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# A Distance-Regular Graph with Strongly Closed Subgraphs

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**Abstract.** Let $\Gamma$ be a distance-regular graph of diameter $d$, valency $k$ and $r := \max\{i \mid (c_i, b_i) = (c_1, b_1)\}$. Let $q$ be an integer with $r + 1 \leq q \leq d - 1$.

In this paper we prove the following results:

**Theorem 1** *Suppose for any pair of vertices at distance $q$ there exists a strongly closed subgraph of diameter $q$ containing them. Then for any integer $i$ with $1 \leq i \leq q$ and for any pair of vertices at distance $i$ there exists a strongly closed subgraph of diameter $i$ containing them.*

**Theorem 2** *If $r \geq 2$, then $c_{2r+3} \neq 1$.*

As a corollary of Theorem 2 we have $d \leq k^2(r + 1)$ if $r \geq 2$.

**Keywords:** distance-regular graph, strongly closed subgraph

## 1. Introduction

First we recall our notation and terminology.

All graphs considered are undirected finite graphs without loops or multiple edges. Let $\Gamma$ be a connected graph with usual distance $\partial_\Gamma$. We identify $\Gamma$ with the set of vertices. The *diameter* of $\Gamma$, denoted by $d_\Gamma$, is the maximal distance of two vertices in $\Gamma$. Let $u \in \Gamma$. We denote by $\Gamma_j(u)$ the set of vertices which are at distance $j$ from $u$.

For two vertices $u$ and $v$ in $\Gamma$ with $\partial_\Gamma(u, v) = j$, let

$$C(u, v) := \Gamma_{j-1}(u) \cap \Gamma_1(v),\ A(u, v) := \Gamma_j(u) \cap \Gamma_1(v),\ B(u, v) := \Gamma_{j+1}(u) \cap \Gamma_1(v).$$

We denote by $c(u, v)$, $a(u, v)$ and $b(u, v)$ their cardinalities, respectively.

We say $c_i$ *exists* if $c_i = c(x, y)$ does not depend on the choice of $x$ and $y$ under the condition $\partial_\Gamma(x, y) = i$. Similarly, we say $a_i$ *exists*, or $b_i$ *exists*.

A graph $\Gamma$ is said to be *distance-regular* if $c_i$, $a_i$ and $b_{i-1}$ exist for all $1 \leq i \leq d_\Gamma$. Then $c_i$, $a_i$ and $b_i$ are called the *intersection numbers* of $\Gamma$. In particular, $k_\Gamma = b_0$ is called the *valency* of $\Gamma$. Let $r(\Gamma) := \max\{i \mid (c_i, b_i) = (c_1, b_1)\}$.

A bipartite graph $\Gamma$ with bipartition $\Gamma^+ \cup \Gamma^-$ is called *distance-biregular* if $c(x, y)$ and $b(x, y)$ depend only on $i = \partial_\Gamma(x, y)$ and the part that the vertex $x$ belongs to.

The reader is referred to [1, 2] for more detailed descriptions of distance-regular graphs.

Let $\emptyset \neq \Delta \subseteq \Gamma$. We identify $\Delta$ with the induced subgraph on it. $\Delta$ is called *strongly closed* if $C(u, v) \cup A(u, v) \subseteq \Delta$ for any $u, v \in \Delta$.

We say *the condition* $(SC)_q$ *holds* if there exists a strongly closed subgraph of diameter $q$ containing any given pair of vertices at distance $q$.

Throughout this paper $\Gamma$ denotes a distance-regular graph of diameter $d_\Gamma = d$, valency $k_\Gamma = k \geq 3$ and $r(\Gamma) = r$. Let $q$ be an integer with $r + 1 \leq q \leq d - 1$.

In [5] the author conjectured that if $r \geq 2$, then $\eta_1 := \max\{i \mid c_i = 1\} \leq 2r + 2$. Except for the remaining case $(a_1, a_{r+1}) = (0, 1)$ this conjecture was proved by showing the existence of strongly closed subgraphs. A shorter and easier proof was given in [6].

In [6, 7] we proved that if $c_{q+r} = 1$, then $(SC)_q$ holds.

In this paper we investigate distance-regular graphs satisfying the condition $(SC)_q$ in order to resolved our remaining case.

The following are our results.

**Theorem 1** *If* $(SC)_q$ *holds, then* $(SC)_i$ *holds for all* $1 \leq i \leq q$.

**Theorem 2** *If* $r \geq 2$, *then* $\eta_1 := \max\{i \mid c_i = 1\} \leq 2r + 2$.

In [10] Koolen and the author improved the so-called Ivanov diameter bound. Their result is $d \leq k^2\eta_1/2 < k^3 r/2$.

