Journal of Algebraic Combinatorics **16** (2002), 83–96

Quotient Complexes and Lexicographic Shellability

AXEL HULTMAN axel@math.kth.se
Kungl. Tekniska Hogskolan, SE-100 44, Stockholm, Sweden

Received October 13, 2000; Revised August 22, 2001

Abstract. Let $\Pi_{n,k,k}$ and $\Pi_{n,k,h}$, $h < k$, denote the intersection lattices of the k-equal subspace arrangement of type $\mathcal{D}_n$ and the k, h-equal subspace arrangement of type $\mathcal{B}_n$ respectively. Denote by S_n^B the group of signed permutations. We show that $\Delta(\Pi_{n,k,k})/S_n^B$ is collapsible. For $\Delta(\Pi_{n,k,h})/S_n^B$, $h < k$, we show the following. If $n \equiv 0 \pmod{k}$, then it is homotopy equivalent to a sphere of dimension $\frac{2n}{k} - 2$. If $n \equiv h \pmod{k}$, then it is homotopy equivalent to a sphere of dimension $2\frac{n-h}{k} - 1$. Otherwise, it is contractible. Immediate consequences for the multiplicity of the trivial characters in the representations of S_n^B on the homology groups of $\Delta(\Pi_{n,k,k})$ and $\Delta(\Pi_{n,k,h})$ are stated.

The collapsibility of $\Delta(\Pi_{n,k,k})/S_n^B$ is established using a discrete Morse function. The same method is used to show that $\Delta(\Pi_{n,k,h})/S_n^B$, $h < k$, is homotopy equivalent to a certain subcomplex. The homotopy type of this subcomplex is calculated by showing that it is shellable. To do this, we are led to introduce a lexicographic shelling condition for balanced cell complexes of boolean type. This extends to the non-pure case work of P. Hersh (Preprint, 2001) and specializes to the CL-shellability of A. Björner and M. Wachs (*Trans. Amer. Math. Soc.* **4** (1996), 1299–1327) when the cell complex is an order complex of a poset.

Keywords: quotient complex, cell complex of boolean type, lexicographic shellability, coxeter subspace arrangement, homotopy

1. Introduction

Kozlov [17] studied the complex $\Delta(\Pi_n)/S_n$, i.e. the order complex of the partition lattice modulo the symmetric group, and showed that it is collapsible. The partition lattice occurs in a variety of combinatorial subjects. Of interest here is that it is (isomorphic to) the intersection lattice $L(\mathcal{A}_n)$ of the braid arrangement. This is the arrangement of reflecting hyperplanes of a Coxeter group of type A_{n-1} (for Coxeter group terminology, see Humphreys [15]). In fact, Kozlov used a larger collection of subspace arrangements, including the k-equal braid arrangement $\mathcal{A}_{n,k}$. It seems natural to consider complexes originating from other Coxeter groups.

The aim of this paper is to determine the homotopy type of two families of quotient complexes $\Delta(L(\mathcal{H}))/G$, namely when $\mathcal{H} = \mathcal{D}_{n,k}$, the k-equal subspace arrangement of type D_n, and when $\mathcal{H} = \mathcal{B}_{n,k,h}$, the k, h-equal subspace arrangement of type B_n. In particular, the arrangements of reflecting hyperplanes of Coxeter groups of type B_n and D_n are special cases ($\mathcal{B}_{n,2,1}$ and $\mathcal{D}_{n,2}$ respectively). In our case, G will be the group of signed permutations, S_n^B, which has a natural action on these arrangements.

To establish our results we proceed in two steps. First, we apply discrete Morse theory to show that there is a sequence of elementary collapses leading from our original complexes to certain subcomplexes. In the type D case, these subcomplexes are just simplices, and the collapsibility result follows. This is very similar to what Kozlov did in [17]. In the type B case, however, the remaining subcomplexes are more difficult to understand. We determine their homotopy type by proving that they are shellable. To facilitate this, we introduce a lexicographic shellability condition for balanced (pure or non-pure) cell complexes of boolean type. This technique generalizes to the non-pure case a method which recently was introduced by Hersh [14].

It should be pointed out that it is an open question whether or not the original complexes are themselves shellable. Thus, we provide a model (in the type B case) for how one can use discrete Morse theory in conjunction with lexicographic shellability where it is not clear how to proceed solely by either method.

The material is organized as follows. After reviewing some necessary notation and tools in Section 2, we introduce the aforementioned lexicographic shelling condition in Section 3. In Section 4, we define the complexes we wish to study, and they are described using a combinatorial model in terms of trees in Section 5. This model is then used to establish the main results in Section 6; we determine the homotopy type of $\Delta(L(\mathcal{D}_{n,k}))/S_n^B$ and $\Delta(L(\mathcal{B}_{n,k,h}))/S_n^B$. Following the beaten track and work of e.g. Babson and Kozlov [1], Hersh [14] and Kozlov [17], we use these results to draw conclusions concerning representations of S_n^B.

2. Basic definitions and notation

In this section we collect basic definitions and agree on notation. For anything not explained here, we refer to the standard textbooks by Stanley [22] (combinatorics) and Munkres [19] (topology).

2.1. *Shelling cell complexes of boolean type*

A *cell complex of boolean type*, or *boolean cell complex* for short, is a regular cell complex whose face poset is a *simplicial poset*, i.e. a poset, equipped with a minimal element, in which every interval is a boolean algebra. Hence, a boolean cell complex is almost a simplicial complex, except that several simplices may share the same vertex set. Cell complexes of boolean type were introduced by Björner [3] and by Garsia and Stanton [13]. Boolean cell complexes and simplicial posets have since received considerable attention e.g. from Stanley [21], Reiner [20], Duval [11] and Hersh [14].

A cell complex is *pure* if all its *facets*, i.e. inclusion-maximal cells, are equidimensional. Björner [3] defined shellability for pure regular cell complexes. The natural translation to non-pure complexes was given by Björner and Wachs [8]. Specializing to boolean cell complexes gives the following definition.

Definition 2.1 An ordering $F_1, \ldots, F_t$ of the facets of a boolean cell complex Δ is a *shelling order* of Δ if $F_j \cap (\bigcup_{\alpha<j} F_\alpha)$ is pure of codimension 1 in F_j for all $2 \leq j \leq t$. If there exists a shelling order of Δ, then Δ is *shellable*.

