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Derangements and Tensor Powers of Adjoint Modules for $\mathfrak{sl}_n$

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Abstract. We obtain the decomposition of the tensor space $\mathfrak{sl}_n^{\otimes k}$ as a module for $\mathfrak{sl}_n$, find an explicit formula for the multiplicities of its irreducible summands, and (when $n \geq 2k$) describe the centralizer algebra $\mathcal{C} = \mathrm{End}_{\mathfrak{sl}_n}(\mathfrak{sl}_n^{\otimes k})$ and its representations. The multiplicities of the irreducible summands are derangement numbers in several important instances, and the dimension of $\mathcal{C}$ is given by the number of derangements of a set of $2k$ elements.

Keywords: derangements, centralizer algebras, walled Brauer algebras, tensor powers, adjoint representation

Introduction

Weyl's celebrated theorem on complete reducibility says that a finite-dimensional module X for a finite-dimensional simple complex Lie algebra $\mathfrak{g}$ is a direct sum of irreducible $\mathfrak{g}$-modules. However, to determine an explicit expression for the multiplicities of the irreducible $\mathfrak{g}$-summands of X often is a very challenging task. In this note we assume $\mathfrak{g} = \mathfrak{sl}_n$, the simple Lie algebra of $n \times n$ matrices of trace 0 over $\mathbb{C}$, and view $\mathfrak{sl}_n$ as a $\mathfrak{g}$-module under the adjoint action $x \cdot y = [x, y]$. We take X to be the k-fold tensor power of $\mathfrak{sl}_n$. Using combinatorial methods and results developed in [2], we establish an explicit description of the irreducible $\mathfrak{g}$-summands of $\mathfrak{sl}_n^{\otimes k}$ (Theorem 1.16) and determine an expression for their multiplicities (Theorem 2.2). As a consequence of our formula, we obtain the following results, expressed in terms of the number D_k of derangements of $\{1, \ldots, k\}$: For $n \geq 2k$, the dimension of the space of $\mathfrak{g}$-invariants in $\mathfrak{sl}_n^{\otimes k}$ is D_k; the multiplicity of $\mathfrak{sl}_n$ in $\mathfrak{sl}_n^{\otimes k}$ is D_{k+1}; and the dimension of the centralizer algebra $\mathcal{C} = \mathrm{End}_{\mathfrak{g}}(\mathfrak{sl}_n^{\otimes k})$ is D_{2k}.

In Section 3, we identify the centralizer algebra $\mathcal{C}$ with a certain subalgebra of the walled Brauer algebra $B_{k,k}(n)$. This subalgebra has a basis indexed by derangements of $\{1, \ldots, 2k\}$. We then give a description (for $n \geq 2k$) of the irreducible modules for $\mathcal{C}$, and obtain the "double centralizer" decomposition of the tensor space $\mathfrak{sl}_n^{\otimes k}$ as a bimodule for $\mathcal{C} \times \mathfrak{g}$.

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1. The tensor product realization

The general linear Lie algebra $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}I$ of all $n \times n$ complex matrices acts on $\mathfrak{sl}_n$ via the adjoint action, and the identity matrix I acts trivially. Hence, there is no harm in assuming that $\mathfrak{g}$ is $\mathfrak{gl}_n$ rather than $\mathfrak{sl}_n$ acting on $\mathfrak{sl}_n^{\otimes k}$ in what follows; the results are exactly the same. This enables us to label the irreducible summands by pairs of partitions and to apply known results on the decomposition of tensor products for $\mathfrak{gl}_n$.

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g} = \mathfrak{gl}_n$ of diagonal matrices, and let $\epsilon_i : \mathfrak{h} \to \mathbb{C}$ be the projection of a diagonal matrix onto its (i, i)-entry. The irreducible finite-dimensional $\mathfrak{g}$-modules are labeled by their highest weight, which is an integral linear combination $\sum_{i=1}^{n} \kappa_i \epsilon_i$ with $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$. By letting $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ denote the sequence of positive κ_i and $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ be the partition determined by the negative κ_i, we may associate to each highest weight a pair of partitions (λ, μ). For example, for $\mathfrak{g} = \mathfrak{gl}_{12}$ the highest weight

$$3\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 + \epsilon_5 - 4\epsilon_{10} - 5\epsilon_{11} - 5\epsilon_{12}$$

is identified with the pair of partitions $\lambda = (3, 2, 2, 2, 1) \vdash 10$ and $\mu = (5, 5, 4) \vdash 14$. Therefore, the set of highest weights for $\mathfrak{g}$-modules is in bijection with the set of pairs of partitions such that the total number of nonzero parts does not exceed n.

