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A Weighted Enumeration of Maximal Chains in the Bruhat Order

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Abstract. Given a finite Weyl group W with root system Φ, assign the weight $\alpha \in \Phi$ to each covering pair in the Bruhat order related by the reflection corresponding to α. Extending this multiplicatively to chains, we prove that the sum of the weights of all maximal chains in the Bruhat order has an explicit product formula, and prove a similar result for a weighted sum over maximal chains in the Bruhat ordering of any parabolic quotient of W. Several variations and open problems are discussed.

Keywords: Bruhat order, Weyl group, root system

Introduction

In the Bruhat ordering of a Weyl group, the covering edges relate certain pairs of elements that differ by a reflection. If we assign a weight to each edge that is a linear function f of the corresponding root (a normal vector for the hyperplane fixed by the reflection), we can extend this multiplicatively to chains, thereby obtaining a polynomial weight function on (unrefinable) chains in the Bruhat order. Our main results consist of an explicit product formula for the sum of the weights of all maximal chains in the Bruhat ordering of a finite Weyl group W (Theorem 1.1), and a more general result for the Bruhat ordering of any parabolic quotient W/W_J (Theorem 2.2). The only restriction on the weight function f is that it should be W_J-invariant; in the case of the full Bruhat order (i.e., when W_J is trivial), this is no restriction at all.

One motivation for this work can be traced to a 1984 paper of Proctor [7]. In this paper, Proctor used a basis theorem from the Standard Monomial Theory of Lakshmibai and Seshadri to derive a number of interesting combinatorial identities related to the Bruhat ordering of those parabolic quotients that correspond to minuscule W-orbits. It is natural then to investigate what happens in general quotients.

Although Standard Monomial Theory is significantly more complicated in the non-minuscule case (see [4] for example), the results in this paper can be viewed as generalizing a limiting case of Proctor's identities to arbitrary quotients. Indeed, in a minuscule quotient, all covering edges connect pairs of elements that differ by *simple* reflections, and f can be chosen so that each edge has unit weight. Hence in this case, the weighted sum over maximal chains counts the number of ways to express the longest element of the quotient

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as a minimal product of simple reflections, and Theorem 2.2 provides a product formula for this number, recovering one of the original results in [7].

We should point out that the proofs in this paper do not use Standard Monomial Theory. We use the combinatorial structures of that theory, but the only property we require (a basis theorem due to Littelmann [5]) can be proved by elementary methods.

1. Maximal chains in the full Bruhat order

Let Φ be a finite crystallographic root system embedded in a Euclidean space $\mathbf{E}$ with inner product $\langle\, ,\rangle$. We let $\{\alpha_i : i \in I\} \subset \Phi$ denote a choice of simple roots, and Φ^+ the corresponding set of positive roots; i.e., the roots in the nonnegative linear span of the simple roots. One knows that Φ is the disjoint union of Φ^+ and $-\Phi^+$. (For this and other standard facts about finite root systems, we refer the reader to [1] or [3].)

For each $\alpha \in \Phi$, we let $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ denote the co-root and $t_\alpha : \mathbf{E} \to \mathbf{E}$ the reflection corresponding to α, so that $t_\alpha(\lambda) = \lambda - \langle\lambda,\alpha^\vee\rangle\alpha$. The reflection corresponding to α_i is denoted s_i, and the Weyl group W is the subgroup of $GL(\mathbf{E})$ generated by $\{s_i : i \in I\}$ and includes all of the reflections t_α for $\alpha \in \Phi$. For all $w \in W$, the length $\ell(w)$ is defined to be the minimum l such that w is expressible in the form $s_{i_1} \cdots s_{i_l}$.

The height of a root α, denoted $\mathrm{ht}(\alpha)$, is the sum of its simple root coordinates; more generally, $\mathrm{ht}(\cdot)$ can be viewed as a linear functional on the root lattice $\mathbf{Z}\Phi$. It is well-known and easy to show that if $\rho = \sum_{\alpha\in\Phi^+} \alpha/2$, then

$$\langle\rho, \alpha_i^\vee\rangle = 1 \quad (i \in I), \tag{1.1}$$

so $\langle\rho,\cdot\rangle$ is the height function for the co-root system $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$.