We think of a shelling order as a way of putting together Δ facet by facet. Therefore we say that F_j *attaches* over $F_j \cap (\bigcup_{\alpha<j} F_\alpha)$. If Δ is shellable, then it has the homotopy type of a wedge of spheres, the spheres of dimension i being indexed by the i-dimensional facets that attach over their entire boundary. In the pure case, this was proven by Björner [3], and the proof can easily be modified to the non-pure case.

2.2. *Discrete Morse theory*

Let Δ be a regular cell complex. A *matching* on the face poset $P(\Delta)$ is a partition of $P(\Delta)$ into three sets X, Y and Z, such that there exists a bijection $\phi : Y \to X$ with the property that y is covered by $\phi(y)$ for all $y \in Y$. The remaining set Z is called the *critical set* of the matching. The matching is *acyclic* if there exists no sequence $y_1, \dots, y_q \in Y$ such that $y_q = y_1$, $y_i \neq y_{i+1}$ and $\phi(y_i)$ covers y_{i+1} for all $i \in [q-1]$.

From Forman's work [12], the next result follows. See also Kozlov [17] for a direct combinatorial proof. We formulate the result in terms of matchings rather than discrete Morse functions. The connection between the two points of view is given by Chari [10].

Theorem 2.2 *Suppose we have an acyclic matching on $P(\Delta)$ with critical set Z. If Z is a subcomplex of Δ, then Z can be obtained from Δ by a sequence of elementary collapses. In particular, Z and Δ are homotopy equivalent.*

Remark We wish to emphasize the requirement of Theorem 2.2 that Z be a *subcomplex* of Δ. This ensures that the incidences between the simplices in Z are left unchanged during the collapsing, and this is vital for our applications.

2.3. *Quotient complexes*

Throughout we will assume that all posets we consider are finite. We will not make any notational distinctions between a simplicial complex and its geometric realization. Given a poset P equipped with a maximal element $\hat{1}$ and a minimal element $\hat{0}$, we let $\bar{P}$ denote the *proper part* $P \backslash \{\hat{0}, \hat{1}\}$. The *order complex* $\Delta(P)$ is the simplicial complex having the chains of $\bar{P}$ as simplices. If G is a group acting on P in an order-preserving way, we may define $\Delta(P)/G$ as the boolean cell complex whose simplices are the G-orbits of simplices of $\Delta(P)$. In general, $\Delta(P)/G$ is not a simplicial complex, since there may be more than one simplex on the same set of vertices. Babson and Kozlov [1] give conditions under which $\Delta(P)/G \cong \Delta(P/G)$. Earlier, Welker [23] had given specific examples of posets and groups with this property.

3. Lexicographic shellings of balanced boolean cell complexes

A d-dimensional boolean cell complex Δ is *balanced* if there exists a coloring $f : \mathit{vert}(\Delta) \to [d+1]$ of the vertices of Δ whose restrictions to all simplices are injective. An order complex $\Delta(P)$ of a poset P is balanced (define $f(v)$ to be the maximal cardinality of a P-chain with v as maximal element). Furthermore, if a group G acts on P order-preservingly, then this balancing is inherited by $\Delta(P)/G$.

In [14], Hersh gave a lexicographic shelling condition for pure balanced boolean cell complexes. In this section we extend her work to the non-pure case.

Consider a simplex c in a balanced boolean cell complex with coloring f. Suppose that $f(vert(c)) = \{f_0 < \cdots < f_r\}$. To shorten notation, we let $c^{i\to j}$, $-1 \leq i < j \leq r+1$, denote the unique simplex contained in c with colors $\{f_0, \ldots, f_i, f_j, \ldots, f_r\}$. We also let $c^{i\to} := c^{i\to r+1}$, $c^i := c^{i-1\to i+1}$ and $c^{i_1\to j_1,\ldots,i_m\to j_m} := (\ldots(c^{i_m\to j_m})\ldots)^{i_1\to j_1}$.

For a d-dimensional boolean cell complex Δ, let $\hat{\Delta}$ denote the complex whose facets are $\{F \cup \{\hat{0}, \hat{1}\} \mid F \text{ is a facet in } \Delta\}$. This is the *join* of Δ and the one-dimensional simplex on $\{\hat{0}, \hat{1}\}$ (see e.g. Björner [5, Section 9]). If Δ is balanced by $f : vert(\Delta) \to [d+1]$, we extend the balancing to $\hat{\Delta}$ by defining $f(\hat{0}) = 0$ and $f(\hat{1}) = d+2$.

Suppose that $\{F_1, \ldots, F_t\}$ is the set of facets in $\hat{\Delta}$. A *root simplex* of $\hat{\Delta}$ is a simplex of the form $F_\alpha^{i\to}$ for some $\alpha, i \geq 1$. (In particular, all facets are root simplices.) Furthermore, a *rooted interval* is determined by a simplex of the form $c = F_\alpha^{i\to j,j\to}$, $i+2 \leq j$. It consists of all minimal root simplices that contain c.

Let $R(\hat{\Delta})$ be the set of root simplices of $\hat{\Delta}$. A *chain labelling* of Δ is a map $\lambda : R(\hat{\Delta}) \to \Lambda$, where Λ is some poset of labels.

Pick a root simplex $c = F_\alpha^{r\to} \in R(\hat{\Delta})$. Given a chain labelling λ, we define the *descent set* of c to be $D(c) := \{i \in [r-1] \mid \lambda(c^{i\to}) \not\leq \lambda(c^{i+1\to})\}$. Consider the rooted interval given by $c^{i\to j,j\to}$ for some $i+2 \leq j < r$. We say that c is *falling* on this interval if $D(c) \supseteq \{i+1, \ldots, j-1\}$. If, instead, $D(c) \cap \{i+1, \ldots, j-1\} = \emptyset$, then c is *rising* on the interval.

If two distinct root simplices, b_1 and b_2 contain $c^{i\to j,j\to}$, then we compare them on the rooted interval of $c^{i\to j,j\to}$ using the lexicographical order, i.e. $b_1 <_{lex} b_2$ iff $\lambda(b_1^{t\to}) < \lambda(b_2^{t\to})$, where t is the smallest index such that $i < t \leq j$ and $\lambda(b_1^{t\to}) \neq \lambda(b_2^{t\to})$. If no such t exists, then b_1 and b_2 are incomparable on the interval.

