m = 1$, we get the simple remark that $|a_1|$ is the number of edges of $\mathcal{G}$, regardless of $\mathbf{a}$. An interpretation for a_2 is already subtler: each subgraph with two white and two black vertices contributes with a real number between 0 and 4. Subgraphs which are not matchable of course contribute with 0 and two disjoint edges as well as four points on a line contribute with 1. The interesting part are the squares, which admit two matchings, say μ_1 and μ_2: then $\delta(\mathbf{a} + \mathbf{k}_{\mathcal{H}}, \mathcal{H}) = |1 - \eta(\mathbf{a}(\mu_1 - \mu_2))|^2$; in general, this may be any number between 0 and 2 but if $\mathbf{a} \in H^1(\mathcal{G}, \mathbb{Z}/(2))$ then this is 0 or 4.

In order to prove this theorem, we need an auxiliary result in linear algebra. The proof of Lemma 2.3 is actually a rather straightforward computation.

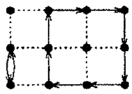


Figure 2. A pipe system.

Lemma 2.3 *Let* $P(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$ *be the singular polynomial of* A (*where* A *is an arbitrary* $n \times n'$ *complex matrix*). *Then*

$$a_m = (-1)^m \sum_B |\det B|^2$$

where B *ranges over all* $m \times m$ *submatrices of* A.

Proof: Since balanced subgraphs of $\mathcal{G}$ with $2m$ elements correspond to $m \times m$ submatrices of A, this is a consequence of Lemma 1.4 and Lemma 2.3. □

We now describe another, more graphical, interpretation for Theorem 2.2. We define a *pipe system* of $\mathcal{G}$ as an oriented pair $\nu = (\mu_1, \mu_2)$ of matchings of a subgraph $\mathcal{H}$ of $\mathcal{G}$; we call $\mathcal{H}$ the *support* of the pipe system. Figure 2 shows an example of a pipe system: we draw the edges of μ_2 oriented from black to white and the edges of μ_1 from white to black (unused edges are represented by dotted lines). A pipe system is thus a collection of pipes (i.e., oriented edges of $\mathcal{G}$) such that, at each vertex, there is either one pipe coming in and one pipe going out or no pipe coming in and no pipe going out (the water that comes in must go out and you can not pipe too much water through a vertex). We define the *size* $|\nu|$ of the pipe system as half the number of vertices in $\mathcal{H}$ (in figure 3(c), $|\nu| = 5$).

The Kasteleyn class $\mathbf{k}_{\mathcal{H}}$ shall be called $\mathbf{k}_\nu$. A pipe system obtains an element of $C_1(\mathcal{H}, \mathbb{Z})$ (and thus of $C_1(\mathcal{G}, \mathbb{Z})$) but must not be confused with it: if two pipes cancel each other homologically, they still have to be taken into account for the pipe system. If $\mathbf{a} \in H^1(\mathcal{H}, \mathbb{S}^1)$, $\eta(\mathbf{a}(\nu))$ and $\eta((\mathbf{a} + \mathbf{k}_\nu)(\nu))$ are well defined complex numbers.

Corollary 2.4 *Let* A *be a generalized Kasteleyn matrix and let* $P_{\mathbf{a}}(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$ *be the singular polynomial of* A. *Then*

$$P_{\mathbf{a}}(t) = \sum_\nu (-1)^{|\nu|} t^{n-|\nu|} \eta((\mathbf{a} + \mathbf{k}_\nu)(\nu)),$$

where ν *ranges over all pipe systems of* $\mathcal{G}$.

Proof: This follows directly from Theorem 2.2 and the definitions. □

3. Singular polynomials of planar graphs

Theorem 2.2 provides an interpretation for the coefficients of singular polynomials of arbitrary generalized Kasteleyn matrices. In this section we take a closer look at the right hand side of Eq. (2) when A is a Kasteleyn matrix.

