Journal of Algebraic Combinatorics **15** (2002), 223–229

The Parameters of Bipartite Q-polynomial Distance-Regular Graphs

JOHN S. CAUGHMAN, IV caughman@mth.pdx.edu
Department of Mathematical Sciences, Portland State University, P.O. Box 751, Portland, OR 97207–0751, USA

Received September 28, 1999; Revised November 26, 2001; Accepted December 6, 2001

Abstract. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Suppose $\theta_0, \theta_1, \ldots, \theta_D$ is a Q-polynomial ordering of the eigenvalues of Γ. This sequence is known to satisfy the recurrence $\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0$ $(0 < i < D)$, for some real scalar β. Let q denote a complex scalar such that $q + q^{-1} = \beta$. Bannai and Ito have conjectured that q is real if the diameter D is sufficiently large.

We settle this conjecture in the bipartite case by showing that q is real if the diameter $D \geq 4$. Moreover, if $D = 3$, then q is not real if and only if θ_1 is the second largest eigenvalue and the pair (μ, k) is one of the following: (1, 3), (1, 4), (1, 5), (1, 6), (2, 4), or (2, 5). We observe that each of these pairs has a unique realization by a known bipartite distance-regular graph of diameter 3.

Keywords: distance-regular graph, bipartite, association scheme, P-polynomial, Q-polynomial

1. Introduction

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$ (definitions appear in Sections 2 and 3 below). Suppose $\theta_0, \theta_1, \ldots, \theta_D$ is a Q-polynomial ordering of the eigenvalues of Γ. By [3, p. 241], this eigenvalue sequence satisfies

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0 \quad (1 \leq i \leq D - 1), \tag{1}$$

for some real scalar β. Let q denote a complex scalar such that $q + q^{-1} = \beta$. In [1, p. 381] Bannai and Ito conjectured that q is real if the diameter D is sufficiently large.

We settle this conjecture in the bipartite case by showing q is real if $D \geq 4$. Moreover, for the case $D = 3$, we describe the conditions under which q fails to be real. Precise statements of these theorems are given below. In future work, we intend to use these results to classify the bipartite Q-polynomial distance-regular graphs.

In stating and proving the present results, it will be convenient to work with the scalar β rather than with q itself. To interpret our results for q, we need only make the following observation.

Lemma 1.1 *Let β be any real number and let q denote a complex scalar such that $q + q^{-1} = \beta$. Then the following hold.*
(i) *Suppose $\beta \leq -2$. Then q is a negative real number.*
(ii) *Suppose $\beta \geq 2$. Then q is a positive real number.*
(iii) *Suppose $-2 < \beta < 2$. Then q is a complex (non real) number with norm $|q| = 1$.*

Proof: Observe q is a root of the polynomial $x^2 - \beta x + 1$. □

We now state our main results, beginning with the case $D \geq 4$.

Theorem 1.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Suppose $\theta_0, \theta_1, \ldots, \theta_D$ is a Q-polynomial ordering of the eigenvalues of Γ, and let β be as in* (1). *Then the following hold.*
(i) *Suppose $\theta_1 < -1$. Then $\beta \leq -2$.*
(ii) *Suppose $\theta_1 > -1$. Then $\beta \geq 2$.*
We remark that $\theta_1 \neq -1$ (cf. Lemma 3.2(i)).

We point out that the conditions on θ_1 in Theorem 1.2(i) and (ii) are in fact sufficient to determine the full ordering of the eigenvalues. For more information on the possible Q-polynomial orderings for a bipartite distance-regular graph, we refer the reader to [4].