Let $V = \mathbb{C}^n$ be the natural representation of $\mathfrak{g} = \mathfrak{gl}_n$ on $n \times 1$ matrices by matrix multiplication. The dual module V^* may be identified with $1 \times n$ matrices, where the $\mathfrak{g}$-action is by right multiplication by the negative of an element $x \in \mathfrak{g}$. The matrix product

$$V \otimes V^* \to \mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}I, \quad u \otimes w^* \mapsto uw^* \tag{1.1}$$

is a $\mathfrak{g}$-module isomorphism which allows us to identify $\mathfrak{gl}_n$ with $V \otimes V^*$.

Let $\{v_1, \ldots, v_n\}$ denote the standard basis of V, where v_i is the matrix having 1 in the ith row and 0 everywhere else. Assume $\{v_1^*, \ldots, v_n^*\}$ is the dual basis in V^*, so that v_i^* has 1 in its ith column and 0 elsewhere. The *contraction mapping* $c : V \otimes V^* \to V \otimes V^*$ is defined using the trace by

$$c(u \otimes w^*) = \mathrm{tr}(uw^*) \sum_{\ell=1}^{n} v_\ell \otimes v_\ell^*. \tag{1.2}$$

Under the isomorphism in (1.1), $v_\ell \otimes v_\ell^*$ is mapped to the matrix unit $E_{\ell,\ell} \in \mathfrak{gl}_n$. Therefore, we may identify the image of c with $\mathbb{C}I$, and the kernel of c with $\mathfrak{sl}_n$.

As $c^2 = nc$, the mapping $p = (1/n)c$ is an idempotent. It is the projection onto the trivial summand $\mathbb{C}I$, and $\mathrm{id} - p$ is the projection onto $\mathfrak{sl}_n$. These idempotents are orthogonal,

$$p(\mathrm{id} - p) = 0 = (\mathrm{id} - p)p,$$

and satisfy $\mathrm{id} = (\mathrm{id} - p) + p$. (Here id is the identity map on $V \otimes V^*$.)

In order to identify $\mathfrak{sl}_n^{\otimes k}$ with a summand of

$$M = V^{\otimes k} \otimes (V^*)^{\otimes k} \cong (V \otimes V^*)^{\otimes k} \cong \mathfrak{gl}_n^{\otimes k}, \tag{1.3}$$

we define the contraction map $c_{i,j}$ to be the contraction c applied to the ith factor of $V^{\otimes k}$ and the jth factor of $(V^*)^{\otimes k}$ according to

$$\begin{aligned} &c_{i,j}(u_1 \otimes \cdots \otimes u_k \otimes w_1^* \otimes \cdots \otimes w_k^*) \\ &\quad = \operatorname{tr}(u_i w_j^*) \sum_{\ell=1}^{n} u_1 \otimes \cdots v_\ell \otimes \cdots \otimes v_k \otimes w_1^* \otimes \cdots \otimes v_\ell^* \cdots \otimes w_k^*, \end{aligned}$$

where v_ℓ is placed in the ith slot of $V^{\otimes k}$ and v_ℓ^* in the jth slot of $(V^*)^{\otimes k}$. As before, $c_{i,j}^2 = nc_{i,j}$, so that

$$p_i = \frac{1}{n} c_{i,i} \tag{1.4}$$

is an idempotent.

Proposition 1.5 $\ker p_1 \cap \ker p_2 \cap \cdots \cap \ker p_k = (\mathrm{id} - p_1)(\mathrm{id} - p_2) \cdots (\mathrm{id} - p_k)M$.

Proof: The idempotents p_i commute and satisfy $p_i(\mathrm{id} - p_i) = 0$. For J a subset of $\{1, \ldots, k\}$, let $p_J = \prod_{j \in J} p_j$. Set $q_j = \mathrm{id} - p_j$ and $q_J = \prod_{j \in J} q_j$. Then

$$M = \bigoplus_{J \subseteq \{1,\ldots,k\}} p_{J^c} q_J M,$$

where $J^c = \{1, \ldots, k\} \backslash J$. This can be argued by induction on k. Note that the sum is direct because for any fixed choice of subset J', the idempotent $p_{J'^c} q_{J'}$ acts as the identity on $p_{J'^c} q_{J'} M$ and annihilates the remaining terms $p_{J^c} q_J M$ with $J \neq J'$. Whenever $j \in J^c$, then $p_{J^c} q_J M$ is not contained in $\ker p_j$. Therefore, from the decomposition of M above, it is easy to see that $\ker p_1 \cap \ker p_2 \cap \cdots \cap \ker p_k = (\mathrm{id} - p_1)(\mathrm{id} - p_2) \ldots (\mathrm{id} - p_k)M$. □

