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Quasi-Minuscule Quotients and Reduced Words for Reflections

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Abstract. We study the reduced expressions for reflections in Coxeter groups, with particular emphasis on finite Weyl groups. For example, the number of reduced expressions for any reflection can be expressed as the sum of the squares of the number of reduced expressions for certain elements naturally associated to the reflection. In the case of the longest reflection in a Weyl group, we use a theorem of Dale Peterson to provide an explicit formula for the number of reduced expressions. We also show that the reduced expressions for any Weyl group reflection are in bijection with the linear extensions of a natural partial ordering of a subset of the positive roots or co-roots.

Keywords: Coxeter group, reflection, minuscule, reduced word, weak order

0. Introduction

Let W be a Coxeter group with distinguished generating set $s_1, \ldots, s_n$. Any such group has a faithful representation in which the generators s_i act as reflections on some real vector space V, and the conjugates of the generators form the set of all reflections in W.

In this paper, we study the structure of the reduced expressions for the reflections in W, with particular emphasis on the finite crystallographic groups. For example, in many cases of interest (including all Coxeter groups with acyclic diagrams), we show that the reduced expressions for a reflection are in one-to-one correspondence with certain chains in a partial ordering (the "Cayley order") of an associated root system for W.

There is a connected component in the Cayley ordering corresponding to each orbit of roots (or conjugacy class of reflections). In the case of a finite orbit, this order is isomorphic to the weak ordering of the quotient W/W', where W' denotes the stabilizer of the dominant root in the orbit. (In an infinite orbit, there is no dominant root.) If W is finite and crystallographic, then the quotient W/W' is "quasi-minuscule," in the sense that there is a representation of a Lie algebra with Weyl group W whose weights consist of 0 and the orbit in question.[1,2] In particular, the maximal chains in the weak ordering of a quasi-minuscule quotient correspond to the reduced expressions for the longest reflection in a given conjugacy class.

It is easy to show (Proposition 2.4) that the number of reduced expressions for any reflection t can be expressed as the sum of the squares of the number of reduced expressions for certain elements of W naturally associated with t. In the case of the longest reflection

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in a finite crystallographic group, these elements turn out to be "dominant minuscule" in the sense of Dale Peterson (see [7, 8] or [13]). Using an unpublished product formula of Peterson for counting reduced expressions of dominant minuscule elements, we obtain an explicit formula for the number of reduced expressions for the longest reflection in any finite Weyl group (Theorem 3.6). It is interesting to note that the elements that occur in this way come from every one of the 15 families of simply-laced dominant minuscule elements in Proctor's classification [8], as well as both multiply-laced families in [13].

Our second main result (Theorem 4.6) shows that in a finite Weyl group, the Cayley (or weak) order associated to any reflection can be deformed ("smashed") into a distributive lattice in a way that preserves the number of maximal chains. Thus the reduced expressions for a reflection are in one-to-one correspondence with the linear extensions of some poset, and this poset turns out to be isomorphic to a natural partial ordering of a subset of the positive roots or co-roots.

This "near distributivity" is related to some earlier work of Proctor. In [6], Proctor shows that the Bruhat ordering (as well as the weak ordering) of a finite Weyl group quotient W/W' is a distributive lattice if and only if the quotient is minuscule, and in that case, he identifies the join-irreducibles with a partial ordering of a subset of the positive co-roots. This and some related conjectures (now theorems) led Proctor to predict that the number maximal chains in the weak ordering of any (parabolic) quotient of Weyl groups should be expressible as the number of linear extensions of a specific partial ordering of a set of positive co-roots. Although this fails in general, Theorem 4.6 confirms this in the quasi-minuscule case, a result obtained independently by Proctor but never published. For further details, see the discussion at the end of Section 4 below.

1. Preliminaries

Continuing the notation established in the introduction, W shall denote a Coxeter group with distinguished generators $s_1, \dots, s_n$. We let Φ denote a root system for W, embedded in some real vector space V with an inner product $\langle \cdot\,, \cdot \rangle$ (not assumed to be positive definite). Standard references are [2] and [5].

For general poset terminology and notation, we follow Chapter 3 of [10].

For each $\alpha \in V$ such that $\langle \alpha, \alpha \rangle > 0$, the reflection through the hyperplane orthogonal to α is denoted σ_α. Thus $\sigma_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for all $\lambda \in V$, where $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$.

Corresponding to each generator s_i is a simple root $\alpha_i \in \Phi$ such that the map $s_i \mapsto \sigma_{\alpha_i}$ defines a faithful representation of W as a group of isometries of V. Henceforth we will identify W with this representation. It should be noted that W is finite if and only if the inner product is positive definite on the span of the simple roots.

One may partition Φ into positive and negative roots Φ^+ and $\Phi^- = -\Phi^+$. The former are those roots in the nonnegative linear span of the simple roots. The root system is said to be *crystallographic* if $\langle \alpha, \beta^\vee \rangle \in \mathbf{Z}$ for all $\alpha, \beta \in \Phi$. In that case, every root is in the $\mathbf{Z}$-linear span of the simple roots.

If α is a root, then $\alpha^\vee$ is said to be a *co-root*. The set of co-roots, denoted $\Phi^\vee$, is also a root system for W, with $\alpha_1^\vee, \dots, \alpha_n^\vee$ serving as simple roots. The co-root system is crystallographic if and only if the original root system is crystallographic.

If Φ is crystallographic, a vector $\lambda \in V$ is said to be an *integral weight* if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbf{Z}$ for $1 \leq i \leq n$. The integral weights are partially ordered by the rule

$$\mu \leq \lambda \ \Leftrightarrow \lambda - \mu \in \mathbf{N}\alpha_1 + \cdots + \mathbf{N}\alpha_n,$$

where $\mathbf{N}$ denotes the nonnegative integers. We call this the *standard ordering*. Of particular importance will be the standard ordering of Φ and the analogous ordering of $\Phi^\vee$.

The *Coxeter graph*, denoted Γ, is a weighted graph with vertex set $[n] := \{1, 2, \ldots, n\}$ and an edge between i and j if s_i and s_j do not commute. If $s_i s_j$ has order $m \geq 3$ in W, then the corresponding edge of Γ is assigned the weight m. The Coxeter group is said to be *irreducible* if Γ is connected.

Given $w \in W$, an expression $w = s_{i_1} \cdots s_{i_l}$ is said to be *reduced* if the length l is minimal; in this case, we write $l = \ell(w)$. The number of reduced expressions for w is denoted $r(w)$.

The (*left*) *weak ordering* of W is the partial order obtained by taking the transitive closure of the relations $x <_L s_i x$ whenever $\ell(x) < \ell(s_i x)$. Equivalently, one has

$$y \leq_L xy \Leftrightarrow \ell(xy) = \ell(x) + \ell(y)$$

for all $x, y \in W$. Note that $r(w)$ is the number of maximal chains in the weak order from the identity element to w.

A vector $\lambda \in V$ is said to be *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for $1 \leq i \leq n$. In that case, the stabilizer of λ is a parabolic subgroup of W; i.e., a subgroup generated by a subset of $\{s_1, \ldots, s_n\}$ (namely, $\{s_i : \langle \lambda, \alpha_i \rangle = 0\}$). Every W-orbit in V has at most one dominant member, and W is finite if and only if every orbit has a dominant member.

Every parabolic subgroup W' has the property that each left coset xW' has a unique element of minimum length, and these minimum-length representatives form an order ideal of $(W, <_L)$ (e.g., see Proposition 2.5 of [12]). When referring to the weak ordering of W/W', it is this order ideal that we have in mind.

2. Reflections in general Coxeter groups

It is well-known that every reflection in the Coxeter group W is of the form σ_β for some root $\beta \in \Phi^+$. Furthermore, since $w\sigma_\beta w^{-1} = \sigma_{w\beta}$, it follows that two reflections are conjugate in W if and only if the corresponding roots belong to the same W-orbit.

We define a directed graph on Φ by assigning edges

$$\beta \to \gamma \quad \text{when } \langle \beta, \alpha_j \rangle > 0 \quad \text{and} \quad s_j\beta = \gamma \text{ for some } j \in [n].$$

If we regard the edge $\beta \to s_j\beta$ as having label j (or s_j), then the connected components of this graph can be viewed as (oriented) Cayley graphs for the action of W on the various orbits of roots. For example, the graphs corresponding to A_4 and the two orbits in B_4 are illustrated in figure 1. We should caution the reader that in the general (infinite) case, an orbit of roots need not have a dominant member, so the Cayley graph of Φ lacks a number of features that are sometimes taken for granted in the finite case. (For example, see Remark 2.3 below).