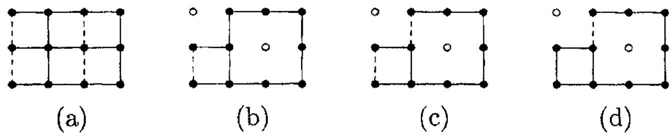


Figure 3. A graph, a subgraph and Kasteleyn classes.

For planar balanced bipartite graphs $\mathcal{H} \subseteq \mathcal{G}$, define $\mathbf{p}_{\mathcal{G},\mathcal{H}} \in H^1(\mathcal{H}, \mathbb{Z}/(2))$ by $\mathbf{p}_{\mathcal{G},\mathcal{H}} = \pi_{\mathcal{G},\mathcal{H}}(\mathbf{k}_{\mathcal{G}}) - \mathbf{k}_{\mathcal{H}}$. In figure 3 we illustrate the several objects involved in this definition: (a), (b), (c) and (d) represent $\mathbf{k}_{\mathcal{G}}$, $\pi_{\mathcal{G},\mathcal{H}}(\mathbf{k}_{\mathcal{G}})$, $\mathbf{k}_{\mathcal{H}}$ and $\mathbf{p}_{\mathcal{G},\mathcal{H}}$, respectively, where again solid lines stand for a label 1 and dashed lines stand for a lable -1. The following lemma provides an alternate definition for this class.

Lemma 3.1 *Let $\mathcal{H} \subset \mathcal{G}$ be balanced planar graphs and let C be a cycle in $\mathcal{H}$. Let q be the number of vertices of $\mathcal{G}$ not belonging to $\mathcal{H}$ which are inside C. Then*

$$\mathbf{p}_{\mathcal{G},\mathcal{H}}(C) \equiv q \pmod 2.$$

Proof: This follows directly from Eq. (1) in Lemma 1.1. □

In the hope of making the intuitive meaning of this definition clearer, especially for adjacency graphs of quadriculated or triangulated disks, we introduce some extra structure.

Let $\bar{\mathcal{G}}$ be the CW-complex obtained from $\mathcal{G}$ by closing each hole with a 2-cell; $\bar{\mathcal{G}}$ is thus always homeomorphic to a disk. For $\mathcal{H} \subseteq \mathcal{G}$, let $\bar{\mathcal{H}} \subseteq \bar{\mathcal{G}}$ be the obtained from $\mathcal{H}$ by adding the 2-cells of $\bar{\mathcal{G}}$ whose boundaries are contained in $\mathcal{H}$; in other words, we close the holes of $\mathcal{H}$ which contain no points of $\mathcal{G}$. The inclusion $\mathcal{H} \subseteq \bar{\mathcal{H}}$ induces an injective map from $H^1(\bar{\mathcal{H}}, \mathbb{Z}/(2))$ to $H^1(\mathcal{H}, \mathbb{Z}/(2))$ which allows for a natural identification of $H^1(\bar{\mathcal{H}}, \mathbb{Z}/(2))$ with a subset of $H^1(\mathcal{H}, \mathbb{Z}/(2))$.

Lemma 3.2 $\mathbf{p}_{\mathcal{G},\mathcal{H}}$ *belongs to $H^1(\bar{\mathcal{H}}, \mathbb{Z}/(2))$.*

Proof: This follows easily from Lemma 3.1. □

Recall that two tilings by dominoes of a quadriculated region are said to differ by a *flip* if they coincide except for two dominoes; in other words, their difference (in the homological sense) is a square. If $\mathcal{G}$ is the graph of a quadriculated planar region, the difference between two tilings of $\mathcal{H}$ differing by a flip is 0 in $H_1(\bar{\mathcal{H}}, \mathbb{Z})$; thus, tilings mutually accessible by flips always have the same $\mathbf{p}_{\mathcal{G},\mathcal{H}}$-parity.

We may now state the promised interpretation for the coefficients of singular polynomial $P_{\mathbf{k}}$ of K. Since $P_{\mathbf{k}}$ is well defined from $\mathcal{G}$, we may adopt a lighter notation and call it $P_{\mathcal{G}}$, the *singular polynomial* of $\mathcal{G}$.