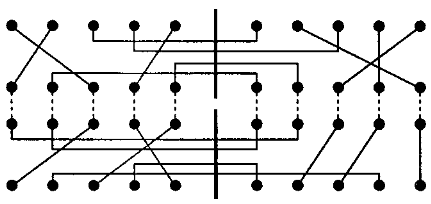
Henceforth, let

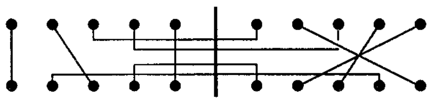
$$e = (\mathrm{id} - p_1)(\mathrm{id} - p_2) \ldots (\mathrm{id} - p_k) \tag{1.6}$$

so that

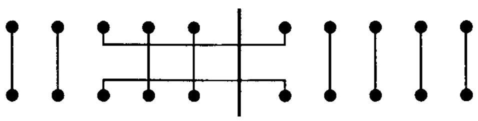
$$eM \cong \mathfrak{sl}_n^{\otimes k}. \tag{1.7}$$

The centralizer algebra $\mathrm{End}_{\mathfrak{g}}(M)$ of transformations commuting with the action of $\mathfrak{g} = \mathfrak{gl}_n$ on $M = V^{\otimes k} \otimes (V^*)^{\otimes k}$ was investigated in [2], where it was shown to be a homomorphic image of a certain algebra $B_{k,k}(n)$ of diagrams with walls. A diagram in $B_{k,k}(n)$ consists of two rows of vertices with $2k$ vertices in each row. There is a wall separating the first k vertices on the left in each row from the k vertices on the right. Each vertex is connected to precisely one edge but with the requirement that horizontal edges must cross the wall, but vertical edges cannot cross. The product d_1d_2 of two diagrams d_1 and d_2 is obtained by placing d_1 above d_2, identifying the bottom row of d_1 with the top row of d_2, and following the resulting paths. Cycles in the middle are deleted, but there is a scalar factor, which is n to the number of middle cycles. For example, in $B_{5,5}(n)$ we would have the following product,

$d_1 d_2 =$ 

$= n^1$ 

The group $S_k \times S_k$ acts on M, where the first copy of the symmetric group S_k acts on the first k factors and the second copy on the next k factors by place permutation. These actions commute with the $\mathfrak{g}$-action, and so afford transformations in $\mathrm{End}_{\mathfrak{g}}(M)$. There is a representation $\phi : B_{k,k}(n) \to \mathrm{End}_{\mathfrak{g}}(M)$ of the algebra $B_{k,k}(n)$ on M which commutes with the $\mathfrak{g}$-action. Under this representation, the diagrams in $B_{k,k}(n)$ having no horizontal edges are mapped to the place permutations coming from $S_k \times S_k$. The identity element in $B_{k,k}(n)$ is just the diagram with each node in the top row connected to the one directly below it in the second row, and it maps to the identity transformation in $\mathrm{End}_{\mathfrak{g}}(M)$. Under ϕ, a diagram such as the one pictured below is mapped to a contraction mapping (in this case to $c_{3,1}$).



(1.8)

It is shown in [2] that the algebra $\mathrm{End}_{\mathfrak{g}}(M)$ is generated by $S_k \times S_k$ and the contraction maps $c_{i,j}$, and the above mapping ϕ is an isomorphism if $n \geq 2k$. Moreover [2] describes the projection maps onto the irreducible summands of M in the following way.

Suppose for some integer r satisfying $0 \leq r \leq k$ that $\underline{s} = \{s_1, \ldots, s_{k-r}\}$ and $\underline{t} = \{t_1, \ldots, t_{k-r}\}$ are ordered subsets of $\{1, \ldots, k\}$ of cardinality $k-r$, and define the following product

$$c_{\underline{s},\underline{t}} \stackrel{\text{def}}{=} c_{s_1,t_1} \cdots c_{s_{k-r},t_{k-r}} \tag{1.9}$$

of the contraction maps c_{s_i,t_i}. Then $c_{\underline{s},\underline{t}}$ belongs to the centralizer algebra $\mathrm{End}_{\mathfrak{g}}(M)$. There is a corresponding product of diagrams in $B_{k,k}(n)$ like the one displayed in (1.8), which ϕ maps onto $c_{\underline{s},\underline{t}}$.

Assume $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ is a partition of r. Associated to λ is its Young frame or Ferrers diagram having λ_i boxes in the ith row. A standard tableau is a filling of the boxes in the diagram of λ in such a way that the entries increase from left to right across each row and down each column. Let T be a standard tableau of shape λ with entries in $\underline{s}^c = \{1, \ldots, k\} \backslash \{s_1, \ldots, s_{k-r}\}$. Associated to T is its Young symmetrizer

$$y_T = \left(\sum_{\rho \in R_T} \rho\right)\left(\sum_{\gamma \in C_T} \mathrm{sgn}(\gamma)\gamma\right), \tag{1.10}$$

where the first sum ranges over the row group of T, which consists of all permutations in S_k that transform each entry of T to an entry in the same row, and the second sum is over the column group of T of permutations that move each entry of T to an entry in the same column. For example,

$$y_{\begin{smallmatrix}1&5\\4&\end{smallmatrix}} = (\mathrm{id} + (1\,5))(\mathrm{id} - (1\,4)),$$

which belongs to the group algebra $\mathbb{C}S_k$ of the symmetric group S_k. The map y_T is an *essential idempotent*, that is, there is an integer m so that $y_T^2 = my_T$.