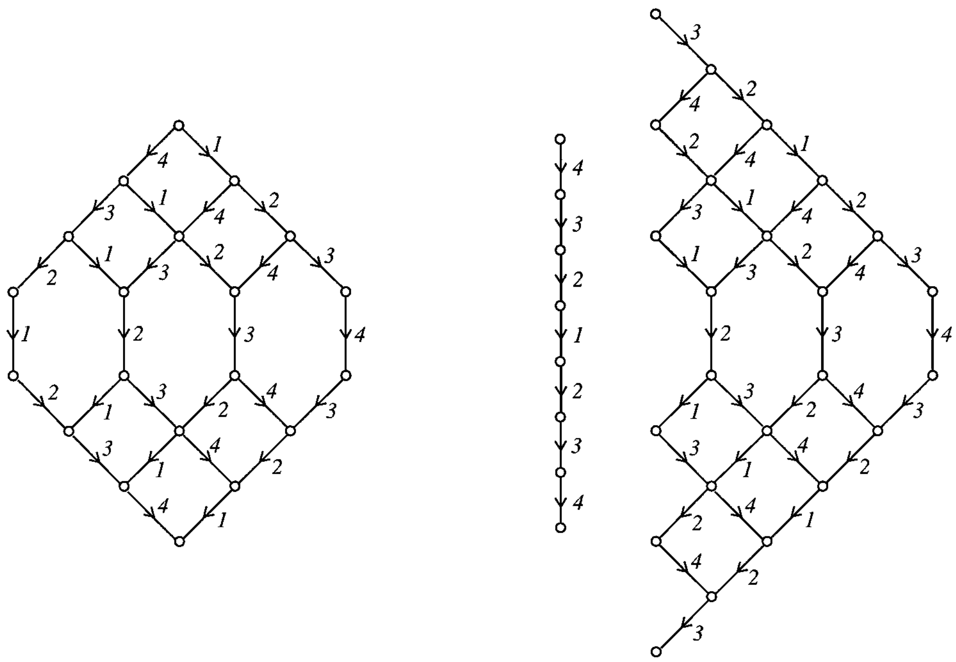

Figure 1. Cayley orderings of A_4 and B_4.

Note that if $\beta \to \gamma$ is an edge, then $\beta - \gamma$ is a positive multiple of a simple root, so the graph is acyclic. Also, the map $\beta \mapsto -\beta$ is an orientation-reversing automorphism.

If β is dominant, then $\langle \beta, \alpha_i \rangle \geq 0$ for all i, so β is a source (i.e., edges are directed away from β), and conversely. Since every W-orbit has at most one dominant member, it follows that each connected component of the graph has at most one source and one sink.

Given a root β, we define the *depth* of β to be $d(\beta) := \ell(\sigma_\beta)/2$ if β is positive, and $-\ell(\sigma_\beta)/2$ if β is negative. Thus a root is simple if and only if it has depth 1/2.

Proposition 2.1 *If $\beta \to \gamma$, then $d(\beta) - d(\gamma) = 1$.*

Proof: Replacing β and γ with $-\gamma$ and $-\beta$ if necessary, we may assume that β is positive. We must have $\gamma = s_j\beta$ and $\langle \beta, \alpha_j \rangle > 0$ for some j. Setting $t = \sigma_\beta$, we claim that $t\alpha_j$ must be negative. If $\beta = \alpha_j$ this is clear. Otherwise, β must be supported on at least one simple root distinct from α_j, so $t\alpha_j = \alpha_j - \langle \alpha_j, \beta^\vee \rangle \beta$ cannot be in the positive linear span of the simple roots and hence must be negative.

Given the claim, it follows that $\ell(ts_j) < \ell(t)$ [5, 5.4], so there is a reduced expression $t = s_{i_1} \cdots s_{i_l}$ that ends with s_j. Since $t = t^{-1}$, we may reverse the expression and assume $i_1 = j$. By the Exchange Property [5, 5.8], we must have $t = s_{i_1} \cdots \hat{s}_{i_k} \cdots s_{i_l} s_j$ for some k, where $\hat{}$ denotes omission. If $k > 1$, then we have obtained a reduced expression that starts and ends with s_j, so we have $d(\beta) - d(\gamma) = (\ell(t) - \ell(s_j t s_j))/2 = 1$. Otherwise, $k = 1$

and $s_j t s_j = t$. In that case, the reflections corresponding to β and $s_j\beta$ are the same; i.e., $s_j\beta = \pm\beta$. However, $\langle\beta, \alpha_j\rangle > 0$, so this is possible only if $\beta = \alpha_j$ and $\gamma = -\alpha_j$. □

Corollary 2.2
(a) *Every (directed) path from β to γ has length $d(\beta) - d(\gamma)$.*
(b) *If $\beta \to s_{i_1}\beta \to s_{i_2}s_{i_1}\beta \to \cdots \to s_{i_l}\cdots s_{i_1}\beta$, then $s_{i_l}\cdots s_{i_1}$ is reduced.*

Proof: Part (a) follows immediately. For (b), note that the application of *any* sequence of reflections to a root β traces a path in the unoriented Cayley graph, possibly remaining stationary at certain moments. Thus if the expression failed to be reduced, then there would be a shorter (undirected) path from β to $\gamma = s_{i_l}\cdots s_{i_1}\beta$. However, Proposition 2.1 shows that each step in the path can decrease depth by at most 1, so we cannot reach γ in fewer steps whether or not the path respects the orientation. □

It follows from Corollary 2.2(a) that the edges of the Cayley graph form the covering pairs of a partial ordering with rank function $d(\cdot)$. We call this the *Cayley ordering* of Φ, and denote it by $<_C$. Thus $\beta \leq_C \gamma$ if there is a directed path from γ to β. Furthermore, if $\beta \leq_C \gamma$, then $d(\gamma) - d(\beta)$ is the minimum length among all elements w such that $w\gamma = \beta$.

Remark 2.3 Given $\beta \leq_C \gamma$, there need not be a *unique* element of minimum length such that $w\gamma = \beta$. For example, consider the affine Weyl group of type A_2, an infinite Coxeter group with relations $(s_1s_2)^3 = (s_2s_3)^3 = (s_3s_1)^3 = 1$. It is not hard to show that if $x = s_3s_1s_2s_3s_1$, then x and x^{-1} are distinct elements of minimum length that map $3\alpha_1 + 2\alpha_2 + 3\alpha_3$ (a root of depth $11/2$) to α_2.

Given $\beta \in \Phi^+$, we define s_i to be a *center* of the reflection σ_β if $\alpha_i \leq_C \beta$. Every reflection has at least one center, since there are no sinks among the positive roots. An element $x \in W$ of length $d(\beta) - d(\alpha_i) = (\ell(\sigma_\beta) - 1)/2$ such that $x\beta = \alpha_i$ is said to be an *agent* of σ_β for the center s_i. The example in the previous remark shows that a reflection can have more than one agent for a given center.

The following result is a generalization of the well-known fact that every reflection has a palindromic reduced expression.

Proposition 2.4 *Every reduced expression for a reflection t is obtained by inserting s_i between reduced expressions for x^{-1} and x, where s_i is a center for t and x is an agent of t for s_i. In particular, $r(t) = \sum r(x)^2$, where x ranges over all agent s of t.*

Proof: Assume $t = \sigma_\beta$ ($\beta \in \Phi^+$), and consider the path in the Cayley graph traced by a reduced expression $t = s_{i_1}\cdots s_{i_l}$, starting at β. This path terminates at $t\beta = -\beta$, at distance $2d(\beta) = \ell(t)$ from β, so each step in the path must decrease depth by 1; i.e., the path is a maximal chain in the Cayley ordering. In particular, after $d = (\ell(t) - 1)/2$ steps, the path reaches a simple root α_i, then s_i is applied, and then the path travels from $-\alpha_i$ to $-\beta$ in the final d steps. It follows that $x = s_{i_{d+2}}\cdots s_{i_l}$ and $y = s_{i_d}\cdots s_{i_1}$ are both agents of t for the center s_i, and $t = y^{-1}s_ix$. However, $x\beta = \alpha_i$ implies $t = x^{-1}s_ix$, so $x = y$. □

One knows (e.g., [2, IV.1.5]) that any reduced expression for an element $w \in W$ can be obtained from any other by means of a sequence of braid relations; i.e., replacing a subword $s_i s_j s_i \cdots$ of length m with $s_j s_i s_j \cdots$, where m denotes the order of $s_i s_j$ in W.

Lemma 2.5 *Let t be a reflection. If two reduced expressions for t having centers s_i and s_j and agents x and y differ by the application of a single braid relation, then either $x = y$ and $s_i = s_j$, or $y = (s_i s_j)^k x$ and $s_i s_j$ has order $2k + 1$.*

Proof: Assume that $t = s_{i_1} \cdots s_{i_l}$ is a reduced expression with center s_i and corresponding agent $x = s_{i_d} \cdots s_{i_1}$, where $d = (l - 1)/2$. If the braid relation involves the first or last d positions, then there is clearly no effect on the center or the agent. Otherwise, suppose that the relation involves changing the center from s_i to s_j, as well as changing the adjacent k terms to the right and $m - k - 1$ terms to the left (a total of m terms). It must be the case that $m = 2k + 1$, for if (say) $k > m - k - 1$, then we could replace the first d terms with another reduced expression for x^{-1} that ends with an $s_i s_j$-subword of length k, and hence obtain a reduced expression for t that contains an $s_i s_j$-subword of length $2k + 1 > m$, a contradiction. Thus the center of the reduced expression must also be at the center of the braid relation, and the first k terms of $s_{i_d} \cdots s_{i_1}$ change from $s_j s_i \cdots$ to $s_i s_j \cdots$, which corresponds to multiplication by $(s_i s_j)^k$. □

We define the *transformation graph* of a reflection t to be the graph G with a vertex i for each center s_i, and an edge between i and j if t has a pair of reduced expressions with centers s_i and s_j that differ by the application of a single braid relation. The above result shows that G is a connected subgraph of 'Γ mod 2', the graph obtained by deleting all edges of the Coxeter graph having even (or infinite) weight.