Theorem 3.3 *Let $\mathcal{G}$ be a planar bipartite graph and let $P_{\mathcal{G}} = t^n + k_1 t^{n-1} + \cdots + k_{n-1}t + k_n$ be the singular polynomial of $\mathcal{G}$. Then*

$$k_m = (-1)^m \sum_{|\mathcal{H}|=2m} \delta(\mathbf{p}_{\mathcal{G},\mathcal{H}}, \mathcal{H}) \tag{3}$$

where $\mathcal{H}$ ranges over all balanced subgraphs with $2m$ vertices.

Proof: This is a corollary of Theorem 2.2 and the definition of $\mathbf{p}_{\mathcal{G},\mathcal{H}}$. □

Recall that if $\mathbf{p}_{\mathcal{G},\mathcal{H}} = 0$ then $\delta(\mathbf{p}_{\mathcal{G},\mathcal{H}}, \mathcal{H})$ is just the square of the number of matchings of $\mathcal{H}$. This always happens if $\bar{\mathcal{H}}$ is simply connected. As a corollary, if $\mathcal{G}$ is the graph of a quadriculated planar region and $m \leq 3$, or if $\mathcal{G}$ is the graph of a triangulated planar region and $m \leq 5$, then

$$k_m = (-1)^m \sum_{|\mathcal{H}|=2m} \delta(0, \mathcal{H})$$

where $\mathcal{H}$ ranges over all balanced subgraphs (subregions) with $2m$ vertices (squares, triangles).

Notice that $\mathbf{p}_{\mathcal{G},\mathcal{H}}$, and thus the right hand side of Eq. (3), depends on the way $\mathcal{G}$ is drawn in the plane. Examples (a) and (c) in figure 1 show that $P_{\mathcal{G}}$ indeed depends on the way $\mathcal{G}$ is drawn: for (a) we have $k_2 = 9$ but for (b) we have $k_2 = 5$. This causes the singular values to change in a complicated way: for (a) the singular values are approximately 0.5549581321, 0.8019377358, 2.246979604 while for (c) they are 0.3472963553, 1.532088886, 1.879385242.

It is natural to conjecture that the number of non-zero singular values coincides with the size of a maximal partial matching of $\mathcal{G}$. In figure 4(a) we present an example to show that this is not always true: there are partial matchings of size 3 but since the singular polynomial of $\mathcal{G}$ is $t^3 - 7t^2 + 10t$ there are only two non-zero singular values: $\sqrt{2}$ and $\sqrt{5}$. In figure 4(b) we draw the same graph in a different way and we now have three non-zero singular values: 1, $\sqrt{2}$ and 2.

We state Theorem 3.3 in the language of pipe systems. Denote $\mathbf{p}_{\mathcal{G},\mathcal{H}}(\nu)$ (where $\mathcal{H}$ is the support of ν) by $\mathbf{p}(\nu) \in \mathbb{Z}/(2)$. We describe an elementary definition of $\mathbf{p}(\nu)$. Join pairs of vertices not in $\mathcal{H}$, matching black vertices with white vertices in an arbitrary way; if all

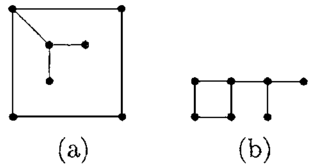


Figure 4. Two graphs with different singular values.

intersections are transversal, $\mathbf{p}(\nu)$ is the parity of the number of intersections of such new lines with the pipes.

Corollary 3.4 *Let $\mathcal{G}$ be a planar graph and $P_{\mathcal{G}}(t)$ its singular polynomial. Then*

$$P_{\mathcal{G}}(t) = \sum_{\nu} (-1)^{|\nu|} t^{n-|\nu|} \eta(\mathbf{p}(\nu)),$$

where ν ranges over all pipe systems of $\mathcal{G}$.

Proof: This follows directly from Theorem 3.3 and Corollary 2.4. □

4. q-counting

In several branches of combinatorics, q-analogues or quantizations of classical problems have been seen to be interesting and useful. There are often several interpretations for the q-analogue of a given concept, some sophisticated (involving quantum groups and the like) and some elementary. In this section we briefly consider a q-analogue of Kasteleyn matrices in a very naïve way and extend the results of the previous sections to this setting; our interest in doing so is that the methods of the previous sections extend very easily to this more general context and the coefficients will actually have a natural interpretation.