For each $J \subseteq I$, we let Φ_J denote the root subsystem of Φ generated by $\{\alpha_j : j \in J\}$, and W_J the corresponding Weyl subgroup. One knows that

$$W^J := \{w \in W : \ell(ws_j) > \ell(w), \quad \text{all } j \in J\}$$

is a set of coset representatives for W/W_J, and moreover, each element $w \in W^J$ is the shortest member of the coset wW_J.

We let $<$ denote the Bruhat ordering of W; i.e., the transitive closure of the relations

$$w < wt_\alpha \quad \text{if } \ell(w) < \ell(wt_\alpha) \quad (w \in W,\ \alpha \in \Phi^+).$$

One has $\ell(w) < \ell(t_\alpha w)$ if and only if $w^{-1}\alpha \in \Phi^+$ (e.g., [3, Section 5.7]), and it is easy to check that $t_\alpha w = wt_{w^{-1}\alpha}$, so

$$w < t_\alpha w \Leftrightarrow \ell(w) < \ell(t_\alpha w) \Leftrightarrow w^{-1}\alpha \in \Phi^+ \quad (w \in W,\ \alpha \in \Phi^+), \tag{1.2}$$

and it follows that the Bruhat ordering is also the transitive closure of these relations.

Define a weight function on the relations of the form $w < wt_\alpha$ ($\alpha \in \Phi^+$) by setting

$$\mathrm{wt}(w < wt_\alpha) = \alpha,$$

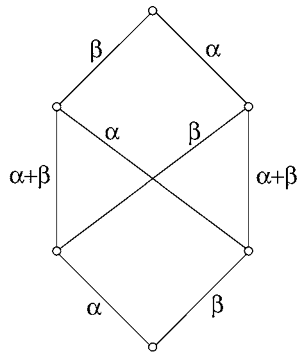

Figure 1. The Bruhat ordering of $W(\mathcal{A}_2)$.

and extend this multiplicatively to (some) Bruhat chains by defining

$$\mathrm{wt}(x_0 < x_1 < \cdots < x_l) = \mathrm{wt}(x_0 < x_1) \cdots \mathrm{wt}(x_{l-1} < x_l),$$

provided that each step $x_{i-1} < x_i$ is of the form $w < wt_\alpha$ for some $w \in W$ and $\alpha \in \Phi^+$. For example, this applies to all unrefinable chains in $(W, <)$. The weights of these chains may be understood as elements of the polynomial ring $\mathbf{Z}[\alpha_i : i \in I]$.

Theorem 1.1 *We have*

$$\sum \mathrm{wt}(x_0 < \cdots < x_l) = |\Phi^+|! \prod_{\alpha \in \Phi^+} \frac{\alpha}{\mathrm{ht}(\alpha)},$$

where the sum ranges over all maximum-length chains in the Bruhat ordering of W.

The Bruhat order is known to be graded, so maximum-length chains are the same as maximal chains.

For example, the root system $\Phi = \mathcal{A}_2$ has three positive roots, α, β, and $\alpha + \beta$. The Bruhat ordering of W (see figure 1) has four maximal chains, the weights of these chains being $\alpha\beta\alpha$, $\beta\alpha\beta$, and $\alpha(\alpha+\beta)\beta$ (twice). Hence the above identity takes the form

$$\alpha\beta\alpha + \beta\alpha\beta + 2\alpha(\alpha+\beta)\beta = 3!\,\alpha\beta(\alpha+\beta)/2.$$

2. Maximal chains in a quotient

If $\lambda \in \mathbf{E}$ is dominant vector (i.e., $\langle\lambda, \alpha_i\rangle \geq 0$ for all $i \in I$), then the W-stabilizer of λ is the parabolic subgroup W_J, where $J = \{i \in I : \langle\lambda, \alpha_i\rangle = 0\}$ (e.g., see [3, Section 1.15]). Thus the W-orbit of λ can be identified with W/W_J, and the restriction of the Bruhat ordering from W to W^J can (we claim) be converted to an ordering on $W\lambda$ by taking the transitive closure of the relations

$$t_\alpha\mu < \mu \quad \text{if } \langle\mu, \alpha\rangle > 0 \quad (\mu \in \mathrm{W}\lambda, \alpha \in \Phi^+).$$

Note that this places λ at the top of the partial ordering, whereas the identity element is at the bottom of $(W, <)$.