Similarly, assume for some partition $\mu \vdash r$ that T^* is a standard tableau of shape μ with entries chosen from $\underline{t}^c = \{1, \ldots, k\} \backslash \{t_1, \ldots, t_{k-r}\}$. The mapping

$$y_T y_{T^*} c_{\underline{s},\underline{t}} \tag{1.11}$$

is an essential idempotent in $\mathrm{End}_{\mathfrak{g}}(M)$. (Note that here we are supposing that y_T acts on the factors in $V^{\otimes k}$ and y_{T^*} on the factors in $(V^*)^{\otimes k}$ by place permutations, and that id is the identity map on $V^{\otimes k}$ or $(V^*)^{\otimes k}$, respectively.) Moreover, $y_T y_{T^*} c_{\underline{s},\underline{t}} M$ is isomorphic to the irreducible $\mathfrak{g}$-module $L(\lambda, \mu)$ having highest weight given by the partitions λ and μ. The collection of all maps $y_T y_{T^*} c_{\underline{s},\underline{t}}$ as $r = 0, 1, \ldots, k$; $\underline{s}$, $\underline{t}$ range over all possible choices of ordered subsets of cardinality $k - r$ in $\{1, \ldots, k\}$; λ and μ range over all partitions of r; and T (resp. T^*) ranges over all standard tableaux of shape λ (resp. μ) with entries in $\underline{s}^c$ (resp. in $\underline{t}^c$), give all the projections onto the irreducible summands of M (this can be found in [2]).

Now for the idempotent e in (1.6) we may apply the standard result,

$$\mathrm{End}_{\mathfrak{g}}\big(\mathfrak{sl}_n^{\otimes k}\big) \cong \mathrm{End}_{\mathfrak{g}}(eM) = e\mathrm{End}_{\mathfrak{g}}(M)e\,|_{\,eM}, \tag{1.12}$$

(see for example, [4, Lemma 26.7] or [1, Proposition 1.1]).

Lemma 1.13 *Assume $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$. If $c_{\underline{s},\underline{t}}$ contains one of the contraction maps $c_{j,j}$ for some $j = 1, \ldots, k$, then $ey = 0 = ye$.*

Proof: The mappings $y_T, y_{T^*}, c_{s_i,t_i}, i = 1, \ldots, k-r$, all commute with one another as they operate on different tensor factors. If one of the contraction maps in y equals $c_{j,j} = np_j$, then moving it to the far right produces a product $p_j e = p_j(\mathrm{id} - p_j)\prod_{\ell \neq j}(\mathrm{id} - p_\ell) = 0$ in ye, so $ye = 0$. The argument for ey is similar. □

In [2, Definition 2.4] (compare also [6]) a certain simple tensor $x_{T,T^*,\underline{s},\underline{t}} = u_1 \otimes \cdots \otimes u_k \otimes w_1^* \otimes \cdots w_k^*$ of M is constructed via the algorithm

$$\begin{aligned} u_p &= \begin{cases} v_1 & \text{if } p \in \underline{s} \\ v_j & \text{if } p \in \underline{s}^c \text{ and } p \text{ is in the } j\text{th row of } T \end{cases} \\ w_p^* &= \begin{cases} v_1^* & \text{if } p \in \underline{t} \\ v_{n-j+1}^* & \text{if } p \in \underline{t}^c \text{ and } p \text{ is in the } j\text{th row of } T^* \end{cases} \end{aligned} \tag{1.14}$$

When $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ is applied to the simple tensor $x = x_{T,T^*,\underline{s},\underline{t}}$ the result yx is a nonzero highest weight vector for yM. Moreover, all the highest weight vectors in M are produced in this fashion.

Observe that the factors in x lie in $\{v_1, \ldots, v_r, v_1^*, v_n^*, \ldots, v_{n+1-r}^*\}$. When the pair (s_i, t_i) belongs to $(\underline{s}, \underline{t})$, then the vector v_1 lies in slot s_i in $V^{\otimes k}$, and v_1^* lies in slot t_i in $(V^*)^{\otimes k}$. Replace v_1 by v_{r+i} and v_1^* by v_{r+i}^* in slots s_i and t_i for $i = 1, \ldots, k-r$, to produce a new simple tensor x'. Then $yx = yx'$, as the effect of applying a contraction to $v_1 \otimes v_1^*$ or to $v_{r+i} \otimes v_{r+i}^*$ is the same. However, if $s_i \neq t_i$ for any $i = 1, \ldots, k-r$, then $p_j x' = 0$ for all j. The reason for this is that the vector factors of x' form a subset of $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_k, v_n^*, \ldots, v_{n+1-r}^*, v_k^*, \ldots, v_{r+1}^*\}$. If $n \geq 2k$, these are all distinct. As $s_i \neq t_i$ for any $i = 1, \ldots, k-r$, slot j on the left and slot j on the right do not contain a pair of dual vectors (of the form v_ℓ, v_ℓ^*). Therefore $p_j x' = 0$ for all j and $ex' = x'$.