Theorem 2.6 *Given a reflection $t = \sigma_\beta (\beta \in \Phi^+)$, the following are equivalent.*
(a) *The transformation graph of t is acyclic.*
(b) *Each center of t has a unique agent.*
(c) *The reflection t is the unique element of W of minimum length such that $t\beta = -\beta$.*
(d) *The number of reduced expressions for t is the number of maximal chains from β to $-\beta$ in the Cayley ordering.*
(e) *The Cayley ordering of $\{\alpha \in \Phi : -\beta \leq_C \alpha \leq_C \beta\}$ is isomorphic to the weak ordering of $\{w \in W : 1 \leq_L w \leq_L t\}$.*
In particular, these conditions hold if Γ mod 2 is acyclic. (This includes when W is finite.)

Proof: We first prove the equivalence of (b)–(e), and then (a) and (b).

(e) ⇒ (d) It is clear from the definition that the maximal chains from 1 to t in the weak ordering are in one-to-one correspondence with reduced expressions for t. Given an isomorphism with the Cayley interval from $-\beta$ to β, (d) follows.

(d) ⇒ (b) Every reduced expression for t determines a unique maximal chain from β to $-\beta$, since the depth must drop by 1 at each step. If there were two agents x and y for the center s_i, then a reduced expression for $y^{-1} s_i x$ would yield another maximal chain, contradicting (d).

(b) ⇒ (c) If $w\beta = -\beta$ and $\ell(w) = 2d(\beta)$, then the same reasoning used in the proof of Proposition 2.4 shows that $w = y^{-1}s_i x$, where x and y are agents for some center s_i. However if s_i has a unique agent, then $x = y$ and $w = x^{-1}s_i x = t$.

(c) ⇒ (e) We claim that the map $w \mapsto -w\beta$ is an order isomorphism between the two intervals. Indeed, for any $w \leq_L t$, we may obtain a reduced expression for t by prepending terms to any reduced expression for w, so $w\beta$ appears along some maximal chain between β and $-\beta$; i.e., $-\beta \leq_C -w\beta \leq_C \beta$ and $\ell(w) = d(\beta) - d(w\beta)$. Conversely, any root α (assumed positive, say) between $-\beta$ and β appears along a maximal chain from β to some simple root α_i, and hence appears as the trailing portion of a reduced expression for an agent of t (as well as t itself), so the map is surjective. To prove injectivity, note that if $w_1\beta = \alpha = w_2\beta$ and $w_1, w_2 \leq_L t$, then $\ell(w_1) = d(\beta) - d(\alpha) = \ell(w_2)$ and

$$\ell\big(tw_1^{-1}\big) + \ell(w_1) = \ell(t) = \ell\big(tw_2^{-1}\big) + \ell(w_2).$$

Thus $tw_1^{-1}w_2$ is an element of length at most $\ell(t)$ that sends β to $-\beta$, hence (c) implies $w_1 = w_2$. Finally, having established that the map is bijective and rank-preserving, it must be an order isomorphism, since two elements of either order form a covering pair (in some direction) if and only if they differ by a simple reflection.

(a) ⇒ (b) If there were two agents for some center, then there would exist a sequence of braid relations that generate (via the centers) a closed path in the transformation graph G. If a portion of this path travels from s_i to s_j and then back to s_i, then the corresponding agent changes from (say) x to $(s_is_j)^k x$ back to x, by Lemma 2.5. Thus we may "contract" this part of the path and still have a valid braid sequence. Given that G is acyclic, the entire path can be contracted to a point, so the agents corresponding to the endpoints of the original path must have been the same.

(b) ⇒ (a) Define x_i to be the (unique) agent for the center s_i. If i and j are adjacent in the transformation graph G, then Lemma 2.5 implies $x_j = (s_is_j)^k x_i$, where $2k+1$ denotes the order of s_is_j in W. It follows that if there were a circuit in G, then there would be relations in W of either of the equivalent forms

$$\begin{aligned}&\big(s_{i_l}s_{i_1}\big)^{k_l}\cdots\big(s_{i_2}s_{i_3}\big)^{k_2}\big(s_{i_1}s_{i_2}\big)^{k_1} = 1,\\ &\big(s_{i_l}s_{i_1}\big)^{k_l}\cdots\big(s_{i_3}s_{i_4}\big)^{k_3} = \big(s_{i_2}s_{i_1}\big)^{k_1}\big(s_{i_3}s_{i_2}\big)^{k_2},\end{aligned}$$

where $i_1, \ldots, i_l$ are distinct and $2k_r + 1$ denotes the order of $s_{i_r}s_{i_{r+1}}$ in W. Note that s_{i_2} does not occur on the left side of the second relation, so the right expression cannot be reduced. However, there are no opportunities to apply any braid relations other than interchanging the innermost s_{i_1} and s_{i_3} (assuming they commute), so the right expression is reduced (in fact "fully commutative" in the sense of [12]). □

Example 2.7 In the affine Weyl group of type A_2, the reflection corresponding to the root $2\alpha_1 + \alpha_2 + 2\alpha_3$ has centers s_1, s_2 and s_3, with corresponding agents $s_2s_3s_1$, $s_1s_3s_1 = s_3s_1s_3$ and $s_2s_1s_3$. The transformation graph includes the edges $\{1, 2\}$ and $\{2, 3\}$, but not $\{1, 3\}$. Thus the transformation graph may be acyclic even when the corresponding induced subgraph of Γ mod 2 has a circuit.

Remark 2.8 It can be difficult to determine the set of centers of a given reflection σ_β without exhaustive calculation. Clearly, if s_i is a center of σ_β, then α_i must occur in the support of β (or equivalently, s_i must appear in every reduced expression for σ_β) and belong to the same W-orbit. In the next section, we will see that these necessary conditions are sufficient if W is finite and crystallographic. On the other hand, in the (non-crystallographic) dihedral group $I_2(5)$, the reflection $t = s_1s_2s_1$ has only one reduced expression. Similarly, in the affine Weyl group of type A_2, the reflection corresponding to $\alpha_1 + 2\alpha_2 + \alpha_3$ has only two centers: s_1 and s_3.

3. The longest short reflection

Henceforth, we restrict our attention to Coxeter groups and root systems that are finite, irreducible, and crystallographic. In such cases, there are at most two orbits of roots, distinguishable by their length ("short" and "long"). If there is only one orbit, we consider the roots to be short by convention. We say that a reflection is short or long according to the status of the corresponding root. In the co-root system, the roles of long and short are interchanged, so there is no loss of generality in studying only short reflections.

The main advantage in using short reflections is contained in the following basic result. For a proof, see [2, VI.1.3], or simply analyze the root systems of rank two.

Lemma 3.1 *Given $\alpha, \beta \in \Phi$ with α short, we have $-2 \leq \langle \alpha, \beta^\vee \rangle \leq 2$, with equality if and only if $\alpha = \pm\beta$.*

Let $h(\beta) = c_1 + \cdots + c_n$ denote the height of the root $\beta = c_1\alpha_1 + \cdots + c_n\alpha_n$. Since Φ is assumed to be crystallographic, the coordinates c_i and the height are integers. In the following, Φ_s^+ denotes the set of short positive roots.

Proposition 3.2 *If Φ is finite and crystallographic, then the Cayley ordering of Φ_s^+ is identical to the standard ordering of Φ_s^+, and $h(\cdot)$ is a rank function on $(\Phi_s^+, <_C)$.*

Proof: Since $\beta \to \gamma$ only if $\beta - \gamma$ is a positive multiple of a simple root, it is clear that $\gamma \leq_C \beta$ implies $\gamma \leq \beta$. Conversely, suppose $\beta, \gamma \in \Phi_s^+$ and $\gamma < \beta$. It follows that $\beta - \gamma = \sum c_i\alpha_i$ for certain integers $c_i \geq 0$. Since $\langle \beta - \gamma, \beta - \gamma \rangle > 0$, we must have $\langle \beta - \gamma, \alpha_i \rangle > 0$ for some i such that $c_i > 0$. Hence $\langle \beta, \alpha_i \rangle > 0$ or $\langle \gamma, \alpha_i \rangle < 0$, so $s_i\beta <_C \beta$ or $\gamma <_C s_i\gamma$. However by Lemma 3.1, we must have $\langle \beta, \alpha_i^\vee \rangle = 1$ or $\langle \gamma, \alpha_i^\vee \rangle = -1$, so $\gamma \leq \beta - \alpha_i = s_i\beta <_C \beta$ or $\gamma <_C s_i\gamma = \gamma + \alpha_i \leq \beta$. By induction, it follows that there is a maximal chain from γ to β in the Cayley ordering, and the length of this chain is $h(\beta) - h(\gamma)$. □

Since s_i is a center of the reflection σ_β if and only if $\alpha_i \leq_C \beta$, we obtain the following.