Proposition 2.1 *Assume $\lambda \in \mathbf{E}$ is dominant and $\mu, \nu \in W\lambda$.*
(a) *The map $w \mapsto w\lambda$ is order-reversing $(W, <) \to (W\lambda, <)$.*
(b) *We have $\mu \leq \nu$ if and only if $x \geq y$, where x, y denote the shortest elements of W such that $x\lambda = \mu$ and $y\lambda = \nu$.*

Thus $(W^J, <)$ and $(W\lambda, <)$ are dual-isomorphic.

Proof: (a) Consider a covering relation in $(W, <)$; say $w < t_\alpha w$, where $w \in W, \alpha \in \Phi^+$. We must have $w^{-1}\alpha \in \Phi^+$ (see (1.2)), whence $\langle w\lambda, \alpha\rangle = \langle \lambda, w^{-1}\alpha\rangle \geq 0$ and $t_\alpha w\lambda \leq w\lambda$.

(b) Suppose that $\mu < \nu$ is a covering relation in $(W\lambda, <)$; thus $\nu = t_\alpha\mu$ for some $\alpha \in \Phi^+$ such that $\langle \mu, \alpha\rangle < 0$. Since $x\lambda = \mu$, we have $\langle \lambda, x^{-1}\alpha\rangle = \langle \mu, \alpha\rangle < 0$, so $x^{-1}\alpha$ is necessarily a negative root, $\ell(t_\alpha x) < \ell(x)$ (again by (1.2)), and $t_\alpha x < x$. If $t_\alpha x$ is not the shortest element such that $t_\alpha x\lambda = \nu$ (i.e., $t_\alpha x \notin W^J$), then we must have $\ell(t_\alpha x s_j) < \ell(t_\alpha x)$ for some $j \in J$, in which case $t_\alpha x s_j < t_\alpha x < x$. By iteration, it follows that $y < x$, and the converse is a consequence of (a). ☐

Define a weight function on the relations $t_\alpha\mu < \mu$ in $(W\lambda, <)$ by setting

$$\mathrm{wt}_\lambda(t_\alpha\mu < \mu) = \langle \lambda, w^{-1}\alpha\rangle = \langle \mu, \alpha\rangle > 0 \quad (\mu = w\lambda,\ \alpha \in \Phi^+),$$

and extend this multiplicatively to some (but certainly all unrefinable) chains in $(W\lambda, <)$. Given the anti-isomorphism $w \mapsto w\lambda$, this can be viewed as a specialization $\beta \mapsto \langle \lambda, \beta\rangle$ of the weight function $\mathrm{wt}(w < t_\alpha w) = \mathrm{wt}(w < wt_{w^{-1}\alpha}) = w^{-1}\alpha$ we defined previously.

Theorem 2.2 *If $\lambda \in \mathbf{E}$ is dominant and $\Phi_\lambda := \{\alpha \in \Phi : \langle \lambda, \alpha\rangle > 0\}$, then*

$$\sum \mathrm{wt}_\lambda(\mu_0 < \cdots < \mu_l) = |\Phi_\lambda|! \prod_{\alpha \in \Phi_\lambda} \frac{\langle \lambda, \alpha\rangle}{\mathrm{ht}(\alpha)},$$

where the sum ranges over all maximum-length chains in $(W\lambda, <)$.

Remark 2.3 If we fix $J \subseteq I$ and let $\lambda \in \mathbf{E}$ vary over all dominant vectors with stabilizer W_J, both sides of the identity can be viewed as polynomial functions in the variables $\{\langle \lambda, \alpha_i\rangle : i \in I - J\}$. In this way, Theorem 2.2 is equivalent to a polynomial identity in $n = |I - J|$ variables, and Theorem 1.1 is the special case $J = \varnothing$.

For example, consider the root system $\Phi = \mathcal{A}_4$, with the simple roots α_i $(i = 1, 2, 3, 4)$ numbered according to the path structure of the Dynkin diagram. If we choose a dominant $\lambda \in \mathbf{E}$ so that $\langle \lambda, \alpha_i\rangle = m, n, 0, 0$ for $i = 1, 2, 3, 4$ (where $m, n > 0$), then the W-stabilizer of λ is isomorphic to the Weyl group of $\mathcal{A}_2$, the W-orbit of λ has 20 elements, and the Bruhat ordering of this orbit is displayed on the left in figure 2. The λ-weights on the edges of this poset are all equal to one of m, n, or $m + n$. The posets on display in the center and