These calculations show that $yeM \neq 0$, as it contains $yex' = yx' = yx$, which is a maximal vector of weight (λ, μ). But then $yeM \subseteq yM$, and the irreducibility of yM forces $yeM = yM$. To summarize we have

Proposition 1.15 *Assume $n \geq 2k$. If $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ and $s_i \neq t_i$ for any pair (s_i, t_i) in $(\underline{s}, \underline{t})$, then $yeM = yM$, an irreducible $\mathfrak{g}$-module of highest weight (λ, μ), where λ is the shape of T and μ is the shape of T^*.*

Let c_j denote the diagram in $B_{k,k}(n)$ corresponding to the contraction $c_{j,j}$, but scaled by a factor of $1/n$. Then under the representation $\phi : B_{k,k}(n) \to \mathrm{End}_{\mathfrak{g}}(M)$, c_j is sent to p_j, and $b = \prod_{j=1}^k (1 - c_j)$ is mapped to the idempotent e.

Let us consider the subspace A spanned by the diagrams d having no forbidden pairs. By a forbidden pair we mean that the ith node on the left is connected to the ith node on the right of the wall either in the top or in the bottom row of d for some $i = 1, \ldots, k$.

We claim that the map $B_{k,k}(n) \to bB_{k,k}(n)b$ is injective on the subspace A of diagrams with no forbidden pairs. Indeed, $\sum_{d \in A} a_d d \mapsto \sum_{d \in A} a_d bdb = \sum_{d \in A} a_d d + f$, where $a_d \in \mathbb{C}$ and f is a linear combination of diagrams in $B_{k,k}(n)$ having at least one forbidden pair. The reason for this is that when diagrams are multiplied, the horizontal edges in the top row of the top diagram and the horizontal edges in the bottom row of the bottom diagram always appear in the resulting product diagram.

Assume $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ is such that $s_i \neq t_i$ for any $i = 1, \ldots, k-r$, and consider the map $yM = yeM \to eyeM$ given by $yem \mapsto eyem$. We have argued that $yeM = yM$, an irreducible $\mathfrak{g}$-module. Therefore, this map either is an injection or is identically zero. In the latter case, eye must be the zero transformation in $\mathrm{End}_{\mathfrak{g}}(M)$. But there is a linear combination z of diagrams in A which maps to y under the representation ϕ, and $bzb \mapsto eye$. Because the product bzb is nonzero, and the representation ϕ is faithful on M if $n \geq 2k$, eye must be nonzero. Therefore $yM = yeM \to eyeM$ is an injection. But it is clearly surjective and a $\mathfrak{g}$-module map, so $eyeM \cong yM$, an irreducible $\mathfrak{g}$-module with highest weight (λ, μ). We have proved part (1) of the following:

Theorem 1.16 *Assume $n \geq 2k$, $\mathfrak{g} = \mathfrak{gl}_n$, and $M = V^{\otimes k} \otimes (V^*)^{\otimes k}$.*

(1) *Let $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$, where $s_i \neq t_i$ for any $i = 1, \ldots, k-r$. Then $eye(eM) = eyeM$ is an irreducible $\mathfrak{g}$-submodule of eM of highest weight (λ, μ) where λ is the shape of T and μ is the shape of T^*.*

(2) *$\mathfrak{sl}_n^{\otimes k} \cong eM = \bigoplus_y eyeM$, where the sum is over all $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ such that $s_i \neq t_i$ for any i.*

Proof: What remains to be shown is that $eM = \bigoplus_y eyeM$. Observe that because

$$M = \bigoplus_{T, T^*, \underline{s}, \underline{t}} y_T y_{T^*} c_{\underline{s},\underline{t}} M, \tag{1.17}$$

([2, Theorem 2.11], compare also [7]),

$$eM = \sum_y eyM = \sum_y eyeM, \tag{1.18}$$

where the sum is over all $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ such that $s_i \neq t_i$ for any i. We need to argue that the decomposition $eM = \sum_y (eye)eM$ (over such y) is direct.