Applying our theorem to this bound we have the following corollary.

**Corollary 3** *If* $r \geq 2$, *then* $d \leq k^2(r + 1)$.

## 2. Proof of the theorems

Let $G$ be a connected graph. We define the *n-subdivision graph of* $G$, denote by ${}^{(n)}G$, the graph obtained from $G$ by replacing each edge by a path of length $n$. □

It is not hard to show the following lemmas from our definition.

**Lemma 4** *Let* $G$ *be a connected graph and* $\Delta_1, \ldots, \Delta_t$ *be strongly closed subgraphs of* $G$. *Then their intersection* $\cap_{i=1}^{t} \Delta_i$ *is also strongly closed, unless it is the empty set.*

**Lemma 5** *Let* $\Omega$ *be the 3-subdivision graph* ${}^{(3)}K_{k+1}$ *of a complete graph, or the 3-subdivision graph* ${}^{(3)}M_k$ *of a Moore graph, where* $k \geq 3$. *Let* $m := d_\Omega - 2$. *Then* $c_i$ *and* $a_i$ *of* $\Omega$ *exist for* $1 \leq i \leq m + 2$ *with* $a_1 = \cdots = a_m = 0$ *and* $c_1 = \cdots = c_{m+2} = a_{m+1} = a_{m+2} = 1$.

*We denote by* $P$ *the set of vertices contained in the original graph, and by* $L$ *the set of vertices which are added by replacing each edge by a path. Let* $x, y \in \Omega$ *with* $\partial_\Omega(x, y) = d_\Omega - 1$ *such that* $B(x, y) = B(y, x) = \emptyset$. *Then* $x, y \in L$ *and* $A(x, y) \subseteq P$.

The following result is proved by H. Suzuki [12, Theorem 1.1].

**Proposition 6** *Let $\Delta$ be a strongly closed subgraph of $\Gamma$ of diameter $d_\Delta$ with $r+1 \le d_\Delta \le d-1$. Then one of the following holds.*

(1) *$\Delta$ is a distance-regular graph.*
(2) *$\Delta$ is a distance-biregular graph. Moreover $r \equiv d_\Delta \equiv 0 (\mathrm{mod} 2)$ and $c_{2i-1} = c_{2i}$ for all $i$ with $1 \le i \le d_\Delta/2$.*
(3) *$\Delta$ is the 3-subdivision graph ${}^{(3)}K_{k+1}$ of a complete graph or the 3-subdivision graph ${}^{(3)}M_k$ of a Moore graph. Moreover $d_\Delta = r+2 \in \{5, 8\}$ and $c_{r+1} = c_{r+2} = a_{r+1} = a_{r+2} = 1$.*

**Definition** Suppose $\Gamma$ satisfies the condition $(SC)_q$. For any pair $(u, v)$ of vertices at distance $q$ in $\Gamma$ there exist strongly closed subgraphs of diameter $q$ containing them. Let $\Delta(u, v)$ be their intersection. Then $\Delta(u, v)$ is the smallest strongly closed subgraph of diameter $q$ containing $u$ and $v$.

**Remarks**

(1) Let $\Delta$ be a strongly closed subgraph of $\Gamma$ of diameter $q$. Then $c_i$ and $a_i$ of $\Delta$ exist for all $1 \le i \le q$ which are the same as those of $\Gamma$. In particular, $\Delta$ is distance-regular iff $b_{q-1} > b_q$.
(2) Suppose the condition $(SC)_q$ holds and $b_{q-1} > b_q$. For any pair $(u, v)$ of vertices at distance $q$ a strongly closed subgraph of diameter $q$ containing them is distance-regular with the same intersection numbers of $\Delta(u, v)$. It follows that $\Delta(u, v)$ is the unique strongly closed subgraph of diameter $q$ containing $u$ and $v$.
(3) If $\Gamma$ has no induced subgraph $K_{2,1,1}$, then $(SC)_i$ always holds for all $1 \le i \le r$. In this case for any pair $(u, v)$ of vertices at distance at most $r$, $\Delta(u, v)$ is the graph induced by the set of vertices on singular lines on each edge of the unique shortest path between $u$ and $v$.
(4) If $(SC)_q$ holds, then $\Gamma$ has no induced subgraph $K_{2,1,1}$. (See [8, Lemma 3.6].) In particular, $(SC)_i$ holds for all $1 \le i \le r$.

**Proof of Theorem 1:** We assume $r+2 \le q$ and prove that the condition $(SC)_{q-1}$ holds. Then the assertion is proved by an easy induction.