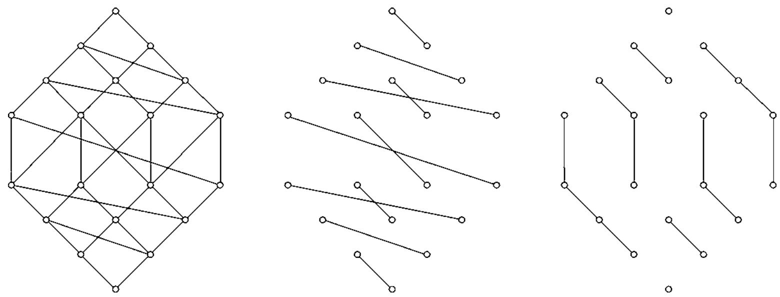

Figure 2. The Bruhat ordering of a W-orbit in $\mathcal{A}_4$.

on the right in figure 2 show the edges of weight m and $m + n$, respectively. The remaining 12 edges have weight n.

There are seven roots in Φ_λ; their respective heights and weights are 1, 1, 2, 2, 3, 3, 4 and $m, n, m + n, n, m + n, n, m + n$. On the other hand, one can check that $(W\lambda, <)$ has 74 maximal chains, the weights of these chains being $mn^3(m + n)^3$, $mn^4(m + n)^2$, and $m^2n^3(m + n)^2$ (14 times each); $m^2n^4(m + n)$ (12 times); $mn^5(m + n)$ and $m^3n^3(m + n)$ (6 times each); m^2n^5 and m^3n^4 (3 times each); and mn^6 and m^4n^3 (once each). The assertion of Theorem 2.2 is that in this case, the weights sum to $35mn^3(m + n)^3$.

3. Asymptotic standard monomials

Having recognized the polynomial nature of Theorem 2.2, it suffices to prove it for the special cases corresponding to all λ in the semigroup of dominant integral weights; i.e.,

$$\Lambda^+ = \{\lambda \in \mathbf{E} : \langle \lambda, \alpha_i^\vee \rangle \in \mathbf{Z}^{\geq 0},\ \text{all } i \in \mathbf{I}\}.$$

At the same time, it will be more convenient to replace Φ with the co-root system $\Phi^\vee$; this has the effect of modifying the weight function slightly so that

$$\mathrm{wt}_\lambda(t_\alpha \mu < \mu) = \langle \mu, \alpha^\vee \rangle \in \mathbf{Z}^{>0},$$

and similar replacements $\alpha \to \alpha^\vee$ are necessary on the product side of the identity.

The advantage of restricting our attention to dominant integral weights is that there is an irreducible finite-dimensional representation $V(\lambda)$ of a semisimple Lie algebra with root system Φ corresponding to each $\lambda \in \Lambda^+$, and Theorem 2.2 can be viewed (we claim) as a comparison of two asymptotic expansions of $\dim V(m\lambda)$ in the limit $m \to \infty$.

For the first expansion, use the Weyl dimension formula (e.g., [2, Section 24.3]) to obtain

$$\dim V(m\lambda) = \prod_{\alpha \in \Phi^+} \frac{\langle m\lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} = \prod_{\alpha \in \Phi_\lambda} \left(m \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} + 1 \right).$$

Bearing in mind (1.1), we deduce

Lemma 3.1 *For all* $\lambda \in \Lambda^+$, *we have*

$$\dim V(m\lambda) = \prod_{\alpha\in\Phi_\lambda} \frac{\langle\lambda, \alpha^\vee\rangle}{\mathrm{ht}(\alpha^\vee)} m^l + O(m^{l-1}) \quad (l = |\Phi_\lambda|).$$

For the second expansion, we analyze a set of combinatorial objects that index a basis of $V(m\lambda)$. These objects were defined by Lakshmibai and Seshadri in their program of Standard Monomial Theory (e.g., see Section 4 of [4]). Here, we (mostly) follow the notation and terminology in Section 8 of [10].

For any rational $b > 0$, the *b-Bruhat ordering* of $W\lambda$ is defined to be the transitive closure of the relations

$$t_\alpha\mu <_b \mu \quad \text{if } t_\alpha\mu \lessdot \mu \text{ and } b\langle\mu, \alpha^\vee\rangle \in \mathbf{Z} \quad (\mu \in W\lambda,\ \alpha \in \Phi^+),$$

where $\nu \lessdot \mu$ indicates that μ covers ν in $(W\lambda, <)$. Thus $\nu <_b \mu$ is a covering relation of the b-Bruhat order if and only if μ covers ν in $(W\lambda, <)$ and $b(\mu - \nu)$ is a positive integer multiple of a positive root. Notice also that if b is an integer, then the b-Bruhat order coincides with the original Bruhat order.