Note that the the notions of rising and falling, as well as the lexicographical order, are defined in the context of a rooted interval. When we apply them to facets of $\hat{\Delta}$ without referring to a specific interval, we have the interval determined by $\{\hat{0}, \hat{1}\}$ in mind.

We now give a lexicographic shelling condition for balanced boolean cell complexes. Although stated differently, the most significant difference being in the formulation of Condition 4 below, implies the CL-version of [14, Definition 2.4] in the pure case.

Definition 3.1 A balanced boolean cell complex Δ is *CL-shellable* if there exists a chain labelling of Δ such that the following four conditions are fulfilled:

1. Every rooted interval contains a unique simplex which is rising on the interval.
2. In every rooted interval, the rising simplex is lexicographically smaller than all other simplices.

In 3 and 4, let $F_1, \ldots, F_t$ be an ordering of the facets of $\hat{\Delta}$ which is a linear extension of the lexicographical order.

3. Let c be a maximal simplex in $F_p \cap F_r$, where $p < r$. Write $c = F_r^{s_1\to t_1,\ldots,s_m\to t_m}$, where $s_i \leq t_i - 2$ and $t_{i-1} \leq s_i$ for all i. (There is a unique way to do this.) Let $j \in [m]$ be

maximal such that F_r is rising on all rooted intervals given by $F_r^{s_i \to t_i, t_i \to}$, $i < j$. Then, for some $q < r$ and $i \leq j$, we have $F_r^{s_i \to t_i} \subseteq F_q \cap F_r$.
4. Let c be a maximal simplex in $F_q \cap (\bigcup_{\alpha < q} F_\alpha)$. Suppose $c = F_q^{s \to t}$ and that F_q is rising on the rooted interval given by $F_q^{s \to t, t \to}$. Then $\operatorname{codim}_{F_q}(c) = 1$.

Remark If Δ is the (simplicial) order complex of a poset P, then a chain labelling of Δ is just a chain-edge labelling of $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$ in the sense of Björner and Wachs [7]. In this case, Conditions 3 and 4 are trivially satisfied and Definition 3.1 defines ordinary CL-shellability (see [7, Definition 5.2]) for $\hat{P}$. Condition 3 is satisfied since if two maximal P-chains $c_1 <_{lex} c_2$ differ on several intervals, then one can find a chain $d <_{lex} c_2$ which only differs from c_2 on the first of those intervals (simply by letting the first part of d imitate c_1 and the second part c_2). Condition 4 follows since no such c can exist without Condition 1 to be violated.

Theorem 3.2 *If a balanced boolean cell complex Δ is CL-shellable, then it is shellable.*

Proof: Adjusted to fit our formulation of Definition 3.1, the proof of [14, Theorem 2.1] goes through in the non-pure case, too. We sketch it using our notation. Let the ordering $F_1, \ldots, F_t$ be as in Definition 3.1. Condition 3 of Definition 3.1 ensures that a maximal simplex c in $F_j \cap (\bigcup_{\alpha < j} F_\alpha)$ can be written $c = F_j^{l \to m}$. If F_j is rising on the rooted interval of $F_j^{l \to m, m \to}$, then $codim_{F_j}(c) = 1$ by Condition 4. Otherwise, by Conditions 1 and 2, there is a facet preceding F_j which contains $F_j^{p, p+1 \to}$ but not $F_j^{p+1 \to}$ for some $l < p < m$. By Condition 3, $F_j^p \subset F_i$, for some $i < j$. Hence $c = F_j^p$, and we are done. □

The following result is reminiscent of [14, Proposition 2.1]. It will be of use to us later.

Proposition 3.3 *Let G be a group acting on the poset P in an order preserving way. Then $\Delta = \Delta(P)/G$ is CL-shellable if and only if it has a chain labelling satisfying Conditions* 1 *and* 2 *of Definition* 3.1 *together with the following condition*:
3′. *Let $F_1, \ldots, F_t$ be an ordering of the facets of $\hat{\Delta}$ which is a linear extension of the lexicographical order. Let c be a simplex in $F_r \cap (\bigcup_{\alpha < r} F_\alpha)$. Write $c = F_r^{s_1 \to t_1, \ldots, s_m \to t_m}$, where $s_i \leq t_i - 2$ and $t_{i-1} \leq s_i$ for all i. (Again, there is a unique way to do this.) Suppose F_r is rising on all rooted intervals given by $F_r^{s_i \to t_i, t_i \to}$. Then $c \subseteq b \subseteq F_r \cap (\bigcup_{\alpha < r} F_\alpha)$ for some simplex b with $codim_{F_r}(b) = 1$.*

Proof: The only if direction follows immediately from Theorem 3.2 and the definition of shellability.

Now suppose that we have a chain labelling of Δ satisfying Conditions 1, 2 and 3′. Then Condition 4 is immediate. Let c and F_r be as in Condition 3. If F_r is rising on all rooted intervals given by $F_r^{s_i \to t_i, t_i \to}$, then Condition 3 follows since c is contained in a codimension 1 simplex in $F_r \cap (\bigcup_{\alpha < r} F_\alpha)$. Otherwise, Condition 3 follows via an argument similar to Hersh's proof of [14, Proposition 2.1]. □

As with simplicial complexes, if a boolean cell complex Δ is shellable, then it is homotopy equivalent to a wedge of spheres. Just as in lexicographic shellings of posets, the falling facets correspond to simplices attached over their entire boundaries. However, unlike the ordinary case, there may exist other simplices that attach over their entire boundaries. Consider, e.g., two facets, F and G, on the same vertex set. Even if, say, $1 \notin D(F) \cup D(G)$, we may still have $F^1 = G^1$. Hence, the last attached of F and G will be attached over the boundary simplex $F^1 = G^1$ even though 1 is not a descent. For an example, see the proof of Theorem 6.5 and the illustration in figure 2. This motivates the following definition:

Definition 3.4 Let $F_1, \ldots, F_t$ be an ordering of the facets of Δ which is a linear extension of the lexicographical order. We say that F_j has a *topological descent* at i if $F_j^i \subset \bigcup_{\alpha<j} F_\alpha$. Otherwise, i is a *topological ascent*.