Corollary 3.3 *If β is short, then s_i is a center of σ_β if and only if α_i is short and appears in the support of β.*

Let $\bar{\alpha}$ denote the unique dominant root in Φ_s. Since $\bar{\alpha}$ is the maximum element of the Cayley ordering of Φ_s, it follows that s_i is a center of $\sigma_{\bar{\alpha}}$ if and only if α_i is short. We define x_i to be the agent corresponding to s_i; i.e., the unique element of W of minimum length such that $x_i\bar{\alpha} = \alpha_i$.

If λ is a (dominant) integral weight, then $w \in W$ is "λ-minuscule" in the sense of Dale Peterson (see [7, 8] or [13]) if there is a reduced expression $w = s_{i_1} \cdots s_{i_l}$ such that each member of the sequence

$$\lambda,\ s_{i_l}\lambda,\ s_{i_{l-1}}s_{i_l}\lambda, \ldots,\ s_{i_1}\cdots s_{i_l}\lambda$$

differs from the previous one by the subtraction of a simple root.

In case $\lambda = \bar{\alpha}$ and $w = x_i$, the above weight sequence corresponds to a maximal chain in the Cayley ordering from $\bar{\alpha}$ to α_i. Since Proposition 3.2 shows that each step in a chain decreases the height by one, we obtain the following.

Corollary 3.4 *Each agent x_i is $\bar{\alpha}$-minuscule.*

The following result of Dale Peterson is unpublished, but a generalization will appear in a forthcoming paper of Peterson and Proctor.

Theorem 3.5 [Peterson, ca. 1989] *If λ is dominant and w is λ-minuscule, then*

$$r(w) = \ell(w)! \prod_{\beta\in\Psi} \frac{1}{h(\beta)},$$

where $\Psi = w\Phi^- \cap \Phi^+ = \{\beta \in \Phi^+ : w^{-1}\beta \in \Phi^-\}$.

The preceding results yield an explicit formula for the number of reduced expressions for $\sigma_{\bar{\alpha}}$, the longest short reflection. Equivalently, this is also the number of maximal chains in the weak ordering of the quasi-minuscule quotient of W corresponding to the orbit of short roots. First let us introduce the notation

$$\Psi_i := \{\beta \in \Phi^+ : \langle\alpha_i, \beta^\vee\rangle = -1\}$$

for each short simple root α_i.

Theorem 3.6 *We have $r(\sigma_{\bar{\alpha}}) = \sum N_i^2$ (summed over i such that α_i is short), where*

$$N_i = r(x_i) = (h(\bar{\alpha}) - 1)! \prod_{\beta\in\Psi_i} \frac{1}{h(\beta)}.$$

Proof: We know that $r(\sigma_{\bar{\alpha}}) = \sum r(x_i)^2$ (Proposition 2.4), and $\ell(x_i) = h(\bar{\alpha}) - h(\alpha_i)$ is the length of a maximal chain from $\bar{\alpha}$ to α_i (Proposition 3.2), so by Theorem 3.5, it suffices to show that $\Psi_i = x_i\Phi^- \cap \Phi^+$; i.e.,

$$\langle\alpha_i, \beta^\vee\rangle = -1 \quad \Leftrightarrow \quad x_i^{-1}\beta \in \Phi^-$$

for all positive roots β. For this, observe that since α_i and β are both positive, Lemma 3.1 shows that we may replace the condition $\langle \alpha_i, \beta^\vee \rangle = -1$ with $\langle \alpha_i, \beta \rangle < 0$.

Since $\langle \alpha_i, \beta \rangle = \langle x_i \bar{\alpha}, \beta \rangle = \langle \bar{\alpha}, x_i^{-1} \beta \rangle$ and $\bar{\alpha}$ is dominant, it is clear that if $\langle \alpha_i, \beta \rangle < 0$, then $x_i^{-1}\beta$ must be a negative root. Conversely, if $x_i^{-1}\beta$ is negative, then $\gamma = -x_i^{-1}\beta$ is positive and $x_i\gamma$ is negative. Hence $w\gamma$ must be negative for all $w \geq_L x_i$. This follows by induction from the fact that (i) if $\ell(s_i w) > \ell(w)$, then $w^{-1}\alpha_i \in \Phi^+$ [5, 5.4], and (ii) s_i permutes the positive roots other than α_i [5, 5.6]. In particular, since $\sigma_{\bar{\alpha}} \geq_L x_i$, we obtain that $\sigma_{\bar{\alpha}}\gamma = \gamma - \langle \gamma, \bar{\alpha}^\vee \rangle \bar{\alpha}$ is negative, so $\langle \gamma, \bar{\alpha} \rangle = -\langle \beta, \alpha_i \rangle > 0$. □

Example 3.7 Consider the case of D_n. The standard realization of the root system is $\{ \pm\varepsilon_i \pm \varepsilon_j : 0 \leq j < i < n\}$, where $\varepsilon_0, \ldots, \varepsilon_{n-1}$ are orthonormal. Choosing simple roots $\alpha_1 = \varepsilon_1 + \varepsilon_0$ and $\alpha_{i+1} = \varepsilon_i - \varepsilon_{i-1}$ ($i \geq 1$), the height of root $\varepsilon_i \pm \varepsilon_j$ is $i \pm j$. When $i \geq 1$,

$$\Psi_{i+1} = \{\varepsilon_j - \varepsilon_i : j > i\} \cup \{\varepsilon_j + \varepsilon_{i-1} : j > i\} \cup \{\varepsilon_{i-1} \pm \varepsilon_j : j < i-1\},$$

and the heights of these roots are $1, \ldots, n-i-1$, $2i, \ldots, n+i-2$, and $1, \ldots, 2i-3$ (with $i-1$ occurring twice). Taking particular care in the case $i = 1$, Theorem 3.6 yields

$$r(x_{i+1}) = N_{i+1} = \begin{cases} M_i & \text{if } i = 1, \\ 2M_i & \text{if } i > 1, \end{cases} \quad \text{where } M_i = \frac{2i-1}{2n-3}\binom{2n-3}{n-i-1}.$$

We also have $N_1 = N_2$, thanks to the diagram automorphism of D_n. The quantities M_i can be recognized as ballot numbers; namely, the number of minimum-length lattice paths in the region $R = \{(u, v) \in \mathbf{Z}^2 : u > v\}$ from $(1, 0)$ to $(n+i-2, n-i-1)$ [3, 1.8]. In particular, $M_1 = N_2 = N_1$ is the Catalan number $\mathcal{C}_{n-2}$. By symmetry, M_i^2 is the number of paths in R from $(1, 0)$ to $(2n-3, 2n-4)$ that pass through $(n+i-2, n-i-1)$, so $\sum M_i^2$ is the total number of paths in R from $(1, 0)$ to $(2n-3, 2n-4)$; i.e., the Catalan number $\mathcal{C}_{2n-4}$. We conclude that

$$\sum_{i=1}^{n} N_i^2 = 4\sum_{i=1}^{n-1} M_i^2 - 2M_1^2 = 4\mathcal{C}_{2n-4} - 2\mathcal{C}_{n-2}^2$$

is the number of reduced expressions for the longest reflection in D_n.

Tables listing the number of reduced expressions for $\sigma_{\bar{\alpha}}$ and each agent x_i are provided in the Appendix.

4. The smash theorem

Having counted the number of maximal chains in the Cayley ordering of a short orbit, it is natural to analyze the structure of the chains in more detail. The most obvious feature, evident from figure 1, is that every chain that passes through the simple root α_i must subsequently pass through $-\alpha_i$. Thus if we define an equivalence relation $\sim$ on Φ_s by declaring $\alpha_i \sim -\alpha_i$ for all (short) simple roots α_i, then the resulting "smashed" Cayley ordering (i.e., $(\Phi_s/\sim, <_C)$) has the same number of maximal chains as the original.