Before stating the result for $D = 3$ we mention a few basic facts. Let Γ denote any bipartite distance-regular graph with diameter $D = 3$, and let λ denote the positive square root of the intersection number b_2. Then $k, \lambda, -\lambda$, and $-k$ are the distinct eigenvalues of Γ, and the sequence

$$k, \lambda, -\lambda, -k \tag{2}$$

is a Q-polynomial ordering [3, p. 432]. If $b_2 = 1$ then Γ has no further Q-polynomial orderings, but if $b_2 > 1$ then Γ has a second Q-polynomial ordering:

$$k, -\lambda, \lambda, -k. \tag{3}$$

Theorem 1.3 *Let Γ denote a bipartite distance-regular graph with diameter $D = 3$ and valency $k \geq 3$. Set $\mu := c_2$. Then the following hold.*
(i) *For the ordering* (2), *we have $\beta \geq 1$. Furthermore, $\beta < 2$ if and only if the pair (μ, k) is one of the following*: (1, 3), (1, 4), (1, 5), (1, 6), (2, 4), *or* (2, 5).
(ii) *Suppose $b_2 > 1$. For the ordering* (3), *we have $\beta \leq -2$.*

Remark 1.4 Each of the pairs (μ, k) listed in Theorem 1.3(i) above has a unique realization by a bipartite distance-regular graph of diameter 3. In particular, the pair (1, 3) is uniquely realized by the Heawood graph. For $4 \leq k \leq 6$, the pair $(1, k)$ is uniquely realized by the incidence graph of the (unique) projective plane of order $k - 1$. The pair (2, 4) is uniquely realized by the distance 3 graph of the Heawood graph, and the pair (2, 5) is uniquely realized by the incidence graph of the (unique) 2-(11, 5, 2) design. For these facts and more about these graphs, we refer to the book of Brouwer et al. [3].

2. Distance-regular graphs and the Q-polynomial property

In this article we consider only graphs which are finite, connected, undirected, and without loops or multiple edges. Let $\Gamma = (X, R)$ denote a graph with vertex set X and edge set R. Let ∂ denote the path length distance function for Γ, and recall the *diameter* of Γ is the scalar

$D := \max\{\partial(x, y) \mid x, y \in X\}$. Γ is said to be *distance-regular*, with *intersection numbers* $b_i, c_i (0 \le i \le D)$, whenever for all integers $i (0 \le i \le D)$ and for all $x, y \in X$ with $\partial(x, y) = i$,

$$b_i = |\{z \in X \mid \partial(x, z) = i + 1, \partial(y, z) = 1\}|,$$
$$c_i = |\{z \in X \mid \partial(x, z) = i - 1, \partial(y, z) = 1\}|.$$

Following convention, we abbreviate $\mu := c_2$ and $k := b_0$. We refer to k as the *valency*.

Let $\Gamma = (X, R)$ denote any distance-regular with diameter $D \ge 3$. By [3, Proposition 4.1.6], the intersection numbers must satisfy

$$c_i \le b_j \text{ whenever } i + j \le D. \tag{4}$$

We now recall the adjacency algebra of Γ. Let $\mathbb{R}$ denote the field of real numbers, and let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of matrices over $\mathbb{R}$ with rows and columns indexed by X. For $0 \le i \le D$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \ne i \end{cases} \quad (x, y \in X). \tag{5}$$

We abbreviate $A = A_1$; this is the adjacency matrix for Γ. Let $\mathcal{A}$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by A. $\mathcal{A}$ is known as the *adjacency algebra* of Γ. It is well known that $A_0, \ldots, A_D$ is a basis for $\mathcal{A}$ [2, p. 160]. Also, $\mathcal{A}$ is semisimple; in particular, $\mathcal{A}$ has a basis $E_0, \ldots, E_D$ consisting of mutually orthogonal primitive idempotents [3, p. 132]. We refer to $E_0, \ldots, E_D$ as the *primitive idempotents* of Γ. Observe that for each i $(0 \le i \le D)$, there exists a real scalar θ_i such that $AE_i = \theta_i E_i$. We refer to $\theta_0, \ldots, \theta_D$ as the *eigenvalues* of Γ. Note that $\theta_0, \ldots, \theta_D$ are distinct, since A generates $\mathcal{A}$.

We next recall the Q-polynomial property. Let Γ denote any distance-regular graph with diameter $D \ge 3$, and let $\mathcal{A}$ denote the adjacency algebra for Γ. Since $\mathcal{A}$ has a basis $A_0, \ldots, A_D$ of 0–1 matrices, we see $\mathcal{A}$ is closed under entry-wise matrix multiplication. Let $\theta_0, \ldots, \theta_D$ denote an ordering of the eigenvalues of Γ. This ordering is said to be *Q-polynomial* whenever for each integer i $(0 \le i \le D)$, the primitive idempotent E_i is a polynomial of degree exactly i in E_1, in the $\mathbb{R}$-algebra $(\mathcal{A}, \circ)$, where $\circ$ denotes entry-wise multiplication.