We have shown previously that $yM = yeM$ and the map $E : yeM \to eyeM$ given by $yem \mapsto eyem$ is an isomorphism of $\mathfrak{g}$-modules for $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ such that $s_i \neq t_i$ for any i. Fix one such idempotent y' and consider the intersection

$$ey'eM \cap \sum_{y \neq y'} eyeM$$

of $ey'eM$ with the sum over the remaining ones. Then

$$ey'eM \cap \sum_{y \neq y'} eyeM \stackrel{E^{-1}}{\to} y'eM \cap \sum_{y \neq y'} yeM = y'M \cap \sum_{y \neq y'} yM$$

But $y'M \cap \sum_{y \neq y'} yM = 0$ by (1.17). Thus, the sum in (1.18) is direct and we have (2). □

2. Multiplicities

Knowing that

$$\mathfrak{sl}_n^{\otimes k} \cong eM = \bigoplus_y eyeM,$$

where the sum is over all $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ such that $s_i \neq t_i$ for any i, we may deduce the multiplicity of a particular irreducible summand in $\mathfrak{sl}_n^{\otimes k}$ labelled by (λ, μ), where $\lambda, \mu \vdash r$ and $r = 0, 1, \ldots, k$. That multiplicity is the number of $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ with T having shape λ, T^* having shape μ, and $c_{\underline{s},\underline{t}}$ having no pairs $s_i = t_i$. Counting the number of $c_{\underline{s},\underline{t}}$ with at least j factors of the form $c_{\ell,\ell}$, we have $\binom{k}{j}$ for the choice of those contractions, $\binom{k-j}{k-r-j}$ choices for the remaining s_i's in $\underline{s}$, and $\binom{k-j}{k-r-j}$ for the rest of the t_i's in $\underline{t}$, and $(k-r-j)!$ for the number of ways to pair the chosen s_i's with the chosen t_i's. Thus, the number of such $y_T y_{T^*} c_{\underline{s},\underline{t}}$ with at least j contractions of the form $c_{\ell,\ell}$ is

$$\binom{k}{j}\binom{k-j}{k-r-j}^2 (k-r-j)!\, f^\lambda f^\mu = \binom{k}{j}\binom{k-j}{r}^2 (k-r-j)!\, f^\lambda f^\mu, \tag{2.1}$$

where f^λ (resp. f^μ) is the number of standard tableaux of shape λ, (resp. μ). Therefore, by the inclusion-exclusion principle, we have the following result.

Theorem 2.2 *When $n \geq 2k$, the multiplicity $m^k_{\lambda,\mu}$ in $\mathfrak{sl}_n^{\otimes k}$ of the irreducible $\mathfrak{g} = \mathfrak{gl}_n$-module $L(\lambda, \mu)$ with highest weight (λ, μ), where $\lambda, \mu \vdash r$, is*

$$m^k_{\lambda,\mu} = f^\lambda f^\mu \left(\sum_{j=0}^{k-r} (-1)^j \binom{k}{j}\binom{k-j}{r}^2 (k-r-j)! \right). \tag{2.3}$$

For a partition λ of r, the number f^λ of standard tableaux of shape λ is given by the well-known hook length formula

$$f^\lambda = \frac{r!}{h(\lambda)},$$

where $h(\lambda) = \prod_{(i,j)\in\lambda} h_{i,j}$, the product of the *hook lengths* of the boxes of λ. Thus, $h_{i,j}$ is the number of boxes in the (i, j) hook of λ: the number of boxes to the right of (i, j) plus the number of boxes below (i, j) plus 1.

As a result, the expression for the multiplicity of the summand labelled by (λ, μ) also can be written as

$$m^k_{\lambda,\mu} = \frac{1}{h(\lambda)h(\mu)} \sum_{j=0}^{k-r} (-1)^j \frac{k!(k-j)!}{j!(k-r-j)!}. \tag{2.4}$$

Let us consider a few interesting special cases. The multiplicity of the trivial $\mathfrak{g}$-module in $\mathfrak{sl}_n^{\otimes k}$ (that is, the dimension of the space of $\mathfrak{g}$-invariants) is

$$m^k_{\emptyset,\emptyset} = \sum_{j=0}^{k}(-1)^j \binom{k}{j}(k-j)! = k! \sum_{j=0}^{k}(-1)^j \frac{1}{j!} = D_k, \tag{2.5}$$

which is the number of *derangements* on the set $\{1, \dots, k\}$ (permutations with no fixed elements). For small values of k, this number is given by

$$\begin{array}{ccccccccc} k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ D_k & 0 & 1 & 2 & 9 & 44 & 265 & 1854 & 14{,}833 \end{array} \tag{2.6}$$