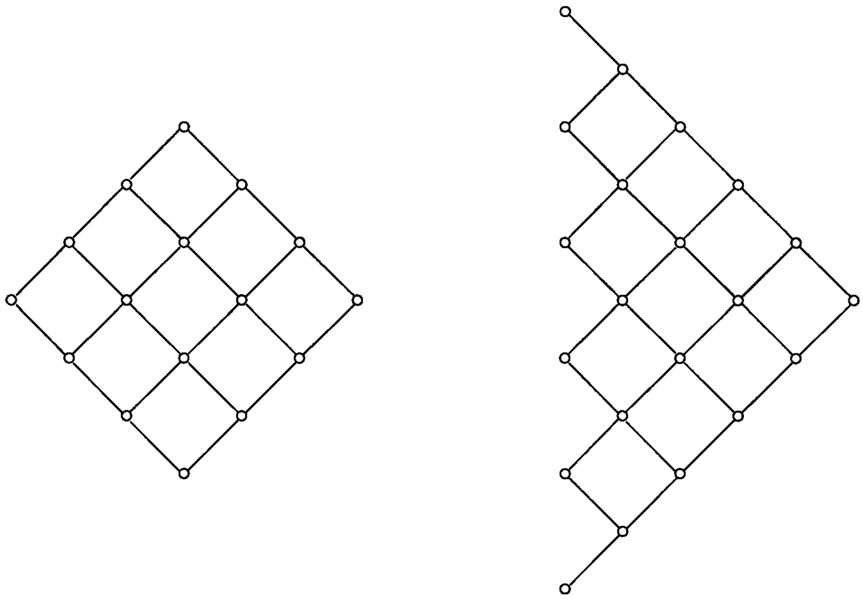

Figure 2. Smashed Cayley orderings for A_4 and C_4.

For example, in figure 2 we have illustrated the smashed orderings for A_4 and the short orbit of C_4. These two examples make it clear that the number of reduced expressions for the longest short reflection is $\binom{2n-2}{n-1}$ in A_n and $\mathcal{C}_{2n-3}$ in C_n.

Continuing the finite and crystallographic hypotheses, our aim in this section is to prove that the smashed Cayley ordering is a distributive lattice, and thus representable as the lattice of order ideals of a suitable poset. As noted in the introduction, this is analogous to similar results of Proctor for minuscule quotients of finite Weyl groups [6].

Our starting point is the inversion-set representation of the weak ordering of a Coxeter group. Continuing the notation of [13], we let

$$\Phi^\vee(w) := \{\gamma^\vee : \gamma \in \Phi^+,\ w\gamma \in \Phi^-\}$$

denote the inversion set of $w \in W$. The use of co-roots here, rather than roots, turns out to be crucial.

The following folkloric result is valid for general Coxeter groups. An equivalent result is stated by Björner in Proposition 2 of [1].

Lemma 4.1 *For all $x, y \in W$, we have $x \leq_L y$ if and only if $\Phi^\vee(x) \subseteq \Phi^\vee(y)$.*

Proof: Consider a covering pair in the weak order; say $x <_L s_i x$. Since s_i permutes the positive roots other than α_i, it follows that every co-root $\gamma^\vee \in \Phi^\vee(x)$ is also in $\Phi^\vee(s_i x)$, except possibly when $\gamma = -x^{-1}\alpha_i$. However, $x <_L s_i x$ (i.e., $\ell(x) < \ell(s_i x)$) is equivalent to $x^{-1}\alpha_i$ being positive. Thus $x <_L s_i x$ implies $\Phi^\vee(s_i x) = \Phi^\vee(x) \cup \{x^{-1}\alpha_i^\vee\}$, and more generally, $x \leq_L y$ implies $\Phi^\vee(x) \subseteq \Phi^\vee(y)$.

Conversely, suppose $\Phi^\vee(x) \subseteq \Phi^\vee(y)$. If $x = 1$ the result is trivial, so assume $\ell(x) > 1$ and choose a generator s_i that appears at the end of a reduced expression for x. In that case, $\alpha_i^\vee \in \Phi^\vee(x)$, hence $\alpha_i^\vee \in \Phi^\vee(y)$, so there is also a reduced expression for y that ends with s_i. Again making use of the fact that s_i permutes $\Phi^+ - \{\alpha_i\}$, it follows that

$$\Phi^\vee(xs_i) = s_i\big(\Phi^\vee(x) - \{\alpha_i^\vee\}\big) \subseteq s_i\big(\Phi^\vee(y) - \{\alpha_i^\vee\}\big) = \Phi^\vee(ys_i).$$

By induction, we obtain $xs_i \leq_L ys_i$, and hence $x \leq_L y$. □

The following result characterizes inversion sets. There are similar characterizations available for any Coxeter group (cf. Proposition 3 of [1] in the finite case), however the one we give here is specific to the finite crystallographic case.

Lemma 4.2 *Given $\Psi \subseteq (\Phi^\vee)^+$, we have $\Psi = \Phi^\vee(w)$ for some $w \in W$ if and only if for all triples of positive co-roots $\alpha^\vee, \beta^\vee, \alpha^\vee + \beta^\vee$, we have*
(i) *$\alpha^\vee, \beta^\vee \in \Psi$ implies $\alpha^\vee + \beta^\vee \in \Psi$, and*
(ii) *$\alpha^\vee + \beta^\vee \in \Psi$ implies $\alpha^\vee \in \Psi$ or $\beta^\vee \in \Psi$.*

Proof: The necessity of these conditions follows from the fact that if $w\alpha^\vee$ and $w\beta^\vee$ are both positive (or both negative), then the same must be true of $w(\alpha^\vee + \beta^\vee)$. For sufficiency, assume Ψ is nonempty and satisfies (i) and (ii). Choose $\gamma^\vee \in \Psi$ of minimum height. If γ is not simple, then by choosing a simple root α_i satisfying $\langle \alpha_i, \gamma^\vee\rangle > 0$, we obtain that $\gamma^\vee - \alpha_i^\vee$ is a co-root [2, VI.1.3]. Hence (ii) implies $\alpha_i^\vee \in \Psi$ or $\gamma^\vee - \alpha_i^\vee \in \Psi$. Either way we contradict the minimality of $\gamma^\vee$, so we have (say) $\gamma = \alpha_i$.

We claim that $\Psi' := s_i(\Psi - \{\alpha_i^\vee\})$ satisfies (i) and (ii). Given a triple $\alpha^\vee, \beta^\vee, \alpha^\vee + \beta^\vee$ that does not involve $\alpha_i^\vee$, this is clear, since s_i permutes $\Phi^+ - \{\alpha_i\}$. Otherwise, if (say) $\alpha = \alpha_i$, then (i) is vacuous and in (ii), if $\alpha_i^\vee + \beta^\vee \in \Psi'$, then $s_i\beta^\vee - \alpha_i^\vee \in \Psi - \{\alpha_i^\vee\}$, so $s_i\beta^\vee \in \Psi - \{\alpha_i^\vee\}$ (using (i) for Ψ), so $\beta^\vee \in \Psi'$, proving the claim.

By induction on $|\Psi|$, it follows that $\Phi^\vee(w) = \Psi' = s_i(\Psi - \{\alpha_i^\vee\})$ for some $w \in W$. Hence $w\alpha_i^\vee$ is positive and $\Phi^\vee(ws_i) = \Psi$ (cf. the proof of Lemma 4.1). □

Since we know that the Cayley ordering of Φ_s is isomorphic to the weak ordering of W below $\sigma_{\tilde{\alpha}}$ (Theorem 2.6), the previous two lemmas provide a representation of the Cayley ordering as a family of subsets of $\Phi^\vee(\sigma_{\tilde{\alpha}})$, partially ordered by inclusion. More generally, the subinterval of $(\Phi_s, <_C)$ from $-\alpha$ to α is representable in terms of subsets of $\Phi^\vee(\sigma_\alpha)$.

Lemma 4.3 *If $\alpha \in \Phi^+$ is short, then $\Phi^\vee(\sigma_\alpha) = \{\alpha^\vee\} \cup \Phi_\alpha^\vee$, where*

$$\Phi_\alpha^\vee := \{\beta^\vee \in (\Phi^\vee)^+ : \beta^\vee \leq \alpha^\vee, \langle \alpha, \beta^\vee\rangle = 1\} = \{\beta^\vee \in (\Phi^\vee)^+ : \alpha^\vee - \beta^\vee \in (\Phi^\vee)^+\}.$$

Proof: We have $\sigma_\alpha\beta^\vee = \beta^\vee - \langle\alpha, \beta^\vee\rangle\alpha^\vee$, so $\sigma_\alpha\beta^\vee$ cannot be negative unless $\langle\alpha, \beta^\vee\rangle > 0$, whence $\langle\alpha, \beta^\vee\rangle = 1$ or $\alpha = \beta$, by Lemma 3.1. Conversely, if $\beta^\vee \leq \alpha^\vee$ and $\langle\alpha, \beta^\vee\rangle = 1$, then $\sigma_\alpha\beta^\vee = \beta^\vee - \alpha^\vee$ is negative.

That the two definitions of $\Phi_\alpha^\vee$ are equivalent can be seen from the fact that if $\beta^\vee$ and $\alpha^\vee - \beta^\vee$ are both positive co-roots, then neither can equal $\alpha^\vee$, so $\langle\alpha, \beta^\vee\rangle \leq 1$ and

$\langle\alpha, \alpha^\vee - \beta^\vee\rangle \leq 1$ by Lemma 3.1. On the other hand, $\langle\alpha, \beta^\vee\rangle + \langle\alpha, \alpha^\vee - \beta^\vee\rangle = 2$, so $\langle\alpha, \beta^\vee\rangle = \langle\alpha, \alpha^\vee - \beta^\vee\rangle = 1$. □

Henceforth, we assume that α is a short positive root.