Fix any eigenvalue θ of Γ, and let E denote the associated primitive idempotent. Write $E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$ for some scalars θ_i^* $(0 \le i \le D)$. We refer to $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ as the *dual eigenvalue sequence* associated with θ. Note θ_0^* equals the rank of E, and is therefore nonzero [1, p. 62]. If $\theta_0, \ldots, \theta_D$ is a Q-polynomial ordering of the eigenvalues of Γ, then $\theta_0 = k$ and the dual eigenvalues associated with θ_1 are distinct [1, pp. 193, 197].

Lemma 2.1 ([3, p. 237]) *Let Γ denote any distance-regular graph with diameter $D \ge 3$. Suppose $\theta_0, \theta_1, \ldots, \theta_D$ is a Q-polynomial ordering of the eigenvalues of Γ, and let $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ denote the dual eigenvalue sequence associated with θ_1. Then there exists a unique $\beta \in \mathbb{R}$ such that*
(i) $\theta_{i-1} - \beta\theta_i + \theta_{i+1}$ *is independent of* i $(1 \le i \le D - 1)$, *and*
(ii) $\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^*$ *is independent of* i $(1 \le i \le D - 1)$.

3. Bipartite distance-regular graphs

Recall that a graph $\Gamma = (X, R)$ is *bipartite* whenever there exists a partition of the vertices $X = X^+ \cup X^-$ such that X^+ and X^- contain no edges. Let Γ denote a distance-regular graph with diameter $D \geq 3$, and valency $k \geq 3$. Assume Γ is bipartite. Then it is easily shown that

$$c_i + b_i = k \quad (0 \leq i \leq D). \tag{6}$$

Since $b_D = 0$, it follows that $c_D = k$. By [8, p. 399], the valency k is the largest eigenvalue of Γ, and $-k$ is the minimal eigenvalue. We refer to k and $-k$ as the *trivial eigenvalues*.

Let θ denote any nontrivial eigenvalue for Γ and set $\mu := c_2$. In [5, Theorem 18], Curtin gives the following bound:

$$\theta^2(\mu - 1) \leq (k - \mu)(k - 2). \tag{7}$$

Furthermore, by [5, Lemma 4], the dual eigenvalue sequence associated with θ satisfies

$$c_i \times \theta^*_{i-1} + b_i\theta^*_{i+1} = \theta\theta^*_i \quad (0 \leq i \leq D), \tag{8}$$

where $\theta^*_{-1}, \theta^*_{D+1}$ are indeterminates. When Γ is Q-polynomial, we have the following.

Lemma 3.1 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$. Suppose $\theta_0, \theta_1, \ldots, \theta_D$ is a Q-polynomial ordering of the eigenvalues of Γ. Let β be as in Lemma* 2.1. *Then the following hold.*

(i) [3, *p*. 241]

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0 \quad (1 \leq i \leq D - 1). \tag{9}$$

(ii) [4, *Theorem* 9.6]

$$\theta_i = -\theta_{D-i} \quad (0 \leq i \leq D). \tag{10}$$

Lemma 3.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$. Suppose $\theta_0, \theta_1, \ldots, \theta_D$ is a Q-polynomial ordering of the eigenvalues of Γ. Let β be as in Lemma* 2.1. *Then the following hold.*

(i) $\theta_1 \neq -1$, *and*

$$\beta = \frac{\theta_1^2 + \mu\theta_1 + (k - \mu)(k - 2)}{(k - \mu)(\theta_1 + 1)}. \tag{11}$$

(ii) $$\theta_1^3(b_2 - b_3) + \theta_1^2(b_2 - \mu b_3) + \theta_1 b_2(2b_3 - \mu b_3 - b_2) + b_2^2(b_3 - 1) = 0. \tag{12}$$

Proof: (i) If $\theta_1 = -1$ then $\theta_1^* = \theta_2^*$ by (8), contradicting the fact that the dual eigenvalues are distinct. Observe that by Lemma 2.1,

$$\theta_0^* - \beta\theta_1^* + \theta_2^* = \theta_1^* - \beta\theta_2^* + \theta_3^*. \tag{13}$$

Divide both sides of (13) by θ_0^* and eliminate the dual eigenvalues using (8) and simplify to obtain (11).