Next, we compute the number of times the adjoint module $\mathfrak{sl}_n = L(\square, \square)$ occurs in $\mathfrak{sl}_n^{\otimes k}$. Using the fact that $\mathfrak{sl}_n$ is self-dual as a $\mathfrak{g}$-module, we see that the number of times $\mathfrak{sl}_n$ appears in $\mathfrak{sl}_n^{\otimes k}$ is the number of times the trivial module appears in $\mathfrak{sl}_n^{\otimes k} \otimes \mathfrak{sl}_n = \mathfrak{sl}_n^{\otimes(k+1)}$. Hence, the number of times $\mathfrak{sl}_n$ appears in $\mathfrak{sl}_n^{\otimes k}$ is

$$m^k_{\square,\square} = D_{k+1}. \tag{2.7}$$

This can also be derived from (2.4) which gives

$$\begin{aligned} m^k_{\square,\square} &= \sum_{j=0}^{k-1}(-1)^j \frac{k!(k-j)}{j!} = \sum_{j=0}^{k}(-1)^j \frac{k!(k-j)}{j!} \\ &= k\sum_{j=0}^{k}(-1)^j \frac{k!}{j!} + \sum_{j=1}^{k}(-1)^{j-1} \frac{k!}{(j-1)!} \\ &= k\sum_{j=0}^{k}(-1)^j \frac{k!}{j!} + k\sum_{j=0}^{k-1}(-1)^j \frac{(k-1)!}{j!} \\ &= k(D_k + D_{k-1}) = D_{k+1}. \end{aligned} \tag{2.8}$$

The last equality in (2.8) is a linear recurrence relation satisfied by the derangement numbers (see for example, [3, (6.5)]).

For any $\mathfrak{g}$-module X,

$$X \otimes X^* \cong \mathrm{End}(X)$$

where the action on the right is $(g \cdot \psi)(x) = g\psi(x) - \psi(gx)$ for all $g \in \mathfrak{g}$, $\psi \in \mathrm{End}(X)$, and $x \in X$. Considering the $\mathfrak{g}$-invariants on both sides, we see that

$$(X \otimes X^*)^{\mathfrak{g}} \cong \mathrm{End}(X)^{\mathfrak{g}} = \mathrm{End}_{\mathfrak{g}}(X). \tag{2.9}$$

Now applying this to $X = \mathfrak{sl}_n^{\otimes k} \cong X^*$, we have

$$\mathrm{End}_{\mathfrak{g}}\big(\mathfrak{sl}_n^{\otimes k}\big) \cong \big(\mathfrak{sl}_n^{\otimes 2k}\big)^{\mathfrak{g}} \tag{2.10}$$

Consequently,

$$\dim \operatorname{End}_{\mathfrak{g}}\big(\mathfrak{sl}_n^{\otimes k}\big) = m_{\emptyset,\emptyset}^{2k} = D_{2k}, \tag{2.11}$$

the number of derangements on a set of $2k$ elements.

We conclude by displaying the multiplicities $m_{\lambda,\mu}^{k}$ for $k = 4$. By double centralizer theory, it follows that

$$\dim \operatorname{End}_{\mathfrak{g}}\big(\mathfrak{sl}_n^{\otimes k}\big) = \sum_{\lambda,\mu \vdash r \leq k} \big(m_{\lambda,\mu}^{k}\big)^2.$$

The reader can verify that the squares of the numbers in the following tables do indeed sum to D_8 =14,833.

Example $m_{\lambda,\mu}^{4}$:

	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
(4)	1	3	2	3	1
$(3,1)$	3	9	6	9	3
$(2,2)$	2	6	4	6	2
$(2,1,1)$	3	9	6	9	6
$(1,1,1,1)$	1	3	2	3	1

	(3)	$(2,1)$	$(1,1,1)$
(3)	12	24	12
$(2,1)$	24	48	24
$(1,1,1)$	12	24	12

	(2)	$(1,1)$
(2)	42	42
$(1,1)$	42	42

	(1)
(1)	44

	$\emptyset$
$\emptyset$	9

3. The centralizer algebra

Now we consider the centralizer algebra $\mathcal{C} = \operatorname{End}_{\mathfrak{g}}(\mathfrak{sl}_n^{\otimes k}) = \operatorname{End}_{\mathfrak{sl}_n}(\mathfrak{sl}_n^{\otimes k})$ and its representation theory. As has already been pointed out in (1.12), we have an isomorphism

$$\mathcal{C} \cong e\operatorname{End}_{\mathfrak{g}}(M)e \tag{3.1}$$

where e is the idempotent defined in (1.6). We also have a representation $\phi : B_{k,k}(n) \to \mathrm{End}(M)$ which commutes with the $\mathfrak{g}$-action on M. Thus the image of this representation lies in the commuting algebra $\mathrm{End}_{\mathfrak{g}}(M)$. In [2, Theorem 5.8] it was shown that ϕ induces an algebra isomorphism

$$B_{k,k}(n) \cong \mathrm{End}_{\mathfrak{g}}(M) \tag{3.2}$$

for $n \geq 2k$. Let $b \in B_{k,k}(n)$ be given as above, $b = \prod_{j=1}^{k}(1 - c_j)$ where c_j is the diagram corresponding to the contraction map $c_{j,j}$ but scaled by $1/n$. Then ϕ maps b onto e, and we obtain the following.