We remark that the interval $\{w \in W : w \leq_L \sigma_\alpha\}$ has an order-reversing involution given by $w \mapsto w\sigma_\alpha$. Indeed, since $w \leq_L \sigma_\alpha$ if and only if $\ell(\sigma_\alpha w^{-1}) + \ell(w) = \ell(\sigma_\alpha)$, it follows easily that $w \leq_L \sigma_\alpha$ implies $w\sigma_\alpha \leq_L \sigma_\alpha$, and

$$\Phi^\vee(w\sigma_\alpha) = \Phi^\vee(\sigma_\alpha) - \{-\sigma_\alpha\beta^\vee : \beta^\vee \in \Phi^\vee(w)\}. \tag{4.1}$$

This involution corresponds (via the isomorphism $w \mapsto -w\alpha$) to the map $\beta \mapsto -\beta$ on the Cayley interval $-\alpha \leq_C \beta \leq_C \alpha$.

Lemma 4.4 *If $\beta^\vee, \gamma^\vee \in \Phi^\vee_\alpha$, then $\beta^\vee - \gamma^\vee \in \Phi^\vee \cup \{0\}$ or $\alpha^\vee - \beta^\vee - \gamma^\vee \in \Phi^\vee \cup \{0\}$.*

Proof: We have $\langle\gamma, \beta^\vee\rangle + \langle\gamma, \alpha^\vee - \beta^\vee\rangle = \langle\gamma, \alpha^\vee\rangle > 0$, so $\langle\gamma, \beta^\vee\rangle > 0$ or $\langle\gamma, \alpha^\vee - \beta^\vee\rangle > 0$. In the former case, either $\beta^\vee - \gamma^\vee$ is a co-root or $\beta^\vee = \gamma^\vee$ [2, VI.1.3]. In the latter case, similar reasoning implies that $\alpha^\vee - \beta^\vee - \gamma^\vee$ is a co-root or $\gamma^\vee = \alpha^\vee - \beta^\vee$. □

Lemma 4.5 *If $w \leq_L \sigma_\alpha$ and $\alpha^\vee \notin \Phi^\vee(w)$, then $\Phi^\vee(w)$ is an order ideal of $\Phi^\vee_\alpha$ relative to the standard order.*

Our proof of this lemma is postponed to the next section; however, it is worth noting here that the special case $\alpha = \tilde{\alpha}$ can be handled easily. Indeed, it follows from Proposition 5.1 of [13] and Lemma 4.3 that w is α-minuscule, and Remark 5.6(a) of [13] shows if λ is dominant, then the inversion set of any λ-minuscule element is an order ideal of $\{\beta^\vee : \langle\lambda, \beta^\vee\rangle = 1\}$ relative to the standard order.

Theorem 4.6 *The smashed Cayley ordering of $\{\beta \in \Phi_s : -\alpha \leq_C \beta \leq_C \alpha\}$ is isomorphic to the lattice of order ideals of the standard ordering of $\Phi^\vee_\alpha$. In particular, $(\Phi_s/\sim, <_C) \cong J(\Phi^\vee_{\tilde{\alpha}}, <)$.*

Proof: For each $w \leq_L \sigma_\alpha$, define $\phi(w) = \Phi^\vee(w) - \{\alpha^\vee\} \subseteq \Phi^\vee_\alpha$. We claim that $\phi(w)$ is an order ideal of $\Phi^\vee_\alpha$ relative to the standard ordering. If $\alpha^\vee \notin \Phi^\vee(w)$, this is the assertion of Lemma 4.5. Otherwise, we have $\alpha^\vee \in \Phi^\vee(w)$ and $\alpha^\vee \notin \Phi^\vee(w\sigma_\alpha)$, so Lemma 4.5 shows that $\phi(w\sigma_\alpha)$ is an order ideal of $\Phi^\vee_\alpha$. However, $\phi(w\sigma_\alpha) = \{\alpha^\vee - \beta^\vee : \beta^\vee \in \Phi^\vee_\alpha - \phi(w)\}$ by (4.1), so $\{\alpha^\vee - \beta^\vee : \beta^\vee \in \phi(w)\}$ is an order filter and $\phi(w)$ is an order ideal.

Given an order ideal Ψ of $\Phi^\vee_\alpha$, we define $\bar{\Psi} = \Psi \cup \{\alpha^\vee\}$ if there exists a pair $\beta^\vee, \gamma^\vee \in \Psi$ such that $\beta^\vee + \gamma^\vee = \alpha^\vee$, and set $\bar{\Psi} = \Psi$ otherwise. We claim that $\bar{\Psi}$ satisfies the criterion of Lemma 4.2. Indeed, suppose $\beta^\vee, \gamma^\vee, \beta^\vee + \gamma^\vee$ are positive co-roots. If $\beta^\vee, \gamma^\vee \in \bar{\Psi}$, then we have $\langle\alpha, \beta^\vee\rangle, \langle\alpha, \gamma^\vee\rangle \geq 1$, so $\langle\alpha, \beta^\vee + \gamma^\vee\rangle \geq 2$. This forces $\beta^\vee + \gamma^\vee = \alpha^\vee$ and hence $\beta^\vee + \gamma^\vee \in \bar{\Psi}$ by construction. Conversely, suppose $\beta^\vee + \gamma^\vee \in \bar{\Psi}$. If $\beta^\vee + \gamma^\vee \neq \alpha^\vee$, then $\langle\alpha, \beta^\vee\rangle + \langle\alpha, \gamma^\vee\rangle = 1$ by Lemma 4.3, whence $\langle\alpha, \beta^\vee\rangle = 1$ and $\langle\alpha, \gamma^\vee\rangle = 0$ or vice-versa, by Lemma 3.1. Assuming the former, we have $\beta^\vee \in \Phi^\vee_\alpha$ and hence $\beta^\vee \in \Psi$, since Ψ is an order ideal and $\beta^\vee < \beta^\vee + \gamma^\vee$. The remaining possibility is that $\beta^\vee + \gamma^\vee = \alpha^\vee$. In that case, we must have $\beta^\vee, \gamma^\vee \in \Phi^\vee_\alpha$ by Lemma 4.3, and since $\alpha^\vee \in \bar{\Psi}$ in this case, there must (by the definition of $\bar{\Psi}$) be a pair $\beta^\vee_0, \gamma^\vee_0 \in \Psi$ such that $\beta^\vee_0 + \gamma^\vee_0 = \alpha^\vee$. We seek to establish that $\beta^\vee \in \Psi$ or $\gamma^\vee \in \Psi$. If $\beta = \beta_0$ or $\beta = \gamma_0$, this is clear. Otherwise, Lemma 4.4 implies that $\beta^\vee_0 - \beta^\vee = \gamma^\vee - \gamma^\vee_0$ or $\gamma^\vee_0 - \beta^\vee = \gamma^\vee - \beta^\vee_0$ must be a co-root. Since Ψ is an order ideal, it follows that if either is positive, we obtain $\beta^\vee \in \Psi$; if either is negative, we obtain $\gamma^\vee \in \Psi$ as desired. Having established the claim, it now follows that $\bar{\Psi} = \Phi^\vee(w)$ and $\Psi = \phi(w)$ for some $w \in W$, and we must have $w \leq_L \sigma_\alpha$ by Lemmas 4.1 and 4.3.

We have now shown that ϕ is a map from $\{w \in W : w \leq_L \sigma_\alpha\}$ onto $J(\Phi^\vee_\alpha, <)$. Moreover, ϕ is clearly order-preserving, since Lemma 4.1 shows that $x \leq_L y$ implies $\Phi^\vee(x) \subseteq \Phi^\vee(y)$, and hence $\phi(x) \subseteq \phi(y)$. To complete the proof, it suffices to show that $\phi(x) \subseteq \phi(y)$ implies either $x \leq_L y$ or that x and y correspond to elements that have been identified in the smashed Cayley ordering. Recall that the isomorphism between the weak ordering and the Cayley ordering is $w \mapsto -w\alpha$ (see the proof of Theorem 2.6), so the latter case occurs when $x\alpha = -\alpha_i$ and $y\alpha = \alpha_i$ for some i.