The concepts *topologically falling* and *topologically rising* facets are defined in the obvious way. Our definitions are tailor-made for the following proposition to hold:

Proposition 3.5 *If a balanced boolean cell complex Δ is CL-shellable, then it is homotopy equivalent to a wedge of spheres. For all i, its (reduced) Betti numbers satisfy*

$$\tilde{\beta}_i(\Delta) = \#\textit{topologically falling facets on } i+1 \textit{ vertices}.$$

Remark If "rising" is replaced by "topologically rising" in Definition 3.1 and Theorem 3.2 (and Proposition 3.3), one obtains a more general shellability called CC-shellability. In the pure case, this was done by Hersh [14], and the reason for the name is that it is modelled after the CC-shellability for posets that was introduced by Kozlov [16]. Apart from being more general, it has the advantage of making Proposition 3.5 less artificial. For our applications, though, CL-shellability is sufficient.

4. The objects of study

Throughout the rest of the paper we will frequently encounter the triple (n, k, h). Whenever these integers appear, it will be assumed that $1 \le h \le k \le n$ and that $k \ge 2$ if nothing else is explicitly stated.

Definition 4.1 The *k-equal subspace arrangement of type D_n*, $\mathcal{D}_{n,k}$, is the collection of all linear subspaces of the form

$$\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \tau_1 x_{i_1} = \cdots = \tau_k x_{i_k}\right\}$$

for $1 \le i_1 < \cdots < i_k \le n$ and $\tau_i \in \{-1, 1\}$ for all i.

Definition 4.2 For $h < k$, we define $\mathcal{B}_{n,k,h}$, *the k, h-equal subspace arrangement of type B_n*, to be the union of $\mathcal{D}_{n,k}$ and the collection of linear subspaces of the form

$$\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, x_{i_1} = \cdots = x_{i_h} = 0\right\}$$

for $1 \le i_1 < \cdots < i_h \le n$.

These arrangements were introduced by Björner and Sagan [6]. The special cases $\mathcal{B}_n = \mathcal{B}_{n,2,1}$ and $\mathcal{D}_n = \mathcal{D}_{n,2}$ are the ordinary *hyperplane arrangements of types B_n and D_n* respectively.

Zaslavsky's work [24] provides a nice description of the intersection lattices $L(\mathcal{B}_{n,k,h})$ and $L(\mathcal{D}_{n,k})$ in terms of lattices of signed graphs. We will, however, only briefly consider the structure of these lattices, so we settle for a more naive description of $L(\mathcal{B}_{n,k,h})$ and $L(\mathcal{D}_{n,k})$. For more on subspace arrangements and intersection lattices we refer to Björner [4].

Definition 4.3 Let $\Pi_{n,k,h}$ be the lattice of set partitions of the set $\{-1, 1, -2, 2, \ldots, -n, n\}$ such that the following conditions hold:

1. The partitions are sign-symmetric, i.e. if all plus and minus signs are interchanged, then the partition is unchanged.
2. There is at most one self-symmetric block, i.e. block containing both $-a$ and a for some $a \in [n]$.
3. The non-singleton non-self-symmetric blocks have size at least k.
4. The self-symmetric block has size at least $2h$, if it exists.

With the obvious interpretation $\tau_s x_s = \tau_t x_t$ iff $\tau_s s$ and $\tau_t t$ are in the same block (in particular, $x_s = 0$ iff $\pm s$ belong to the self-symmetric block), we see that $\Pi_{n,k,h}$, $h < k$, is isomorphic to $L(\mathcal{B}_{n,k,h})$ and that $\Pi_{n,k,k}$ is isomorphic to $L(\mathcal{D}_{n,k})$.

Note that "most" of the lattices $\Pi_{n,k,h}$ are not graded. It is straightforward to check that when $k > 2$, $\Pi_{n,k,h}$ is graded if and only if $n < k + h$ and $k \in \{h, h+1\}$. In the hyperplane case $k = 2$, $\Pi_{n,k,h}$ is always graded.

To shorten notation, we introduce $\Delta_{n,k,h} := \Delta(\Pi_{n,k,h})/S_n^B$. These are the objects we will study. Here, S_n^B is the group of signed permutations, which acts in a natural way on $\Pi_{n,k,h}$. It is a Coxeter group of type B_n. For our purposes, it suffices to view S_n^B as the group of permutations π of the set $\{-1, 1, -2, 2, \ldots, -n, n\}$ such that $\pi(a) = -\pi(-a)$ for all $a \in [n]$.

A more thorough analysis of the simplex structure of $\Delta_{n,k,h}$ takes place in Section 5. Here we will only describe the vertices, $vert(\Delta_{n,k,h})$, of $\Delta_{n,k,h}$. Given two partitions $\pi, \tau \in \Pi_{n,k,h}$, it is clear that there exists a $g \in S_n^B$ such that $g\tau = \pi$ if and only if there is a bijection between the blocks of τ and the blocks of π which respects block size and commutes with the operation of interchanging all plus and minus signs.

To avoid confusion, the elements of an integer partition will be denoted *parts*, as opposed to the *blocks* of a set partition. Let N_n denote the set of integer partitions of n in which we allow at most one part, the *null* part, to be distinguished. Define the *nullity*, $null(\lambda)$, of λ to be the size of the null part of λ, or zero, if λ has no null part. Order N_n by the rule: $\lambda \leq \kappa$ if λ refines κ as an integer partition and $null(\lambda) \leq null(\kappa)$.

Definition 4.4 We let $N_{n,k,h}$ denote the subposet of N_n induced by the elements λ with $null(\lambda) \notin [h-1]$ in which all non-null parts are either singletons or have size at least k.

Consider the map *type* : $vert(\Delta_{n,k,h}) \to \overline{N_{n,k,h}}$ defined as follows. Pick $v \in vert(\Delta_{n,k,h})$ and $\tau_v \in \Pi_{n,k,h}$ such that $orb(\tau_v) = v$, where $orb(\ )$ denotes S_n^B-orbit. A size $2s$ self-symmetric block in τ_v gives rise to a size s null part in $type(v)$. The other blocks in τ_v occur in sign-reflected pairs. Each such pair of blocks of size s gives rise to a size s (non-null) part in $type(v)$. For example, with $(n, k, h) = (4, 2, 1)$, we have $type(orb(\{\{-1, 1\}, \{-2, 3\}, \{4\}, \{2, -3\}, \{-4\}\})) = \{\bar{1}, 2, 1\}$, where the bar indicates null part. Obviously, in the light of the discussion above, *type* is well-defined and bijective. Hereafter, we will consider the vertices of $\Delta_{n,k,h}$ to be elements of $N_{n,k,h}$.