Let $(u, v)$ be a pair of vertices at distance $q-1$. Let

$$\Omega := \left( \bigcap_{x \in B(u,v)} \Delta(u, x) \right) \cap \left( \bigcap_{y \in B(v,u)} \Delta(v, y) \right).$$

Then $\Omega$ is a strongly closed subgraph containing $u$ and $v$. We prove $d_\Omega = q-1$.

It is clear that $q-1 \le d_\Omega \le q$. Assume $d_\Omega = q$ to derive a contradiction.

Suppose there exists $z \in B(u, v) \cap \Omega$. Take any $w \in B(z, u) \subseteq B(v, u)$. Then we have $z \in \Omega \subseteq \Delta(v, w) =: \Sigma$ and hence $q+1 = \partial_\Gamma(z, w) \le d_\Sigma = q$. This is a contradiction.

By symmetry we may assume $B(u, v) \cap \Omega = B(v, u) \cap \Omega = \emptyset$.

This implies $\Omega$ is the 3-subdivision graph of a complete graph, or the 3-subdivision graph of a Moore graph from Proposition 6. In particular, $c_{r+1} = c_{r+2} = a_{r+1} = a_{r+2} = 1$. It follows, by Lemma 5, that $u, v \in L$ and $\{\alpha\} := A(u, v) \subseteq P$. Let $\beta \in B(u, \alpha) \subseteq \Omega$ and $\gamma \in B(\beta, u)$. Then $\gamma \in B(\beta, u) = B(\alpha, u) = B(v, u)$ and $\beta \in \Omega \subseteq \Delta(v, \gamma) =: \Pi$. This is a contradiction as $q + 1 = \partial_\Gamma(\beta, \gamma) \leq d_\Pi = q$. The theorem is proved. □

Next we collect several results to prove Theorem 2.

**Lemma 7**

(1) *If* $b_{q-1} > b_q$ *and* $c_{q+r} = 1$, *then* $(SC)_q$ *holds.*
(2) *If* $a_1 = 0$ *and* $c_{r+4} = 1$, *then* $s := |\{i | (c_i, a_i) = (1, 1)\}| \leq 2$.
(3) *If* $(a_1, a_{r+1}, c_{r+2}) = (0, 1, 1)$ *and* $d = r + 2$, *then no such* $\Gamma$ *exists.*
(4) *If* $a_1 = 0$ *and* $r \in \{3, 6\}$, *then* $c_{2r+3} \neq 1$.

**Proof:** See [7, Theorem 1.3] [4], [3], and [11, Proposition 4.3], respectively. □

**Proof of Theorem 2:** Suppose $\eta_1 \geq 2r + 3$ to derive a contradiction.

Since $b_r > b_{r+1}$ and $c_{2r+1} = 1$, the condition $(SC)_{r+1}$ holds from Lemma 7(1). Then a strongly closed subgraph of diameter $r+1 \geq 3$ is the collinearity graph of a Moore geometry with valency $1 + a_{r+1}$. Thus $(a_1, a_{r+1}) = (0, 1)$. (See [2, Theorem 6.8.1].) It follows, by Lemma 7 (2), that $s := |\{i | (c_i, a_i) = (1, 1)\}| \in \{1, 2\}$. Since $b_{r+s} > b_{r+s+1}$ and $c_{2r+s+1} = 1$, the condition $(SC)_{r+s+1}$ holds from Lemma 7 (1). Then a strongly closed subgraph $\Delta$ of diameter $d_\Delta = r + s + 1$ has $(a_1, a_{r+1}, c_{r+s+1}) = (0, 1, 1)$.

If $s = 1$, then no such $\Delta$ exists from Lemma 7 (3). We have a contradiction.

Suppose $s = 2$. Then $(SC)_{r+3}$ holds, and thus $(SC)_{r+2}$ holds from Theorem 1. Since $b_{r+1} = b_{r+2}$ and $a_{r+1} = a_{r+2} = 1$, a strongly closed subgraph of diameter $r + 2$ is the 3-subdivision graph of a complete graph, or of a Moore graph from Proposition 6. In particular, we have $r \in \{3, 6\}$. This contradicts Lemma 7 (4).

We complete the proof of Theorem 2. □

**Remark** In the forthcoming paper [9], we investigate a distance-regular graph which satisfies the conditions $(SC)_q$ and $(SC)_{q+1}$ for some $r + 1 \leq q \leq d - 1$ and a strongly closed subgraphs of diameter $q$ is a non-regular distance-biregular graph. We prove that such a graph is either the doubled Grassmann graph, the doubled Odd graph, or the Odd graph.

We will be able to classify distance-regular graphs satisfying the condition $(SC)_q$.

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