Let $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$. We extend the usual complex conjugation to $\mathbb{C}_q$ by postulating $\bar{q} = q^{-1}$; q may be thought of as an unknown complex number of absolute value 1. Let $\mathcal{G}$ be a planar bipartite graph with n white vertices and n' black vertices. A *generalized Kasteleyn q-matrix* for $\mathcal{G}$ is an $n \times n'$ matrix A with coefficients in $\mathbb{C}_q$ such that

$$a_{ij}\bar{a}_{ij} = \begin{cases} 1, & \text{if the } i\text{-th white vertex and the } j\text{-th black vertex are adjacent,} \\ 0 & \text{otherwise;} \end{cases}$$

thus, the entries are always monomials and substituting q by a complex number of absolute value 1 changes a generalized Kasteleyn q-matrix into an ordinary generalized Kasteleyn matrix.

Consider the additive group $\mathbb{S}_q = \mathbb{S}^1 \oplus \mathbb{Z}$ and let $\mathbf{q}$ be the canonical generator of the $\mathbb{Z}$ component. If we extend the classical η to $\eta : \mathbb{S}^1 \oplus \mathbb{Z} \to \mathbb{C}_q$ by postulating $\eta(\mathbf{q}) = q$ we may identify a generalized Kasteleyn q-matrix A with a cocomplex $\mathbf{A} \in C^1(\mathcal{G}, \mathbb{S}_q)$. Again, $C^2(\mathcal{G}, \mathbb{S}_q) = 0$ and A defines an element $\mathbf{a} \in H^1(\mathcal{G}, \mathbb{S}_q)$.

Let $\mathcal{G}$ be a planar graph. Bounded connected components of the complement of $\mathcal{G}$ have well defined positively oriented boundaries β in $H_1(\mathcal{G}, \mathbb{Z})$. We define a Kasteleyn q-matrix to be a generalized Kasteleyn q-matrix A such that $\mathbf{a}(\beta) = \mathbf{q}$ for all such boundaries β. We define the *singular q-polynomial* of $\mathcal{G}$ to be the singular polynomial of a Kasteleyn q-matrix of $\mathcal{G}$. As before, singular q-polynomials are easuly seen to be well defined but now they are of course polynomials in $\mathbb{C}_q[t]$, or, rather equivalently, polynomials in two variables q and t. Finally, define the area of a pipe system ν, $A(\nu)$ to be the number of bounded connected components of the complement of $\mathcal{G}$ positively surrounded by ν, counted with sign and multiplicity.

With these definitions we have the following theorem.

Theorem 4.1 *Let $\mathcal{G}$ be a planar graph and $P_{\mathcal{G}}(q,t)$ its singular q-polynomial. Then*

$$P_{\mathcal{G}}(q,t) = \sum_{\nu} (-1)^{|\nu|} q^{A(\nu)} t^{n-|\nu|} \eta(\mathbf{p}(\nu)),$$

where ν ranges over all pipe systems of $\mathcal{G}$.

Since the proof is entirely analogous to that of Corollary 3.4, we leave the details to the reader.

5. Rectangles and other examples

Kasteleyn [2] computes the determinant of K for rectangles essentially by computing its singular values. For the reader's convenience, we repeat that part of his work in our language. In order to simplify notation in the statement and proof, let

$$X^{+}_{M,N} = \left\{ (k,\ell) \in \mathbb{Z}^2 \mid 1 \le k \le \frac{M+1}{2};\ \text{if } k = \frac{M+1}{2} \text{ then } 1 \le \ell \le \frac{N+1}{2} \right\},$$
$$X^{-}_{M,N} = \left\{ (k,\ell) \in \mathbb{Z}^2 \mid 1 \le k \le \frac{M+1}{2};\ \text{if } k = \frac{M+1}{2} \text{ then } 1 \le \ell < \frac{N+1}{2} \right\}.$$