5. Describing $\Delta_{n,k,h}$ using trees

In this section we give a description of the simplices of $\Delta_{n,k,h}$ in terms of a certain kind of trees. With modifications, we follow Kozlov's [18, Section 4] description of some complexes related to the order complex of the partition lattice modulo the symmetric group.

5.1. *The trees*

In the following, we will suppose that all trees are finite. Given a tree T, let $V(T)$ denote the vertex set of T. For a rooted tree T, let $l_i(T)$ be the number of vertices at distance i from the root. A rooted tree T is called a *graded tree of rank r* if the distance from an arbitrary leaf to the root is $r + 1$ and $1 = l_0(T) < l_1(T) < \cdots < l_{r+1}(T)$. In such a tree, the *depth* of a vertex v is the distance from the root to v.

Let $\langle\bar{n}\rangle$ denote the set $\{\bar{0}, \bar{1}, \dots, \bar{n}\}$. Define $\bar{i} + j := \overline{i + j}$ for integers i and j, and extend the definition by associativity and commutativity. (Sums of the type $\bar{i} + \bar{j}$ are not defined.)

Definition 5.1 An *(n, k, h)-tree of rank r* is a pair (T, η), where T is a graded tree of rank r and $\eta : V(T) \to \{\bar{0}, \bar{h}, \overline{h+1}, \dots, \bar{n}\} \cup \{1, k, k+1, \dots, n\}$ is a labelling of the vertices of T such that

1. $\eta(\rho) = \bar{n}$, where ρ is the root of T.
2. For all non-leaf vertices $v \in V(T)$, we have $\eta(v) = \sum \eta(w)$ (sum over all children w of v).
3. On every depth, there is at least one non-trivial vertex v, i.e. $\eta(v) \notin \{\bar{0}, 1\}$.

We often abuse notation and write T for (T, η).

Remark In particular, the second condition implies that if $\eta(v) \in \langle\bar{n}\rangle$ for a non-leaf v, then $\eta(w) \in \langle\bar{n}\rangle$ for exactly one child w of v (figure 1).

Let $T^r_{n,k,h}$ denote the set of (n, k, h)-trees of rank r (we will not distinguish between isomorphic trees), and let $T_{n,k,h}$ be the set of all (n, k, h)-trees so that, in particular, $T_{n,k,h} = \bigcup_r T^r_{n,k,h}$.

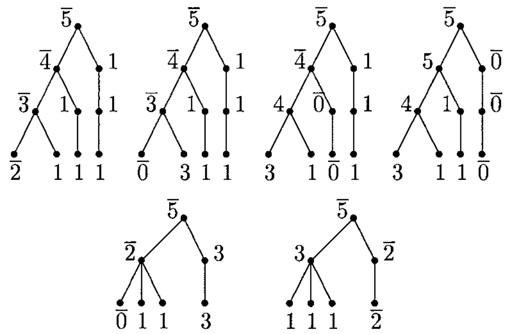

Figure 1. All maximal (5, 3, 2)-trees. The upper ones are of rank 2 and the lower of rank 1.

5.2. The deletion operator

For an integer i, $0 < i \leq r + 1$, we define the deletion operator $\delta^i : T^r_{n,k,h} \to T^{r-1}_{n,k,h}$ by $\delta^i((T, \eta)) := (T^i, \eta^i)$, where T^i is the tree obtained from T by deleting all vertices of depth i and letting the grandchildren of the depth $i - 1$ vertices be their new children. The labelling η^i is the restriction of η to $V(T^i)$. For convenience, we introduce $\delta^{i\to j} := \delta^{i+1} \circ \delta^{i+2} \circ \cdots \circ \delta^{j-1}$ for $i < j+1$. We will also use the notation $\delta^{i\to} := \delta^{i+1} \circ \delta^{i+2} \circ \cdots \circ \delta^{r+1}$.

5.3. A description of $\Delta_{n,k,h}$

Recall from Section 4 the description of the vertices of $\Delta_{n,k,h}$ in terms of the elements of $N_{n,k,h}$. Note that each level, i.e. set of vertices of some fixed depth, of $T \in T_{n,k,h}$ can be viewed as an element $\lambda \in N_{n,k,h}$. The (labelled) vertices of T correspond to parts in λ. A vertex v such that $\eta(v) \in \langle \bar{n} \rangle$ corresponds to a null part. (We interpret $\eta(v) = \bar{0}$ as the non-existence of a null part.)

Let $\Delta^r_{n,k,h}$ be the set of r-dimensional simplices in $\Delta_{n,k,h}$. We describe a mapping $\psi : \Delta^r_{n,k,h} \to T^r_{n,k,h}$ as follows:

Take a simplex $c \in \Delta^r_{n,k,h}$ and let the chain $\pi = \{\pi_{r+1} < \pi_r \ldots < \pi_1\} \subseteq \Pi_{n,k,h}$ be a representative of c. We construct $\psi(c)$ level by level. The (labelled) vertices at depth i of $\psi(c)$ are the parts in $orb(\pi_i) \in N_{n,k,h}$. We put an edge between two vertices a and b, of depths i and $i + 1$ respectively, if the block of π_i corresponding to a is refined in π by the block of π_{i+1} corresponding to b. Finally, we add the root $\bar{n}$ and put edges from it to all vertices of depth 1. It follows immediately from the construction that ψ is well-defined, i.e. $\psi(c)$ does not depend on the choice of representative of c.

The following result is analogous to [18, Theorem 4.4], and the same straightforward proof applies with obvious modifications. It allows us to use $T_{n,k,h}$ as a model of $\Delta_{n,k,h}$.

Proposition 5.2 *The mapping $\psi : \Delta^r_{n,k,h} \to T^r_{n,k,h}$ is bijective. Furthermore, under ψ, inclusion in $\Delta_{n,k,h}$ corresponds to deletion in $T_{n,k,h}$. In other words, for two simplices $c_1, c_2 \in \Delta_{n,k,h}$ we have $c_1 \subseteq c_2$ iff $\psi(c_1) = \delta^{i_1} \circ \cdots \circ \delta^{i_t}(\psi(c_2))$ for some $i_1, \ldots, i_t$.*

6. The homotopy type of $\Delta_{n,k,h}$

In the following we will need two projection maps $\mu, \nu : N_{n,k,h} \to N_{n,k,h}$.