(ii) First suppose $D = 3$. Then $b_3 = 0$, so the left side of (12) becomes $b_2(\theta_1 + 1)(\theta_1^2 - b_2)$, which is 0 since $\theta_1^2 = b_2$ (cf., [3, p. 432]). Now assume $D \geq 4$. By Lemma 2.1,

$$\theta_0^* - \beta\theta_1^* + \theta_2^* = \theta_2^* - \beta\theta_3^* + \theta_4^*. \tag{14}$$

Divide both sides of (13) by θ_0^* and eliminate the dual eigenvalues using (8). Eliminate β using (11). Then simplify, noting that $(\theta_1^2 - k^2)$ is a factor, to obtain Eq. (12). □

Lemma 3.3 *With the notation and assumptions of Theorem* 1.2, *the following hold.*
(i) [6, *Theorem* 8.1.3] *Suppose $D \geq 5$. Then $\theta_0, \theta_1, \ldots, \theta_D$ are integers.*
(ii) *Suppose $D = 4$. If $\theta_0, \theta_1, \ldots, \theta_D$ are not all integers, then $b_3 = 1$ and*

$$\beta^2 = \theta_1^2 = k = 2\mu. \tag{15}$$

Proof: (ii) Recall that $\theta_0 = k$. By Lemma 3.1(ii), $\theta_4 = -k$ and $\theta_2 = 0$. The remaining eigenvalues can be computed directly from the intersection matrix (cf. [2, p. 165]) to obtain

$$\{\theta_1, \theta_3\} = \{\pm\sqrt{c_2(b_3 - 1) + k}\}. \tag{16}$$

First suppose $b_3 \neq 1$. Then (12) and (16) imply that θ_1 is rational, so (16) forces θ_1 and θ_3 to be integers, as desired. Now suppose $b_3 = 1$. Then (16) implies $\theta_1^2 = k$. But $\beta = k/\theta_1$ by (9) at $i = 1$. Substituting these values into (11), we find that $k = 2\mu$, as desired. □

4. Proofs of the main results

Proof of Theorem 1.2(i): Let $\theta := \theta_1$. By assumption, $\theta < -1$. So by (11),

$$\beta + 2 = \frac{(\theta + k)(\theta + k - \mu)}{(k - \mu)(\theta + 1)}. \tag{17}$$

We distinguish two cases.

Case $\mu \geq 2$. Consider the expression on the right side of (17). Observe $\theta + k$ is positive, and by assumption, $\theta + 1$ is negative. Also, $k - \mu = b_2$ is positive. Finally, since $\mu \geq 2$, line (7) implies that $\theta + k - \mu$ is nonnegative. It now follows by (17) that $\beta \leq -2$ as desired.

Case $\mu = 1$. Again consider the expression on the right side of (17). Since $\mu = 1$ and $k > 2, \theta$ is an integer by Lemma 3.3, and the numerator of (17) is nonnegative. Also, $k - \mu = b_2$ is positive, and by assumption, $\theta + 1$ is negative. It now follows by (17) that $\beta \leq -2$ as desired. □

Proof of Theorem 1.2(ii): Let $\theta := \theta_1$. By assumption, $\theta > -1$. So by (11),

$$\beta - 2 = \frac{(2\theta - 2k + 3\mu)^2 + 8(k-\mu)(\mu - 2) - \mu^2}{4(k-\mu)(\theta+1)}. \tag{18}$$