If we specialize the example in Section 2 and take $m = 2, n = 1$, then one can check that the $(1/2)$-Bruhat and $(1/3)$-Bruhat orderings of $W\lambda$ are the posets displayed in the center and on the right in figure 2.

A *Lakshmibai-Seshadri chain* (or *LS chain*) $\boldsymbol{\mu}$ of *type* λ and *degree* m is a pair consisting of a chain in $(W\lambda, <)$ of any length $l \geq 0$, say $\mu_0 < \mu_1 < \cdots < \mu_l$, and an increasing sequence of rationals $0 < b_1 < \cdots < b_l < m$ such that

$$\mu_0 <_{b_1} \mu_1 <_{b_2} \cdots <_{b_l} \mu_l.$$

It is useful to record the information carried by $\boldsymbol{\mu}$ in a (noncommutative) monomial

$$\boldsymbol{\mu} = \mu_0^{a_0}\mu_1^{a_1}\cdots\mu_l^{a_l} \quad (a_0 = b_1,\ a_i = b_{i+1} - b_i,\ a_l = m - b_l)$$

of degree $m = a_0 + \cdots + a_l$. There is no harm in allowing repetitions in the terms of $\boldsymbol{\mu}$, say $\mu_i = \mu_{i+1}$, but in such circumstances, we must identify $\boldsymbol{\mu}$ with the LS chain obtained by deleting b_i and μ_i. In terms of monomials, this means for example that $\mu_0^a\mu_1^{b+c}\mu_2^d$ and $\mu_0^a\mu_1^b\mu_1^c\mu_2^d$ label identical objects.

Let $C_m(\lambda)$ denote the set of all LS chains of type λ and degree m. The only property we require of LS chains, due to Littelmann [5], is that $C_1(\lambda)$ indexes a basis of $V(\lambda)$. (For another proof, see Theorem 8.3 of [10].) Bearing in mind that $\mu \leq_{mb} \nu$ if and only if $m\mu \leq_b m\nu$, we may rescale $\boldsymbol{\mu} \in C_m(\lambda)$, replacing μ_i with $m\mu_i$ and b_i with b_i/m, thereby obtaining a bijection between $C_m(\lambda)$ and $C_1(m\lambda)$. To summarize, we have

Lemma 3.2 *For all* $\lambda \in \Lambda^+$, *we have* $\dim V(m\lambda) = |C_1(m\lambda)| = |C_m(\lambda)|$.

Proof of Theorem 2.2: Define a binary relation $\leftarrow$ on $C(\lambda) = C_1(\lambda)$ by setting

$$\left(\nu_0^{c_0}\cdots\nu_k^{c_k}\right) \leftarrow \left(\mu_0^{a_0}\cdots\mu_l^{a_l}\right) \quad \text{if } \nu_k \leq \mu_0.$$

This imposes a "partially reflexive" order structure on $C(\lambda)$ in the sense that $\leftarrow$ is transitive, asymmetric ($\boldsymbol{\nu} \leftarrow \boldsymbol{\mu}$ and $\boldsymbol{\mu} \leftarrow \boldsymbol{\nu}$ implies $\boldsymbol{\mu} = \boldsymbol{\nu}$), but not necessarily reflexive. (We propose that such structures should be called "prosets.") Indeed, we have $\boldsymbol{\mu} \leftarrow \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ is a singleton (i.e., $\boldsymbol{\mu} = (\mu^1)$ for some μ).

Choose an m-tuple $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m \in C(\lambda)$, and let μ_i^+ and μ_i^- denote the top and bottom elements of $\boldsymbol{\mu}_i$. We have $\mu_i^+ \leq_i \mu_{i+1}^-$ if and only if $\boldsymbol{\mu}_i \leftarrow \boldsymbol{\mu}_{i+1}$, so the concatenation of $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m$ is an LS chain of degree m if and only if $\boldsymbol{\mu}_1 \leftarrow \cdots \leftarrow \boldsymbol{\mu}_m$. Conversely, it is easy to see that any LS chain of degree m factors (uniquely) into LS chains of degree 1, so we can identify $C_m(\lambda)$ with the m-element multichains $\boldsymbol{\mu}_1 \leftarrow \cdots \leftarrow \boldsymbol{\mu}_m$ in $C(\lambda)$.