Now if $\phi(x) \subseteq \phi(y)$, then either $\Phi^\vee(x) \subseteq \Phi^\vee(y)$ (whence $x \leq_L y$, by Lemma 4.1), or else we claim that $\phi(x) = \phi(y)$ and $\Phi^\vee(x) = \Phi^\vee(y) \cup \{\alpha^\vee\}$. Indeed, if the former case does not hold, then $\alpha^\vee \in \Phi^\vee(x)$ and $\alpha^\vee \notin \Phi^\vee(y)$. So if there were any $\beta^\vee \in \phi(y) - \phi(x)$, then we would have $\alpha^\vee - \beta^\vee \in \phi(x)$ (Lemma 4.2), hence $\alpha^\vee - \beta^\vee \in \phi(y)$, which forces $\alpha^\vee \in \Phi^\vee(y)$ (Lemma 4.2 again), a contradiction. Having established $\Phi^\vee(x) = \Phi^\vee(y) \cup \{\alpha^\vee\}$, it now follows that $y <_L x = s_i y$ for some i (Lemma 4.1), and hence $\Phi^\vee(x) = \Phi^\vee(y) \cup \{y^{-1}\alpha^\vee_i\}$; i.e., $y\alpha = \alpha_i$ and $x\alpha = -\alpha_i$, so the proof is complete. □

Since $\alpha^\vee > \beta^\vee$ for all $\beta^\vee \in \Phi^\vee_\alpha$, the number of maximal chains in $J(\Phi^\vee_\alpha, <)$ (or equivalently, linear extensions of $(\Phi^\vee_\alpha, <)$) is unaffected by the addition of $\alpha^\vee$. Since maximal chains in the Cayley order correspond to reduced expressions, we obtain the following.

Corollary 4.7 *The number of reduced expressions for any short reflection t equals the number of linear extensions of $(\Phi^\vee(t), <)$.*

The standard ordering of $(\Phi^\vee_{\bar{\alpha}}, <)$ (i.e., the poset of join-irreducibles for the smashed Cayley ordering of Φ_s) is displayed in figure 3 for each of D_5, E_6, and F_4.

Remark 4.8 (a) The dominant case of Theorem 4.6 (i.e., the special case $\alpha = \bar{\alpha}$) occurs in some unpublished notes of Proctor, with a different proof. Also, the dominant case of Corollary 4.7 is mentioned (without proof) at the end of Section 11 in [6].

(b) Once the dominant case of Theorem 4.6 is established, it follows immediately that any subinterval of the smashed Cayley order, such as from $-\alpha$ to α, is isomorphic to the lattice of order ideals of some convex subposet of $(\Phi^\vee_{\bar{\alpha}}, <)$. What is not clear *a priori*, and is perhaps even surprising, is that this subposet is isomorphic to $(\Phi^\vee_\alpha, <)$.

It would be interesting to investigate the extent to which Corollary 4.7 generalizes to Weyl group elements that are not reflections; i.e., identify those $w \in W$ for which $r(w)$ equals the number of linear extensions of $(\Phi^\vee(w), <)$. We say that such elements are *inversion-orderable*.

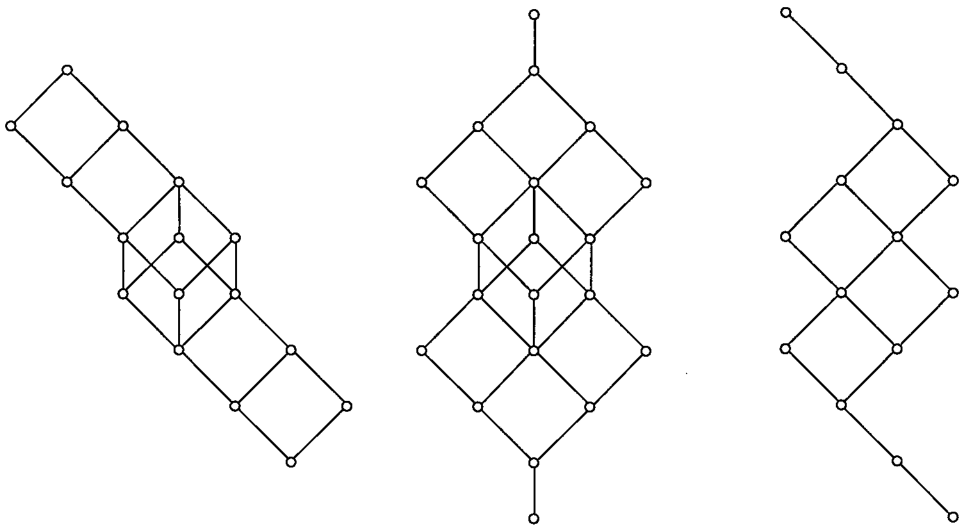

Figure 3. $(\Phi^\vee_{\tilde\alpha}, <)$ for D_5, E_6, and F_4.

This is somewhat related to Theorem 3.2 of [12], which shows that w is fully commutative if and only if one may construct a partial ordering $P = (X, <)$ and a labeling of the elements of P by simple reflections s_i so that the words corresponding to the linear extensions of P are the reduced expressions for w. In contrast, reflections are rarely fully commutative, and here our only requirement is that the linear extensions and reduced expressions should be equinumerous.

This is also somewhat related to the notion of "vexillary" permutations in the symmetric group. If w is vexillary, then $r(w)$ is the number of standard Young tableaux of some shape of size $\ell(w)$ (Corollary 4.2 of [9]), and thus is the number of linear extensions of a poset with $\ell(w)$ elements. However, this poset need not be isomorphic to $(\Phi^\vee(w), <)$. For example, among the permutations of four objects, all except 3241 and 4132 are inversion-orderable, whereas 2143 is the only one that is not vexillary.

By Theorem 5.5 of [13], one knows that every dominant minuscule element is inversion-orderable (and fully commutative). At the opposite extreme, the longest elements in A_n [9] and B_n [4] are known to be inversion-orderable, and this and other data led Proctor to suggest (while these results were still conjectures) that the longest element of every parabolic quotient of a finite Weyl group should be inversion-orderable (see Section 7 of [9]). This conjecture turns out to be false in general, and recent computer searches show that $\sigma_{\tilde\alpha}$ is the longest inversion-orderable element in D_5, E_6, and F_4.

5. Inversions and the standard ordering

In this section we prove Lemma 4.5, thereby completing the proof of Theorem 4.6. Towards this goal, we assume to the contrary that there is a short positive root α for which the lemma fails. Among all such counterexamples, choose one that is minimal in the Cayley order.

Given such a counterexample, there exists some $w \leq_L \sigma_\alpha$ such that $\alpha^\vee \notin \Phi^\vee(w)$, and a pair $\beta^\vee < \gamma^\vee$ in $\Phi^\vee_\alpha$ such that $\gamma^\vee \in \Phi^\vee(w)$ and $\beta^\vee \notin \Phi^\vee(w)$. Among the available choices for $\gamma^\vee$, take one that is minimal in the standard ordering.

Claim 1 $\gamma^\vee - \beta^\vee \notin \Phi^\vee$. Otherwise, by Lemma 4.2, it would have to be the case that $\gamma^\vee - \beta^\vee \in \Phi^\vee(w) \subset \Phi^\vee_\alpha$. However $\langle \alpha, \gamma^\vee - \beta^\vee \rangle = 0$, so this contradicts Lemma 4.3.

It is well-known that the standard ordering of the positive roots is ranked by the height function. In the simply-laced case, this is part of Proposition 3.2. In general, this follows (for example) from the adjoint case of Proposition 2.4 in [11].

Thus we may select a simple root α_i so that $\gamma^\vee - \alpha_i^\vee$ is a co-root and $\gamma^\vee - \alpha_i^\vee > \beta^\vee$. The fact that the inequality must be strict is a consequence of Claim 1.

Claim 2 $\langle \alpha, \alpha_i^\vee \rangle = 1$ *and* $\alpha_i^\vee \in \Phi^\vee(w)$. By Lemma 4.2, either $\gamma^\vee - \alpha_i^\vee \in \Phi^\vee(w)$ or $\alpha_i^\vee \in \Phi^\vee(w)$. However the former cannot occur, or else we contradict the rule for choosing $\gamma^\vee$. In the latter case we also obtain $\alpha_i^\vee \in \Phi^\vee_\alpha$, so the claim follows.

Claim 3 $\langle \alpha_i, \beta^\vee \rangle \geq 0$ *and* $\beta \neq \alpha_i$. If $\langle \alpha_i, \beta^\vee \rangle < 0$ then $\beta^\vee + \alpha_i^\vee$ is a co-root, and the inner product of this co-root with α is 2, so $\beta^\vee + \alpha_i^\vee = \alpha^\vee$, by Lemma 3.1. However, this contradicts the fact that $\beta^\vee < \gamma^\vee < \alpha^\vee$. If $\beta = \alpha_i$, we contradict Claim 1.

Claim 4 $s_i\beta^\vee \not\leq s_i\gamma^\vee$. Since $\beta^\vee < \gamma^\vee < \alpha^\vee$ and $\beta \neq \alpha_i$, it follows that each of $s_i\beta$, $s_i\gamma$ and $\alpha' := s_i\alpha$ are positive roots distinct from α_i. Furthermore, since $\alpha_i^\vee \in \Phi^\vee(w)$ (Claim 2), we have $\Phi^\vee(ws_i) = s_i(\Phi^\vee(w) - \{\alpha_i^\vee\})$ and similarly

$$\Phi^\vee(\sigma_{\alpha'}) = \Phi^\vee(s_i\sigma_\alpha s_i) = s_i\big(\Phi^\vee(\sigma_\alpha) - \big\{\alpha_i^\vee, \alpha^\vee - \alpha_i^\vee\big\}\big).$$

We have $\alpha^\vee - \alpha_i^\vee \notin \Phi^\vee(w)$ (otherwise Claim 2 and Lemma 4.2 would contradict the fact that $\alpha^\vee \notin \Phi^\vee(w)$), so $\Phi^\vee(ws_i) \subset \Phi^\vee(\sigma_{\alpha'})$ and $ws_i \leq_L \sigma_{\alpha'}$ (Lemma 4.1). However, since $\alpha' <_C \alpha$, the minimality of α forces Lemma 4.5 to be valid for α'. In particular, $\Phi^\vee(ws_i)$ must be an order ideal of $\Phi^\vee_{\alpha'}$. Therefore, it cannot be the case that $s_i\beta^\vee \leq s_i\gamma^\vee$; otherwise we would have $s_i\beta^\vee \in \Phi^\vee(ws_i)$ and $\beta^\vee \in \Phi^\vee(w)$, a contradiction.