Definition 6.1 Let $\lambda \in N_{n,k,h}$ be given. Recall that *null*(λ) is the size of the null part of λ.

1. Define $\mu(\lambda)$ to be the maximal element $\mu \leq \lambda$ such that $null(\mu) = 0$ and all non-singleton parts of μ have size k.
2. Define $\nu(\lambda)$ to be the maximal element $\nu \leq \lambda$ such that $null(\nu) = null(\lambda)$ and all non-singleton non-null parts of ν have size k.
3. We say that λ is *μ-like* (*ν-like*) if λ is fixed by μ (ν).

For example, with $n = 7$ and $k = 2$, $\mu(\{\bar{3}, 3, 1\}) = \{2, 2, 1, 1, 1\}$, which is a μ-like partition, and $\nu(\{\bar{3}, 3, 1\}) = \{\bar{3}, 2, 1, 1\}$, which is ν-like.

Note that $\lambda \in N_{n,k,h}$ is μ-like iff $null(\lambda) = 0$ and all non-singleton parts of λ have size k. Similarly, λ is ν-like iff all non-singleton non-null parts of λ have size k.

6.1. The $\mathcal{D}_{n,k}$ case

Theorem 6.2 *The complex $\Delta_{n,k,k}$ is collapsible.*

Proof: The idea is the same as in Kozlov's proof of [17, Theorem 4.1]. We will give an acyclic matching on the face poset $P(\Delta_{n,k,k})$. In view of Proposition 5.2, we will consider the elements of $P(\Delta_{n,k,k})$ to be (n, k, k)-trees. Recall from Section 5.3 that the levels of a tree $T \in P(\Delta_{n,k,k})$ are viewed as elements of $N_{n,k,k}$. Let $\lambda_i(T) \in N_{n,k,k}$ denote the element obtained from depth i in T.

$P(\Delta_{n,k,k})$ is partitioned into the sets X, Y and Z in the following way. Pick a tree $T \in P(\Delta_{n,k,k})$. If $\lambda_i(T)$ is μ-like for all i, then T belongs to Z. Otherwise, let i be the largest index such that $\lambda_i(T)$ is not μ-like. If $\mu(\lambda_i(T)) = \lambda_{i+1}(T)$, then T belongs to X. If, on the other hand, $\mu(\lambda_i(T)) \neq \lambda_{i+1}(T)$, then T belongs to Y.

Let $T \in Y$ be given, and let i be the largest index such that $\lambda_i(T)$ is not μ-like. It is easily seen that there is a unique way (up to tree-isomorphisms) to insert a new level corresponding to $\mu(\lambda_i(T))$ directly beneath level i in T. The tree $\tilde{T}$ thus obtained covers T (since $\delta^{i+1}(\tilde{T}) = T$) and belongs to X. Conversely, given $\tilde{T} \in X$, we construct $T \in Y$ uniquely by deleting the level below the deepest not μ-like level in $\tilde{T}$. Hence we have a bijection$\tilde{\ }:Y \to X$ such that $\tilde{T}$ covers T.

It remains to show that our matching is acyclic. To this end, suppose $T_1, T_2 \in Y$, $T_1 \neq T_2$ and that $\tilde{T}_1$ covers T_2. Since $T_2 \neq T_1$, T_2 must be obtained from $\tilde{T}_1$ by deleting some other level than the one below the deepest not μ-like level. This other level must in fact be the deepest not μ-like level; otherwise we would have $T_2 \in X$. Hence the number of μ-like levels in T_2 is one larger than in T_1. Repeating this argument shows the non-existence of a sequence $T_1, \ldots, T_t \in Y$ such that $T_1 = T_t$, $T_i \neq T_{i+1}$ and $\tilde{T}_i$ covers T_{i+1} for $i \in [t-1]$. Hence the matching is acyclic.

Since the property that all levels are μ-like is preserved under deletion of levels, Z is a subcomplex of $\Delta_{n,k,k}$. Theorem 2.2 shows that there is a sequence of elementary collapses transforming $\Delta_{n,k,k}$ to Z. This subcomplex consists of the (n, k, k)-trees in which every level is μ-like. There is obviously exactly one maximal such tree. Hence, the subcomplex given by Z is a simplicial complex generated by one simplex and thus collapsible. □

6.2. *The $\mathcal{B}_{n,k,h}$ case*

When $h < k$, the proof of Theorem 6.2 fails. The reason is that for $h \le i < k$, we have $\mu(\{\bar{i}, 1, 1, \dots, 1\}) = \{1, 1, \dots, 1\}$ which is not in the proper part of $N_{n,k,h}$. To overcome this, we will use ν instead of μ. As before, we will use simplices of $\Delta_{n,k,h}$ and their tree representations interchangeably.

Definition 6.3 Let $U_{n,k,h}$ be the subcomplex of $\Delta_{n,k,h}$ consisting of the (n, k, h)-trees in which every level is ν-like.

Lemma 6.4 *The complexes $\Delta_{n,k,h}$ and $U_{n,k,h}$ are homotopy equivalent.*

Proof: Except for the last two sentences, the proof of Theorem 6.2 goes through with obvious modifications if we replace μ with ν. □

Theorem 6.5 *Let $h < k$. If $n \equiv 0 \pmod{k}$, then $\Delta_{n,k,h}$ is homotopy equivalent to a sphere of dimension $\frac{2n}{k} - 2$. If $n \equiv h \pmod{k}$, then $\Delta_{n,k,h}$ is homotopy equivalent to a sphere of dimension $2\frac{n-h}{k} - 1$. Otherwise, $\Delta_{n,k,h}$ is contractible.*

Proof: In the light of Lemma 6.4, we restrict our attention to $U_{n,k,h}$. We let P be the set of ν-like elements in $N_{n,k,h}$ and we choose to order them in *reverse fashion* to $N_{n,k,h}$. That is, $\kappa \le_P \lambda$ if λ refines κ as number partitions and $null(\lambda) \le null(\kappa)$. The map $f : vert(U_{n,k,h}) \to [dim(U_{n,k,h}) + 1]$, where $f(v)$ is the maximal cardinality of a P-chain with v as largest element clearly balances $U_{n,k,h}$. We identify $\hat{0} = \{\bar{n}\}$ and $\hat{1} = \{1, 1, \dots, 1\}$ in $\hat{U}_{n,k,h}$.