If we choose an r-chain $\boldsymbol{\nu}_1 \leftarrow \cdots \leftarrow \boldsymbol{\nu}_r$ of *distinct* elements from $C(\lambda)$, then the m-multichains with support set $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_r\}$ can be obtained by choosing integers $k_i \geq 1$ such that $k_1 + \cdots + k_r = m$ and concatenating k_i copies of $\boldsymbol{\nu}_i$ in the order $i = 1, \ldots, r$. However, as noted previously, the only terms that can occur more than once in a multichain are the singleton LS chains. Thus if $\boldsymbol{\nu}_i$ is not a singleton, we must choose $k_i = 1$.

If there are $l + 1$ singletons among $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_r$, then there are

$$\binom{m - r + l + 1}{l} = \frac{1}{l\,!} m^l + O(m^{l-1})$$

ways to choose $k_1, \ldots, k_r$. Combining this with Lemma 3.2, we obtain

$$\dim V(m\lambda) = |C_m(\lambda)| = \frac{1}{\ell!} N_\lambda m^\ell + O(m^{\ell-1}),$$

where $\ell(\lambda) + 1 = \ell + 1$ denotes the maximum number of singletons that can appear in a strict chain $\boldsymbol{\nu}_1 \leftarrow \cdots \leftarrow \boldsymbol{\nu}_r$ in $C(\lambda)$, and N_λ denotes the number of such chains. Comparing this with the expansion in Lemma 3.1, it must be the case that $\ell(\lambda) = |\Phi_\lambda|$ and

$$N_\lambda = \ell(\lambda)! \prod_{\alpha \in \Phi_\lambda} \langle \lambda, \alpha^\vee \rangle / \mathrm{ht}(\alpha^\vee).$$

By extracting the top element ν_i^+ from each member of a strict chain $\boldsymbol{\nu}_1 \leftarrow \cdots \leftarrow \boldsymbol{\nu}_r$, we obtain a Bruhat multichain $\nu_1^+ \leq \cdots \leq \nu_r^+$. Hence $\ell(\lambda)$ is the maximum length of a (strict) chain in $(W\lambda, <)$. It also follows that for each non-singleton $\boldsymbol{\nu}_i$ occurring in a chain counted by N_λ, it must be the case that $\nu_i^- \lessdot \nu_i^+$. Otherwise, the strict Bruhat chain obtained by extracting the set of distinct top elements that occur could not have maximum length. Thus the only non-singletons that may occur have the form $\boldsymbol{\mu} = \mu_0^a \mu_1^{1-a}$ for some covering pair $\mu_0 \lessdot \mu_1$ with $\mu_0 <_a \mu_1$. In particular, this requires $\mu_0 = t_\alpha \mu_1$ for some $\alpha \in \Phi^+$ such that $ka \in \mathbf{Z}$, where $k = \langle \mu_1, \alpha^\vee \rangle > 0$. Hence there are $k - 1$ possible choices for a; namely, $a = i/k$ for $i = 1, \ldots, k-1$. Therefore, as we traverse a maximum-length singleton chain through $(W\lambda, <)$, each (necessarily covering) edge $t_\alpha \mu < \mu$ provides an opportunity to insert one of $\langle \mu, \alpha^\vee \rangle - 1$ possible doubleton chains, or no doubleton chain at all, for a total of $\langle \mu, \alpha^\vee \rangle = \mathrm{wt}_\lambda(t_\alpha \mu < \mu)$ choices. It follows that

$$N_\lambda = \sum \mathrm{wt}_\lambda(\mu_0 < \cdots < \mu_l),$$

where the sum ranges over all maximum-length chains in $(W\lambda, <)$. □

Remark 3.3 The above proof appears to rely on representation theory. However the arguments in [5] and [10] show by elementary methods that Weyl's formula for the character of $V(\lambda)$ can be viewed as a generating function for LS chains of type λ. Furthermore, it is easy to deduce the Weyl dimension formula from the character formula, so in fact both Lemmas 3.1 and 3.2 can be formulated in a way that involves no representation theory, and proved by elementary methods.