Theorem 5.1 *Let $\mathcal{G}$ be a $M \times N$ rectangular grid and let K be its Kasteleyn matrix. Then the non-zero singular values of K are $\sigma_{k,\ell}$, $(k,\ell) \in X^{-}_{M,N}$, where*

$$\sigma^2_{k,\ell} = (\alpha^k + \alpha^{-k})^2 + (\beta^\ell + \beta^{-\ell})^2, \quad \alpha = \exp\left(\frac{\pi i}{M+1}\right), \quad \beta = \exp\left(\frac{\pi i}{N+1}\right).$$

The complicated description of the allowed values of the indices k and ℓ is necessary in order to avoid zeroes and duplications in a way which is correct for all possible parities of M and N (Kasteleyn has a simpler formula since he assumes N to be even). Notice that

$$\sigma_{k,\ell} = 2\left(\cos^2 \frac{k\pi}{M+1} + \cos^2 \frac{\ell\pi}{N+1}\right)^{1/2},$$

(an expression closer to Kasteleyn's),

$$\sigma_{k,\ell} = \sigma_{M+1-k,\ell} = \sigma_{k,N+1-\ell} = \sigma_{M+1-k,N+1-\ell}$$

and that $\sigma_{k,\ell} = 0$ if and only if M and N are both odd, $k = (M+1)/2$ and $\ell = (N+1)/2$. Thus, if we just demand $1 \le k \le M$ and $1 \le \ell \le N$ then all non-zero singular values are counted twice and we occasionally introduce a 0.

Proof: We index vertices by pairs (k', ℓ'), $1 \leq k' \leq m$, $1 \leq \ell' \leq n$. The vertex (k', ℓ') is called white when $k' + \ell'$ is even. Define K as the Kasteleyn matrix with entries 1 for horizontal edges and i for vertical edges: K defines a linear transformation from the "black space" to the "white space". Consider the white vectors

$$w_{k,l} = (\alpha^{kk'} - \alpha^{-kk'})(\beta^{\ell\ell'} - \beta^{-\ell\ell'}),$$

$(k, \ell) \in X^+_{m,n}$: they clearly form an orthogonal basis for the white space (this is where a careful choice of $X^+_{m,n}$ becomes necessary). Similarly, the black vectors $b_{k,l}$ defined by the same formula with $(k, \ell) \in X^-_{m,n}$ form an orthogonal basis for the black space. A simple computation yields $|w_{k,l}| = |b_{k,l}|$ for $(k, l) \in X^-_{m,n}$ and

$$\begin{aligned} Kb_{k,\ell} &= ((\alpha^k + \alpha^{-k}) + i(\beta^\ell + \beta^{-\ell}))\, w_{k,\ell}, \\ K^* w_{k,\ell} &= ((\alpha^k + \alpha^{-k}) + i(\beta^\ell + \beta^{-\ell}))\, b_{k,\ell}. \end{aligned}$$

Thus, $w_{k,\ell}$ and $b_{k,\ell}$ are singular vectors and $\sigma_{k,\ell}$ are singular values. □

Corollary 5.2 *Let $\mathcal{G}$ be a $m \times n$ rectangular grid. Let*

$$\alpha = \exp\left(\frac{\pi i}{m+1}\right), \quad \beta = \exp\left(\frac{\pi i}{n+1}\right), \quad N = \left\lfloor \frac{mn}{2} \right\rfloor.$$

Then

$$\prod_{(k,\ell)\in X^-_{m,n}} (t - (\alpha^k + \alpha^{-k})^2 - (\beta^\ell + \beta^{-\ell})^2) = \sum_{j=0,\ldots,N} t^{N-j}(-1)^j \sum_{|\mathcal{H}|=2j} \delta(\mathbf{p}_{\mathcal{G},\mathcal{H}}, \mathcal{H}),$$

where $\mathcal{H}$ ranges over all balanced subgraphs with $2j$ vertices.