We distinguish three cases.
Case $\mu \geq 3$. By (4), $k - \mu = b_2 \geq \mu$. Therefore, since $\mu \geq 3$,

$$8(k-\mu)(\mu-2) - \mu^2 \geq 0, \tag{19}$$

and the numerator in (18) is nonnegative. By our assumptions, the denominator is positive, so it follows that $\beta \geq 2$ as desired.
Case $\mu = 2$. By way of contradiction, suppose $\beta < 2$. Then by (11), $k - 4 < \theta < k - 2$. Since $\mu = 2$, Lemma 3.3 implies θ is an integer, so $\theta = k - 3$. So by (9) with $i = 1$, and (11) with $\mu = 2$ and $\theta = k - 3$,

$$\theta_2 = k - 6 - (k-2)^{-1} + (k-2)^{-2}. \tag{20}$$

It follows that $(k-2)^{-1} = 1 + (k-2)(\theta_2 - k + 6)$, which is an integer, so $k = 3$. Now (20) implies $\theta_2 = -k = \theta_D$, forcing $D = 2$, a contradiction.
Case $\mu = 1$. By (11), with $\mu = 1$,

$$\beta - 2 = \frac{\theta^2 + (3-2k)\theta + (k-4)(k-1)}{(k-1)(\theta+1)}. \tag{21}$$

The denominator in (21) is positive, so $\beta > 2$ whenever the numerator is positive. Now we consider (12). Setting $b_2 = k - \mu$, $b_3 = k - c_3$, and $\mu = 1$, line (12) becomes

$$\theta^3(c_3 - 1) + \theta^2(c_3 - 1) - \theta(k-1)(c_3-1) + (k-1)^2(k - c_3 - 1) = 0. \tag{22}$$

Line (22) implies $c_3 \neq 1$, since $k \geq 3$. Also $k > c_3$ since $D \geq 4$, so (22) implies

$$(k-1)^2 + (k-1)\theta - \theta^2 - \theta^3 = \frac{(k-2)(k-1)^2}{c_3 - 1} \geq (k-1)^2. \tag{23}$$

Since the θ_i are distinct, Lemma 3.1(ii) implies $\theta_1 \neq 0$. So by Lemma 3.3 and our assumptions, θ is a positive integer. Now (23) implies $\theta < \sqrt{k}$. When $k \geq 11$,

$$\frac{(k-4)(k-1)}{2k-3} > \sqrt{k} > \theta. \tag{24}$$

Line (24) implies the numerator in (21) is positive, so $\beta > 2$ as desired. It remains to consider the case $k \leq 10$. Recall θ is a positive integer. The only pairs of integers θ, k with $1 \leq \theta < k \leq 10$ for which $c_3 - 1$ as given in (22) is a positive integer less than $k - 1$ are the pairs $(\theta, k) = (2, 7)$ and $(\theta, k) = (1, 3)$. If $(\theta, k) = (2, 7)$, then $\beta = 2$ by (11). And if

$(\theta, k) = (1, 3)$, then (21) implies $\beta = 1$. But (9) implies $\theta_3 = -k = \theta_D$, forcing $D = 3$, a contradiction. □

Proof of Theorem 1.3(i): By (9), (10) at $i = 1$, we find $\beta = k\lambda^{-1} - 1$. Since $k \geq 3$ and $\lambda = \sqrt{k - \mu}$, it follows that $\beta \geq 1$. Moreover, when $k \geq 8$, β is apparently greater than 2. It is readily verified that the only pairs of integers (μ, k) which satisfy $1 \leq \mu < k \leq 7$ and for which $\beta < 2$ are (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 4), and (2, 5). As noted in [3, p. 432], the existence of a graph with array (1, 7) is equivalent to the existence of a 2-(43, 7, 1) design, which is impossible by the Bruck-Ryser-Chowla Theorem [7, p. 391]. This completes the proof. □

Proof of Theorem 1.3(ii): By (9), (10) at $i = 1$, we find $\beta = -k\lambda^{-1} - 1$, which is clearly less than -2. □

References

1. E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, London, 1984.
2. N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1974.
3. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.
4. J.S. Caughman, IV, "Spectra of bipartite P- and Q-polynomial association schemes," *Graphs Combin.* **14** (1998), 321–343.
5. B. Curtin, "2-homogeneous bipartite distance-regular graphs," *Discrete Math.* **187** (1998), 39–70.
6. G. Dickie, "Q-polynomial structures for association schemes and distance-regular graphs," Ph.D. Thesis, University of Wisconsin, 1995.
7. F.S. Roberts, *Applied Combinatorics*, Prentice-Hall, New Jersey, 1984.
8. J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.