We give a chain-labelling of $U_{n,k,h}$ which induces a CL-shelling. Our poset of labels is $\Lambda = \{A < B < C_1 < \cdots < C_{n-1} < D\}$. The label of a root simplex c of $\hat{U}_{n,k,h}$ is determined by the refinement taking place between level $r = rank(T_c)$ and the leaves of its corresponding tree $T_c = \psi(c\backslash\{\hat{0}\})$. (If c contains $\hat{1}$, and hence is a facet, we let T_c be the tree $\psi(c\backslash\{\hat{0}, \hat{1}\})$ to which we have attached a leaf-level corresponding to $\hat{1}$ in the obvious way.) Let $x = \lambda_r(T_c)$ and $y = \lambda_{r+1}(T_c)$ (notation as in the proof of Theorem 6.2). Define the labelling $\omega : R(\hat{U}_{n,k,h}) \to \Lambda$ by:

- ○ $\omega(c) = A$, if $null(x) = null(y) + 1$, i.e. if y is obtained from x by cutting a singleton off the null part.
- ○ $\omega(c) = B$, if $h > 1$, $null(x) = h$ and $null(y) = 0$, i.e. if the null part in x is of size h and is split into singletons in y.
- ○ $\omega(c) = C_i$, if a non-null k-part in x is split into singletons in y and the k-part first appeared at depth i in T_c. By this we mean that the k-part was cut off the null part in the refinement process between depth $i - 1$ and depth i in T_c.

- $\circ$ $\omega(c) = D$, if $null(x) = null(y) + k$, i.e. if y is obtained from x by cutting a k-part off the null part.

Remark If $h = 1$, then label B is never used. However, A plays its role in this case.

The simplices determining root intervals in $\hat{U}_{n,k,h}$ correspond to trees which are saturated except between some level r and the leaves. It is not hard to see, that in every rooted interval there is a unique rising simplex and that this simplex is lexicographically least on the interval. Hence, the first and second conditions of Definition 3.1 are fulfilled.

In the following, we will not make distinctions between simplices and their corresponding trees. Let $F_1, \dots, F_t$ be the lexicographical ordering of the facets of $\hat{U}_{n,k,h}$. To verify Condition 3′ of Proposition 3.3, pick c in $F_j \cap (\bigcup_{\alpha<j} F_\alpha)$. Write $c = \delta^{a_1 \to b_1} \circ \cdots \circ \delta^{a_m \to b_m}(F_j)$ for some appropriate a_x, b_x with $a_x + 2 \leq b_x$ and $b_{x-1} \leq a_x$. Choose F_i, $i < j$, such that $c = \delta^{a'_1 \to b'_1} \circ \cdots \circ \delta^{a'_m \to b'_m}(F_i)$ for some a'_x, b'_x such that $a'_x + 2 \leq b'_x$ and $b'_{x-1} \leq a'_x$. (Since $\hat{U}_{n,k,h}$ possibly is nonpure, we do not a priori have $a_x = a'_x$ and $b_x = b'_x$.) Suppose that F_j is rising on every rooted interval given by $\delta^{a_x \to b_x} \circ \delta^{b_x \to}(F_j)$. We must show that c is contained in some simplex $b \subseteq F_j \cap (\bigcup_{\alpha<j} F_\alpha)$ with $codim_{F_j}(b) = 1$.

Let a be minimal such that $\omega(\delta^{a\to}(F_j)) \neq \omega(\delta^{a\to}(F_i))$. We cannot have $a_x < a < b_x$ for any x, because this would imply $\delta^{a_x\to}(F_j) = \delta^{a_x\to}(F_i)$ and $\delta^{b_x\to}(F_j) \neq \delta^{b'_x\to}(F_i)$, which is impossible since $i < j$ and F_j is rising on the rooted interval of $\delta^{a_x \to b_x} \circ \delta^{b_x\to}(F_j) = \delta^{a_x \to b'_x} \circ \delta^{b'_x\to}(F_i)$. Thus, the refinement process which determines $\omega(\delta^{a\to}(F_j))$ occurs within $c \subseteq F_i \cap F_j$. By the definition of ω, this implies $\omega(\delta^{a\to}(F_j)) = C_{x_j}$ and $\omega(\delta^{a\to}(F_i)) = C_{x_i}$ for some $x_i < x_j$. Moreover, we know that $a_x < x_i < x_j < b_x$ for some x and that $\omega(\delta^{x_i\to}(F_j)) = \omega(\delta^{x_i+1\to}(F_j)) = \cdots = \omega(\delta^{x_j\to}(F_j)) = D$; otherwise C_{x_i} and C_{x_j} would correspond to different refinement processes within c. The label C_{x_j} precedes C_{x_i} in the sequence of labels of F_j, so there is some x_s such that $x_i \leq x_s < x_s + 1 \leq x_j$ and C_{x_s+1} precedes C_{x_s} in this sequence. Now create the facet F_s which has every ω-label in common with F_j except that C_{x_s} and C_{x_s+1} are interchanged. Then $s < j$ and $\delta^{x_s}(F_s) = \delta^{x_s}(F_j) \supseteq c$. Thus c is contained in a simplex of codimension 1. Hence $U_{n,k,h}$ is CL-shellable.

In order to determine the homotopy type of $U_{n,k,h}$, we will identify the topologically falling facets in the described shelling. The homotopy type is then given by Proposition 3.5. To this end, consider a facet F and its boundary simplex $\delta^i(F)$. Let ω_i and ω_{i+1} denote the labels of the root simplices $\delta^{i\to}(F)$ and $\delta^{i+1\to}(F)$ respectively. If i is a descent, i.e. $\omega_i > \omega_{i+1}$ (since Λ is a total ordering), then it is also a topological descent. Now suppose $\omega_i \leq \omega_{i+1}$. If $\omega_i \neq D$, then no facet preceding F contains $\delta^i(F)$. This is because all labels below depth i depend only on refinement processes within $\delta^i(F)$ in this case. The only case left to check is thus $\omega_i = \omega_{i+1} = D$. As above, it is seen that $\delta^i(F)$ is contained in some facet preceding F if and only if the label C_{i+1} precedes the label C_i, or, in other words, if the (unique) root simplex contained in F with label C_{i+1} is contained in the one labelled C_i. Hence, the only possible facets attaching over $\delta^i(F)$ for all i must be labelled either $DD \dots DC_tC_{t-1} \dots C_1$ or $DD \dots DC_tC_{t-1} \dots C_1B$ (B is replaced by A if $h = 1$), where t is equal to the number of D:s in the sequence. For an example, see figure 2. Clearly, such a facet exists (and is unique) if and only if $n \equiv 0 \pmod{k}$ (the first kind) or $n \equiv h \pmod{k}$