Proof: This follows directly from Theorem 3.3 and Theorem 4.1. □

These results show that the characteristic polynomial of KK^* usually factors a lot if $\mathcal{G}$ is a rectangle. If ζ is a root of unity whose order M is the least common multiple of $2(m+1)$ and $2(n+1)$ then all the roots $\sigma^2_{k,\ell}$ of this polynomial are in $\mathbb{R} \cap \mathbb{Z}[\zeta]$, a ring of degree $\phi(M)/2$ over $\mathbb{Z}$. In particular, for square grids of order n, irreducible factors of the characteristic polynomial of KK^* have degree at most n. Here are a few sample examples; we give the polynomial $\det(tI - KK^*) = t^n + k_1 t^{n-1} + \cdots + k_{n-1}t + k_n$ (whose roots are the squares of singular values) factored in $\mathbb{Z}$.

[5, 5] $(t-1)^2(t-2)^2(t-3)^2(t-6)^2(t-4)^4$
[6, 6] $(t^3 - 10t^2 + 24t - 8)^2(t^3 - 10t^2 + 31t - 29)^4$
[7, 7] $(t-2)^2(t^2 - 4t + 2)^2(t^2 - 8t + 8)^2(t^2 - 8t + 14)^4(t-4)^6$
[8, 8] $(t-2)^2(t^3 - 12t^2 + 36t - 8)^2(t^3 - 9t^2 + 24t - 17)^4(t^3 - 12t^2 + 45t - 53)^4$

Although Aztec diamonds have so many interesting properties (see [1] and [4]), the characteristic polynomial of KK^* does not factor very much:

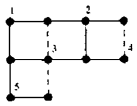
3-Aztec diamond $(t^4 - 10t^3 + 28t^2 - 24t + 4)^2(t - 4)^4$

4-Aztec diamond $(t^{10} - 32t^9 + 441t^8 - 3424t^7 + 16432t^6 - 50240t^5 + 97041t^4 - 112896t^3 + 70921t^2 - 18784t + 1024)^2$

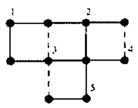
5-Aztec diamond $(t^{11} - 34t^{10} + 496t^9 - 4064t^8 + 20562t^7 - 66524t^6 + 137728t^5 - 177120t^4 + 131825t^3 - 49066t^2 + 6576t - 128)^2(t - 4)^8$

The fact that these polynomials are always squares follows from symmetry. The factor $(t-4)^{4k}$ seems to appear in the $2k+1$-Aztec diamond, a fact for which we have no explanation.

Finally, here are a couple of "irregular" examples. A possible real Kasteleyn matrix is indicated by the dashed lines (the -1's).



$t^5 - 13t^4 + 63t^3 - 140t^2 + 140t - 49$
$= (t^2 - 6t + 7)(t^3 - 7t^2 + 14t - 7)$
2.101003, 1.949856, 1.563663, 1.259280, 0.867768



$t^5 - 13t^4 + 62t^3 - 132t^2 + 121t - 36$
$= (t-4)(t^4 - 9t^3 + 26t^2 - 28t + 9)$
2.126757, 2, 1.576415, 1.197126, 0.747468

Acknowledgment

The author gratefully acknowledges the support of CNPq, Faperj, Finep and ENS-Lyon.

References

1. N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, "Alternating-sign matrices and domino tilings," *Journal of Algebraic Combinatorics* **1** (1992), 111–132 and 219–234.
2. P.W. Kasteleyn, "The statistics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice," *Phisica* **27** (1961), 1209–1225.
3. E.H. Lieb and M. Loss, "Fluxes, Laplacians and Kasteleyn's theorem," *Duke Math. Jour.* **71** (1993), 337–363.
4. J. Propp, "Enumeration of matchings, problems and progress," in *New Perspectives in Algebraic Combinatorics*, Louis J. Billera, Anders Bjrner, Curtis Greene, Rodica Simion, and Richard P. Stanley (Eds.), MSRI Publications, Vol. 38, 1999.
5. N.C. Saldanha and C. Tomei, "An overview of domino and lozenge tilings," *Resenhas IME-USP* **2**(2) (1995), 239–252.