4. Variations and questions

4.1. Asymptotic characters

Let $\chi(\lambda)$ denote the character of $V(\lambda)$, an element of the group ring $\mathbf{Z}[e^\mu : \mu \in \mathbf{E}]$. Using the Weyl character formula, one can show that the limit of $e^{-m\lambda}\chi(m\lambda)$ as $m \to \infty$ is well-defined as a formal power series, and in fact

$$\lim_{m\to\infty} e^{-m\lambda}\chi(m\lambda) = \prod_{\alpha\in\Phi_\lambda} \frac{1}{1-e^{-\alpha}}.$$

However, as hinted in Remark 3.3, this series can also be identified as a generating series for "stable" LS chains of type λ and unbounded degree, or alternatively, as a weighted sum over (strict) chains of arbitrary length in $(W\lambda, <)$. In the latter form, the weight function has a rather complicated description. However, in case λ is minuscule, LS chains of type λ and degree m are simply the m-multichains $\mu_1 \leq \cdots \leq \mu_m$ in $(W\lambda, <)$, and each such chain contributes $e^{\mu_1+\cdots+\mu_m}$ to $\chi(m\lambda)$. This observation played a critical role in Proctor's study of "Bruhat lattices" in [7].

4.2. Asymptotic tableaux

There are many combinatorial objects that have been used to index bases of representations of semisimple Lie algebras, and for each of these, an asymptotic analysis of $\dim V(m\lambda)$ analogous to our proof of Theorem 2.2 has the potential to produce an interesting combinatorial identity.

The most familiar of these objects are the semistandard tableaux associated with the root system $\Phi = \mathcal{A}_{n-1}$ (for definitions, see [6, Section I.5]). In this case, dominant weights can be viewed as integer partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $< n$ parts (i.e., $\lambda_n = 0$), and the quantity $c_i := \langle\lambda, \alpha_i^\vee\rangle = \lambda_i - \lambda_{i+1}$ measures the number of columns of length i in λ.

A semistandard tableau can be viewed as a sequence of subsets of $[n] = \{1, \ldots, n\}$ (i.e., columns), subject to certain rules about compatibility between adjacent columns. The tableau has shape λ if the number of i-sets in the sequence is c_i for $1 \leq i < n$, and these tableaux index a basis for $V(\lambda)$.

Let P denote the partial ordering on subsets of $[n]$ obtained by declaring $X \leq Y$ if X and Y can occur consecutively as columns of a semistandard tableau. (One should check that this does define a true partial order, unlike the "proset" in Section 3.) In this way, semistandard tableaux are multichains in P. It is an interesting coincidence that P turns out to be isomorphic to the Bruhat ordering of $W(\mathcal{B}_n)/W(\mathcal{A}_{n-1})$.

Given λ, let $J = \{i : c_i > 0\} = \{i : \langle\lambda, \alpha_i^\vee\rangle > 0\}$, and define P_J to be the subposet of P formed by the subsets whose cardinalities are in J. An asymptotic analysis of the number of semistandard tableaux of shape $m\lambda$ leads to the following analogue of Theorem 2.2:

$$\sum \mathrm{wt}(X_0 < \cdots < X_l) = \prod_{\alpha\in\Phi_\lambda} \langle\lambda, \alpha^\vee\rangle/\mathrm{ht}(\alpha^\vee),$$

where the sum ranges over all maximum-length chains in P_J,

$$\mathrm{wt}(X_0 < \cdots < X_l) = \prod_{i\in J} \frac{\langle\lambda, \alpha_i^\vee\rangle^{\ell_i}}{\ell_i!},$$

and ℓ_i denotes the length of the subchain of $X_0 < \cdots < X_l$ formed by the i-subsets. Note that this weight function is not multiplicative.

4.3. *Chains in subintervals*

It would be interesting to investigate the elements $w \in W$ such that the sum over all maximum-length chains in the Bruhat order from 1 to w, weighted as in Theorem 1.1, factors completely into linear factors. More generally, one should also investigate when the analogous sums in the parabolic quotients $(W^J, <)$ factor completely.