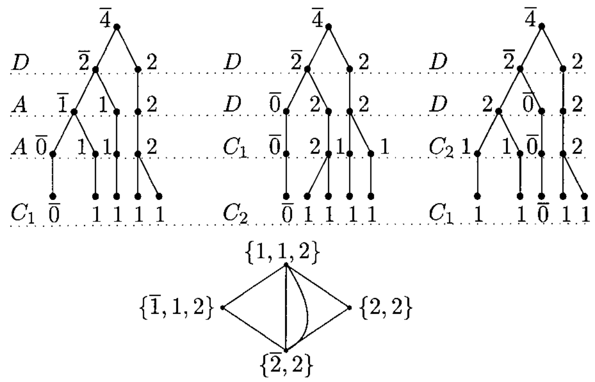

Figure 2. A subcomplex of $U_{4,2,1}$ containing three of the ten facets (bottom) and the corresponding facets of $\hat{U}_{4,2,1}$ seen as trees (top). The labelling ω on the successive rooted subsimplices is indicated. Note that the third facet attaches over its entire boundary (hence is topologically falling) even though it is not falling.

(the second kind). When it exists, it contains $\frac{2n}{k} - 1$ vertices in the former case and $2\frac{n-h}{k}$ vertices in the latter. Hence the theorem. □

Remark The hypothesis $h < k$ is necessary (hence Theorem 6.5 does not contradict Theorem 6.2). To see this, note that to get rid of a null part in the $h = k$ case, one must at some place let a D-label precede a C_x-label, thereby violating Condition 1 of Definition 3.1.

We now state some immediate consequences of Theorems 6.2 and 6.5 concerning S_n^B-representations. The key is the following well-known fact, which follows, e.g., from Bredon [9, Theorem 2.4, p. 120].

Lemma 6.6 *Let K be a field. If Δ is a finite simplicial complex acted upon by a finite group G, where char(K) does not divide $|G|$, then the multiplicity of the trivial character of the induced representation of G on $\tilde{H}_i(\Delta, K)$ equals $\tilde{\beta}_i(\Delta/G, K)$.*

Corollary 6.7 *Let K be a field with char$(K) \notin [n]$. Then*
1. *The trivial character of the induced representation of S_n^B on the vector space $\tilde{H}_i(\Delta(\Pi_{n,k,k}), K)$ has multiplicity zero for all i.*
2. *Let $h < k$. The trivial character of the induced representation of S_n^B on the vector space $\tilde{H}_i(\Delta(\Pi_{n,k,h}), K)$ has multiplicity one if $n \equiv 0 \pmod{k}$ and $i = \frac{2n}{k} - 2$ or if $n \equiv h \pmod{k}$ and $i = 2\frac{n-h}{k} - 1$. Otherwise it has multiplicity zero.*

Proof: Note that $|S_n^B| = 2^n n!$. The corollary now follows from Theorem 6.2, Theorem 6.5 and Lemma 6.6. □

Acknowledgment

I am indebted to Dmitry Kozlov for suggesting the problem and for providing valuable comments. I would also like to thank the anonymous referee for the numerous suggestions that have improved this paper.

References

1. E. Babson and D. Kozlov, "Group actions on posets," preprint, 2000.
2. A. Björner, "Shellable and Cohen-Macaulay partially ordered sets," *Trans. Amer. Math. Soc.* **1** (1980), 159–183.
3. A. Björner, "Posets, regular CW-complexes and Bruhat order," *European J. Combin.* **5** (1984), 7–16.
4. A. Björner, "Subspace arrangements," *First European Congress of Math.*, Paris, 1992. A. Joseph et al. (Eds.), Birkhäuser, Basel, 1994, pp. 321–370; *Progr. Math*. **119**.
5. A. Björner, "Topological methods," in *Handbook of Combinatorics*, R. Graham, M. Grötschel, and L. Lovász (Eds.), North-Holland, 1995, pp. 1819–1872.
6. A. Björner and B. Sagan, "Subspace arrangements of type B_n and D_n," *J. Alg. Combin.* **5** (1996), 291–314.
7. A. Björner and M. Wachs, "Shellable nonpure complexes and posets I," *Trans. Amer. Math. Soc.* **4** (1996), 1299–1327.
8. A. Björner and M. Wachs, "Shellable nonpure complexes and posets II," *Trans. Amer. Math. Soc.* **10** (1997), 3945–3975.
9. G.E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.
10. M. Chari, "On discrete Morse functions and combinatorial decompositions," *Discrete Math.* **217** (2000), 101–113.
11. A. Duval, "Free resolutions of simplicial posets," *J. Algebra* **188** (1997), 363–399.
12. R. Forman, "Morse theory for cell complexes," *Adv. in Math.* **134** (1998), 90–145.
13. A. Garsia and D. Stanton, "Group actions of Stanley-Reisner rings and invariants of permutation groups," *Adv. in Math.* **51** (1984), 107–201.
14. P. Hersh," Lexicographic shellability for balanced complexes," preprint, 2001.
15. J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990.
16. D. Kozlov, "General lexicographic shellability and orbit arrangements," *Ann. Comb.* **1** (1997), 67–90.
17. D. Kozlov, "Collapsibility of $\Delta(\Pi_n)/S_n$ and some related CW complexes," *Proc. Amer. Math. Soc.* **128** (2000), 2253–2259.
18. D. Kozlov, "Rational homology of spaces of complex monic polynomials with multiple roots," *Math. CO/0111167*.
19. J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park, 1984.
20. V. Reiner, "Quotients of Coxeter complexes and P-partitions," *Memoirs. Amer. Math. Soc.* **95** (1992).
21. R.P. Stanley, "f-vectors and h-vectors of simplicial posets," *J. Pure and Applied Algebra* **71** (1991), 319–331.
22. R.P. Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge University Press, 1997.
23. V. Welker, "The topology of posets and orbit posets," in *Coding Theory, Design Theory, Group Theory*, Burlington, VT, 1990, Wiley-Interscience, Wiley, New York, 1993, pp. 283–289.
24. T. Zaslavsky, "Signed graphs," *Discrete Appl. Math.* **4** (1982), 47–74.