Claim 5 $\langle \alpha_i, \gamma^\vee \rangle = 2$ *and* $\langle \alpha_i, \beta^\vee \rangle = 0$. Let $p = \langle \alpha_i, \gamma^\vee \rangle$ and $q = \langle \alpha_i, \beta^\vee \rangle$. We know that $q \geq 0$ by Claim 3. Furthermore, we have $p > q + 1$, since otherwise

$$s_i\gamma^\vee - s_i\beta^\vee = \gamma^\vee - \beta^\vee - (p - q)\alpha_i^\vee \geq \gamma^\vee - \beta^\vee - \alpha_i^\vee \geq 0,$$

contrary to Claim 4. Bearing in mind that the largest possible Cartan integer in a finite crystallographic root system is 3, the only possibilities are $(p, q) = (3, 1)$, $(3, 0)$, or $(2, 0)$. In particular, γ is short, α_i is long, $\langle \gamma, \alpha_i^\vee \rangle = 1$, $\langle \alpha_i, \alpha^\vee \rangle = p$ (Claim 2), and $\langle \gamma, \alpha^\vee \rangle = 1$. It follows that if $p = 3$, then $\alpha^\vee + \gamma^\vee - 3\alpha_i^\vee$ is orthogonal to each of α, γ, and α_i. Since $\langle\, , \rangle$ is positive definite on the span of the simple roots when Φ is finite, this is possible only if $\alpha^\vee + \gamma^\vee = 3\alpha_i^\vee$. However this is an absurdity, since the support of two distinct positive co-roots must involve at least two simple co-roots.

Since $\gamma^\vee - \alpha_i^\vee > \beta^\vee$, there is a simple root α_j such that $\gamma^\vee - \alpha_i^\vee - \alpha_j^\vee$ is a co-root and

$$\gamma^\vee - \alpha_i^\vee - \alpha_j^\vee \geq \beta^\vee. \tag{5.1}$$

It cannot be the case that $i = j$. Otherwise, Claim 5 would imply

$$s_i\gamma^\vee = \gamma^\vee - 2\alpha_i^\vee \geq \beta^\vee = s_i\beta^\vee,$$

contrary to Claim 4.

Claim 6 $\langle \alpha_i, \alpha_j \rangle = 0$ *and* $\alpha_i^\vee + \alpha_j^\vee$ *is not a co-root*. Writing $\gamma^\vee - \beta^\vee = \sum c_k \alpha_k^\vee$, we know that c_k is a nonnegative integer for all k. Moreover, $c_i = 1$ (or else Claims 4 and 5 are contradictory) and $c_j \geq 1$, by (5.1). Since $\langle \alpha_i, \alpha_k \rangle \leq 0$ for all $k \neq i$ [5, 1.3], it follows that if $\langle \alpha_i, \alpha_j \rangle < 0$, then

$$\langle \alpha_i, \gamma^\vee - \beta^\vee \rangle = \sum c_k \langle \alpha_i, \alpha_k^\vee \rangle \leq 2 - c_j \leq 1,$$

contrary to Claim 5. The remaining possibility is that $\langle \alpha_i, \alpha_j \rangle = 0$. In that case, $\alpha_i^\vee + \alpha_j^\vee$ cannot be a co-root, since $s_j(\alpha_i^\vee + \alpha_j^\vee) = \alpha_i^\vee - \alpha_j^\vee$ would also be a co-root. This is an absurdity, since the expression is neither positive nor negative.

Claim 7 $\gamma^\vee - \alpha_j^\vee$ *is a co-root*. We must have $\langle \gamma, \gamma^\vee - \alpha_i^\vee - \alpha_j^\vee \rangle \leq 0$, or else it would follow that $\gamma^\vee - (\gamma^\vee - \alpha_i^\vee - \alpha_j^\vee)$ is a co-root, contrary to Claim 6. On the other hand, recalling that γ is short (see the proof of Claim 5), we have

$$\langle \gamma, \gamma^\vee - \alpha_i^\vee - \alpha_j^\vee \rangle = 1 - \langle \gamma, \alpha_j^\vee \rangle.$$

Hence $\langle \gamma, \alpha_j^\vee \rangle > 0$ and the claim follows.

Finally, bearing in mind that $\gamma^\vee - \alpha_j^\vee > \beta^\vee$, Claim 7 shows that all of the deductions pursuant to the choice of α_i apply equally well to α_j, *mutatis mutandis*. In particular, $\langle \alpha_j, \gamma^\vee \rangle = 2$ (Claim 5), from which it follows that $\gamma^\vee - \alpha_i^\vee - \alpha_j^\vee$ is orthogonal to each of γ, α_i and α_j. Hence $\gamma^\vee = \alpha_i^\vee + \alpha_j^\vee$, contrary to Claim 6, so the proof is complete.

Appendix

Here we provide the number of reduced expressions for the longest short reflection in each irreducible finite Weyl group, along with the number of reduced expressions for each agent and the structure of the smashed Cayley ordering. Here $J(P)$ denotes the lattice of order ideals of the poset P.

A_n

Diagram labeling: $12 \cdots n$
Structure of $(\Phi_s/\sim, <_C)$: $[n] \times [n]$
$r(\sigma_{\tilde{\alpha}}) = \binom{2n-2}{n-1}$; $r(x_i) = \binom{n-1}{i-1}$

B_n

Diagram labeling: $1\langle 23\cdots n$
Structure of $(\Phi_s/\sim, <_C)$: $[2n-1]$
$r(\sigma_{\tilde\alpha}) = r(x_1) = 1$

C_n

Diagram labeling: $1\rangle 23\cdots n$
Structure of $(\Phi_s/\sim, <_C)$: $J([2]\times[2n-3])$
$r(\sigma_{\tilde\alpha}) = \mathcal{C}_{2n-3}$; $r(x_i) = \dfrac{i-1}{n-1}\dbinom{2n-2}{n-i}$ $(i \geq 2)$

D_n

Diagram labeling: $\overset{\;2}{134}\cdots n$
Structure of $(\Phi_s/\sim, <_C)$: see figure 3
$r(\sigma_{\tilde\alpha}) = 4\mathcal{C}_{2n-4} - 2\mathcal{C}_{n-2}^2$; $r(x_i) = \dfrac{4i-6}{2n-3}\dbinom{2n-3}{n-i}$ $(i \geq 3)$; $r(x_1) = r(x_2) = \mathcal{C}_{n-2}$

E_6

Diagram labeling: $13\overset{2}{4}56$
Structure of $(\Phi_s/\sim, <_C)$: $J^2([3]\times[3])$
$r(\sigma_{\tilde\alpha}) = 41526$; $(r(x_1),\ldots,r(x_6)) = (12,\ 42,\ 75,\ 168,\ 75,\ 12)$

E_7

Diagram labeling: $13\overset{2}{4}567$
Structure of $(\Phi_s/\sim, <_C)$: $J^3([2]\times[4])$
$r(\sigma_{\tilde\alpha}) = 38491408$; $(r(x_1),\ldots,r(x_7)) = (286,\ 1176,\ 2002,\ 4992,\ 2750,\ 728,\ 78)$

E_8

Diagram labeling: $13\overset{2}{4}5678$
Structure of $(\Phi_s/\sim, <_C)$: $J^5([2]\times[3])$
$r(\sigma_{\tilde\alpha}) = 34331977926000$; $(r(x_1),\ldots,r(x_8)) =$
$(233220,\ 1046520,\ 1739010,\ 4601610,\ 2838660,\ 946680,\ 170430,\ 13110)$

F_4

Diagram labeling: $12\langle 34$
Structure of $(\Phi_s/\sim, <_C)$: see figure 3
$r(\sigma_{\tilde\alpha}) = 29$; $(r(x_1), r(x_2)) = (2, 5)$

G_2

Diagram labeling: $1\langle 2$
Structure of $(\Phi_s/\sim, <_C)$: $[5]$
$r(\sigma_{\tilde\alpha}) = r(x_1) = 1$

Notes

1. In the simply-laced case, the representation involved is the adjoint representation.
2. A Weyl group quotient W/W' is minuscule if there is a representation of a Lie algebra with Weyl group W whose weights form a single W-orbit, with W' being the stabilizer of the highest weight.

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