Proposition 3.3 *Let $n \geq 2k$. The map ϕ induces an algebra isomorphism between $bB_{k,k}(n)b$ and $\mathcal{C} = \mathrm{End}_{\mathfrak{g}}(\mathfrak{sl}_n^{\otimes k})$. Moreover, the set of all elements of the form bdb, as d ranges over all diagrams with no forbidden pairs, is a basis for $bB_{k,k}(n)b$.*

Proof: The first claim follows from the remarks above, so only the second claim remains to be proved. We observe that left (resp., right) multiplication by b kills any diagram with a forbidden pair in its top (resp., bottom) row. Since the diagrams form a basis for $B_{k,k}(n)$, the result follows. □

The basis statement of Proposition 3.3 provides another proof of (2.11), that the dimension of the centralizer algebra $\mathcal{C}$ is D_{2k}. Indeed, the diagrams with no forbidden pairs are easily seen to be in bijective correspondence with the permutations σ on the set $\{1, \dots, 2k\}$ such that $\sigma(i) \neq i$ for all $i = 1, \dots, 2k$. This correspondence is given by performing two "flips", which take a walled Brauer diagram to the diagram obtained by first interchanging the rightmost k dots in its top and bottom rows and then switching corresponding dots on the two sides of the wall on the top row while retaining the edges.

Let $r \leq k$ and let λ, μ be fixed partitions of r. In [2] $M_{\lambda,\mu}$ was defined to be the space spanned by all maximal vectors yx, where $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ and $x = x_{T,T^*,\underline{s},\underline{t}}$ (notation of (1.14)), for all pairs $\underline{s} = \{s_1, \dots, s_{k-r}\}$, $\underline{t} = \{t_1, \dots, t_{k-r}\}$ of ordered subsets of $\{1, \dots, k\}$, and all standard tableaux T (resp., T^*) of shape λ (resp., μ) with entries from $\underline{s}^c$ (resp., $\underline{t}^c$). Moreover, for $n \geq 2k$, the $M_{\lambda,\mu}$ provide a complete set of pairwise nonisomorphic irreducible modules for the algebra $\mathrm{End}_{\mathfrak{g}}(M)$ (and hence also for $B_{k,k}(n)$).

Lemma 3.4 *Assume $n \geq 2k$ and let $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$, $x = x_{T,T^*,\underline{s},\underline{t}}$. Then $ey \neq 0$ if and only if $s_i \neq t_i$ for all pairs (s_i, t_i) in $(\underline{s}, \underline{t})$. Hence $eM_{\lambda,\mu} \neq 0$ precisely when this condition can be satisfied, and in that case, $eM_{\lambda,\mu}$ is the linear span of all the nonzero eyx, y and x as above.*

Proof: This follows from results in [2], Lemma 1.13, and its converse, which is in the paragraph before Theorem 1.16. □

We remark that for y, x as in the preceding lemma, we have $eyx = eyx' = eyex'$, where x' is the modified simple tensor described in Section 1.

Moreover, it is easy to see that $eM_{\lambda,\mu} = 0$ when $\lambda = \mu = \emptyset$ and $k = 1$, for in that case it is impossible to construct a $y = y_T y_{T^*} c_{\underline{s},\underline{t}}$ satisfying the condition $s_i \neq t_i$ for all pairs (s_i, t_i) in $(\underline{s}, \underline{t})$. In all other cases $eM_{\lambda,\mu} \neq 0$ when $n \geq 2k$.

Theorem 3.5 *Assume $n \geq 2k$. The collection of all nonzero $eM_{\lambda,\mu}$ for λ, μ partitions of r, $r = 0, 1, \ldots, k$, forms a complete set of pairwise nonisomorphic irreducible modules for the algebra $\mathcal{C} \cong bB_{k,k}(n)b$.*

Proof: It is well-known that if u is an idempotent in an algebra A, the functor $u(-)$ (sometimes called the Schur functor; see [5, 6.2]) taking A-modules to uAu-modules is an exact covariant functor which maps an irreducible module to either an irreducible module or zero. In the particular case that $A = B_{k,k}(n)$ and $u = b$, this functor takes the irreducible module $M_{\lambda,\mu}$ to $bM_{\lambda,\mu} = eM_{\lambda,\mu}$. □

Theorem 3.6 *Assume $n \geq 2k$. Then as a bimodule for $\mathcal{C} \times \mathfrak{g}$,*

$$\mathfrak{sl}_n^{\otimes k} \cong eM \cong \bigoplus_{r=0}^{k} \bigoplus_{\lambda,\mu \vdash r} eM_{\lambda,\mu} \otimes L(\lambda, \mu),$$

where the decomposition is into pairwise nonisomorphic irreducible modules for $\mathcal{C} \times \mathfrak{g}$.

Proof: This follows from the previous results and standard double-centralizer theory. □

For $n \geq 2k$ the dimension of the irreducible $\mathcal{C}$-module $eM_{\lambda,\mu}$ is given by $m^k_{\lambda,\mu}$ (see Theorem 2.2).

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