One class of elements with this property are the "dominant minuscule" elements first studied by Dale Peterson (see [8] and [9]). Indeed, an element $w \in W$ is dominant minuscule if and only if there exists $\lambda \in \Lambda^+$ so that

$$\ell(w) = \mathrm{ht}(\lambda - w\lambda) \quad \text{and} \quad w \in W^J,$$

where W_J denotes the W-stabilizer of λ. The key point is that in each maximum-length chain from λ to $w\lambda$ in $(W\lambda, <)$, each term must differ from the next by a (positive) integer multiple of a root, so the length of the chain (namely, $\ell(w)$) is always at most $\mathrm{ht}(\lambda - w\lambda)$. However, given that equality occurs, each term must differ from the next by a simple root, so every step in every maximal chain from 1 to w in $(W^J, <)$ must arise from left multiplication by a simple reflection. (This fact is also noted in Section 10 of [8].)

For each unrefinable chain $1 < w_1 < \cdots < w_l = w$ whose steps correspond to left multiplication by simple reflections, it is an easy exercise to show that

$$\mathrm{wt}(1 < w_1 < \cdots < w_l) = Q(w) := \prod_{\alpha\in\Phi(w)} \alpha,$$

where $\Phi(w) = \{\alpha \in \Phi^+ : w\alpha \in -\Phi^+\}$. Thus if w is dominant minuscule, the sum of the weights of all maximum-length chains from 1 to w is $r(w)Q(w)$, where $r(w)$ denotes the number of reduced expressions for w as a product of simple reflections. Furthermore, it is interesting to note that by some unpublished work of Peterson, it is known that

$$r(w) = \ell(w)! \prod_{\alpha \in \Phi(w^{-1})} \frac{1}{\mathrm{ht}(\alpha)},$$

so in the dominant minuscule case, there is an identity strikingly similar to Theorem 1.1.

If W_J is the stabilizer of a minuscule weight for Φ or $\Phi^\vee$, then the longest element of W^J is dominant minuscule relative to Φ or $\Phi^\vee$, so in either case, the above reasoning shows that the sum of the weights of all maximal chains in $(W^J, <)$ must factor completely. Computations suggest that the same is true if W_J is the stabilizer of a dominant root (i.e., a quasi-minuscule weight relative to Φ or $\Phi^\vee$). However in general, the maximal chains of parabolic quotients tend not to have this property; aside from a few small cases, such as *all* quotients of $W(\mathcal{A}_3)$, we are aware of no other examples. In contrast, Theorem 2.2 shows that the weighted sum for every quotient has a specialization that factors completely.

4.4. Non-crystallographic root systems

It would be interesting to determine if there are analogues of Theorems 1.1 and 2.2 for the non-crystallographic root systems. Certainly the formulations of both results make sense whether or not the root system is crystallographic, and computer investigations suggest that the results *are* valid, aside from a scalar factor.

One indication that the constant of proportionality is problematic can be seen among the crystallographic root systems with more than one W-orbit of roots. In such root systems, each orbit can be rescaled independently and arbitrarily (at the expense of dropping the crystallographic condition), and it is not hard to show that the effect of these rescalings on the weight of each maximal chain is exactly the same as the effect on the product $Q = \prod_{\alpha \in \Phi^+} \alpha$ (in Theorem 1.1) or $Q_\lambda = \prod_{\alpha \in \Phi_\lambda} \langle \lambda, \alpha \rangle$ (in Theorem 2.2). However, assuming that the root system is irreducible, these rescalings also have an effect on the heights of roots, so the two sides of both identities diverge under rescaling.

On the other hand, using the relationship between the exponents $e_1, e_2, \ldots$ of W and the heights of roots (see [3, Section 3.20]), one knows that

$$e(W) = \prod_{\alpha \in \Phi^+} \mathrm{ht}(\alpha) \quad \text{and} \quad e(W)/e(W_J) = \prod_{\alpha \in \Phi_\lambda} \mathrm{ht}(\alpha),$$

where $e(W) = e_1! e_2! \cdots$. This allows one to rewrite the identities in a way that transforms correctly under rescaling, and computations suggest that these modified identities are valid even in the non-crystallographic cases.

Leaving aside the constant of proportionality, one possible approach to a version of Theorem 1.1 that includes non-crystallographic cases would be to exploit the fact that Q freely generates the module of skew-invariants; it is the polynomial of lowest degree that is skew-symmetric with respect to the action of W, and all such polynomials are divisible by Q. Since the weighted sum of maximal chains clearly has the same degree as Q, proving that the sum is skew-symmetric would imply that it is a scalar multiple of Q.

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